Revised Version

# QUANTUM GAUGED NEURAL NETWORK: U(1) GAUGE THEORY

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## ABSTRACT

A quantum model of neural network is introduced and its phase structure is examined. The model is an extension of the classical Z(2) gauged neural network of learning and recalling to a quantum model by replacing the Z(2) variables, $S_i = \pm 1$ of neurons and $J_{ij} = \pm 1$ of synaptic connections, to the U(1) phase variables, $S_i = \exp(i\varphi_i)$ and $J_{ij} = \exp(i\theta_{ij})$. These U(1) variables describe the phase parts of the wave functions (local order parameters) of neurons and synaptic connections. The model takes the form similar to the U(1) Higgs lattice gauge theory, the continuum limit of which is the well known Ginzburg-Landau theory of superconductivity. Its current may describe the flow of electric voltage along axons and chemical materials transfered via synaptic connections. The phase structure of the model at finite temperatures is examined by the mean-field theory, and Coulomb, Higgs and confinement phases are obtained. By comparing with the result of the $Z(2)$ model, the quantum effects is shown to weaken the ability of learning and recalling.

## 1. INTRODUCTION

To study rich activities of human brains, there are various approaches. A typical one is neural networks. Various models of neural networks have been proposed. The Hopfield model of associative memory[1] has offered us a good explanation of the mechanism how we recall patterns. On the other hand, the perceptron or its improvement, the back-propagation model[2], may be a representative model of learning.

In Ref.[3, 4], yet another network model is proposed, which is an extension of the Hopfield model to a model of learning by treating the strength $J_{ij}$ of the synaptic connection between $i$-th and $j$-th neurons as an independent dynamical variable. Both the neuron variables $S_i = \pm 1$ and the new variables $J_{ij} = \pm 1$ are treated on an equal footing. $J_{ij}$ is viewed as a "connection" of gauge theory[5], and the energy $E(\{S_i\}, \{J_{ij}\})$ is postulated to possess the local $Z(2)$ gauge symmetry. The gauge symmetry assures us that the time evolutions of $S_i$ and $J_{ij}$ occur through local (contact) interactions as they should be.

Among approaches other than neural networks, there is a quantum-theoretical approach to the brain activities. Stuart, Takahashi and Umezawa[6] proposed a microscopic quantum field theory by using operators expressing neurons and intermediating bosons. They proposed that memory should be stored in low-energy modes like Goldstone bosons. Jibu and Yasue[7] argued that their quantum brain model may be regarded as a practical model of dipoles of ordered water and evanescent (massive) photons in the brain.

Another quantum approach is advocated by Penrose. [8] He insists on the relevance of quantum theory like the problem of observations in quantum mechanics, coupling to quantum gravity, and so on. It seems ambitious, but interesting and worth enough to scrutinize its validity. Hameroff and Penrose[9] proposed a quantum theory of consciousness. They claim that objective reductions of wave functions of microtubules, main building blocks of axons connecting neurons, are relevant for our consciousness. The central physical quantity in their theory is the so called decoherence time $\tau$, the average time interval between successive reductions. $\tau$ corresponds to each "moment" of the stream of one's consciousness. There are several estimates of $\tau$[10, 11], but they seem to be still controvercial each other.

In this paper, we introduce a quantum version of the gauged neural network of learning and recalling[3, 4]. This quantum neural network is regarded as an effective (phenomenological) model at macroscopic scales derived from the underlying microscopic quantum theory of brain. The purpose of this neural network model is to explore the difference between classical and quantum neural networks and eventually to find the possible relevance of quantum natures in the activities of human brains. The structure of the paper is as follows; In Sect.2, we introduce the quantum gauged model. In Sect.3, we study the phase structure of the model at finite "temperatures" $T$. In Sect.4 we present conclusions and future problems.

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## 2. QUANTUM GAUGE MODEL

In this section, we first explain the relevance of gauge symmetry. Next we discuss the possible ways to include quantum effects. Then we propose an explicit model and the rule of time evolution.

### 2.1. Gauge Symmetry

In the Hopfield model, the state of $i$-th neuron (active or inactive) is described by the $Z(2)$ variable $S_i(=\pm 1)$, and the state of the synaptic connection between $i$-th and $j$-th neurons is expressed by its strength $J_{ij}$, which is a preassigned constant. The signal at the $j$-th site at time $t$, $S_j(t)$, propagetes to the $i$-th site through the axon and the synaptic connection in the form $V_{ij}S_j$ to affect the state $S_i$ in the next time step $t+\Delta t$;

