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Entanglement Entropy and Higher Spin Holography in AdS$_3$

Jan de Boer and Juan I. Jottar

Institute for Theoretical Physics, University of Amsterdam,
Science Park 904, Postbus 94485, 1090 GL Amsterdam, The Netherlands

`J.deBoer@uva.nl, J.I.Jottar@uva.nl`

Abstract

A holographic correspondence has been recently developed between higher spin theories in three-dimensional anti-de Sitter space (AdS$_3$) and two-dimensional Conformal Field Theories (CFTs) with extended symmetries. A class of such dualities involves $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ Chern-Simons gauge theories in the (2+1)-dimensional bulk spacetime, and CFTs with $\mathcal{W}_N$ symmetry algebras on the (1+1)-dimensional boundary. The topological character of the Chern-Simons theory forces one to reconsider standard geometric notions such as black hole horizons and entropy, as well as the usual holographic dictionary. Motivated by this challenge, in this note we present a proposal to compute entanglement entropy in the $\mathcal{W}_N$ CFTs via holographic methods. In particular, we introduce a functional constructed from Wilson lines in the bulk Chern-Simons theory that captures the entanglement entropy in the CFTs dual to standard AdS$_3$ gravity, corresponding to $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ gauge group, and admits an immediate generalization to the higher spin case. We explicitly evaluate this functional for several known solutions of the Chern-Simons theory, including charged black holes dual to thermal CFT states carrying higher spin charge, and show that it reproduces expected features of entanglement entropy, study whether it obeys strong subadditivity, and moreover show that it reduces to the thermal entropy in the appropriate limit.

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1 Introduction

Over the last two decades, the holographic principle [1, 2] has become a cornerstone of theoretical physics. Put simply, it asserts that certain theories of gravity in $(d+1)$ dimensions can be described as quantum field theories in d dimensions, and vice versa. To date, the most successful and concrete realization of holography is the Anti-de Sitter (AdS)/Conformal Field Theory (CFT) correspondence of [3] (see [4, 5] also), which proposes a duality between conformally-invariant gauge theories, and string theory on a space in one higher dimension that asymptotes to AdS, the maximally-symmetric solution of Einstein's equations with negative cosmological constant. Notably, the ideas behind the AdS/CFT correspondence have shed light on the structure of quantum gravity and gauge theories alike, while providing powerful computational techniques to explore the connections between them.

The holographic correspondence is an example of strong-weak duality; so far, the vast majority of the work in AdS/CFT has focused on the regime where the gauge theory is strongly-coupled and the string theory description becomes weakly-coupled, effectively reducing to classical (super)gravity. To a high extent, the power of the correspondence lies in the fact that many problems can be addressed analytically in the latter theory, providing insight into the regime where the quantum field theory becomes intractable with the standard perturbative techniques. Quite remarkably, in light of this feature holographic techniques have found a niche of applications in condensed matter physics, where strongly-correlated systems are routinely engineered and studied in the laboratory. It is however of considerable theoretical importance to understand and test the holographic duality in other regimes of couplings as well. A particularly interesting example which is outside the scope of the usual AdS/CFT correspondence is the conjecture [6] of Klebanov and Polyakov relating critical $O(N)$ vector models in the large-N limit to the higher spin Fradkin-Vasiliev theory in AdS$_4$ [7, 8], for which a considerable amount of evidence has been provided recently (see [9] and references therein).

Although of a somewhat different flavor, similar dualities have been put forward in lower dimensions, an interesting example being the proposal of [10] relating the three-dimensional Vasiliev higher spin theory and the large-N limit of $\mathcal{W}_N$ minimal coset CFTs. An aspect that makes the lower-dimensional setup particularly appealing is the fact that universal results for two-dimensional CFTs, such as the Cardy entropy formula [11, 12] and the zero- and finite-temperature entanglement entropy in $1d$ systems [13, 14], are known to be recovered in the framework of the standard AdS$_3$/CFT$_2$ correspondence. In particular, the thermal entropy of the three-dimensional (BTZ) black hole as computed with the standard Bekenstein-Hawking formula precisely matches the form predicted by Cardy's asymptotic growth of states in a unitarity CFT (see e.g. [15, 16, 17]). Similarly, the single-interval entanglement entropy of the CFT state dual to this black hole is reproduced using the Ryu-Takayanagi (R-T) prescription to compute entanglement entropy holographically [18, 19]. Recently, some universal aspects of the AdS$_3$/CFT$_2$ correspondence were further elucidated in [20, 21]. An interesting question is then whether (and how) this universality prevails in the presence of extended symmetries furnished by higher spin operators.

As an added bonus, given that the higher-dimensional theories of interacting massless higher spin fields are technically involved and difficult to work with, it is desirable to work with models that retain their key features while being amenable to study, such as the AdS$_3$ higher spin theories. In fact, in three dimensions it is possible to truncate the tower of massless modes to retain fields of spin $s \leq N$ only [22]. This is to be contrasted with their higher-dimensional counterparts of the Fradkin-Vasiliev type, where an infinite number of higher spin fields must be kept. Furthermore, the corresponding higher spin theories in AdS$_3$ can be cast in the form of an $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ Chern-Simons gauge theory, and many of the familiar techniques to analyze such theories can be brought to bear.

