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A Formalism for the Systematic Treatment of Rapidity Logarithms in Quantum Field Theory

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Jui-yu Chiu,[a] **Ambar Jain,**[a] **Duff Neill,**[a] **Ira Z. Rothstein**[a,b]

[a] *Department of Physics, Carnegie Mellon University, Pittsburgh, PA 15213, U.S.A.*
[b] *California Institute of Technology, Pasadena CA, 91125, U.S.A.*

E-mail: `jychiu@andrew.cmu.edu`, `ambar@andrew.cmu.edu`, `dneill@andrew.cmu.edu`, `izr@andrew.cmu.edu`

Abstract:

Many observables in QCD rely upon the resummation of perturbation theory to retain predictive power. Resummation follows after one factorizes the cross section into the relevant modes. The class of observables which are sensitive to soft recoil effects are particularly challenging to factorize and resum since they involve rapidity logarithms. Such observables include: transverse momentum distributions at p_T much less then the high energy scattering scale, jet broadening, exclusive hadroproduction and decay, as well as the Sudakov form factor. In this paper we will present a formalism which allows one to factorize and resum the perturbative series for such observables in a systematic fashion through the notion of a “rapidity renormalization group”. That is, a Collin-Soper like equation is realized as a renormalization group equation, but has a more universal applicability to observables beyond the traditional transverse momentum dependent parton distribution functions (TMDPDFs) and the Sudakov form factor. This formalism has the feature that it allows one to track the (non-standard) scheme dependence which is inherent in any scenario where one performs a resummation of rapidity divergences. We present a pedagogical introduction to the formalism by applying it to the well-known massive Sudakov form factor. The formalism is then used to study observables of current interest. A factorization theorem for the transverse momentum distribution of Higgs production is presented along with the result for the resummed cross section at NLL. Our formalism allows one to define gauge invariant TMDPDFs which are independent of both the hard scattering amplitude and the soft function, i.e. they are universal. We present details of the factorization and resummation of the jet broadening cross section including a renormalization in $p_\perp$ space. We furthermore show how to regulate and renormalize exclusive processes which are plagued by endpoint singularities in such a way as to allow for a consistent resummation.

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1 Introduction and Motivation

Many observables in high energy collisions suffer from poorly behaved perturbative expansions due to the existence of large logarithms. Logarithms of fixed scales, such as masses, are easily handled by standard renormalization group procedures within the confines of effective field theories. However, when one is interested in less inclusive observables, it is often the case that large logarithms of kinematic factors can arise. In particular, when one is interested in studying corners of phase space large hierarchies can induce large logarithms. In such cases resummations allow us to maintain control over theoretical errors [1].

Within the vast class of observables which require resummation there is a large sub-class which are technically more challenging to handle than others. These observables correspond generically to those for which the recoil of a collinear jet due to soft radiation is an order one effect. We will use the acronym SRSO for Soft Recoil Sensitive Observables. A classic example of an SRSO is jet broadening [2, 3]. Another set of SRSO's are transverse momentum distributions when $p_T/Q \ll 1$, where[1] logarithms of the ratio p_T/Q can invalidate fixed order results.

In this paper we will be employing effective field theory (EFT) techniques to factorize and resum large logarithms in these SRSO's. Traditional EFT's allow for resummation in a systematic fashion by morphing these large logarithms into logarithms associated with UV divergences and then applying standard renormalization group techniques. However, such traditional methods where one sums logs of invariant mass scales, are insufficient for SRSO's. Indeed, as we will see below for such observables, not all the large logarithms are associated with UV divergences in the effective theory.

Standard SCET methods breakdown when the modal decomposition in the EFT involves multiple fields with the same invariant mass scalings (such theories fall under the rubric of what is known as $\mathrm{SCET_{II}}$ [4]). These cases exactly correspond to SRSO's since the soft radiation, in light-cone coordinates, has momentum scaling $(\lambda, \lambda, \lambda)$, where λ is the small power counting parameter, while the collinears scale as $(1, \lambda^2, \lambda)$. For such observables one can run into a new type of divergence which is not associated with singular behavior in the UV or IR, but with limits of large rapidities, as was elucidated in [5]. These "rapidity divergences" have been studied extensively outside the realm of EFT, albeit perhaps with differing nomenclature, especially within the context of transverse momentum distribution functions and Sudakov form factor [6–8].

