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The Sound of Topology in the AdS/CFT Correspondence

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Abstract: Using the gauge/gravity correspondence, we study the properties of 2-point correlation functions of finite-temperature strongly coupled gauge field theories, defined on a curved space of general spatial topology with a dual black hole description. We derive approximate asymptotic expressions for the correlation functions and their poles, supported by exact numerical calculations, and study their dependence on the dimension of spacetime and the spatial topology. The asymptotic structure of the correlation functions depends on the relation between the spatial curvature and the temperature, and is noticeable when they are of the same order. In the case of a hyperbolic topology, a specific temperature is identified for which exact analytical solutions exist for all types of perturbations. The asymptotic structure of the correlation functions poles is found to behave in a non-smooth manner when approaching this temperature.

Keywords: AdS-CFT Correspondence, Black Holes

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1 Introduction

The aim of this paper is to analyze the properties of 2-point correlation functions of finite-temperature strongly coupled (deconfined) gauge field theories defined on spacetimes with general spatial topology, and admit a dual black hole description via the AdS/CFT correspondence. Of specific interest is the dependence of these properties on the topology of the space on which the gauge theory is defined, and its temperature. We will see that the topology is indeed encoded in the properties of the correlation function and its poles, corresponding to quasinormal modes in the dual bulk spacetime: the sound of topology. In particular, the asymptotic "slope" of the poles encodes information about the spatial topology and the temperature, but is "universal" otherwise, i.e. it doesn't depend on the type of operators considered.

One motivation for this work is the fact that important aspects of the strong coupling dynamics of gauge field theories can depend crucially on the topology of space-time. For instance, the confinement-deconfinement phase transition is a property of strongly coupled thermal conformal gauge theories defined on a space-time with spherical spatial topology. The theory is always in the deconfined phase in the planar and hyperbolic cases. Thus, one may hope that insights to the confinement-deconfinement phase transition mechanism can be gained by analyzing the strong coupling dynamics on different spatial topologies. This can be done by using the gravitational dual description. A second motivation is the recently studied correspondence between gravitational dynamics and strongly coupled condensed matter systems. One can naturally envision in and out of equilibrium condensed matter systems realized on different spatial topologies, where our study can be of much relevance. A third motivation, perhaps a more remote one, is the relevance of the interplay between topology and field theory dynamics to astrophysical/cosmological setups.

We distinguish three classes of spatial topology [1]: The spherical, the flat and the hyperbolic (The relevant thermodynamic quantities and their dependence on the topology are given in Appendix B). Consider a gauge theory defined on a $(d-1)$-dimensional spacetime with the topology of $\mathbb{R} \times \Omega^{FT}_{d-2}$, where Ω^{FT}_{d-2} is a constant curvature $(d-2)$-dimensional Riemannian manifold of arbitrary topology[2]. The two basic scales that characterize the theory are the temperature T and the scalar curvature of the Ω_{d-2}^{FT} manifold R_{Ω}^{FT}. In the analysis we will consider two different regimes:

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[1] In the spherical case, a Hawking-Page phase transition exists, which is interpreted in terms of the AdS/CFT correspondence as a transition between a higher temperature deconfined phase and a lower temperature confined phase (see [2], [3]). Phase transitions might exist also for the flat and hyperbolic cases, e.g. the transition between the black hole solution at a higher temperature and the Horowitz-Myers "AdS soliton" at a lower temperature (see [7]).

[2] We assume that $d \geq 4$.

1. The regime of $\frac{\sqrt{|R_{\Omega}^{FT}|}}{T} \to 0$: In this case the curvature of the Ω_{d-2}^{FT} manifold may essentially be neglected, so we expect all physical quantities to behave as they do in the case of a flat ($R_{\Omega}^{FT} = 0$) space. Therefore, in this limit one does not distinguish between different topologies (aside from the known changes of the Laplace operator spectra between different topologies). This limit includes the hydrodynamic limit[3]: $\frac{\omega}{T}, \frac{L_s^{FT}}{T} \to 0$, where ω denotes the frequency and L_s^{FT} is the spatial Laplace operator eigenvalue.

2. The regime of $\frac{\sqrt{|R_{\Omega}^{FT}|}}{T} =$ fixed: In this case it will be possible to distinguish between the different topologies. In particular we will study the limit of $\frac{L_s^{FT}}{T}$ = fixed and $\frac{\omega}{T} \gg 1$, where analytical asymptotic expressions will be derived for the quasinormal modes and the 2-point correlation functions.