Recently, many entries of the holographic dictionary for higher spin AdS$_3$ theories have been established. In particular, much in the same way that standard Einstein gravity with AdS$_3$ boundary conditions has an asymptotic symmetry group generated by two copies of the Virasoro algebra acting on the spacetime boundary [23], the analysis of asymptotic symmetries in the higher spin case [24, 25, 26] has shown them to correspond to two-dimensional CFTs with extended symmetry algebras of (classical) $\mathcal{W}_N$ type, in agreement with earlier expectations [27]. In contrast with the universal results quoted above, not much is known about the CFTs in the presence of deformations by higher spin operators, making the results obtained from holography all the more interesting. Motivated by these facts, in the present paper we initiate the study of entanglement entropy in higher spin holography in AdS$_3$.

Perhaps the main challenge in extending the usual AdS$_3$ holographic dictionary to the higher spin case is that we must surrender the traditional geometric interpretation of notions such as black hole horizons and entropy, which lie at the core of AdS/CFT dualities in higher dimensions, and formulate them in a language which is appropriate in light of the topological character of the bulk theory. Indeed, the standard R-T prescription to compute entanglement entropy holographically is intrinsically geometric in nature: in order to obtain the entanglement entropy of a region A in the boundary theory, one is instructed to find the minimal area bulk surface that is anchored on the boundary of A and "dips" into the bulk spacetime. In order to solve this problem, in the present work we introduce a functional that is naturally defined in terms of Wilson lines in the bulk Chern-Simons theory and captures the entanglement entropy in the situations where an independent field-theoretical result is available (namely in the absence of higher spin charges), while generalizing naturally to the higher spin setup. In the absence of explicit entanglement entropy results for CFTs perturbed by higher spin operators, we apply our proposal in several examples and show that it reproduces the properties that the field theory entanglement is expected to satisfy, such as strong subadditivity (up to some subtleties). Moreover, building on our previous general results for the thermal entropy in higher spin theories [28], we will show that our entanglement functional is constructed so that it approaches the thermal entropy in the high temperature limit in which the entanglement becomes extensive, even in the presence of non-trivial higher spin charges and chemical potentials.

The structure of the paper is as follows. In section 2 we briefly review the formulation of standard Einstein gravity in three dimensions as a Chern-Simons theory and its extension to include a finite number of higher spin fields, regarded as $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$ Chern-Simons theory (with $N>2$). In section 3 we introduce our proposed bulk functional and explicitly show that it reproduces the known CFT entanglement entropy in the absence of higher spins (namely in the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ case corresponding to standard Einstein gravity). We then discuss various properties of this functional in the general case, and in particular apply our previous results [28] for the thermodynamic entropy of higher spin black holes to show how to choose the representation of the algebra in which the Wilson lines are evaluated, in such a way that our entanglement functional also reproduces the thermal entropy of the dual CFTs in the appropriate limit. In section 4 we apply our proposal to some of the known solutions of the spin-3 theory, including the charged black hole solution of [29] which is dual to a CFT ensemble at finite temperature and finite higher spin charge. We conclude in section 5 with a discussion of our results and outlook. The details of our conventions and some useful facts and calculations are collected in the appendices.

Note: While this work was being completed we became aware of [30], where a similar proposal to compute entanglement entropy in three-dimensional higher spin theories will be put forward. Their formulation is based on the observation that, for $N=2$, the geodesic distance on AdS$_3$ interpreted as a group manifold can be computed in terms of Wilson lines in an infinite-dimensional representation of the gauge group.

2 Higher spin theories in AdS$_3$

As it is well-known, three-dimensional gravity with a negative cosmological constant can be formulated as a Chern-Simons theory [31, 32] (see [33] for a modern perspective). The extension to higher spin theories utilizes the Chern-Simons language, and in fact resembles the pure gravity case in many a way. Therefore, we begin with a brief discussion of standard gravity with AdS boundary conditions in the Chern-Simons formulation. Our conventions and some extra details can be found in appendix A.

2.1 AdS$_3$ gravity as a Chern-Simons theory

Let $a, b, \ldots$ denote local Lorentz indices in $(2+1)$ dimensions, and define the dual ω^a of the spin connection as

$$\omega^a \equiv \frac{1}{2}\epsilon^{abc}\omega_{bc} \tag{2.1}$$

or equivalently $\omega_{ab} = -\epsilon_{abc}\,\omega^c$. We can then combine this object with the dreibein or "triad" e^a into $so(2,1) \simeq sl(2,\mathbb{R})$ connections A, $\bar{A}$ defined as

$$A = \omega + \frac{e}{\ell}\,, \qquad \bar{A} = \omega - \frac{e}{\ell}\,, \tag{2.2}$$

where ℓ is the AdS$_3$ radius, namely the length scale set by the cosmological constant ($\Lambda_{cosmo} = -1/\ell^2$). Here, $\omega \equiv \omega^a J_a$ and $e \equiv e^a J_a$, and the generators J_a obey the $so(2,1) \simeq sl(2,\mathbb{R})$ algebra $[J_a, J_b] = \epsilon_{abc}\,\eta^{cd} J_d = \epsilon_{ab}{}^{c} J_c$ (the relation between the $so(2,1)$ generators and the $sl(2,\mathbb{R})$ generators Λ^0, $\Lambda^{\pm}$ is given in (B.2)). We emphasize that the bar notation *does not* denote complex conjugation in Lorentzian signature.