$$S_i(t+\Delta t) = sgn\left[\sum_j J_{ij}S_j(t)\right]. \qquad (1)$$

The time evolution (1) is known to decrease (not increase) the "energy",

$$E = -\frac{1}{2}\sum_{i,j} S_i J_{ij} S_j. \qquad (2)$$

In order to study processes of learning certain patterns of $S_i$, it is necessary to allow for the time variation of $J_{ij}$. There are various proposals how to treat the dynamics of $J_{ij}$. The idea in Ref.[3, 4] is to regard $J_{ij}$ as a connection variable $U_{xy}$ of gauge theory[12]. This is quite natural because the connection describes the way how a pair of two points are connected, i.e., how two internal coordinates are related. In fact, $U_{xy}$ transports a quantity $\varphi_y$ (e.g., a vector) at a point $y$ to another point $x$ via the "parallel-translate, the result being $U_{xy}\varphi_y$. (See Fig.1.) One may view the signal $J_{ij}S_j$ in the Hopfield model as the result of parallel-translating $S_j$ to the $i$-th site by regarding $J_{ij} \to U_{xy}$, $S_j \to \varphi_y$. Since the connection is an independent variable in nature, $J_{ij}$ is no more a constant but should be time-dependent. Both $S_i(t)$ and $J_{ij}(t)$ should be treated on an equal footing. Once the system is regarded as a gauge system, it should possess the gauge symmetry. That implies the energy $E(\{S_i\},\{J_{ij}\})$ is invariant under gauge transformations.

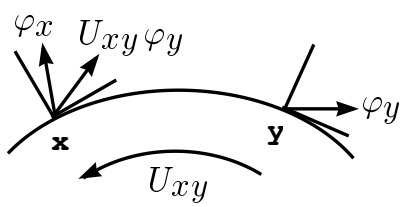



Fig.1. Function of gauge connection $U_{xy}$. $U_{xy}$ parallel-translates a vector $\varphi_y$ at the point $y$ to another point $x$ giving rise to $U_{xy}\varphi_y$. To compare $\varphi_y$ with a vector $\varphi_x$ at $x$, one should take the gauge invariant scalar product $(\varphi_x, U_{xy}\varphi_y) \equiv \varphi_x^\dagger U_{xy}\varphi_y$ instead of $(\varphi_x,\varphi_y) = \varphi_x^\dagger\varphi_y$.

To be general, let us consider a gauge group $G$. Then we prepare gauge variables $J_{ij} \in G$ (unitary representation (unitary matrix) of a group element) for each pair $i,j$ and neuron variables $S_i \in G$ (fundamental representation (vector)) for each point $i$. The gauge symmetry is local, that is the following gauge transformation can be performed at each point $i$ independently;

$$\begin{aligned} S_i &\to S_i' \equiv V_i S_i, \\ J_{ij} &\to J_{ij}' \equiv V_i J_{ij} V_j^\dagger, \\ V_i &\in G, \end{aligned} \qquad (3)$$

where $V_i$ is a unitary matrix ($V_i^\dagger = V_i^{-1}$). The gauge invariance of $E$ is expressed as

$$E(\{S_i'\},\{J_{ij}'\}) = E(\{S_i\},\{J_{ij}\}). \qquad (4)$$

A simple example of the energy is

$$E = -\frac{1}{4}\sum_{i,j}\left(S_i^\dagger J_{ij} S_j + \text{c. c}\right). \qquad (5)$$

Note that the gauge invariance of $E$ holds at $j$, for example, since $V_j$ supplied by $S_j'$ cancels with $V_j^\dagger$ supplied by $J_{ij}'$ (note $V_j^\dagger V_j = 1$). For $G = Z(2)$, this $E$ reduces to the Hopfield energy (2).

To consider generalization of the energy, the principle of gauge symmetry puts severe restrictions on $E$. Actually, the gauge principle implies that the time evolutions of $S_i$, $J_{ij}$, which we shall discuss in Sect.2.4 in details, are controlled only by those signals that have contacts with them. For example, $dS_i/dt$ consists of terms, each of which has the index $i$ like $V_{ij}S_j$. This assures us that the flows (current) of electric voltages and chemical materials change locally through contact intereactions as it should be.

### 2.2. Quantum Effects

Most of the proposed models of neural networks so far is classical in the sense that these models employ real numbers as their dynamical variables. Although there are many successful phenomenological models in the framework of classical physics in various fields of physics, every physical system is necessarily "quantum" in its origin. Neural networks are not an exception at all.

From the microscopic point of view, main functions of our brains should be the result of underlying microscopic systems, basic constituents of which are electrons and various chemical materials. The quantum brain theory of Stuart, Umezawa and Takahashi[6] may be viewed as such a microscopic model.[7] As another approach, the recent quantum-theoretical study of consciousness by Hameroff and Penrose [9] are also interesting since they focus on a microtubule and start form its microscopic model itself. Actually, they consider a two-dimensional system of electrons and its wave function. The time dependence of wave function, particularly its objective reductions, is argued to be important for understanding consciousness. In Ref.[11] a quantum-field-theoretical model of a microtubule is proposed, Hamiltonian of which is described by second-quantized fermionic electron operators. The model resembles familiar strongly-correlated electron systems like Hubbard model, Heisenberg model, t-J model, etc.

