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Taste-split staggered actions: eigenvalues, chiralities and Symanzik improvement

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Abstract

The eigenvalue spectra of staggered fermions with an Adams and/or Hoelbling mass term are studied. The chiralities of the eigenmodes reflect whether the chirality linked to the unflavored approximate $(\gamma_5 \times 1)$ or the flavored exact $(\gamma_5 \times \xi_5)$ staggered symmetry is considered, and which one of the RR, LR, RL, LL eigenmode definitions is used. In either case a sensitivity to the topological charge of the gauge background is found. We discuss how to remove the leading cut-off effects of these actions by means of a properly tuned improvement term and/or the overlap procedure. The combination of Symanzik improvement and link smearing radically improves the properties of the physical branch.

1 Introduction

The so-called fermion doubling problem in lattice field theory has been addressed in many ways; the two most popular approaches are known as Wilson fermions [1] and staggered fermions [2], respectively. A problem being cured in several ways usually indicates that none of the solutions is completely satisfactory in all respects, and the lattice is no exception to this rule.

Wilson fermions represent the spinor components as internal degrees of freedom (i.e. with explicit γ-matrices), such that on a lattice with N grid points the Wilson formulation entails a matrix of size $4N_cN \times 4N_cN$, where N_c is the number of colors. Susskind "staggered" fermions are based on the observation that the eigenvalue spectra of naive fermions on interacting backgrounds are 4-fold degenerate (i.e. not 16-fold in 4 space-time dimensions as one might have naively guessed), and hence a reduction to 1-component spinors is possible (at the price of distributing the spinor degrees of freedom over space-time), such that one ends up with a staggered fermion matrix of size $N_cN \times N_cN$ that corresponds to 4 species in the continuum.

Wilson fermions are convenient for their conceptual simplicity (there is a 1-to-1 correspondence between lattice and continuum flavor). Their main technical disadvantage is the breaking of chiral symmetry through the dimension 5 operator $\bar{\psi}(-\triangle)\psi$ that is added to lift 15 out of the 16 naive species to a mass of order $1/a$, where $\triangle$ is the gauge covariant Laplacian and a is the lattice spacing. Staggered fermions offer the advantage of a truncated form of chiral symmetry and of more speedy simulations (the size of the matrix is smaller). Their main technical disadvantage is that the taste symmetry (among the 4 species) is not exact and that the reduction to a single species proceeds in different ways for sea-quarks (those which come from the functional determinant) and valence-quarks (those which stem from interpolating fields). This renders the rooted staggered formulation (at finite a) non-local and/or non-unitary which, in principle, could affect the universality class of the theory (though there is plenty of analytical and numerical evidence that this does not happen), see e.g. the review [3].

In either approach the main technical disadvantage can be mitigated. Replacing the gauge links $U_\mu(x)$ of the covariant derivative $(\nabla_\mu\phi)(x) = [U_\mu(x)\phi(x{+}\hat{\mu}) - U_\mu^\dagger(x{-}\hat{\mu})\phi(x{-}\hat{\mu})]/2$ in the operator by smeared links $V_\mu(x)$ (see below for details) reduces the amount of chiral symmetry breaking with Wilson fermions (particularly effectively when combined with a clover term) and of taste symmetry breaking with staggered fermions (see [3] for a guide to the literature). In the Wilson case chiral symmetry breaking can be completely removed through the overlap procedure [4,5]; unfortunately this increases the computational requirements by a factor $O(100)$.