Defining the Chern-Simons form CS as

$$CS(A) = A \wedge dA + \frac{2}{3} A \wedge A \wedge A \tag{2.3}$$

one finds that the combination $\mathrm{Tr}\Big[CS(A) - CS(\bar{A})\Big]$ yields the Einstein-Hilbert Lagrangian, up to a total derivative. The precise relation is (c.f. (A.7))

$$\begin{aligned} I &\equiv \frac{k}{4\pi}\int_M \mathrm{Tr}\Big[CS(A) - CS(\bar{A})\Big] \\ &= \frac{1}{16\pi G_3}\left[\int_M d^3x\sqrt{|g|}\left(\mathcal{R} + \frac{2}{\ell^2}\right) - \int_{\partial M}\omega^a \wedge e_a\right], \end{aligned} \tag{2.4}$$

where G_3 is the $3d$ Newton constant. When evaluated on shell, the boundary term amounts to $(1/2)$ times the standard Gibbons-Hawking surface term. Normalizing the $so(2,1)$ generators according to $\mathrm{Tr}\,[J_a J_b] = \eta_{ab}/2$, we identify the Chern-Simons level k as

$$k = \frac{\ell}{4G_3}\,. \tag{2.5}$$

Whether the gauge group is $SO(2,1) \times SO(2,1)$ or some locally isomorphic (but globally inequivalent) cover is a question that has consequences for the quantization of k, and affects the values of the central charges in the dual field theory (see [33, 34], for example).

One can easily establish a dictionary between the standard (metric) and Chern-Simons formulations of the theory. For example, the metric tensor is obtained from the triad as $g_{\mu\nu} = 2\mathrm{Tr}\,[e_\mu e_\nu]$, and Einstein's equations translate into the flatness of the gauge connections,

$$F = dA + A \wedge A = 0\,, \qquad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = 0\,. \tag{2.6}$$

In components, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$ as usual. Similarly, under an infinitesimal gauge transformation $\delta A = d\lambda + [A, \lambda]$, $\delta \bar{A} = d\bar{\lambda} + \left[\bar{A}, \bar{\lambda}\right]$, the dreibein transforms as

$$\delta e_\mu = e_\nu \, \xi^\nu{}_{;\mu} + \frac{1}{2} \left[e_\mu, \lambda + \bar{\lambda}\right] , \tag{2.7}$$

where the infinitesimal generator ξ^μ is defined in terms of the inverse triad as $\xi^\mu = (\ell/2) e_a{}^\mu \left(\lambda^a - \bar{\lambda}^a\right)$. The first term in (2.7) gives rise to the standard infinitesimal diffeomorphisms acting on the metric, while the second term represents a rotation of the local Lorentz frame.

That one can rephrase three-dimensional gravity as a topological theory is a reflection of the fact that the dynamical degrees of freedom in the theory are not local: as it is well-known, all solutions of the three-dimensional Einstein's equations with negative cosmological constant are locally equivalent to AdS$_3$. The non-triviality of the dynamics is rooted in the existence of globally inequivalent solutions, such as black holes, and boundary excitations. Naturally, the latter are intimately tied to the choice of boundary conditions, which are a crucial ingredient in holographic constructions. As first shown by Brown and Henneaux [23], in standard three-dimensional gravity with negative cosmological constant one can choose consistent boundary conditions such that the asymptotic symmetries correspond to two copies of the Virasoro algebra with central charge $c = 6k = 3\ell/(2G_3)$.

Let us briefly review how the Brown-Henneaux result comes about in the Chern-Simons formulation, as first derived by [35]. We consider Chern-Simons theory on a Lorentzian three-dimensional manifold M with topology $\mathbb{R} \times D$, where the $\mathbb{R}$ factor corresponds to the timelike direction and D is a two-dimensional manifold with boundary $\partial D \simeq S^1$. We will introduce coordinates (ρ, t, φ) on M, where ρ is the bulk radial coordinate and the constant-ρ surfaces (in particular the asymptotic boundary ∂M at $\rho \to \infty$) have the topology of a cylinder. Given a set of boundary conditions, the *asymptotic symmetry algebra* is defined as the set of transformations (diffeomorphisms in this case) that respect the boundary conditions, modulo trivial gauge transformations which are generated by constraints. The charges associated with the asymptotic symmetries generate global transformations that take us between distinguishable physical states in phase space (which becomes a Hilbert space upon quantization). Imposing boundary conditions $A_-|_{\partial M} \to 0, \ \bar{A}_+\big|_{\partial M} \to 0$, one finds that the asymptotic symmetries correspond to two copies of an affine $sl(2, \mathbb{R})$ algebra at level k . Equivalently, at this stage the Chern-Simons theory plus boundary conditions becomes a non-chiral Wess-Zumino-Witten (WZW) model. Further imposing that the connection approaches an AdS$_3$ connection at the boundary, $A - A_{AdS_3} \xrightarrow[\rho \to \infty]{} \mathcal{O}(1)$, the asymptotic symmetries reduce to two copies of the Virasoro algebra with central charge $c = 6k = 3\ell/(2G_3)$. This is an example of the so-called Drinfeld-Sokolov reduction [36]: denoting the modes of the Kac-Moody currents by J^a_n , the requirement of AdS asymptotics translates to $J^0_n = 0$, $J^+_n \simeq k\delta^0_n$, reducing the current algebra to the Virasoro symmetries.