The purpose of this paper is to present a renormalization group program for the treatment of rapidity logarithms. In particular, we show how one can regulate and renormalize rapidity divergences, and then using a renormalization group technique resum the associated logarithms, all in a systematic fashion within SCET. To facilitate this procedure we introduce a regulator that necessarily breaks boost invariance in order to distinguish between modes which share a mass shell hyperbola. The regulator preserves eikonal exponentiation and manifest gauge invariance in each sector. However, our formalism is also applicable using other regulators which break boost invariance, as will be discussed below. Once one sums over the soft and collinear sectors, the rapidity divergences cancel at each order in perturbation theory. The regulator introduces a new scale, ν, which leads to a rapidity renormalization group (RRG) flow. The solution to the ensuing differential equation has the effect of summing the large rapidity logarithms which existed in the full theory calculation.

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[1] We will always assume that $p_T \gg \Lambda_{QCD}$, so we are away from the "forward region".

Outline of this paper

We begin in (2) by defining the notion of rapidity logarithms (divergences) and present the necessary and sufficient criteria for their existence. In (3) we present a physical argument which allows us to isolate the type of generic observable where one expects them to arise. In (4) we demonstrate how to regulate the rapidity divergences in context of the massive Sudakov form factor with massless external lines. This section also illustrates how one can resum the rapidity logarithms using the rapidity renormalization group (RRG). We then show how the regulator is applied to generalized soft and collinear jet functions. In (5) we apply the formalism to the transverse momentum distribution in Higgs production at small p_T, resumming logarithms of $p_\perp/m_h$ to next to leading log order. This section includes the definition of a gauge invariant transverse momentum dependent parton distribution function (TMDPDF). At the end of this section we compare our results with previous works on the subject. Section (6) gives a factorization theorem for jet broadening including a next to leading log result for the resummed cross section. Since we have attempted to make this section available to readers not interested in the Higgs production section there is some formal overlap with the previous section. Finally in (7) we show how our formalism can be utilized to renormalize end-point divergences in exclusive decays. We conclude with a summary.

2 Rapidity Divergences

We define a rapidity divergence as arising from momentum region where the invariant mass k^2 is held fixed but the ratio k_+/k_- (or k_-/k_+) diverges, where $k_\pm$ are light-cone momenta. Rapidity divergences are not IR in origin, as they do not show up in the full theory, nor should they be thought of as UV divergences since they can arise from either the upper or lower limit of an integral, as will be shown later. The existence of the divergence stems from the fact that to preserve manifest power counting the EFT must be multipole expanded [9] which leads to eikonal propagators. Furthermore, while rapidity divergences arise in factorized IR sectors of the theory, i.e. collinear or soft, the sum of EFT sectors will have no rapidity divergences. Which is to say that rapidity divergences arise as an artifact of factorization. However, in order to resum logarithms in attempting to save perturbation theory, factorization, and thus rapidity divergences are inevitable for SRSO's. In this sense, they are entirely analogous to the traditional UV divergences which arise in factorization and are necessary for resummations.

To demonstrate how rapidity divergences arise, let us consider an integral of the following form

$$I = \int_{\mu_L}^{Q} \frac{dk_+}{k_+}, \tag{2.1}$$

which may arise when transverse momentum is measured in the real radiation. Q is the scale of hard scattering and μ_L is the relevant low energy scale. Suppose that the n-collinear and soft modes have k_+/Q or order 1 and $\lambda \sim \mu_L/Q$, respectively. Then this integral ranges over both mode regions. To factorize the integral into rapidity regions we introduce a set of cut-offs

$$I = \int_{\mu_L}^{\Lambda} \frac{dk_+}{k_+} + \int_{\Lambda}^{Q} \frac{dk_+}{k_+}, \tag{2.2}$$

corresponding to soft and collinear contributions respectively. In the effective theory, the cut-off should not be finite, to preserve power counting. Or put differently, each sector should depend only upon one relevant scale, which follows after performing the multipole expansion. Taking the limit, $Q >> \Lambda >> \mu_L$, the EFT result reads

$$I = \int_{\mu_L}^{\infty} \frac{dk_+}{k_+} + \int_{0}^{Q} \frac{dk_+}{k_+}, \tag{2.3}$$