Recently, two modifications of the staggered action $S_\mathrm{S} = \bar{\chi}[D_\mathrm{S}+m]\chi$ with $D_\mathrm{S} = \eta_\mu\nabla_\mu$ and $\eta_\mu(x)$ given below were proposed which go by the somewhat confusing name "staggered Wilson fermions". In this article we call them "taste-split staggered actions". In essence the proposal is to replace/augment the usual staggered mass term $m(1\otimes 1)$ in S_S, where the notation is spinor$\otimes$taste [6–8], by taste non-singlet mass terms $\propto (1\otimes\xi)$ which are designed to (partly or fully) lift the staggered near-degeneracy, i.e. some species get masses $2/a$ or $4/a$ such that they decouple in the continuum limit. Such non-standard staggered mass terms were first considered in [8] where it was noticed that they break the remnant chiral symmetry $D_\mathrm{S}(x,y) \to e^{\mathrm{i}\epsilon(x)\theta} D_\mathrm{S}(x,y) e^{\mathrm{i}\epsilon(y)\theta}$ of the massless staggered action [with $\epsilon(x)$ defined below] more severely than the usual mass term does. The Adams proposal [9]

$$S_\mathrm{A} = \bar{\chi}[\eta_\mu\nabla_\mu + r(M_\mathrm{A}{+}1)]\chi \tag{1}$$

builds on a mass term $M_\mathrm{A} \simeq 1\otimes\xi_5$, such that in the naive continuum limit two species stay massless, while two have mass $2r/a$ (where $0 < r < 2$ is akin to the Wilson parameter); the precise form of M_A will be given below. Similarly, Hoelbling proposes the two operators [10]

$$S_\mathrm{Hori} = \bar{\chi}[\eta_\mu\nabla_\mu + r(M_\mathrm{A}{+}M_\mathrm{Hori}{+}2)]\chi \tag{2}$$

$$S_\mathrm{Hsym} = \bar{\chi}[\eta_\mu\nabla_\mu + r(M_\mathrm{Hsym}{+}2)]\chi \tag{3}$$

which fully lift the staggered near-degeneracy, i.e. one species stays massless, while two receive a mass $2r/a$ and the last one is brought up to $4r/a$. The two versions differ on how they break the rotational symmetry group $R_{\mu\nu}$ of the massless staggered action; the precise form of M_Hori and M_Hsym will be given below. It is straight forward to write down the linear combination

$$S_\mathrm{Hnew} = \bar{\chi}[\eta_\mu\nabla_\mu + \frac{r}{2}(M_\mathrm{A}{+}M_\mathrm{Hsym}{+}3)]\chi = \frac{1}{2}\Big[S_\mathrm{A} + S_\mathrm{Hsym}\Big] \tag{4}$$

which lifts all three non-continuum species to the same doubler point $2r/a$.

Evidently, the attractive feature of these operators is that the matrices are just of size $N_cN\times N_cN$, and still only 1 or 2 species survive in the continuum. The question has been raised whether the remaining symmetries of (1) or (2, 3, 4) are sufficient for taking the continuum limit (at a fixed pion mass) by tuning only the standard (relevant) mass term or whether other (relevant, marginal or irrelevant) operators need to be tuned simultaneously [11–17]. In this paper we investigate the eigenvalue spectra of these operators on interacting backgrounds and the chiralities (with respect to $\gamma_5\otimes 1$, $1\otimes\xi_5$ and $\gamma_5\otimes\xi_5$) of their eigenmodes. The idea behind is that the willingness or reluctance of the "bellies" to open up (and hence separate the branches) is indicative of how severe the fermionic operator mixing is – in the Wilson case both link smearing and including the clover term tend to clear the first eigenvalue belly, since they suppress mixing between the dimension 5 Laplacian and the dimension 3 mass operator.

In the proposals [9, 10] (and in [11–16]) a notation is used which obscures the link to the original work [8]. This is why Sec. 2 contains a review of taste non-singlet mass terms using standard staggered notation. The core of the investigation, a look at the staggered eigenvalues and chiralities, is presented in Sec. 3. A brief discussion of some of the peculiarities of the Symanzik improvement program, when applied to these partly or fully undoubled staggered actions, is given in Sec. 4. Results where these taste-split staggered actions serve as kernel to the overlap procedure are shown in Sec. 5. A comparison to the eigenvalue spectra of Wilson-type actions is found in Sec. 6. Finally, Sec. 7 contains a summary.