We will now review some solutions that will play an important role later on. As we mentioned above, all the solutions of three-dimensional Einstein gravity with negative cosmological constant are locally connected to AdS$_3$ by a change of coordinates. In [37] it was pointed out that the metric

$$ds^2 = \ell^2 \left[d\rho^2 + \frac{T(x^+)}{k}(dx^+)^2 + \frac{\bar{T}(x^-)}{k}(dx^-)^2 - \left(e^{2\rho} + \frac{T(x^+)\bar{T}(x^-)}{k^2} e^{-2\rho} \right) dx^+ dx^- \right] , \qquad (2.8)$$

where $x^\pm = t/\ell \pm \varphi$, is a solution of Einstein's equations for any functions $T = T(x^+)$, $\bar{T} = \bar{T}(x^-)$. Furthermore, it represents the whole space of asymptotically AdS$_3$ (AAdS$_3$) solutions with a flat boundary metric at $\rho \to \infty$. In particular, the BTZ black hole [38] with mass M and angular momentum J [1] is obtained for constant T, $\bar{T}$ given as

$$\begin{aligned} T_{BTZ} &= \frac{1}{2}(M\ell - J) = k\frac{\pi^2\ell^2}{\beta_-^2} \\ \bar{T}_{BTZ} &= \frac{1}{2}(M\ell + J) = k\frac{\pi^2\ell^2}{\beta_+^2} , \end{aligned} \qquad (2.9)$$

where we introduced the inverse chiral temperatures $\beta_\pm = 1/T_\pm$. Via the standard holographic dictionary, the functions T and $\bar{T}$ are seen to correspond to the stress tensor modes in the dual CFT (see [17] for a review of the AdS$_3$/CFT$_2$ correspondence); in particular, the zero modes of T and $\bar{T}$ are the eigenvalues of the operators L_0 and $\bar{L}_0$. In this parameterization the global AdS$_3$ solution corresponds to $J = 0$ and $M\ell = -k/2$ ($8GM = -1$), i.e.

$$T_{AdS_3} = \bar{T}_{AdS_3} = -\frac{k}{4} , \qquad (2.10)$$

while the so-called Poincaré patch of AdS$_3$ is obtained with $J = M = 0$ (provided we un-compactify the boundary spatial coordinate).

We will choose a basis of generators $\{\Lambda^0, \Lambda^\pm\}$ for the $sl(2, \mathbb{R})$ algebra, satisfying

$$\left[\Lambda^\pm, \Lambda^0\right] = \pm\Lambda^\pm , \quad \left[\Lambda^+, \Lambda^-\right] = 2\Lambda^0 . \qquad (2.11)$$

In order to write down the above solutions in the Chern-Simons formulation, one first notices that the gauge freedom allows one to fix the radial dependence as

$$\begin{aligned} A &= b^{-1}db + b^{-1}a(x^+, x^-)\, b \\ \bar{A} &= b\, db^{-1} + b\, \bar{a}(x^+, x^-)\, b^{-1} , \end{aligned} \qquad (2.12)$$

with $b = b(\rho) = e^{\rho\Lambda^0}$. The ρ-independent connections a, $\bar{a}$ corresponding to (2.8) are then given by

$$\begin{aligned} a &= \left(\Lambda^+ - \frac{T(x^+)}{k}\Lambda^- \right) dx^+ , \\ \bar{a} &= \left(-\Lambda^- + \frac{\bar{T}(x^-)}{k}\Lambda^+ \right) dx^- . \end{aligned} \qquad (2.13)$$

---

[1] One restricts $|J| \leq \ell M$ in order to avoid naked singularities. The value $|J| = \ell M$ that saturates the bound corresponds to the extremal (zero temperature) BTZ black hole.

Since Chern-Simons theory is a theory of flat connections, we can *locally* write its solutions in terms of group elements g, $\bar{g}$ as follows:

$$A = g^{-1}dg\,, \qquad \bar{A} = \bar{g}^{-1}d\bar{g}\,. \tag{2.14}$$

For example, for the above solutions with constant T, $\bar{T}$ (which include globally-defined black holes) we find

$$\begin{aligned} g &= \exp\left\{\left[\Lambda^+ - \left(\frac{T}{k}\right)\Lambda^-\right]x^+\right\} b(\rho)\,, \\ \bar{g} &= \exp\left\{-\left[\Lambda^- - \left(\frac{\bar{T}}{k}\right)\Lambda^+\right]x^-\right\} b^{-1}(\rho)\,. \end{aligned} \tag{2.15}$$

Additional care must be exercised in the presence of non-contractible cycles, such as the φ circle parameterizing the horizon in three-dimensional black hole geometries: if the connection has non-trivial holonomy, it undergoes a gauge transformation upon transport around the horizon. In other words, the group elements g, $\bar{g}$ in (2.14) are not, in general, single-valued. This will be important for us below when we discuss how to recover the thermodynamic entropy in the limit in which the entanglement entropy becomes extensive.

2.2 The $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ higher spin theory

Having rephrased the standard AdS$_3$ Einstein gravity as an $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons gauge theory, we now introduce higher spins by promoting the gauge group to $SL(N,\mathbb{R})\times SL(N,\mathbb{R})$. When $N > 2$, this theory describes gravity coupled to a tower of fields of spin $s \leq N$ [22]. The precise field content of the gravitational theory (and hence the spectrum and symmetry algebra of the dual CFT) depends on how the $sl(2,\mathbb{R})$ subalgebra associated to the gravity sector is embedded into $sl(N,\mathbb{R})$ (see [39, 40]). The different embeddings are characterized by the way in which the fundamental representation of $sl(N,\mathbb{R})$ decomposes into $sl(2,\mathbb{R})$ representations, and these branching rules are in turn classified by integer partitions of N. As a concrete example, consider the defining representation $\mathbf{3}_3$ of $sl(3,\mathbb{R})$. Denoting the $(2j+1)$-dimensional representation of $sl(2,\mathbb{R})$ by $(\mathbf{2j+1})_2$, the non-trivial inequivalent embeddings are characterized by the branching rules $\mathbf{3}_3 \to \mathbf{3}_2$ and $\mathbf{3}_3 \to \mathbf{2}_2 \oplus \mathbf{1}_2$. The first embedding is the so-called "principal embedding", characterized by the fact that the fundamental representation becomes an irreducible representation of the embedded algebra. The second embedding is called "diagonal embedding", because the embedded $sl(2,\mathbb{R})$ takes a block-diagonal form inside $sl(3,\mathbb{R})$.

