QCD Evolution of Structure Functions at Small x

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Abstract. The status of the resummation of small x contributions to the unpolarized and polarized deep inelastic structure functions is reviewed.

1 Introduction

The measurement of the nucleon structure functions in deep inelastic scattering provides important tests of the predictions of Quantum Chromodynamics (QCD) on the short–distance structure of nucleons. The experiments at the ep–collider HERA allowed to extend the kinematic region to very small values of $x \sim 10^{-4}$ at photon virtualities of $Q^2 \geq 10\mathrm{GeV}^2$ measuring the structure function $F_2(x,Q^2)$ at an accuracy of $O(1\%)$. These precise measurements allow dedicated tests of QCD. Also the polarized deep inelastic experiments approach smaller values of x with a higher accuracy. To obtain a description in this kinematic range potentially large contributions to the evolution kernels were studied during the last two decades and the resummations of 'leading term' contributions were performed. Here mainly two directions were followed.

In one approach [1] a non–linear resummation of fan–diagrams of single ladder cascades is performed in the double logarithmic approximation [2][1]. Corrections of this type may ultimately become important at very small values of x to restore unitarity. Numerical studies of this equation were performed in Refs. [4]. One important assumption in the solution of this equation was that the non-perturbative input distributions for the N–ladder terms are given by the Nth power of the single gluon distribution at some starting scale. This would imply a strong constraint on the hierarchy of higher twist distributions. Later it was found [5] that the approximation [1] has to be supplemented by further color correlations even in the double logarithmic approximation, which cannot be cast into a non–linear equation anymore. As implied by the operator product expansion, the contributions due to different twist renormalize independently. The corresponding input distributions are likely to be unrelated between the different twists. Still saturation effects of the structure functions at very small x may be caused due to higher twist contributions. However, the detailed dynamics is yet unknown.

[1] This approximation has to be considered as qualitative and leads often to an overestimate of the scaling violations, cf. [3].

Ladder equations also form the basis of other approaches. In a physical gauge the emission of gluons along a single ladder–cascade describes[2] in *leading order* (LO) the evolution of a parton density as predicted by the renormalization group equation if the emissions along the ladder are strongly ordered in the transverse momentum $k_{\perp,1} \ll ... k_{\perp,i} \ll k_{\perp,i+1} ...$ If these emissions are evaluated in the approximation $x_1 \gg ... x_i \gg x_{i+1} ...$ instead, using effective vertices, one obtains the BFKL–resummation [6] in LO. This particular aspect led sometimes to the impression that these two approximations were of competing nature. As we will show below this is, however, not the case as far as the description of the scaling violations of structure functions are concerned. One may study this process under a more general point of view and consider angularly ordered emissions covering both the above cases [7], which allows for interesting applications through Monte Carlo studies. Whereas this unified treatment is possible at LO, higher order corrections cannot be cast into this form in general. The renormalization group equation for the mass singularities, on the other hand, allows to perform consistent higher order calculations beyond these approximations accounting for the resummation of the small x contributions in the anomalous dimensions and coefficient functions.

In the second main approach these resummations are studied. Resummations were performed in leading order for the unpolarized singlet case [6,8], the non–singlet structure functions [9,10], and the polarized singlet distributions [11]. Applications were studied in the case of QED for the flavor non–singlet contributions to radiative corrections [12,13]. The quarkonic next–to–leading order (NLO) contributions in the unpolarized singlet case were calculated in [14]. Recently also the NLO resummed gluon anomalous dimension [15,16] in the DIS-Q_0 scheme [17] was obtained. If the evolution kernels are written in terms of a series in $\alpha_s/(N-N_s)$, where N_s denotes the position of the leading pole, the individual terms are large and require resummation.

One of the central questions for the understanding of the deep inelastic structure functions at small x is therefore to analyse the impact and rôle of these small x resummations and their potential corrections in even higher order. These terms have to be viewed in comparison with the known fixed order results used in the current analyses of the scaling violations of the twist–2 contributions to the structure functions.

In the present paper we review the status of the latter resummations and their impact on the scaling violations of deep inelastic structure functions.