Note that the $\frac{\sqrt{|R_{\Omega}^{FT}|}}{T} \gg 1$ regime does not exist in the black hole description in the case of spherical topology (since the temperature has a finite minimal value) - it corresponds to a different (confining) phase on the field theory side. Nor does it exist in the case of flat topology (for which $R_{\Omega}^{FT} = 0$). While this regime does exist for the hyperbolic case, there is nothing separating it qualitatively from the $\frac{\sqrt{|R_{\Omega}^{FT}|}}{T} =$ fixed regime in the context of this paper. For these reasons, we won't consider this regime separately here.

The main new results of this work are (see Appendix A for a list of notations and definitions):

- In the asymptotic limit, where $\frac{L_s^{FT}}{T} \sim$ fixed and $\frac{\omega}{T} \gg 1$, the following holds:
  - For all types of topologies and all the temperatures , the quasinormal frequencies (and therefore the poles of the field theory correlators) are of the form

    $$\lambda_n = \lambda_0 + n\Delta\lambda \ , \tag{1.1}$$

    where $\Delta\lambda$ depends on the topology and temperature.
  - In this limit we derive asymptotic expressions for the field theory 2-point retarded correlation functions of the operators dual to the different perturbation types. These are given in Equations 3.58 and 3.71 for the scalar perturbation modes, in Equations 3.74 and 3.77 for the longitudinal vector perturbation modes, and in Equations 3.81 and 3.84 for the transverse vector perturbation modes.

- In the hyperbolic case there is a specific temperature T_c, where the bulk "black hole" solution has no singularity, and is isometric to AdS. In this special case the following holds:

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[3] The hydrodynamic condition $\frac{L_s^{FT}}{T} \to 0$ requires that $\frac{\sqrt{|R_{\Omega}^{FT}|}}{T} \to 0$ since the minimal eigenvalue $L_{s,min}^{FT}$ is of the same order as $\sqrt{|R_{\Omega}^{FT}|}$.

- At $T = T_c$ we derive exact expressions for the field theory 2-point retarded correlation functions of the operators dual to the different perturbation types. These are given in Equations 3.23 and 3.28 for the scalar perturbation modes, in Equations 3.30 and 3.31 for the longitudinal vector perturbation modes and in Equations 3.33 and 3.34 for the transverse vector perturbation modes.
- For $T > T_c$, the asymptotic $\Delta\lambda$ has a real component $\mathrm{Re}(\Delta\lambda) > 0$. As $T \to T_c$, $\Delta\lambda$ becomes imaginary according to Equation 2.71 for $d = 4$ (as previously calculated in [21]), and according to Equation 2.73 for $d > 4$.

The paper is organized as follows. In Section 2, we will review some known results regarding quasinormal modes (formulated in our notation). We will calculate the QNM equations for massless scalar and vector perturbations, the exact solutions for the "special" case mentioned above (the hyperbolic case with the specific temperature T_c), the approximate quasinormal frequencies in the asymptotic limit ($\frac{L_s^{FT}}{T}$ = fixed and $\frac{\omega}{T} \gg 1$) and the diffusion mode in the hydrodynamic limit. We will discuss the dependence of the asymptotic quasinormal modes on the topology and the temperature, in particular in the hyperbolic case around the temperature T_c, and demonstrate it using some numerical results. In Section 3, we will use AdS/CFT dictionary to calculate the 2-point correlation functions of the dual field theory. We will obtain exact expressions for the hyperbolic case with $T = T_c$ and approximate expressions in the asymptotic limit for general topology and temperature. We will compare the asymptotic expressions with exact numerical solutions. Section 4 is devoted to a discussion. Details of calculations are outlined in the appendices.

2 Quasi-Normal Modes

2.1 QNM Definition

Quasinormal modes of black holes or black branes are defined as the late-time oscillation modes of the black hole metric and the fields coupled to it, satisfying certain boundary conditions. Put differently, these are the eigenmodes of the linearized equations of motion over the black hole background. The boundary conditions are specified at the black hole horizon and at spatial infinity. The boundary condition at the horizon is chosen so that the solution corresponds to a wave ingoing into the horizon, while the boundary condition choice at spatial infinity depends on the asymptotic nature of the spacetime.