The branching of the $\left(N^2-1\right)$-dimensional adjoint representation can be determined from that of the fundamental representation, and one deduces in this way the decomposition of the algebra itself and hence the spectrum [39, 41]. In the principal embedding, $\mathrm{adj}_N \to \mathbf{3}_2 \oplus \mathbf{5}_2 \oplus \ldots \oplus (\mathbf{2N-1})_2$ showing that the $sl(N,\mathbb{R})$ algebra decomposes into $N-1$ representations with $sl(2,\mathbb{R})$ spins ranging from 1 to $N-1$ (the spin 1 multiplet being the $sl(2,\mathbb{R})$ generators themselves). From the perspective of the bulk theory, these representations correspond to the metric ($g_{\mu\nu} \sim \mathrm{Tr}[e_\mu e_\nu]$) and a tower of symmetric tensor fields with spins $3, \ldots, N$ ($\phi_{\mu\nu\rho} \sim \mathrm{Tr}[e_{(\mu} e_\nu e_{\rho)}]$ and so forth). In general, the conformal weight of the corresponding operators in the boundary theory is obtained by adding one to the $sl(2, \mathbb{R})$ spin (see [42] for example). Consequently, in addition to the stress tensor, in the principal embedding one finds primary operators of weight $3, 4 \ldots, N$. In the diagonal embedding one has instead $\mathrm{adj}_N \to \mathbf{3}_2 \oplus 2(N-2) \cdot \mathbf{2}_2 \oplus (N-2)^2 \cdot \mathbf{1}_2$. Hence, the spectrum in the diagonal embedding contains currents of weight 1 and $3/2$. When charged fields are present, there is always a consistent truncation where they are taken to be zero; in the diagonal embedding, this corresponds to setting the spin $3/2$ fields to zero while truncating the weight-one currents to the diagonal subset. From the bulk perspective, the theory in the diagonal embedding then contains a truncation to standard gravity coupled to $U(1)^{2(N-2)}$ gauge fields. Indeed, as discussed in [43], a class of black hole solutions in the diagonal embedding correspond to BTZ black holes charged under Abelian holonomies.

The asymptotic symmetry analysis in the $N > 2$ case was performed in [24, 25] (see [44, 45] for early work), and parallels the $N = 2$ discussion in [35] closely. In particular one imposes "Drinfeld-Sokolov boundary conditions" as before,

$$A_- = 0\,, \qquad A_\rho = b^{-1}(\rho)\partial_\rho b(\rho)\,, \qquad A - A_{AdS_3} \xrightarrow[\rho\to\infty]{} \mathcal{O}(1)\,, \tag{2.16}$$

and similarly for the barred connection (notice that the last condition on A does not imply that a obeys that condition as well, and that (2.13) is compatible with (2.16)). The asymptotic symmetries are then given by two copies of the so-called $\mathcal{W}_N$ algebras [46], which correspond to non-linear extensions of the Virasoro algebra. As a concrete example, for the $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ theory in the principal embedding the corresponding asymptotic symmetry algebra consists of two copies of the $\mathcal{W}_3$ algebra, with classical central charge $c = 6k = 3\ell/(2G_3)$. According to the general features discussed above, the algebra in this case includes the stress tensor and primary operators of weights $(3, 0)$ and $(0, 3)$. For $N = 3$ there is only one other non-trivial inequivalent embedding, i.e. the diagonal embedding. The asymptotic symmetry algebra in this case is identified with the so-called $\mathcal{W}_3^{(2)}$ algebra [24, 25, 47]. Besides the stress tensor, this algebra contains two weight-3/2 primary operators and a weight one current, with classical central charge given by $\hat{c} = c/4 = 3k/2$. Different boundary conditions giving rise to non-AAdS$_3$ higher spin theories have been also considered recently in [48, 49].

Let us write the coefficient of the Chern-Simons action in the higher spin case as $k_{cs}/(4\pi)$. Since the trace in the action is taken in the fundamental representation, matching with the normalization of the Einstein-Hilbert action requires

$$k_{cs} = \frac{k}{2\mathrm{Tr}_N\left[\Lambda^0 \Lambda^0\right]} \tag{2.17}$$

where, according to the above discussion, k is the level of the $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ Chern-Simons theory contained in the full theory through the choice of embedding, and Tr_N denotes the trace in the fundamental (N-dimensional) representation. In terms of the level k_{cs} of the $sl(N,\mathbb{R}) \times sl(N,\mathbb{R})$ theory, the central charge in the boundary CFT is given by

$$c = 12 k_{cs} \mathrm{Tr}_N \left[\Lambda^0 \Lambda^0 \right] . \tag{2.18}$$

Note that, for fixed k_{cs}, the central charge will be different for different embeddings.