We see that we generate a set of divergences which only cancel in the sum of the sectors. It should also be emphasized here that these divergences are not regulated by dimensional regularization, a necessary but not sufficient criteria for rapidity divergences.

Obviously not all observables in QCD will generate rapidity divergences in SCET. A necessary condition for their appearance is that the observable under consideration receive contributions from modes with parametrically distinct rigidities, but whose invariant mass are of the same order. The prototypical observables of this type are transverse momentum distributions. When $p_T \ll Q$, p_T plays the role of μ_L in the example above and rapidity logarithms arise which must be resummed. As such, we will consider in this paper two observables, namely Higgs production at small transverse momentum, and Jet broadening. Rapidity divergences also occur in exclusive processes. In [5] the authors show that the end point singularity problem arises as a consequence of rapidity divergences. We apply our formalism to this issue and show how one can systematically renormalize the divergences in these exclusive decays.

3 What Theories Give Rise to Rapidity Divergences: SCET$_{\rm I}$ vs. SCET$_{\rm II}$

SCET [10–12] (Soft Collinear Effective Theory) is a formalism designed to separate scales in high energy scattering processes for which the hard scattering scale (Q^2) is much greater then the scale of hadronic physics. Here we will not review SCET but only illuminate the points that are germane to the main thrust of this paper. In particular we are interested in the critical issue of (not) double counting regions of phase space. That is, how does one cleanly separate (factorize) the modes which give rise to IR singularities.

As in any well defined EFT, the scale separation is made manifest at the level of the action which systematizes the power counting. Power corrections can be included by adding operators which have definite scalings in powers of μ_L/Q, where μ_L is some low scale of interest. SCET, like its cousin NRQCD, is a “modal” theory whereby fields are decomposed into a set of sub-fields each of which has momenta with definite scalings. For instance, in SCET$_{\rm I}$ the gluon field is written as

$$A_\mu = A_\mu^{c,n} + A_\mu^{c,\bar{n}} + A_\mu^{US} + ... \tag{3.1}$$

where $A_\mu^{c,(n,\bar{n})}$ are collinear fields whose momenta scale as $Q(1,\lambda^2,\lambda)$ and $Q(\lambda^2,1,\lambda)$ respectively. While A_μ^{US} is an ultra-soft (US) field whose momentum scales as $Q(\lambda^2,\lambda^2,\lambda^2)$, where the power counting parameter is $\lambda \equiv \mu_L/Q$ and μ_L is the relevant low energy scale[2]. To cleanly separate the scales it is helpful to use a dynamical label formalism [14] also used in SCET [11]. We re-write the full QCD field as

$$\psi(x) = \sum_{n\cdot p, p_\perp} e^{-in\cdot p\,\bar{n}\cdot x + ip_\perp\cdot x_\perp}\xi_{n\cdot p,p_\perp}(x)\,. \tag{3.2}$$

The purpose of this rescaling is to insure that all derivatives acting on ξ scale as λ^2. Note that (3.2) is written as a sum not an integral. One tessellates the space of possible large momenta into bins whose dimensions scale with the size of the residual momentum of order λ^2 .