It is quite interesting to compare these quantum models and existing classical neural-network models to identify the quantum effects. However, to perform such a comparison explicitly, the present forms of these quantum models are not appropriate; they involve quantum operators and have complicated structures. Thus it is preferable to obtain their effective models (at lower energies, i.e., at macroscopic scales) that take forms similar to the classical neural networks. (See Fig.2.)

At this point, we recall the relation between the BCS (Bardeen-Cooper-Schriefer) model of superconductivity and the GL (Ginzburg-Landau) model of second-order transition. The BCS model is the basic microscopic model of electrons and its variables are electron operators $C_\sigma(x)$ at spatial point $x$, while the GL model is the phenomenological model, and its variables are an

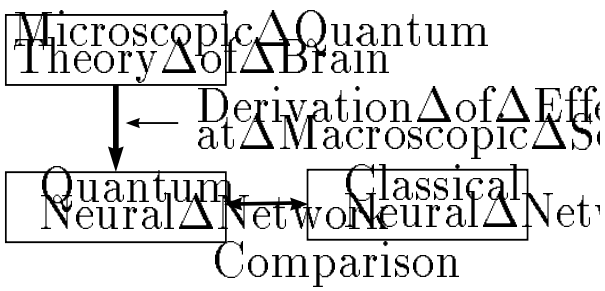



Fig.2. Relation between various models. To study the quantum effects upon human brain, one should compare classical neural network models and quantum neural network models. The latter is derived from the underlying microscopic quantum models as effective (phenomenological) models at macroscopic scales.

order-parameter field $\phi(x)$, i.e., complex numbers that describe quantum amplitudes of Cooper pairs of electrons. The relation between two sets of variables are

$$\phi(x) \quad = \quad \langle C_\uparrow(x) C_\downarrow(x) \rangle, \tag{6}$$

where the brackets implies a statistical average over the canonical ensamble at temperature $T$.

The GL theory was originally introduced as a phenomenological model of superconductivity, but now one can derive it from the BCS model systematically as its effective model by using path-integral techniques[13]. In fact, one starts from the BCS Hamiltonian $H_{\rm BCS}$ to obtain the GL free energy $F_{\rm GL}$ as

$$\begin{aligned} Z_{\rm BCS} &\equiv \mathrm{Tr}\ e^{-\beta H_{\rm BCS}(C)} \\ &= \int [dC] e^{A_{\rm BCS}(C)} = \int [dC][d\phi] e^{A(C,\phi)} \\ &\simeq \int [d\phi] e^{-\beta F_{\rm GL}(\phi)}, \end{aligned} \tag{7}$$

where $\beta = 1/(k_{\rm B}T)$ and the complex field $\phi(x)$ is introduced as an auxiliary field (integration variable) via the Hubbard- Stratonovich transformation.[13]

Because the GL theory may be viewed as a prototype of our neural-netrwork model of Sect.2.3, let us explain it in some detail. If we consider the system in a magnetic field, $F_{\rm GL}$ is written (we set $\hbar = c = 1$) as

$$F_{\rm GL} = \int d^3x \left[ |D_\mu \phi|^2 + a|\phi|^2 + b|\phi|^4 + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \right] \tag{8}$$

where $\mu = 1, 2, 3$ is the direction index, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the magnetic field, $A_\mu(x)$ is the vector potential (connection of gauge theory), and $D_\mu = \partial_\mu - 2ieA_\mu$ is the covariant derivative. The coefficients $a = \alpha(T - T_c)$, $\alpha > 0$, $b$, and the critical temperature $T_c$ are calculable and expressed by the parameters of the BCS model. $F_{\rm GL}$ is invariant under the local $U(1)$ gauge transformation,

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = e^{2ie\alpha(x)} \phi(x), \\ A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x). \end{aligned} \tag{9}$$

The order parameter in zero magnetic field behaves as

$$\langle \phi(x) \rangle \quad = \quad \begin{cases} \left[ \dfrac{\alpha(T_c - T)}{2b} \right]^{1/2}, & T \le T_c, \\ 0, & T_c < T. \end{cases} \tag{10}$$

The equations of motion are obtained from $\delta F_{\rm GL}/\delta\phi(x) = 0$, $\delta F_{\rm GL}/\delta A_\mu(x) = 0$ as

$$\begin{aligned} &\big( - D_\mu D_\mu + a + 2b|\phi|^2 \big)\phi = 0, \\ &\vec{\nabla} \times \vec{B} = \vec{j}, \ \vec{B} = \vec{\nabla} \times \vec{A}, \\ &\vec{j} = -2ie\,(\bar{\phi}\ \vec{\nabla}\phi - \vec{\nabla}\bar{\phi}\ \phi) - 8e^2 |\phi|^2 \vec{A}. \end{aligned} \tag{11}$$