3 A holographic entanglement entropy proposal for higher spin theories

Consider a quantum system described by a density matrix ρ, and divide it into two subsystems A and $B = A^c$. The *reduced density matrix* ρ_A of subsystem A is defined by tracing over the degrees of freedom in B, i.e. $\rho_A = \mathrm{Tr}_B \, \rho$. The *entanglement entropy* S_A of A is then defined as the von Neumann entropy associated with ρ_A:

$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A . \tag{3.1}$$

If the full system was originally in a pure state, i.e. $\rho = |\Psi\rangle\langle\Psi|$, then $S_A = S_B$. This property does not hold if the system was originally in a mixed state, such as a thermal ensemble with density matrix $\rho = e^{-\beta H}$.

From a theoretical standpoint, the entanglement entropy has several interesting properties associated with its non-local nature, and can moreover serve as a useful tool to characterize gapped phases of matter in the absence of classical order parameters and spontaneous symmetry breaking [50, 51]. Unfortunately, field-theoretical calculations of entanglement entropy are in general notoriously difficult to perform, even for free theories. In theories with a (standard) gravity dual, however, entanglement entropies can be computed in a rather straightforward manner using an elegant holographic prescription due to Ryu and Takayanagi [18, 19] (see [52] for a review). Suppose we want to compute the entanglement entropy associated with a spatial region A in the field theory. The R-T recipe instructs us to construct the minimal spacelike surface γ_A that is anchored at the boundary ∂A of A and extends into the bulk spacetime. Then, the corresponding entanglement entropy is obtained in terms of the area of γ_A as $S_A = \mathrm{Area}(\gamma_A)/(4G)$, where G is the Newton constant associated to the bulk spacetime. The prescription correctly reproduces the area law of entanglement entropy, and it has been shown to be strongly-subadditive [53] as well. Moreover, it has been generalized to include cases where the field theory state is time-dependent [54]. Strong evidence for the correctness of the R-T prescription has been given in [55, 56].

In the present context we will focus on situations where the full system is a $(1+1)$-dimensional CFT, and consider subsystems determined by spatial (equal time) intervals. Due to the large amount of symmetry that $(1+1)$-dimensional CFTs enjoy, a variety of quantities can be computed in closed form. Indeed, using CFT techniques, universal results have been derived for the single-interval entanglement entropy at zero and finite temperature [13, 14]. From the point of view of holography, in the particular case of a three-dimensional bulk and a two-dimensional boundary theory the minimal surface prescription of R-T amounts to finding the length of a geodesic in an asymptotically AdS$_3$ (AAdS$_3$) spacetime, and it correctly reproduces the known field theory results. On the other hand, to the extent of our knowledge there are no analytic results for entanglement entropy in the presence of deformations by higher spin currents, or in states carrying non-trivial higher spin charges, and we would therefore like to extend the holographic calculations to encompass these situations.

Since the bulk theory under consideration is topological, a reasonable starting point is to rephrase the geometric statement of the R-T proposal in terms of the natural building blocks at our disposal in the gauge theory, such as Wilson lines. To this end, given two points P and Q in the bulk spacetime, we start by considering the following "composite" Wilson loop

$$W_{\mathcal{R}}(P,Q) \equiv \mathrm{Tr}_{\mathcal{R}}\left[\mathcal{P}\exp\left(\int_Q^P \bar{A}\right)\,\mathcal{P}\exp\left(\int_P^Q A\right)\right] \tag{3.2}$$

where $\mathcal{P}$ denotes the usual path ordering, and the trace is evaluated in a representation that will be specified later on. As we have discussed, the gauge connections undergo a gauge transformation upon transport around a cycle with non-trivial holonomy; locally, however, we can write the flat connections as in (2.14), and $W(P,Q)$ reduces to

$$W_{\mathcal{R}}(P,Q) = \mathrm{Tr}_{\mathcal{R}}\Big[\bar{g}^{-1}(P)\bar{g}(Q)g^{-1}(Q)g(P)\Big]. \tag{3.3}$$

Up to global issues (such as winding around a non-contractible cycle), we see that the result is path-independent, i.e. it depends on the positions of the endpoints P and Q only. One may worry about the lack of obvious gauge invariance of this expression, but as we explain below this is no cause for concern. One may also worry that different, homotopically inequivalent paths may exist that connect P and Q, on which the answer clearly depends. We will address this issue below as well.

To gain some intuition about the significance of the functional (3.2), we first evaluate it for the AAdS$_3$ solutions of the $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ theory. Plugging in the solutions (2.15) with constant

T, $\bar{T}$ and taking the trace in the $2d$ (defining) representation (B.3) of $sl(2,\mathbb{R})$ we obtain

$$\begin{aligned} W_{2d}(P,Q) &= 2\cosh(\rho_P-\rho_Q)\cosh\left[\sqrt{\frac{T}{k}}\left(x_P^+-x_Q^+\right)\right]\cosh\left[\sqrt{\frac{\bar{T}}{k}}\left(x_P^--x_Q^-\right)\right] \\ &-\left(\frac{k}{\sqrt{T\bar{T}}}e^{\rho_P+\rho_Q}+\frac{\sqrt{T\bar{T}}}{k}e^{-(\rho_P+\rho_Q)}\right)\sinh\left[\sqrt{\frac{T}{k}}\left(x_P^+-x_Q^+\right)\right]\sinh\left[\sqrt{\frac{\bar{T}}{k}}\left(x_P^--x_Q^-\right)\right]. \end{aligned} \tag{3.4}$$