The Lagrangian interactions can change the large momentum components of fields. In particular, collinear gluons can split, changing their large light cone momentum. This implies that there are loops in which one must sum over labels. It is then natural to ask what happens in the label sum [3] when one of the labels becomes parametrically small[4] and modes begin to overlap. The existence of these overlap regions, when not treated properly, obscures the physics underlying the effective theory calculations. This is perhaps simplest to see in NRQCD, where the existence of the overlaps leads to pinch singularities, as well as the inability to clearly distinguish between IR and UV divergences. These points were made clear in a seminal paper by Manohar and Stewart [5]. In this paper the authors show that one may exclude these overlap regions by taking a diagram involving a particular mode, Taylor expanding it around the region of the complementary mode one is trying to exclude, and subtracting this contribution from the original diagram. Doing so eliminates the aforementioned problems. It has been shown that in $\mathrm{SCET_I}$ this "zero bin" subtraction [16–18] is *often* equivalent to dividing by matrix element of Wilson lines in non-SCET perturbative QCD factorization formulae [19], which can often be identified with the inverse of the soft-function in the factorization theorem. In this paper we will also have occasion to comment on the role of the what we term the "soft-bin" subtractions, and draw a distinction between the soft-bin and zero-bin, where by zero-bin we always mean an ultra-soft scaling (see appendix (B)).

In $\mathrm{SCET_I}$ all divergences can be regulated using dimensional regularization and/or offshellness, and all modes have distinct virtualities (collinear modes in differing directions being the exception [5]). Zero bin subtractions are relatively simple to utilize to insure that there is no double counting. Thus we should expect that rapidity divergences should not be an issue in $\mathrm{SCET_I}$, and indeed this is in fact the case. Note this is not to say that integrals of the form of (2.2) will not arise in $\mathrm{SCET_I}$. In fact they are ubiquitous, however, they should not be interpreted as rapidity divergences, as they will cancel within each sector. That is to say, there can, and will be such divergences in the collinear (or US) sector, but when one sums over graphs, including zero bins, these divergences will cancel as they must *within each sector.* A classic example arises in the one loop correction to the parton distribution function (PDF). In this case in calculating a real correction one encounters an integral of the form

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[2] The ... represent non-linear terms in the fields that are needed to insure that gauge transformations do not mix orders in the power counting[13].

[3] The fact that these are sums and not integrals is a consequence of separating momenta into labels and residual momenta. This can be thought of as a grid in which labels give the coordinates of a box (or "bin") whose size is of order of the residual momentum. One can combine the sum over bins with the residual momenta integrals when performing loop calculations. See [15] for a discussion.

[4] In $\mathrm{SCET_{II}}$ labels between modes can also overlap by becoming large.

[5] The soft zero bin will eliminate any such overlap. Given the non-existence of a soft mode in $\mathrm{SCET_I}$ such an overlap is necessarily absent.

$$\int \frac{dz}{1-z}, \tag{3.3}$$

where z is the momentum fraction carried by the struck parton. The integral diverges at the upper end point of integration, where the incoming parton and and the struck parton carry the same momentum. To regulate this integral one must introduce a new regulator since dimensional regularization is insufficient. Nonetheless the divergence arises when the emitted gluon goes ultrasoft, and thus by the Kinoshita, Lee and Nauenberg (KLN) theorem we expect it to cancel with the corresponding virtual diagram. Thus the divergence cancels within the collinear sector itself. Note that in this case the zero bin did not play a role, since they actually cancel in this calculation, again by the KLN theorem. An illustrative example of the cancellation of divergences, which one might have thought were rapidity divergences in $\mathrm{SCET_I}$ can be found in [20]. In summary, $\mathrm{SCET_I}$ rapidity divergences do not pose a problem because there is no issue in distinguishing modes of identical virtuality.

For certain observables, such as transverse momentum distributions with $p_\perp \ll Q$, $\mathrm{SCET_I}$ is not the proper effective theory. The reason is that we must account for real radiation with momenta that scale as $(p_\perp, p_\perp, p_\perp)$. Given that $p_\perp$ is the IR scale of the theory the collinear modes scale as either $(Q, \frac{p_\perp^2}{Q}, p_\perp)$ or $(\frac{p_\perp^2}{Q}, Q, p_\perp)$, and both the soft and collinear mode have the same invariant mass. As such, these modes can be interchanged by a boosts and the only real distinction between them is in their relative rapidities. In such cases, the EFT is called $\mathrm{SCET_{II}}$. The need for $\mathrm{SCET_{II}}$ was first pointed out in the context of exclusive B decays [4] where it was necessary to introduce a second effective theory below $\mathrm{SCET_I}$. The equality of invariant masses of the modes in $\mathrm{SCET_{II}}$ leads to complications in the factorization of physical observables since one must break the boost invariance of operators to cleanly distinguish between sectors. As was first pointed out in [21], the process of factorization in $\mathrm{SCET_{II}}$ can lead to additional divergences in sectors that can not be regulated by dimensional regularization, or off-shellness. These divergences will not cancel within each sector as they do in $\mathrm{SCET_I}$. They will cancel only when we sum over sectors, but the lack of cancellation within a sector changes the RG structure of theory. In fact it is the lack of cancellation that allows for the resummation of large rapidity logarithms .