Choosing the incoming wave solution gives the boundary condition:

$$\psi|_{z=0} \sim z^{-\frac{i\omega}{C}} \ , \tag{2.1}$$

where $z = 0$ corresponds to the BH horizon and $z = 1$ corresponds to spatial infinity. As for the boundary condition at spatial infinity, different boundary conditions have been investigated for different asymptotic spacetime geometries. In the context of AdS/CFT, the relevant boundary condition is the one that corresponds to the poles of the retarded correlator of the CFT operators dual to the investigated field. For the fields discussed here, this condition amounts to the Dirichlet boundary condition (see [13], [17] and [22]):

$$\psi|_{z=1} = 0 \ . \tag{2.2}$$

2.2 QNM Equations

We consider the d-dimensional black hole metric given by a metric of the form:

$$\mathrm{d}s^2_{bulk} = -f(r)\,\mathrm{d}t^2 + \frac{1}{f(r)}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{d-2}\ , \tag{2.3}$$

where $\mathrm{d}\Omega^2_{d-2} = \sum_{i,j}(g_\Omega)_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ is the inner metric of a constant curvature, (d-2)-dimensions Riemannian manifold with of same topology as the Ω^{FT}_{d-2} manifold.

From the vacuum Einstein equation with a cosmological constant one can deduce the form of $f(r)$ to be

$$f(r) = k + \frac{r^2}{R^2} - \frac{r_0^{d-3}}{r^{d-3}}\ , \tag{2.4}$$

where R is related to the cosmological constant by $R^2 = -\frac{(d-2)(d-1)}{2\Lambda}$ and k is related to the scalar curvature R_Ω of the manifold Ω_{d-2} by (see, for example, [18]) $k = \frac{R_\Omega}{(d-2)(d-3)}$. In terms of these coordinates, the horizon of the black hole is located at $r = r_+$ where $f(r_+) = 0$, while the boundary is located at $r \to \infty$.

After making the transformation to the coordinate $z = 1 - \frac{r_+}{r}$, the metric takes the form:

$$\mathrm{d}s^2 = -\frac{\rho^2\tilde{g}(z)}{(1-z)^2}\,\mathrm{d}t^2 + \frac{r_+^2}{\rho^2\tilde{g}(z)(1-z)^2}\,\mathrm{d}z^2 + \frac{r_+^2}{(1-z)^2}\,\mathrm{d}\Omega^2_{d-2}\ , \tag{2.5}$$

where $\rho \equiv \frac{r_+}{R}$, $K \equiv \frac{k}{\rho^2}$ and

$$\tilde{g}(z) \equiv 1 + K(1-z)^2 - (1+K)(1-z)^{d-1}\ . \tag{2.6}$$

In terms of the new coordinate z, the horizon is located at $z = 0$ while the boundary is located at $z = 1$.

Let us assume that the spectrum of the scalar and vector Laplace operator Δ_Ω[4] defined on the manifold Ω_{d-2} is given by

$$\Delta_\Omega H_{L^2}(x) = L_s^2 H_{L^2}(x) \tag{2.7}$$

$$\Delta_\Omega \boldsymbol{A}_{L^2}(x) = L_v^2 \boldsymbol{A}_{L^2}(x) \tag{2.8}$$

respectively. In order for a QNM corresponding to the frequency ω and the above Laplace operator eigenvalue to exist, the corresponding boundary conditions problem should have a non-trivial solution. Introducing the dimensionless parameters $\lambda \equiv \frac{\omega r_+}{\rho^2}$, $q_s \equiv \frac{L_s}{\rho}$, $q_v \equiv \frac{L_v}{\rho}$ and $C \equiv \tilde{g}'(0)$, we find the following equations (see Appendix C for a detailed derivation):

- For a (massless, minimally coupled) scalar QNM, the equation is

$$(1-z)^{d-2}\partial_z\left[\frac{\tilde{g}(z)}{(1-z)^{d-2}}\partial_z\psi\right] + \left[\frac{\lambda^2}{\tilde{g}(z)} - q_s^2\right]\psi = 0\ , \tag{2.9}$$

  along with the boundary conditions:

$$\psi|_{z=0} \sim z^{-\frac{i\lambda}{C}} \qquad \psi|_{z=1} = 0\ . \tag{2.10}$$

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[4] In the context of this work we shall define the Laplace operator as: $\Delta \equiv \delta\,\mathrm{d}$ where d and δ are the exterior derivative and codifferential operators respectively.