Bulk quantities in AdS/CFT are usually divergent as $\rho\to\infty$, reflecting the short distance (UV) divergences in the dual field theory. The simplest way to regulate such divergences is to place the boundary on a $\rho=\rho_0$ slice, with finite ρ_0, and take $\rho_0\to\infty$ at the end. Let us then push the points P, Q in (3.4) to the regularized conformal boundary of AdS$_3$, i.e. $\rho_P=\rho_Q=\rho_0\to\infty$. Our basic observation is that the composite loop $W(P,Q)$ in the fundamental representation of $sl(2,\mathbb{R})$ is related to the length $d(P,Q)$ of the geodesic anchored at P and Q as

$$W_{2d}(P,Q)=2\cosh d(P,Q)\xrightarrow[\rho_0\to\infty]{}\exp d(P,Q)\,, \tag{3.5}$$

where we used the fact that the geodesic length becomes large (divergent, in fact) as we push the points to the boundary. As we have discussed, for standard gravitational theories in the bulk the geodesic distance is intimately related to entanglement entropy in the dual theory via the R-T prescription. The functional (3.2) rephrases this result in a language appropriate to the Chern-Simons theory, and it is moreover well-defined in the higher spin theory as well. Motivated by this fact, for points P and Q on a Cauchy slice on the boundary, defining a spacelike interval A, we propose to consider the functional

$$S_A\equiv\frac{k_{cs}}{\sigma_{1/2}}\log\left[\lim_{\rho_0\to\infty}W_{\mathcal{R}}(P,Q)\Big|_{\rho_P=\rho_Q=\rho_0}\right] \tag{3.6}$$

as a candidate entanglement entropy in the $2d$ CFTs dual to the three-dimensional higher spin theories. Here, k_{cs} is the Chern-Simons level defined in (2.17). The constant $\sigma_{1/2}$ takes the value 2 if there are half-integer spin currents in the spectrum, and 1 otherwise; its origin will be explained in section 3.2.2. As we will discuss below, the choice of representation $\mathcal{R}$ in (3.6) depends on both N and the choice of embedding of $sl(2)$ into $sl(N)$. In the $N=2$ case, a change in the chosen representation can be compensated by changing the prefactor in (3.6). For example, evaluating (3.2) in a three-dimensional representation of $sl(2)$, instead of the fundamental, one obtains $W_{3d}(P,Q)=-1+\Big(W_{2d}(P,Q)\Big)^2\xrightarrow[\rho_0\to\infty]{}\Big(W_{2d}(P,Q)\Big)^2$, and (3.6) would remain invariant if we simultaneously divide the prefactor by two. Using the Chern-Simons level as the coefficient (up to $\sigma_{1/2}$) appears as a natural choice from the bulk perspective, which does not rely on details of the representation. Once the coefficient is fixed in this way, we will select the representation based on physical requirements. We emphasize that for $N>2$ the above functional does not have an obvious geometric interpretation: it is determined purely in terms of the gauge connections, as appropriate to the topological character of the bulk theory, and in particular it does not require the identification of a metric tensor.

A comment is in order about the symmetries that (3.6) is expected to have. Under a general gauge transformation, a Wilson line $U(P,Q) = \mathcal{P}\exp\left(\int_Q^P A\right)$ transforms as $U \to h^{-1}(P)U(P,Q)h(Q)$. Since A is valued in the sl_L ("left") algebra and $\bar{A}$ is valued in the sl_R ("right") algebra, it is clear that the composite loop $W_{\mathcal{R}}(P,Q)$ in (3.2) is invariant under the diagonal subgroup parameterized by $h = \bar{h}$. As reviewed above, in the pure gravity case ($N = 2$) the diagonal subgroup corresponds to rotations of the local Lorentz frame (c.f. (2.7)), so such invariance is very natural. Also, in the pure gravity case, off-diagonal gauge transformations correspond to a shift of the endpoints as is clear from (2.7), and the geodesic length is obviously not invariant under such shifts. More importantly, even though (3.2) is not invariant under a general gauge transformation, in (3.6) we are only assigning a field-theoretical interpretation to its leading UV (large ρ_0) divergence. Now recall that the asymptotic behavior of the gauge fields encodes the state of the system, and that gauge transformation that change the asymptotic behavior change the state of the system and are not true symmetries, whereas gauge transformations that leave the asymptotic behavior invariant are true symmetries. Gauge transformations of the first type belong to the so-called asymptotic symmetry group of the system. We therefore see that the entanglement entropy is only invariant under those gauge transformations that act trivially on the state, and not under those that modify the state, exactly as expected.

In the absence of explicit field theory results for entanglement entropy in $2d$ CFTs deformed by higher spin currents (or in non-trivial states carrying higher spin charges), we will content ourselves with testing the plausibility of our proposal. Firstly, we will explicitly check that it allows us to recover the known CFT results in the absence of higher spin charges. Secondly, it will reproduce the thermal entropy in the limit in which the von Neumann entropy becomes extensive. Finally, we will check that the functional S_A satisfies the strong subadditivity property of entanglement entropy (up to subtleties that we will discuss in due course).

3.1 Recovering standard results

We will now show that our prescription, when applied to solutions of pure gravity (i.e. in the absence of higher spin charges), allows us to recover the known results for the single-interval entanglement entropy in $2d$ CFTs [14]. With the result (3.4) for AAdS$_3$ solutions in hand we can easily compute (3.6) for the rotating BTZ black hole, characterized by (2.9), as well as global AdS$_3$ (2.10), and the Poincaré-patch of AdS$_3$ (the latter with $T = \bar{T} = 0$). From the dual field theory point of view, the Poincaré-patch and global AdS$_3$ backgrounds correspond to the CFT ground state (i.e. at zero temperature) on the infinite line and on a system with periodic boundary conditions, respectively.