2 The Evolution Equations

The twist-2 contributions to the structure functions in inclusive deep-inelastic scattering can be described in terms of the QCD-improved parton model. Their scaling violations are governed by renormalization group equations which can be formulated to all orders in the strong coupling constant. All small x resummations are based on perturbative QCD. As in the fixed order calculations one has

[2] The virtual contributions have to be added.

to factorize the collinear or mass singularities, which are absorbed into the non–perturbative input distributions. The soft- and virtual singularities cancel order by order according to the Bloch–Nordsiek theorem. A second renormalization group equation describes the scale dependence of the strong coupling constant $\alpha_s(\mu^2)$. The perturbative all-order small x resummations may turn out to yield important contributions to the *scaling violation* of the deep inelastic structure functions. Predictions on the shape of the parton densities at small x are, however, *beyond* a perturbative treatment, even in resummed form, since generally low scales are involved and partonic approaches have to fail.

The small x resummations can be tested with respect to their prediction on the scaling violations of deep inelastic structure functions as $F_2(x,Q^2)$ and $F_L(x,Q^2)$. The evolution equations for the parton densities $f_i(x,\mu^2)$ are given by

$$\frac{\partial}{\partial \log(\mu^2)} f_i(x,\mu^2) = P_i^j(x,a_s) \otimes f_j(x,\mu^2) \ . \tag{1}$$

Here $\otimes$ denotes the Mellin convolution. The splitting functions $P_{ij}(x,a_s)$ contain besides the completely known LO and NLO contributions the LO and NLO small x resummed terms to all orders in $a_s = \alpha_s/(4\pi)$,

$$P_{ij}(x,a_s) = a_s P_{ij}^{(0)}(x) + a_s^2 P_{ij}^{(1)}(x) + \sum_{k=2}^{\infty} a_s^{k+1} \widehat{P}_{ij,x\to 0}^{(k)}(x) + \sum_{k=2}^{\infty} a_s^{k+2} \widehat{P}_{ij,x\to 0}^{(k)}(x) \ . \tag{2}$$

Similarly, the coefficient functions take the form

$$c_i\left(x,\frac{Q^2}{\mu^2}\right) = \delta_{iq}\delta(1-x) a_s c_i^{(0)}(x) + a_s^2 c_i^{(1)}(x) + \sum_{k=2}^{\infty} a_s^{k+1} \widehat{c}_{i,x\to 0}^{(k)}(x) + \sum_{k=2}^{\infty} a_s^{k+2} \widehat{c}_{i,x\to 0}^{(k)}(x) \ . \tag{3}$$

In this way the effect of the small x resummations is consistently included. As these contributions do not a priori account for Fermion number and energy–momentum conservation these conditions have to be imposed for the contributions beyond $O(a_s^2)$. The structure functions $F_A(x,Q^2)$ are finally obtained as

$$F_A(x,Q^2) = c_{q,A}\left(x,\frac{Q^2}{\mu^2}\right) \otimes f_q\left(x,\frac{\mu^2}{M^2}\right) + c_{g,A}\left(x,\frac{Q^2}{\mu^2}\right) \otimes f_g\left(x,\frac{\mu^2}{M^2}\right) \ . \tag{4}$$

The factorization scale dependence (μ^2) cancels order by order.

3 Small x Resummation of the Anomalous Dimensions

All resummations studied below are based on scale–invariant equations in leading order. If one considers the renormalization group equation for an operator matrix

element E_k^n

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m} + \gamma_{O_k} - n\gamma_\Phi\right] E_k^n = 0 \tag{5}$$

scale invariant solutions are obtained in the massless case ($m = 0$) and if the β–function is set to zero :

$$E_k^n(\mu^2) = E_k^n(\mu_0^2)\left(\frac{\mu^2}{\mu_0^2}\right)^{\frac{1}{2}\left(\gamma_{O_k} - n\gamma_\Phi\right)} . \tag{6}$$

Within this approach the coupling constant a_s is fixed. The scale invariant part of the anomalous dimension has the representation

$$\gamma_{O_k} - n\gamma_\Phi = \sum_{l=1}^{\infty} \gamma_O^{(l)} a_s^l \tag{7}$$

and exponentiates to all orders. The representation (6) applies also for higher order resummations under the above requirements. In this way one may derive in the different subsequent resummations the LO small x resummed anomalous dimensions. In higher than LO scale breaking effects emerge in QCD. Therefore a thorough treatment along these lines is no longer possible. Still one may try to identify those contributions of the anomalous dimension which are scale invariant applying a diagonalization as in (6).