- For a "longitudinal" vector QNM, the equation is

$$\partial_z \left[\tilde{g}(z)(1-z)^{d-4} \partial_z \left(\frac{1}{(1-z)^{d-4}} \psi \right) \right] + \left[\frac{\lambda^2}{\tilde{g}(z)} - q_s^2 \right] \psi = 0 \ , \tag{2.11}$$

  along with the boundary conditions:

$$\begin{aligned} &\psi|_{z=0} \sim z^{-\frac{i\lambda}{C}} \\ &(1-z)^{d-4}\partial_z \left[\frac{1}{(1-z)^{d-4}} \psi \right] \bigg|_{z=1} = \left(\partial_z \psi + \frac{d-4}{1-z} \psi \right) \bigg|_{z=1} = 0 \ , \end{aligned} \tag{2.12}$$

  or *equivalently* the equation

$$\begin{aligned} &(1-z)^{d-4}\partial_z \left[\frac{\tilde{g}(z)}{(1-z)^{d-4}} \partial_z \psi \right] \\ &\qquad + \frac{q_s^2}{\lambda^2 - q_s^2 \tilde{g}(z)} \tilde{g}(z) \partial_z \tilde{g}(z) \partial_z \psi + \left[\frac{\lambda^2}{\tilde{g}(z)} - q_s^2 \right] \psi = 0 \ , \end{aligned} \tag{2.13}$$

  along with the boundary conditions:

$$\psi|_{z=0} \sim z^{-\frac{i\lambda}{C}} \qquad \psi|_{z=1} = 0 \ . \tag{2.14}$$

- For a "transverse" vector QNM, the equation is

$$(1-z)^{d-4}\partial_z \left[\frac{\tilde{g}(z)}{(1-z)^{d-4}} \partial_z \psi \right] + \left[\frac{\lambda^2}{\tilde{g}(z)} - q_v^2 \right] \psi = 0 \ , \tag{2.15}$$

  along with the boundary conditions:

$$\psi|_{z=0} \sim z^{-\frac{i\lambda}{C}} \qquad \psi|_{z=1} = 0 \ . \tag{2.16}$$

It can be easily seen from the above equations that if $\psi(z)$ is as a solution for any of the equations (satisfying the corresponding boundary conditions) with frequency λ, then $\psi^*(z)$ is a solution for the same equation (with the same boundary conditions) with frequency $-\lambda^*$. This means that the QNM frequencies are always symmetric with respect to the imaginary axis.

A detailed derivation of the QNM Equations can be found in Appendix C.

2.3 Exact solutions for the $K = -1$ case

In the case where the Ω_{d-2} manifold is hyperbolic, we consider the special state corresponding to the temperature:

$$T_C \equiv \frac{\sqrt{|k|}}{2\pi R} = \frac{1}{2\pi} \sqrt{\frac{|R_\Omega^{FT}|}{(d-2)(d-3)}} \ , \tag{2.17}$$

for which $K = -1$, and the horizon radius is given by $\rho_C = \sqrt{|k|}$. In this case, the QNM equations (Equations 2.9, 2.11 and 2.15) can be solved analytically. This has been done in [19], [20] and [12]. Here we summarize these solutions in our notation.

Making the transformation:

$$w \equiv \tilde{g}(z) = 1 - (1-z)^2 \tag{2.18}$$

$$\psi \equiv w^{\gamma}(1-w)^{\delta}\phi \tag{2.19}$$

(where γ and δ are defined in Appendix E for each perturbation type), one obtains the hypergeometric equation. The solution to the equation satisfying the incoming-wave boundary condition at the horizon is ${}_2F_1\left(a,b;c;w\right)$, where a, b and c are given in Appendix E for each case.