The rotating black hole background, in turn, corresponds to computing the entanglement entropy in a finite temperature state with a potential for angular momentum. In the $N = 2$ case (2.18) implies $c = 6k_{cs}$ and (3.6) yields

$$\text{Poincaré-patch:} \qquad S_{PAdS_3} = \frac{c}{3} \log \left[\frac{\Delta x}{a} \right] \tag{3.7}$$

$$\text{global:} \qquad S_{AdS_3} = \frac{c}{3} \log \left[\frac{\ell}{a} \sin \left(\frac{\Delta \varphi}{2} \right) \right] \tag{3.8}$$

$$\text{black hole:} \qquad S_{BTZ} = \frac{c}{6} \log \left[\frac{\beta_+ \beta_-}{\pi^2 a^2} \sinh\left(\pi \frac{\Delta x}{\beta_+} \right) \sinh\left(\pi \frac{\Delta x}{\beta_-} \right) \right] \tag{3.9}$$

where we defined the "lattice spacing" a in terms of the radial cutoff ρ_0 as

$$a = \ell e^{-\rho_0} \,, \tag{3.10}$$

and dropped contributions that are subleading as $\rho_0 \to \infty$, as instructed by the limit in (3.6). In the results for Poincaré AdS and the BTZ black hole we have defined $\ell\varphi \to x$. Similarly, $\Delta x \equiv x_P - x_Q$ and $\Delta\varphi \equiv \varphi_P - \varphi_Q$. The zero temperature results, as well as the finite temperature result in the absence of rotation were first reproduced using holography in [18]. The rotating case lies beyond the scope of the original R-T prescription, however, because in a stationary (but not static) spacetime the extremal surface anchored at the spacelike interval in the boundary does not necessarily lie on a constant-time slice in the bulk. The corresponding result was later obtained with the refined covariant prescription put forward in [54]. It is reassuring that our prescription in terms of Wilson lines encompasses all these cases simultaneously, and with a minimal calculational effort.

A comment is in order regarding the black hole result (3.9). As written, this result is valid for "planar" black holes only. The same result holds for globally-defined black holes where the boundary is S^1, but only for sufficiently small Δx, see section 3.2. It is somewhat remarkable that a universal answer exists even in this case, where the result is not determined by conformal symmetry. Since the black hole also has a temperature, the corresponding CFT is defined on a torus. Field-theoretical calculations of entanglement in $2d$ CFTs are usually performed using the so-called *replica trick*, and (for the single-interval case) they effectively boil down to the calculation of a two-point function of twist operators (see e.g. [14]). In the cylinder (or the plane) the form of this two-point function is completely fixed by symmetry (Ward identities), and hence universal. As pointed out in [57], however, the corresponding correlators on the torus depend not only on the conformal weights, but on specific details of the theory such as the operator content. From the holographic point of view, it is conceivable that these non-universalities are washed out in the semiclassical (large-c) limit, much in the same way that non-universalities associated with multiple intervals (rather than finite size effects) have been recently shown to be subleading in the large central charge regime [20, 21].

We stress that even though our formula (3.6) correctly reproduces the entanglement entropy in the absence of higher spin charges and chemical potentials, there is no *a priori* guarantee that it will still compute the entanglement entropy in the higher spin cases. In what follows we will amass more evidence in favor of this interpretation.

3.2 Thermal entropy and the choice of representation

When computed at finite temperature, the von Neumann entropy associated with the reduced density matrix ρ_A receives contributions from classical correlations that mix with the quantum correlations due to "true" entanglement. For fixed temperature, as subsystem A grows in size the reduced density matrix approaches the thermal density matrix of the full system; by definition, the entanglement entropy then becomes the thermal entropy (up to subtraction of short distance divergences). In our discussion of this limit we will distinguish between a system that is infinitely-extended in the spatial direction, and one that is compact (i.e. a circle). In the holographic context the former arises as the boundary of a planar black hole, while the latter corresponds to a globally-defined black hole geometry.

Let us start with the case of a compact system. The cycle parameterizing the horizon of a global black hole is non-contractible, and the non-trivial topology of the bulk manifold in this case makes the definition of holographic entanglement entropy subtle. Let us first review this issue in the context of the R-T proposal, and the global BTZ black hole. As depicted in figure 1, in the presence of a black hole there are in general two geodesic configurations that are homologous to an interval in the boundary. For a fixed temperature (i.e. fixed horizon size), if the boundary interval is small, the corresponding minimal surface is a connected geodesic that does not wrap the horizon. For a sufficiently large boundary interval, the minimal surface will instead be a disconnected sum of two components, one of which is a loop around the black hole horizon [58]. The length of the latter curve effectively computes the black hole horizon area (length), and hence its thermal entropy via the Bekenstein-Hawing formula. Equivalently, for a fixed interval size, the minimal surface can change from connected to disconnected as a function of the size of the horizon (temperature). Incidentally, this shows that the limits of high temperature and large subsystem size do not commute.[2]

We now ask how is the thermal entropy recovered in the context of our proposal, focusing on the global BTZ solution (c.f. (2.9)) as an example. The entropy of the BTZ black hole (and of the dual theory defined on a torus) is given by

$$S_{thermal} = 2\pi\sqrt{kT} + 2\pi\sqrt{k\bar{T}} = 2\pi\sqrt{\frac{c}{6}T} + 2\pi\sqrt{\frac{c}{6}\bar{T}}\,, \tag{3.11}$$

which moreover takes the form predicted by Cardy's asymptotic growth of states in a unitary CFT.

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[2] We thank Mukund Rangamani for emphasizing the importance of this issue to us.