4 Less Singular Terms

For most of the applications only the resummation of the leading singular terms is known. The contributions which are less singular by one or more powers in N may yield substantial contributions. This has been known for long [18,19], cf. also [20], and can easily be seen in the case of $F_L(x, Q^2)$ in $O(\alpha_s)$ as an example. If one disregards the second factor in the leading order coefficient function $c_g^{(0)} = C_g^{(0)} x^2(1-x)$ in view of a small x approximation the value of F_L may be overestimated by a factor of four [18].

To get an estimate of the effect of the terms suppressed by one order or more orders in the Mellin moment N one may study some models. The possible size of these terms may be inferred expanding the LO and NLO anomalous dimensions and coefficient functions into series in $1/N$ comparing the expansion coefficients. Estimates of this kind were performed in [3,10,12,13,21–25]. Possible ansätze for the next order terms are

$$\begin{aligned}
&\Gamma(N, a_s) \to \Gamma(N, a_s) - \Gamma(1, a_s) \\
&\Gamma(N, a_s) \to \Gamma(N, a_s)(1-N) \\
&\Gamma(N, a_s) \to \Gamma(N, a_s)(1-N)^2 \\
&\Gamma(N, a_s) \to \Gamma(N, a_s)(1-2N+N^3) \; .
\end{aligned} \tag{8}$$

If one formally expands the LO and NLO anomalous dimensions in the above manner one finds [3], irrespectively of the factorization scheme, that at least four expansion terms are needed to represent the exact result on the 5% level, cf. figure 1. If one compares the respective NLO resummed coefficients with the LO resummed ones in the cases they were calculated even larger effects than indicated by the above estimate are found (see below).

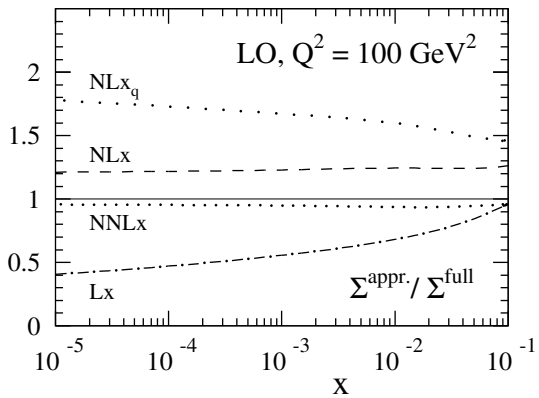

Fig. 1. Different approximation steps in the $1/(N-1)$ expansion of the complete LO unpolarized singlet distribution over four orders (Lx, NLx$_q$, NLx, NNLx), cf. Ref. [3].

5 Non–Singlet Structure Functions

The most singular contributions to the Mellin transforms of the structure–function evolution kernels $K^{\pm}(x,a)$ at all orders in a can be obtained from the positive and negative signature amplitudes $f_0^{\pm}(N,a)$ studied in [9] for QCD via

$$\mathcal{M}\left[K^{\pm}_{x\to 0}(a)\right](N) \equiv \int_0^1 dx\, x^{N-1} K^{\pm}_{x\to 0}(x,a) \equiv -\frac{1}{2}\,\boldsymbol{\Gamma}^{\pm}_{x\to 0}(N,a) = \frac{1}{8\pi^2} f_0^{\pm}(N,a)\,. \tag{9}$$

These amplitudes are subject to the quadratic equations:

$$f_0^{+}(N,a) = 16\pi^2 a_0 \frac{a}{N} + \frac{1}{8\pi^2}\frac{1}{N}\left[f_0^{+}(N,a)\right]^2\,, \tag{10}$$

$$f_0^{-}(N,a) = 16\pi^2 a_0 \frac{a}{N} + 8 b_0^{-}\frac{a}{N^2} f_V^{+}(N,a) + \frac{1}{8\pi^2}\frac{1}{N}\left[f_0^{-}(N,a)\right]^2\,. \tag{11}$$

Here $f_V^{+}(N,a)$ is obtained as the solution of the Riccati differential equation

$$f_V^{+}(N,a) = 16\pi^2 a_V \frac{a}{N} + 2 b_V \frac{a}{N}\frac{d}{dN} f_V^{+}(N,a) + \frac{1}{8\pi^2}\frac{1}{N}\left[f_V^{+}(N,a)\right]^2\,. \tag{12}$$