2 Review of staggered mass terms

Define the Golterman-Smit staggered phase factors [8]

$$\eta_\mu(x) = (-1)^{\sum_{\nu<\mu} x_\nu} \,, \qquad \zeta_\mu(x) = (-1)^{\sum_{\mu<\nu} x_\nu} \tag{5}$$

which multiply to give

$$\eta_5(x) = \eta_1\eta_2\eta_3\eta_4 = (-1)^{x_1+x_3} \,, \qquad \zeta_5(x) = \zeta_1\zeta_2\zeta_3\zeta_4 = (-1)^{x_2+x_4} \tag{6}$$

respectively. The product of the latter two phase factors reads (throughout the spinor⊗taste identification holds up to $O(a)$ corrections [6–8])

$$\epsilon(x) = \eta_5\zeta_5 = (-1)^{x_1+x_2+x_3+x_4} \qquad \longleftrightarrow \qquad \gamma_5 \otimes \xi_5 \tag{7}$$

and induces the $U(1)_\epsilon$-symmetry and ϵ-hermiticity $\epsilon(x)D_\mathrm{S}(x,y)\epsilon(y) = D_\mathrm{S}(x,y)^\dagger$ of the massless staggered action. After being lifted to a site-diagonal operator

$$\epsilon(x,y) = \Gamma_{55}(x,y) = (-1)^{x_1+x_2+x_3+x_4}\delta_{x,y} \tag{8}$$

it couples to the $\gamma_5 \otimes \xi_5$ "Goldstone" state when used as an interpolating field [8].

The $(\gamma_\mu \otimes 1)$ and $(\gamma_5 \otimes 1)$ "taste singlet" operators are defined by

$$\Gamma_{\mu 0}(x,y) \equiv \Gamma_\mu(x,y) = \frac{1}{2}\eta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat\mu,y} + U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}\Big] \tag{9}$$

$$\Gamma_{50}(x,y) \equiv \Gamma_5(x,y) = \frac{1}{4!}\sum_\mathrm{perm} \epsilon_\mathrm{perm}\Gamma_1\Gamma_2\Gamma_3\Gamma_4 \tag{10}$$

and the $(1 \otimes \xi_\mu)$ and $(1 \otimes \xi_5)$ "spinor singlet" operators are defined by

$$\Gamma_{0\mu}(x,y) \equiv \Xi_\mu(x,y) = \frac{1}{2}\zeta_\mu(x)\Big[U_\mu(x)\delta_{x+\hat\mu,y} + U_\mu^\dagger(x-\hat\mu)\delta_{x-\hat\mu,y}\Big] \tag{11}$$

$$\Gamma_{05}(x,y) \equiv \Xi_5(x,y) = \frac{1}{4!}\sum_\mathrm{perm} \epsilon_\mathrm{perm}\Xi_1\Xi_2\Xi_3\Xi_4 \tag{12}$$

with the consequence that both Γ_{50} and Γ_{05} are 4-hop operators. Furthermore, the latter two operators relate to each other by a simple Γ_{55} operation (either from the left or from the right).

In practice it can be advantageous to introduce the commutators in spinor and taste space

$$\Gamma_{\mu\nu}(x,y) \equiv \frac{\mathrm{i}}{2}\Big(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu\Big) \qquad \longleftrightarrow \qquad \gamma_{\mu\nu}\otimes 1 \tag{13}$$

$$\Xi_{\mu\nu}(x,y) \equiv \frac{\mathrm{i}}{2}\Big(\Xi_\mu\Xi_\nu - \Xi_\nu\Xi_\mu\Big) \qquad \longleftrightarrow \qquad 1\otimes\xi_{\mu\nu} \tag{14}$$

respectively, with $\gamma_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma_\mu,\gamma_\nu]$ known as $\sigma_{\mu\nu}$ and $\xi_{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\xi_\mu,\xi_\nu]$. Based on these one finds