Thus, we seek for an effective neural-network model that corresponds to the GL model of the BCS model, although we don't specify the details of the underlying quantum model of the brain. Explicitly, we introduce a $U(1)$ variable $S_i = \exp(i\varphi_i) \in U(1)$ to describe the quantum state of the $i$-th neuron and a $U(1)$ variable $J_{ij} = \exp(i\theta_{ij}) \in U(1)$ to describe the quantum state of the axon (synaptic connection) connecting $i$ and $j$-th neurons. Physically, $S_i$ may be viewed as a wave function of the $i$-th neuron which is in a quantum-mechanical coherent state of its microscopic constituents, i.e., electrons, chemical materials, and so on. Likewise, $J_{ij}$ is a wave function of a coherent state of the axon. We note that the idea that an axon is well expressed by a single coherent state is consistent with the theory of Hameroff and Penrose.[9]

Furthermore, the requirement of gauge symmetry is naturally incorporated into this assignement of $U(1)$ variables by regarding $S_i$ as a charged matter field and $J_{ij}$ as an exponentiated gauge connection.

### 2.3. U(1) Gauge Model

Let us formulate the model on a three-dimensional cubic lattice. We specify each site by the site-index $x$ and use $\mu = 1, 2, 3$ as the direction index. We use $\mu$ also as the unit vector in the $\mu$-th direction. We set the lattice spacing $a = 1$ for simplicity. As explained in Sect.2.2, for each site $x$ we put a $U(1)$ variable,

$$S_x = \exp(i\varphi_x), \tag{12}$$

and for each link $(x\mu) \equiv (x, x+\mu)$, i.e., for nearest-neighbor (NN) pair of sites, we put another $U(1)$ variable,

$$J_{x\mu} = \exp(i\theta_{x\mu}). \tag{13}$$

The local gauge transformation is expressed as

$$\begin{aligned} S_x &\rightarrow V_x S_x, \\ J_{x\mu} &\rightarrow V_{x+\mu} U_{x\mu} \bar{V}_x, \end{aligned} \tag{14}$$

where $V_x = \exp(i\alpha_x)$ and the bar implies the complex conjugate (e.g., $\bar{S}_x \equiv \exp(-i\varphi_x)$).

We propose the following gauge-invariant energy $E$;

$$\begin{aligned} E &= -\frac{c_1}{2}\sum_x\sum_\mu \left(\bar{S}_{x+\mu} J_{x\mu} S_x + \text{c.c}\right) \\ &\quad -\frac{c_2}{2}\sum_x\sum_{\mu>\nu} \left(\bar{J}_{x\nu}\bar{J}_{x+\nu,\mu} J_{x+\mu,\nu} J_{x\mu} + \text{c.c}\right) \\ &\quad -\frac{c_3}{2}\sum_x\sum_\mu\sum_{\nu(\neq\mu)} \left(\bar{S}_{x+\mu}\bar{J}_{x+\mu,\nu} J_{x\nu} J_{x+\nu,\mu} S_x \right. \\ &\quad \left. +\bar{S}_{x+\mu} J_{x-\nu+\mu,\mu} J_{x-\nu,\mu}\bar{J}_{x-\nu,\nu} S_x + \text{c.c}\right). \end{aligned} \tag{15}$$

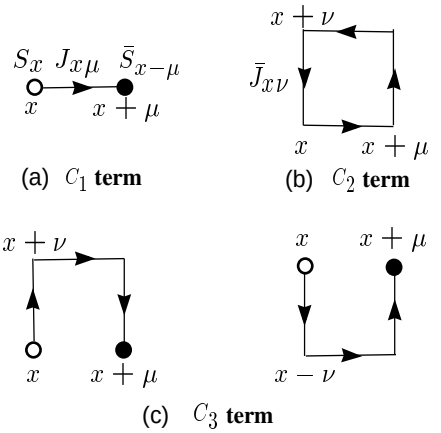



Fig.3. Graphical representation of each term in the energy $E$ of (15). Open circles for $S_x$. Filled circles for $\bar{S}_x$. Straight lines for $J_{x\mu}$. The arrows distinguish $J_{x\mu}$ and $\bar{J}_{x\mu}$. The gauge invariance requires the lines with arrows should (i) start from open circles, (ii) end at filled circles, (iii) continue in a single direction.

Each term in (15) is depicted in Fig.3. $E$ of (15) reduces to the energy of the $Z(2)$ model[4] if we replace $S_x, J_{x\mu}$ by $Z(2)$ variables.