Boost Invariance in $\mathrm{SCET_I}$ and $\mathrm{SCET_{II}}$

In SCET we make a convenient choice of frames where relevant modes are (ultra)soft or collinear, breaking the full Lorentz symmetry of QCD. However, SCET is invariant under boosts along the light-cone direction. This residual boost symmetry of SCET is called $\mathrm{RPI}_{\mathrm{III}}$ [22]. For problems relevant to $\mathrm{SCET}_{\mathrm{I}}$ this symmetry is preserved in each matrix element belonging to the collinear or ultrasoft sectors. The natural distinction between the collinear and ultrasoft sectors of $\mathrm{SCET}_{\mathrm{I}}$ comes from their parametrically differing invariant masses[6]. Since dimensional regularization breaks dilatation symmetry, it is sufficient to distinguish between collinear and ultrasoft modes[7]. Since dimensional regularization preserves boost invariance and is the only regularization required to separate modes in $\mathrm{SCET}_{\mathrm{I}}$, each sector individually remains $\mathrm{RPI}_{\mathrm{III}}$ invariant.

In $\mathrm{SCET}_{\mathrm{II}}$ there is no distinction between the invariant masses of the soft and collinear modes and they can be interchanged via boosts. Dimensional regularization does not distinguish between these modes, so we should introduce a regulator which accomplishes this goal by breaking the boost invariance along the light-cone direction. Due to the soft recoil, "jet functions" describing collinear radiation are not exactly aligned with the preferred light-cone direction used in factorization, in that they carry transverse momentum *w.r.t.* the light-cone direction[8]. A jet function will usually depend upon the transverse momentum and the large light cone component of the momentum $Q_\pm$ carried by all the collinear particles constituting the jet. While the transverse momentum is boost invariant, $Q_\pm$ is not and hence one shouldn't expect the jet function describing collinear radiation in a physical process to be boost invariant. In contrast, the jets in $\mathrm{SCET}_{\mathrm{I}}$ are aligned with the preferred light-cone direction since ultrasofts cannot recoil jets in transverse momentum.

In a back-to-back jet scenario for SRSO's, like the jet broadening event shape, a small boost will reduce the number of particles in one jet and increase the number of particles in the other jet while keeping the number density unchanged in the soft region, when averaged over all events. Thus it is expected that in a renormalized factorized cross-section boost invariance will be broken in each sector and will only be restored when all sectors are added. In problems pertaining to $\mathrm{SCET}_{\mathrm{I}}$, boosts alone cannot interchange ultrasoft and collinear excitations and hence $\mathrm{RPI}_{\mathrm{III}}$ must be preserved in each sector. We see that in order to factorize one must distinguish between soft and collinear radiation, and $\mathrm{RPI}_{\mathrm{III}}$ must be broken via regularization of $\mathrm{SCET}_{\mathrm{II}}$ matrix elements. This point was emphasized in [24].

4 Rapidity Divergences in $\mathrm{SCET}_{\mathrm{II}}$

To understand the nature of rapidity divergences we consider how they arise in the effective theory. Thus we will begin by considering perhaps the simplest case where such divergences arise, the Sudakov form factor. In particular we will renormalize and resum the logarithms in the on-shell, massive gauge boson, space-like Sudakov form factor [19] [9] [10]. As opposed to the massless form factor, which in isolation, is unphysical, the massive case is IR safe and in principle observable. This form factor is also relevant to for summing large electro-weak corrections at energy far above the gauge boson masses.