The coefficients a_i and b_i in the above relations read for the case of QED, cf. sect. 7,

$$a_0 = 1, \quad b_0^- = 1, \quad a_V = 1, \quad b_V = 0, \tag{13}$$

and for QCD [9]

$$a_0 = C_F, \quad b_0^- = C_F, \quad a_V = -\frac{1}{2N_c}, \quad b_V = C_A, \tag{14}$$

with $C_F = 4/3$ and $C_A = N_c = 3$. In QED Eq. (12) further simplifies to an algebraic equation with the same coefficients as (10). The solutions of (10) and (11) were derived in [9] for the QCD case[3]. They are given by

$$\boldsymbol{\Gamma}^+_{x\to 0}(N,a) = -N\left\{1 - \sqrt{1 - \frac{8aC_F}{N^2}}\right\} \tag{15}$$

$$\boldsymbol{\Gamma}^-_{x\to 0}(N,a) = -N\left\{1 - \sqrt{1 - \frac{8aC_F}{N^2}\left[1 - \frac{8aN_c}{N}\frac{d}{dN}\ln\left(e^{z^2/4}D_{-1/[2N_c^2]}(z)\right)\right]}\right\}$$

where $z = N/\sqrt{2N_c a}$, and $D_p(z)$ denotes the function of the parabolic cylinder.

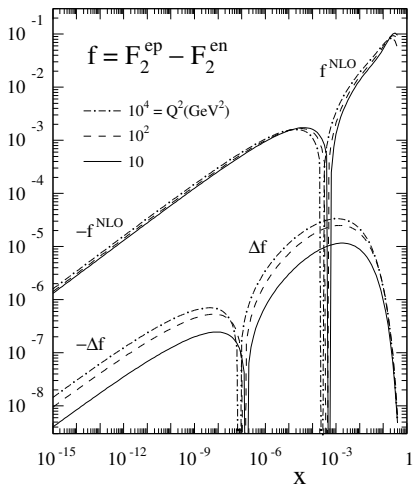

Fig. 2. The small-x Q^2–evolution of the unpolarized non–singlet structure function combination $F_2^{ep} - F_2^{en}$ in NLO and the absolute corrections to these results due to the resummed kernel. The initial distributions were chosen at $Q_0^2 = 4\,\mathrm{GeV}^2$, cf. Refs. [10,12].

Numerical results on the impact of the small x resummations were obtained in Refs. [10,12]. The new terms contribute at $O(a_s^3)$ and higher. If compared to the fixed order contributions in NLO the effect is of $O(1\%)$ or less, which is shown for the scaling violations of $F_2^{ep} - F_2^{en}$ in Fig. 2. Other examples as for xF_3

[3] Note a few misprints in Eq. (4.7) of ref. [9].

and the $\pm$–evolutions for polarized non–singlet structure functions show a rather similar behaviour. Comparable results were obtained in Ref. [26]. The effect is expected to be rather small due to the typical shape of the input distributions in the non–singlet case. The size of the (small) correction does further vary significantly in dependence of the inclusion of less singular terms, cf. sect. 4, or if conservation laws are imposed. Large effects as anticipated in Refs. [27,28] are not confirmed.

6 Polarized Singlet Structure Functions

The LO small x evolution kernels in the case of the polarized singlet evolution were derived in [11]. The resummed splitting function is given by

$$\boldsymbol{P}(x,a_s) \equiv \sum_{l=0}^{\infty} \boldsymbol{P}^{(l)}_{x\to 0} a_s^{l+1} \log^{2l} x = \frac{1}{8\pi^2} \mathcal{M}^{-1}[\boldsymbol{F}_0(N,a_s)](x). \tag{16}$$

The matrix valued function $\boldsymbol{F}_0(N,a_s)$ is obtained as the solution of

$$\boldsymbol{F}_0(N,a_s) = 16\pi^2 \frac{a_s}{N} \boldsymbol{M}_0 - \frac{8a_s}{N^2} \boldsymbol{F}_8(N,a_s)\, \boldsymbol{G}_0 + \frac{1}{8\pi^2}\frac{1}{N} \boldsymbol{F}_0^2(N,a_s) \tag{17}$$

with

$$\boldsymbol{F}_8(N,a_s) = 16\pi^2 \frac{a_s}{N} \boldsymbol{M}_8 + \frac{2a_s}{N} C_A \frac{d}{dN} \boldsymbol{F}_8(N,a_s) + \frac{1}{8\pi^2}\frac{1}{N} \boldsymbol{F}_8^2(N,a_s), \tag{18}$$

where

$$\boldsymbol{M}_0 = \begin{pmatrix} C_F & -2T_R N_f \\ 2C_F & 4C_A \end{pmatrix}, \quad \boldsymbol{M}_8 = \begin{pmatrix} C_F - C_A/2 & -T_R N_f \\ C_A & 2C_A \end{pmatrix}, \quad \boldsymbol{G}_0 = \begin{pmatrix} C_F & 0 \\ 0 & C_A \end{pmatrix}. \tag{19}$$