Let us discuss the continuum limit ($a \rightarrow 0$) of $E$. Following Wilson[5] we write $\theta_{x\mu}$, the phase of $J_{x\mu}$, as $\theta_{x\mu} = gaA_\mu(x)$ where $g$ is the gauge coupling constant, $a$ is t. To take the limit $a \rightarrow 0$, we expand $J_{x\mu}$ w.r.t. $a$ as $J_{x\mu} = \exp(igaA_\mu(x)) \simeq 1 + igaA_\mu(x) - g^2a^2A_\mu(x)^2/2 + O(a^3)$. Also we scale $S_x \propto a^{1/2}\phi(x)$. Then, by taking $c_i$ appropriately, we find

$$\begin{aligned} c_1,\ c_3 \text{ terms} &\rightarrow |D_\mu\phi|^2 \text{ and } |\phi|^2 \text{ terms}, \\ c_2 \text{ term} &\rightarrow F_{\mu\nu}F_{\mu\nu} \text{ term}. \end{aligned} \tag{16}$$

Although we introduced the $U(1)$ GL theory (8) just as a typical example of an effective theory of a microscopic quantum model, it now serves as a continuum limit of the present lattice model.[14] One can draw some useful informations using this relation. For example, one may obtain a "current" $j_{x\mu}$ on the lattice as

$$\begin{aligned} j_{x\mu} &= \frac{\delta E|_{c_1 \text{and} c_3 \text{terms}}}{\delta\theta_{x\mu}} \\ &= -\frac{i}{2}\left(c_1\bar{S}_{x+\mu}J_{x\mu}S_x - \text{c.c.}\right) + \cdots, \end{aligned} \tag{17}$$

which reduces to $\vec{j}(x)$ of (11) in the continuum limit. $j_{x\mu}$ is gauge invariant and may be useful to describe the state of the system.

### 2.4. Time Evolution

Let us consider the dynamics of $S_x(t)$ and $J_{x\mu}(t)$. As in the Hopfield model and Z(2) gauge model, we let

the energy $E$ basically decreases as the time increases with some rate of failures. These failures are caused by misfunctioning of signal processings due to noises, etc., and may be controlled by the "temperature" $T$; For higher(lower) $T$, failures occur more(less). This $T$ should not be confused with the physical temperature of the brain, although there may be some correlations among them.

As explicit rules of time evolution, the following two are possible;

**(I) Metropolis algorithm (MA):**

MA[15] is a standard algorithm to calculate the thermal averages $\langle O(\{S_x\},\{J_{x\mu}\})\rangle$ over Boltzmann distribution,

$$\langle O\rangle \;=\; \frac{1}{Z}\int [dS][dJ]\; O\;\exp(-\beta E), \qquad (18)$$

by generating a Markov(stochastic) process $\{S_x(\ell\Delta t)\}$, $\{J_{x\mu}(\ell\Delta t)\}$ $(\ell = 1,2,\cdots,M)$ as

$$\langle O\rangle \;=\; \lim_{M\to\infty}\frac{1}{M}\sum_{\ell=1}^{M} O(\{S_x(\ell\Delta t)\},\{J_{x\mu}(\ell\Delta t)\}). \qquad (19)$$

By identifying $\ell\Delta t$ as the real time $t$, one may use this Markov process $\{S_x(\ell\Delta t)\},\{J_{x\mu}(\ell\Delta t)\}$ itself just as their time evolutions as proposed in the $Z(2)$ gauge model.[3, 4] The rates of changes in variables are controllable by adjusting some parameters contained in MA.[4] In particular, $S_x$ and $J_{x\mu}$ may have different rates. If $J_{x\mu}$ change much slower than $S_x$, it may be more suitable to first take an ensamble average over $S_x$ for fixed $J_{x\mu}$ and then take average over different $J_{x\mu}$ as in the theory of spin glass.[16]

**(II) Langevin equation:**

Langevin equations[17] are stochastic equations for continuous variables. Since $U(1)$ variables are constrained (e.g., $\bar{S}_x S_x = 1$), it is preferable to focus on their phases, i.e. angles (mod$(2\pi)$) as independent variables. Then one has

$$\begin{aligned}\alpha_\varphi\frac{d\varphi_x}{dt} &= -\frac{\partial E}{\partial\varphi_x}+\sqrt{2T}\eta_{\varphi_x},\\ \alpha_\theta\frac{d\theta_{x\mu}}{dt} &= -\frac{\partial E}{\partial\theta_{x\mu}}+\sqrt{2T}\eta_{\theta_{x\mu}},\end{aligned} \qquad (20)$$

where $\alpha_{\varphi,\theta}$ are parameters to fix the time scales, and $\eta_{\varphi,\theta}$ are random white noises specified by their averages,

$$\begin{aligned}\langle\eta_a(t)\rangle &= 0,\\ \langle\eta_a(t_1)\eta_b(t_2)\rangle &= \delta_{ab}\delta(t_1-t_2).\end{aligned} \qquad (21)$$

In the energy $E$, the term $c_1$, which corresponds to the energy of the Hopfield model, describes the direct transfer of signal from $x$ to $x+\mu$. The term $c_2$ describes the self energy after the transfer of signal through the contour $(x\to x+\mu\to x+\mu+\nu\to x+\nu\to x)$. It may express the energy of circular currents. The term $c_3$ describes indirect transfers of signal from $x$ to $x+\mu$ via the bypath, $(x\to x+\nu\to x+\nu+\mu\to x)$.