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[6] Distinction between two collinear sectors arise essentially from different light-cone directions, so they need not have different invariant mass for distinction.

[7] An ultrasoft mode can be transformed into a collinear mode by a dilatation and a boost.

[8] For example, jet functions in the jet broadening factorization theorem [23] carry non-zero transverse momentum *w.r.t.* the thrust axis.

We defined our power counting parameter as $\lambda \equiv M/Q$. The factorization formula is composed of decomposition in terms of modes which can contribute to the non-analytic structure of the matrix element. The relevant modes for this observable are the soft with light-cone momenta scaling as $(\lambda, \lambda, \lambda)$, and collinear and anti-collinear with momenta scaling as $(1, \lambda^2, \lambda)$ and $(\lambda^2, 1, \lambda)$ respectively. There are no ultra-soft contributions since their momentum $(\lambda^2, \lambda^2, \lambda^2)$ will decouple from all the other lines. Given these modes, this defines an $\mathrm{SCET_{II}}$ process. The jets will recoil against the soft virtual emissions and hence, despite the exclusive nature of this process, it is still an SRSO. However, in the Breit frame the net transverse momentum exchanged between the jets must vanish.

The factorization of the massless Sudakov form factor in SCET was performed in [26][11]. At leading power we have

$$\begin{aligned} J_\mu \equiv \bar{u}(p_n)\gamma_\mu^\perp u(p_{\bar{n}})\, F(Q^2, M^2) &\approx \langle p_n|\bar{\xi}_n W_n S_n^\dagger \gamma_\mu^\perp C(n\cdot\mathcal{P}, \bar{n}\cdot\mathcal{P}) S_{\bar{n}} W_{\bar{n}}^\dagger \xi_{\bar{n}}|p_{\bar{n}}\rangle \\ &\equiv H(Q^2,\mu) J_n(M;\mu,\nu/Q)\gamma_\mu^\perp J_{\bar{n}}(M;\mu,\nu/Q) S(M;\mu,\nu/M)\,. \end{aligned} \tag{4.1}$$

In the last line, we have factorized the form factor F in terms of $\mathrm{SCET_{II}}$ matrix elements $J_n, J_{\bar{n}}, S$ which are defined as

$$\begin{aligned} S(M;\mu,\nu/M) &= \langle 0 \mid S^\dagger S \mid 0\rangle \\ J_n(M;\mu,\nu/Q) &= \langle p_n|\bar{\xi}_n W_n|0\rangle \\ J_{\bar{n}}(M;\mu,\nu/Q) &= \langle 0|\bar{W}_{\bar{n}}^\dagger \xi_{\bar{n}} \mid p_{\bar{n}}\rangle \end{aligned} \tag{4.2}$$

S and W are Wilson lines composed of soft and collinear lines respectively.

Finally we must mention the cumbersome issue of the Glauber modes. These modes, which scale as $(\lambda^2, \lambda^2, \lambda)$ will in general contribute IR singular pieces at the level of the amplitude. While they have been shown to cancel in certain processes outside the realm of effective field theory [27, 28], a systematic treatment of such modes [12] within a self-consistent EFT treatment is still lacking. Here we will assume, as do all SCET treatments, that Glaubers will not contribute.

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[9] The logarithms in the Sudakov form factor are distinguished from "Sudakov logarithms" which can arise in running currents in that they contain rapidity divergences.

[10] The Sudakov form factor can be regulated by going off-shell in which case the systematics may change [25].

[11] This factorization was for the massless case where gauge invariance uniquely fixes the form of the factorization theorem. However, the gauge boson mass does not alter the result. Note that this result was formal in that the IR scales were not clearly delineated. As such, it was not sharply defined to be living in $\mathrm{SCET_I}$ or $\mathrm{SCET_{II}}$and the factorization formula contained both soft and ultra-soft Wilson lines.

[12] For a discussion of these modes within the context of SCET see [26, 29, 30].