Eq. (16) obeys [21]

$$P^{(l)}_{qg}/(T_R N_f) = -P^{(l)}_{gq}/C_F \tag{20}$$

to all orders, where $T_R = 1/2$ and N_f denotes the number of flavors. The leading contributions of the fixed order results in LO and NLO ($\overline{\rm MS}$) are correctly described. In the supersymmetric limit $C_A = C_F = N_f = 1, T_R = 1/2$ the relations

$$P^{(l)}_{qq} + P^{(l)}_{gq} = P^{(l)}_{qg} + P^{(l)}_{gg} \tag{21}$$

are obeyed for all l and Eq. (16) can be given in a simple analytic form [21].

The impact of the resummation (16) on the evolution of the polarized singlet and gluon density and the structure function $g_1(x,Q^2)$ have been studied in Ref. [21]. As shown in Fig. 3 the corrections are much larger than the

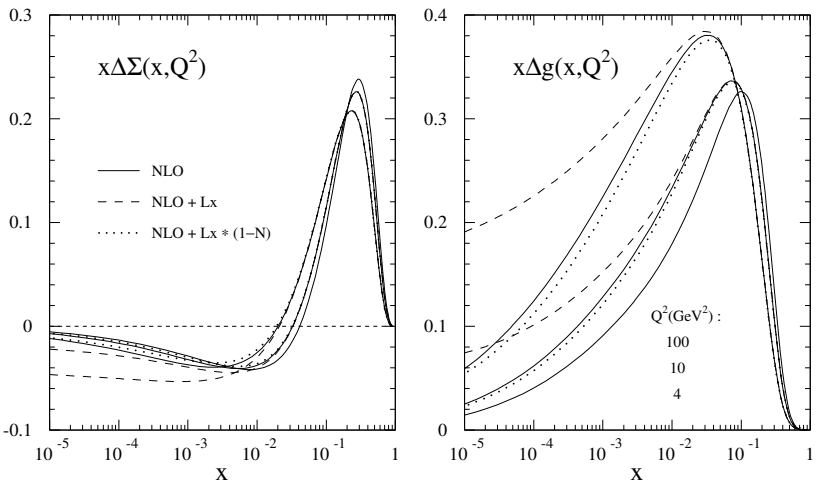

Fig. 3. The Q^2 evolution of the polarized quark singlet and gluon momentum distributions evolving from $Q_0^2 = 4\mathrm{GeV}^2$, Ref. [21].

$O((a_s \ln^2 x)^l)$ corrections in the non–singlet case. Taking into account less singular terms of the type

$$P_{ij}^{(l>1)} \to P_{ij}^{(l>1)} \cdot (1-N) \ , \tag{22}$$

as suggested by the analytic structure of the fixed order LO and NLO anomalous dimensions, this enhancement reduces, however, again to the value of the fixed order evolution in NLO (dotted line in Fig. 3).

7 QED Corrections

The non–singlet small x resummation was applied to resum the $O((\alpha \log^2 x)^l)$ terms in the QED corrections to deep inelastic scattering in Ref. [13]. These corrections are negative and amount to $O(10\%)$ in the high y range for $x = 10^{-4}...10^{-2}$, see Fig. 4.

They diminish the $O((\alpha \log(Q^2/m_e^2))^l)$ corrections which are very large in this domain. The first non-trivial contribution of $O(\alpha^2 \log^2 x)$ is in agreement with the result found in [29]. From the latter calculation also the next less singular term of $O(\alpha^2 \log(x))$ can be derived. Up to this term the evolution kernel reads $(a = \alpha/(4\pi))$

$$\mathcal{M}[P_{z\to 0}](N,a) = \frac{2a}{N} - 12\frac{a^2}{N^3}\left(1 - \frac{2}{9}N\right) + ... \tag{23}$$

If compared to the case of QCD this less–singular term is stronger suppressed.