$$\Gamma_{50}(x,y) \simeq -\frac{1}{6}\Big(\Gamma_{12}\Gamma_{34} - \Gamma_{13}\Gamma_{24} + \Gamma_{14}\Gamma_{23} + \Gamma_{23}\Gamma_{14} - \Gamma_{24}\Gamma_{13} + \Gamma_{34}\Gamma_{12}\Big) \tag{15}$$

$$\Gamma_{05}(x,y) \simeq -\frac{1}{6}\Big(\Xi_{12}\Xi_{34} - \Xi_{13}\Xi_{24} + \Xi_{14}\Xi_{23} + \Xi_{23}\Xi_{14} - \Xi_{24}\Xi_{13} + \Xi_{34}\Xi_{12}\Big) \tag{16}$$

where the near-equality is exact if no SU(3) backprojection in intermediate steps is applied. The main advantage of the form (15,16) is that one can apply SU(3) backprojection twice, in each case the argument being a sum over products of just two color matrices.

Following Golterman and Smit [8], everything which is a singlet in spinor space is called a "mass term". This leads to the categorization of potential mass terms as

$$1\otimes 1 \quad \longleftrightarrow \quad 1 \quad \text{(0-hop, taste-scalar)} \tag{17}$$

$$1\otimes\xi_\mu \quad \longleftrightarrow \quad \Xi_\mu \quad \text{(1-hop, taste-vector)} \tag{18}$$

$$1\otimes\xi_{\mu\nu} \quad \longleftrightarrow \quad \Xi_{\mu\nu} \quad \text{(2-hop, taste-tensor)} \tag{19}$$

$$1\otimes \mathrm{i}\xi_\mu\xi_5 \quad \longleftrightarrow \quad \mathrm{i}\Xi_\mu\Xi_5 \quad \text{(3-hop, taste-pseudovector)} \tag{20}$$

$$1\otimes\xi_5 \quad \longleftrightarrow \quad \Xi_5 \quad \text{(4-hop, taste-pseudoscalar)} \tag{21}$$

in terms of hermitian operators. As was pointed out in [11], an acceptable mass term is $1\otimes 1$, $1\otimes\xi_{\mu\nu}$, $1\otimes\xi_5$ or a combination thereof. In this terminology the Adams term in (1) is the taste-pseudoscalar mass $\Gamma_{05} = \Xi_5$. And the Hoelbling terms in (2, 3) represent unsymmetrized and symmetrized versions of the taste-tensor mass $\Xi_{\mu\nu}$, respectively, i.e. $M_{\mathrm{Hori}} = \Xi_{12}$ [to be used together with M_{A}, see (2)] and $M_{\mathrm{Hsym}} = (\Xi_{12}+\Xi_{34}+\Xi_{13}-\Xi_{24}+\Xi_{14}+\Xi_{23})/\sqrt{3}$ [to be used alone, see (3)]. The operator (4), finally, involves all three types of valid mass terms.

Note that there is some freedom in the definition of the operators (1–4) and of the taste non-singlet mass terms. For instance with the Adams operator as defined in (1) and the choice $M_A = \gamma_5\otimes\xi_5$ it follows that the physical branch has $\xi_5 = -1$. Hence, with this convention the actions of $\gamma_5\otimes 1$ and $\gamma_5\otimes\xi_5$, in the physical branch, on a topological mode will differ in sign.

3 Eigenvalues and chiralities

The eigenvalue spectrum on a thermalized gauge background is a key to get an impression of how suitable a given fermion discretization is in practical terms. An issue with staggered fermions is their sensitivity to the topological charge of the gauge configuration [18–21].

The ϵ-hermiticity ensures that the eigenvalues of the massless operator occur in pairs $\pm\mathrm{i}\lambda$ on the imaginary axis. There are no exact zero-modes of D_{S}; instead $4|q|$ would-be zero-modes show up. Both their separation from the non-topological modes and their chiralities (see below) depend on how much the staggered action is "improved" through link smearing.