At first, it may look strange that there appear the $c_2$ and $c_3$ terms in $E$, which contain direct contacts (products) of two connection variables like $J_{x\mu}$ and $J_{x+\mu,\nu}$, because each synapse connection necessarily contacts with a neuron but not with a neaby synapse. However, two successive transfers like $S_x\to S_{x+\mu}$ and $S_{x+\mu}\to S_{x+\mu+\nu}$ are described as a product of corresponding factors as

$$\begin{aligned}&\bar{S}_{x+\mu+\nu}J_{x+\mu,\nu}S_{x+\mu}\times\bar{S}_{x+\mu}J_{x\mu}S_x\\ =\;&\bar{S}_{x+\mu+\nu}J_{x+\mu,\nu}J_{x\mu}S_x\end{aligned} \qquad (22)$$

due to $S_{x+\mu}\bar{S}_{x+\mu}=1$. This explains why the terms like $c_2$ and $c_3$-terms may appear in $E$. Another explanation is given in Ref.[4] based on the renormalization group.

## 3. PHASE STRUCTURE

### 3.1. Mean Field Theory

The MFT may be formulated as a variational method[18] fot the Helmholtz free energy $F$;

$$\begin{aligned}Z &= \int[dS][dJ]\exp(-\beta E)\equiv\exp(-\beta F),\\ \int[dS] &\equiv \prod_x\int_0^{2\pi}\frac{d\varphi_x}{2\pi},\;\int[dJ]\equiv\prod_{x\mu}\int_0^{2\pi}\frac{d\theta_{x\mu}}{2\pi}.\end{aligned} \qquad (23)$$

For a variational energy $E_0$ there holds the following relations;

$$\begin{aligned}Z_0 &= \int[dS][dJ]\exp(-\beta E_0)\equiv\exp(-\beta F_0),\\ F &\le F_v\equiv F_0+\langle E-E_0\rangle_0,\\ \langle O\rangle_0 &\equiv Z_0^{-1}\int[dS][dJ]\;O\;\exp(-\beta E_0).\end{aligned} \qquad (24)$$

From this Jensen-Peierls inequality, we adjust the variational parameters contained in $E_0$ so that $F_v$ is minimized.

For the trial energy $E_0$ of the present system, we assume the translational invariance and consider the following sum of single-site and single-link energies;

$$E_0 = -W\sum_{x\mu}J_{x\mu}-h\sum_x S_x, \qquad (25)$$

where $W$ and $h$ are real variational parameters. Then we obtain the following free energy per site, $f_v \equiv F_v/N$, where $N$ is the total number of lattice sites (We present the formulae for $d$-dimensional lattice);

$$\begin{aligned} f_v &= -\frac{d}{\beta}\ln I_0(\beta W) - \frac{1}{\beta}\ln(I_0(\beta h) - c_1 d m^2 p \\ &- c_2\frac{d(d-1)}{2}p^4 - 2c_3 d(d-1)m^2p^3 + dWp + hm, \\ m &\equiv \langle S_x\rangle_0 = \frac{I_1(\beta h)}{I_0(\beta h)}, \quad p \equiv \langle J_{x\mu}\rangle_0 = \frac{I_1(\beta W)}{I_0(\beta W)}, \end{aligned} \tag{26}$$

where $I_n(\gamma)$ ($n$ :integer) is the modefied Bessel function,

$$I_n(\gamma) = \int_0^{2\pi}\frac{d\theta}{2\pi}\exp(\gamma\cos\theta + in\theta), \tag{27}$$

The stationary conditions for $f_v$ w.r.t. $W, h$ read

$$\begin{aligned} W &= c_1 m^2 + 2c_2(d-1)p^3 + 6c_3(d-1)m^2p^2, \\ h &= 2dc_1 mp + 4c_3 d(d-1)mp^3. \end{aligned} \tag{28}$$

For many systems, the MFT is known to become exact for $d \to \infty$. It is proved also for the $Z(2)$ model ($c_3 = 0$) [19] by assuming suitable scaling behaviors of parameters $\beta c_i$ at large $d$.