Applying the QNM boundary condition at the AdS boundary we get for the scalar and transverse vector cases:

$${}_2F_1\left(a,b;c;1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = 0 \ , \tag{2.20}$$

and for the longitudinal vector case (and using known formula fot the hypergeometric function, taking into account $c-a-b = \frac{d-5}{2}$) we get:

$$\begin{aligned}(1-w)^{\frac{d-3}{2}}\partial_w \left[\frac{1}{(1-w)^{\frac{d-5}{2}}}\,{}_2F_1(a,b;c;w)\right]\Bigg|_{w=1} \\ = \frac{(c-a)(c-b)}{c}\,{}_2F_1(a,b;c+1;1) \\ = \frac{(c-a)(c-b)}{c}\frac{\Gamma(c+1)\Gamma(c-a-b+1)}{\Gamma(c-a+1)\Gamma(c-b+1)} = 0 \ .\end{aligned} \tag{2.21}$$

The solution for which is given by $c-a=-n$ or $c-b=-n$ where $n \in \mathbb{Z}$ and $n \geq 0$, so that the quasi-normal frequencies are:

- For the massless scalar case:

$$\lambda_n = -\frac{i}{2}(d+1) \pm \sqrt{q^2 - \frac{1}{4}(d-3)^2 - 2ni} \ . \tag{2.22}$$

- For the longitudinal vector case:

$$\lambda_n = -\frac{i}{2}(d-3) \pm \sqrt{q^2 - \frac{1}{4}(d-3)^2 - 2ni} \ . \tag{2.23}$$

- For the transverse vector case:

$$\lambda_n = -\frac{i}{2}(d-1) \pm \sqrt{q^2 - \frac{1}{4}(d-5)^2 - 2ni} \ . \tag{2.24}$$

2.4 QNM Asymptotics (for $K > -1$)

For the case $K > -1$, one may find an approximate analytical expression for the n-th QNM frequency for $n \to \infty$ - that is, in the asymptotic limit of $\frac{L_s^{FT}}{T}$ = fixed and $\frac{\omega}{T} \gg 1$. This has been done in [10] using the “monodromy” method. Here we summarize this calculation using our notations. The general stages of the calculation are as follows:

1. We bring each of the QNM equations to a Schrödinger form, with some effective potentials, using appropriate coordinate and function transformations.
2. We find the poles of these effective potentials on the complex plane, and write approximate forms for the equations in the vicinity of the poles.
3. We solve the approximate equations around each pole using Bessel functions.
4. Using the asymptotic limit of $|\lambda| \gg 1$, we replace the Bessel functions with their asymptotic forms.
5. We match the solutions on each region of the complex plane, and apply the appropriate boundary conditions, to get equations for values of λ that allow for non-trivial solutions.
6. We solve the equations analytically and obtain an asymptotic expression of the form:

$$\lambda_n = \lambda_0 + n\Delta\lambda \ . \tag{2.25}$$

2.4.1 Calculation of Asymptotic QNMs

Starting from the QNM Equations, we perform the following transformation:

$$\psi(z) = (1-z)^{\frac{\alpha}{2}}\phi(z) \tag{2.26}$$

$$\tilde{z} \equiv \int \frac{1}{\tilde{g}(z)}\,\mathrm{d}z = \sum_{k=1}^{d-1} \gamma_k \ln(z - z_k) + \tilde{z}_0 \ , \tag{2.27}$$

where:

$$\alpha = \begin{cases} d-2 & \text{for scalar,} \\ d-4 & \text{for vector} \end{cases} \ , \tag{2.28}$$

z_k $(k = 1, \ldots, d-1)$ are the zeros of the polynomial $\tilde{g}(z)$ and[5]

$$\gamma_k = \lim_{z \to z_k} \frac{z - z_k}{\tilde{g}(z)} = \frac{1}{\tilde{g}'(z_k)} \tag{2.29}$$

$$\tilde{z}_0 = -\sum_{k=1}^{d-1} \gamma_k \ln(1 - z_k) \tag{2.30}$$

(so that $\tilde{z}(z=1) = 0$). After this transformation the equations take the Schrödinger form:

$$\partial_{\tilde{z}}^2 \phi + \left[\lambda^2 - V(\tilde{z})\right]\phi = 0 \ , \tag{2.31}$$

where the effective potential $V(z)$ is given for each case in Appendix G.

Next we turn to the calculation of the asymptotic QNMs using the monodromy method. We first note that the asymptotic QNM frequencies satisfy (see [9] and [10])

$$\mathrm{Im}(\lambda \tilde{z}_0) = 0 \ . \tag{2.32}$$

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[5] The branch cuts are chosen here so that $0 \leq \arg(z) < 2\pi$.