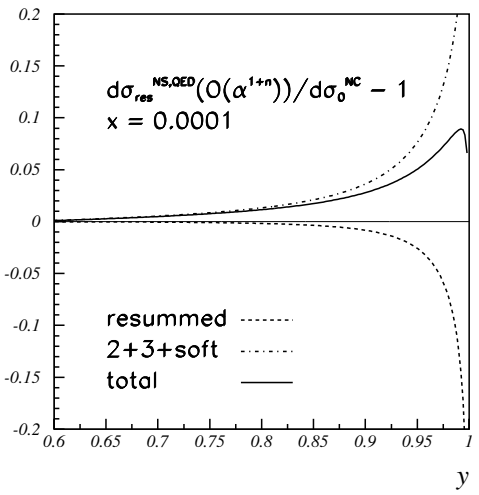

Fig. 4. 2nd and higher order QED initial state radiative corrections to deep inelastic ep scattering. Dashed line: small x resummed contribution, dash-dotted line: LO contributions up to $O(\alpha^3)$ and soft photon exponentiation, full line: resulting correction, Ref. [13].

8 Unpolarized Singlet Distributions

The LO resummation for the evolution kernel of the unpolarized singlet distributions was derived in [6]. Jaroszewicz [8] showed that the eigenvalue

$$(N-1) = \frac{\alpha_s N_c}{\pi}\chi_0(\gamma_L) \equiv \frac{\alpha_s N_c}{\pi}\left[2\psi(1) - \psi(\gamma_L) - \psi(1-\gamma_L)\right] \tag{24}$$

represents the LO resummed gluon-gluon anomalous dimension $\gamma_L = \gamma_{gg}^{(0)}(N, a_s)$. The resummed LO gluon-quark anomalous dimension is given by $\gamma_{gq}^{(0)}(N, a_s) = (C_F/C_A)\gamma_L$ and the quarkonic terms do not contribute in $O((a_s/(N-1))^l$. Eq. (24) can be solved iteratively demanding $\gamma_L(N, a_s) \to \overline{\alpha}_s/(N-1)$ as $|N| \to \infty$ for $N \in \mathbf{C}$, which selects the physical branch of the resummed anomalous dimension,

$$\gamma_L \equiv \gamma_{gg,0}(N, \alpha_s) = \frac{\overline{\alpha}_s}{N-1}\left\{1 + 2\sum_{l=1}^{\infty}\zeta_{2l+1}\gamma_{gg,0}^{2l+1}(N, \alpha_s)\right\}. \tag{25}$$

Here we rewrite $\overline{\alpha}_s = N_c\alpha_s/\pi$. γ_L has the serial representation

$$\gamma_{gg,0}(N, \alpha_s) = \frac{\overline{\alpha}_s}{N-1} + 2\zeta_3\left(\frac{\overline{\alpha}_s}{N-1}\right)^4 + 2\zeta_5\left(\frac{\overline{\alpha}_s}{N-1}\right)^6 + 12\zeta_3^2\left(\frac{\overline{\alpha}_s}{N-1}\right)^7 + \ldots \tag{26}$$

Under the above conditions one may calculate $\gamma_L(N, a_s)$ in the whole complex plane. It is a bounded function of $\rho = (N-1)/\overline{\alpha}_s$, the singularities of which are branch points at [30,22]

$$\rho_1 = 4\log 2, \qquad \rho_{2,3} = -1.41048 \pm 1.97212\, i, \tag{27}$$

cf. [30,3] for detailed representations. The LO BFKL anomalous dimension possesses **no poles**. Since the known NLO resummed anomalous dimensions are functions of $\gamma_L(N, a_s)$ which introduce no further singularities the contour integral around the singularities of the problem has to cover the three BFKL branch points, the singularities of the input distributions along the real axis to the left of 1, and the remaining singularities of the fixed order anomalous dimensions at the non–positive integers [30,3]. Note, that the resummed form of $\gamma_L(N, a_s)$ removes *all* the fixed–order pole singularities of Eq. (26) into branch cuts. Any finite correction to γ_L may thus lead to essential changes of the corresponding numerical results. Early numerical studies on the impact of the LO resummed anomalous dimensions were performed in [31]. More recent analyses have been performed in Refs. [3,22,23].