### 3.2. Phase Structure

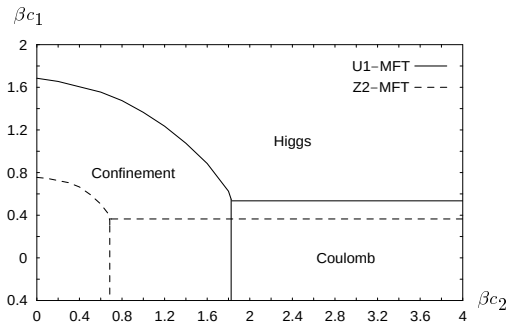
The MFT equations (26-28) for $d = 3$ generate the three phases characterized in the following Table1;

| phase | $\langle J_{x\mu}\rangle$ | $\langle S_x\rangle$ | ability |
|---|---|---|---|
| Higgs | $\neq 0$ | $\neq 0$ | learn and recall |
| Coulomb | $\neq 0$ | 0 | learn |
| Confinement | 0 | 0 | N.A. |

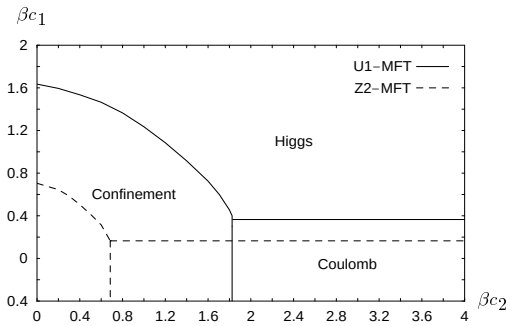
Table1. Phases and order parameters.

In the first column of Table1, the name of each phase is given, which are used in particle physics. The second (third) column shows the order parameter $\langle J_{x\mu}\rangle = p$ ($\langle S_x\rangle = m$). The fourth column shows the properties of each phase characterized by these order parameters.[3, 4] The condition $p \neq 0$ is a necessary condition to learn a pattern of $S_x$ by storing it to $J_{x\mu}$, while $m \neq 0$ is a necessary condition to recall it as in the Hopfield model. We note that the combination $p = 0$ and $m \neq 0$ is missing.

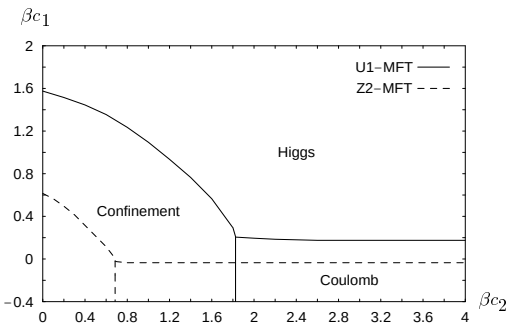
In Fig.4, we plot the phase boundaries obtained from (26-28) by solid curves for various values of $c_3$. We have also superposed the MFT results of the $Z(2)$ gauge model [3, 4] by dashed curves for comparison.



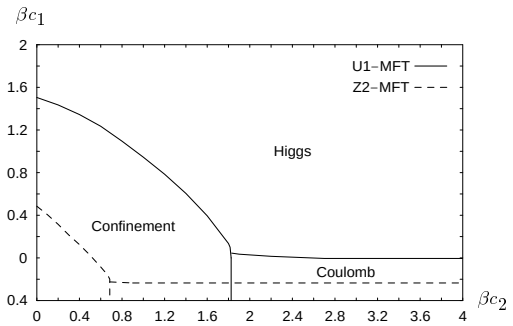

(a) $\beta c_3 = -0.05$




(b) $\beta c_3 = 0.0$




(c) $\beta c_3 = 0.05$




(d) $\beta c_3 = 0.1$

Fig.4. Phase diagrams in MFT. Dashed lines represent the phase boundaries in MFT of $Z(2)$ model.[4]

The phase boundary of MFT between Higgs phase and Coulomb phase is second order, while other two boundaries, Higgs-confinement and confinement-Coulomb, are first order. Across a second-order transition, $p$ and $m$ vary continuously, while across a first-order transition, $p$ and/or $m$ change discontinuously with finite jumps of $\Delta p$ and/or $\Delta m$. For a Higgs-confinement transition, $\Delta p \neq 0$ and $\Delta m \neq 0$, and for a confinement-Coulomb transition, $\Delta p \neq 0$ and $\Delta m = 0$ since $m = 0$ in both phases.

The present MFT may have some inappropriate points, which should be improved by more accurate methods like Monte Carlo simulations;

(i) In the pure gauge case where $c_1 = c_3 = 0$, the system is known to support always the confinement phase for any values of $c_2$.[20] This indicates that the confinement-Coulomb transitions may disappear in some parameter regions (e.g., for $c_3 = 0$). Then the confinement phase should survive and the Coulomb phase should disappear.

(ii) Along the correct Higgs-confinement boundary, the jumps $\Delta p, \Delta m$ may decrease as $c_2$ decreases and disappear at a certain point with $c_2 > 0$. This critical point correponds to the complementarity studied in Ref.[21] for $c_3 = 0$, which states that these two phases are analytically connected via a detour.