Looking at the anti-Stokes lines, defined by

$$\mathrm{Im}(\lambda\tilde{z}) = 0 \ , \tag{2.33}$$

we choose two of them to be the contour for the calculation[6]. We then proceed by developing the equation, and finding its solutions, around points along the chosen contour (the boundary and $z \to \infty$) and "matching" the solutions. The chosen contour is shown in Figure 1[7].

We first look at the equation at $z \to \infty$ or $\tilde{z} \to \tilde{z}_0$. As shown in Appendix G, the equation at $\tilde{z} \to \tilde{z}_0$ is approximately

$$\partial_{\tilde{z}}^2\phi + \left[\lambda^2 - \frac{j_\infty^2 - 1}{4(\tilde{z}-\tilde{z}_0)^2}\right] = 0 \ . \tag{2.34}$$

The solution to this equation can be written as:

$$\phi \approx A_+P_+(\tilde{z}-\tilde{z}_0) + A_-P_-(\tilde{z}-\tilde{z}_0) \ . \tag{2.35}$$

In the non-scalar case where $j_\infty \neq 0$ (and more generally j_∞ isn't an even integer),

$$P_\pm(\tilde{z}-\tilde{z}_0) \equiv \sqrt{2\pi\lambda(\tilde{z}-\tilde{z}_0)}J_{\pm\frac{j_\infty}{2}}\left(\lambda(\tilde{z}-\tilde{z}_0)\right) \ , \tag{2.36}$$

where $J_{\pm\frac{j_\infty}{2}}$ are the Bessel functions. The fact that asymptotically[8] $|\lambda| \gg 1$ allows us to replace the Bessel functions with their corresponding asymptotic forms. For direction (1) (as appears in Figure 1), we have $\lambda(\tilde{z}-\tilde{z}_0) > 0$, so that

$$\begin{aligned}\phi \approx \left[A_+\mathrm{e}^{i(-\lambda\tilde{z}_0-\alpha_+)} + A_-\mathrm{e}^{i(-\lambda\tilde{z}_0-\alpha_-)}\right]\mathrm{e}^{i\lambda\tilde{z}}& \\ + \left[A_+\mathrm{e}^{i(\lambda\tilde{z}_0+\alpha_+)} + A_-\mathrm{e}^{i(\lambda\tilde{z}_0+\alpha_-)}\right]\mathrm{e}^{-i\lambda\tilde{z}} \ ,&\end{aligned} \tag{2.37}$$

where

$$\alpha_\pm = \frac{\pi}{4}(1 \pm j_\infty) \ . \tag{2.38}$$

For direction (2) we have $\lambda(\tilde{z}-\tilde{z}_0) < 0$, so that

$$\begin{aligned}\phi \approx \left[A_+\mathrm{e}^{i(-\lambda\tilde{z}_0+3\alpha_+)} + A_-\mathrm{e}^{i(-\lambda\tilde{z}_0+3\alpha_-)}\right]\mathrm{e}^{i\lambda\tilde{z}}& \\ + \left[A_+\mathrm{e}^{i(\lambda\tilde{z}_0+\alpha_+)} + A_-\mathrm{e}^{i(\lambda\tilde{z}_0+\alpha_-)}\right]\mathrm{e}^{-i\lambda\tilde{z}} \ .&\end{aligned} \tag{2.39}$$

Next we look at the equation at the boundary. As shown in Appendix G, the equation at $\tilde{z} \to 0$ is approximately

$$\partial_{\tilde{z}}^2\phi + \left[\lambda^2 - \frac{j_1^2 - 1}{4(\tilde{z}-\tilde{z}_0)^2}\right] = 0 \ . \tag{2.40}$$

---

[6] The choice of branch cuts so that $0 \leq \arg(z) < 2\pi$ makes sure that the contours don't pass through a branch cut. Choosing $0 < \arg(z) \leq 2\pi$ instead would give the complex conjugates of these contours.

[7] The assumed shape of the anti-Stokes lines is only true in the case of $K > -1$.

[8] Another assumption is that $|\lambda| \gg |q|$, so that the effect of the effective potential can be neglected in the regions between the the "special" points ($z \to 0$, $z \to 1$, $z \to \infty$)