Let us next comment on the Elitzur's theorem.[22] It states that expectation values of *gauge-variant* objects should vanish. Thus $\langle S_x \rangle = \langle J_{x\mu} \rangle = 0$. This sounds to prohibit deconfinement phases like Higgs phase and Coulomb phase in Table 1. However, these deconfinement phases certainly exist. To compromise the MFT results with the Elitzur' theorem, one just needs to average over gauge-rotated copies of a MF solution[19]. Actually, the solution $m = \langle S_x \rangle, p = \langle J_{x\mu} \rangle$ is degenerate in the free energy with their gauge copies $\langle S'_x \rangle$ and $\langle J'_{x\mu} \rangle$, and should be superposed to satisfy the Elitzur's theorem. The location and the nature of phase transitions are unchanged.

## 4. DISCUSSION

We have proposed a quantum model of neural network based on gauge principle. The model resembles lattice $U(1)$ Higgs gauge theory, exhibiting a rich phase structure. The model should be regarded as a phenomenological (effective) model of an underlying microscopic quantum theory of the brain in the sense that the variables of the model, $S_x$ and $J_{x\mu}$, describe coherent quantum states of each neuron at $x$ and axon (or synaptic connection) along $(x, x+\mu)$, respectively.

We have not specified the underlying microscopic theory, although there are some candidates.[6, 9, 11] This point *is not* a flaw but an advantage since the essential characteristics of the effective model at low energies are to be determined by only a few properties of the microscopic model like dimensionality, symmetry, etc. This is known as the universality in renormalization group. The present $U(1)$ model will apply for a wide variety of microscopic models describing "charged" particles and gauge bosons in three dimensions with local $U(1)$ gauge symmetry. The model of Stuart, Takahashi and Umezawa[6] is such a model. Also the model of a microtubule proposed in Ref.[11] can be cast into this category because the Coulomb interactions among electrons can be written as gauge interactions mediated by gauge bosons, i.e., photons.

The model may be regarded as an extension of the classical $Z(2)$ gauge model[3, 4] to the gauge group $U(1)$. This similarity makes it easy to compare these two models and single out the difference between them, which is to be interpreted just as the quantum effects.

On the level of phase structure in MFT, Fig.4 shows that the region of the confinement phase in the $U(1)$ model is wider than that of $Z(2)$ model. This is due to quantum fluctuations; the $U(1)$ variables are continuous while $Z(2)$ variables are discrete. In short, the critical temperatures (both $c_1$ and $c_3$) of the $U(1)$ model is higher than those of the $Z(2)$ model. From the Table 1, this implies that the ability of learning patterns and recalling them is weakened globally by the quantum effects. More detailed study of this point is to be done in simulations of individual learning and recalling processes by using the rule of time evolution in Sect.2.4.

Another significant difference is that $U(1)$ model allows us to define the current $j_{x\mu}$ of (17) as in $\vec{j}(x)$ of (11). This is possible because the $U(1)$ gauge symmetry is not discrete but continuous. For a system with a continuous symmetry, one may obtain conserved current by applying Noether's theorem. It is worth to mention the difference between the present $U(1)$ gauge variables $\theta_{x\mu}$, the exponent of $J_{x\mu}$, and the vector potential $A_\mu(x)$ in (8). Although both are gauge fields, $A_\mu(x)$ describes the usual electromagnetic field, while $\theta_{x\mu}$ describes the synaptic connections. They are independent each other. Thus, $j_{x\mu}$ is *not* the electromagnetic current. We need to scrutinize the physical meaning of $j_{x\mu}$ further, although one expects that it describes the flows of electric voltage along axons and accompanying chemical materials at synaptic connections.

Let us comment here on the usefulness of such current for another network models. In some models that have real continuous $J_{ij} (\in (-\infty, \infty))$, $J_{ij}$ diverges to $\pm\infty$ as time runs. Without imposing artificial and unnatural conditions to avoid divergences of $J_{ij}$, a con-

served current, i.e., local continuous gauge symmetry, may assure us that $J_{ij}$ shall not diverge, since the total amount of chemical materials are finite.

In the present lattice model, the gauge-invariant current $j_{x\mu}$ can be used to scan the network at every time step. By monitoring $j_{x\mu}$ during the processes of learning and recalling, one may study the activities of network as quantum transports systematically. This is an interesting subject in future.

Finally, let us list up other possible problems in future study.
- More realistic phase strucutre by Monte Carlo simulaitons.
- Simulation of processes of learning patterns and recalling them through the time evolution in Sect.2.4.
- Inclusion of long-range interactions into the energy.
- Introduction of another set of gauge variables $\tilde{J}_{x\mu}$ to study the effect associated with the asymmetric couplings $J_{ij}$ and $J_{ji}(\neq J_{ij})$ [3], which is reflected by $\tilde{J}_{x\mu} \neq \bar{J}_{x\mu}$.

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[12] Strictly speaking, the connection $A_{xy}$ connects two nearby points $x, y$. $U_{xy}$ is the path-dependent phase factor $\prod_{\rm path} \exp(iA_{ab})$ where the product is taken along a specified path from $y$ to $x$.

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