The next-to-leading order resummed anomalous dimensions are given by

$$\widehat{\gamma}_{NL}(N, \alpha_s) = -2 \begin{pmatrix} \dfrac{C_F}{C_A}\left[\gamma_{qg}^{NL} - \dfrac{8}{3} a_s T_F\right] & \gamma_{qg}^{NL} \\ \gamma_{gq}^{NL} & \gamma_{gg}^{NL} \end{pmatrix} , \tag{28}$$

with $T_F = T_R N_f$. The quarkonic contributions were calculated in Ref. [14], as well as the resummed coefficient functions $c_2(N, a_s)$ and $c_L(N, a_s)$. Recently γ_{gg}^{NL} was derived in [15,16] and γ_{gq}^{NL} is yet unknown[4]. In the DIS–scheme γ_{qg}^{NL} is found to be an analytic, scale–independent function of $\gamma_L(N, a_s)$ and reads

$$\gamma_{qg}^{NL,DIS}(N, \alpha_s) = T_F \frac{\alpha_s}{6\pi} \frac{2 + 3\gamma_L - 3\gamma_L^2}{3 - 2\gamma_L} \frac{\left[B(1-\gamma_l, 1+\gamma_L)\right]^3}{B(2+2\gamma_L, 2-2\gamma_L)} R(\gamma_L), \tag{29}$$

where

$$R(\gamma) = \left[\frac{\Gamma(1-\gamma)\chi_0(\gamma)}{-\gamma\Gamma(1+\gamma)\chi_0'(\gamma)}\right]^{1/2} \exp\left[\gamma\psi(1) + \int_0^\gamma dz \frac{\psi'(1) - \psi'(1-z)}{\chi_0(z)}\right] . \tag{30}$$

The NLO resummed gluon anomalous dimension γ_{gg}^{NL} was calculated in the Q_0–scheme[5]. One has to solve the Bethe–Salpeter equation

$$(N-1) G_N(q_1, q_2) = \delta^{D-2}(q_1 - q_2) + \int d^{D-2} q_3 K(q_1, q_2) G_N(q_3, q_2) \tag{31}$$

with

$$K(q_1, q_2) = \delta^{D-2}(q_1 - q_2) 2\omega(q_1) + K_{\rm real}(q_1, q_2) + K_{\rm virtual}(q_1, q_2). \tag{32}$$

[4] As pointed out in Ref. [3] its quantitative influence is likely to be minor.
[5] For a transformation into the DIS–scheme cf. [3].

For $q_1^2 \gg q_2^2$ one diagonalizes as in the LO case using *formally* the same ansatz :

$$\int d^{D-2}dq_2 K(q_1,q_2)\left(q_2^2\right)^{\gamma-1} = \overline{\alpha}_s\left[\chi_0(\gamma) - \frac{\overline{\alpha}_s}{4}\delta(\gamma,q_1^2,\mu^2)\right]\left(q_1^2\right)^{\gamma-1} \quad . \tag{33}$$

Here the scale–invariant LO eigenvalue $\overline{\alpha}_s\chi_0(\gamma)$ is supplemented by the NLO correction term $(\overline{\alpha}_s^2/4)\delta(\gamma,q_1^2,\mu^2)$,

$$\begin{aligned}\delta(\gamma,q_1^2,\mu^2) = &-\left(\frac{67}{9} - 2\zeta(2) - \frac{10}{27}N_f\right)\chi_0(\gamma) + 4\Phi(\gamma) - \frac{\pi^3}{\sin^2(\pi\gamma)} \\ &+\frac{\pi^2}{\sin^2(\pi\gamma)}\frac{\cos(\pi\gamma)}{1-2\gamma}\left[(22-\beta_0) + \frac{\gamma(1-\gamma)}{(1+2\gamma)(3-2\gamma)}\left(1+\frac{N_f}{3}\right)\right] \\ &+\frac{\beta_0}{3}\chi_0(\gamma)\log\left(\frac{q_1^2}{\mu^2}\right) + \left[\frac{\beta_0}{6} + \frac{d}{d\gamma}\right]\left[\chi_0^2(\gamma) + \chi_0'(\gamma)\right] - 6\zeta_3, \end{aligned} \tag{34}$$

with

$$\Phi(\gamma) = \int_0^1 \frac{dz}{1+z}\left[z^{\gamma-1} + z^{\gamma}\right]\left[\mathrm{Li}(1) - \mathrm{Li}(z)\right] \quad . \tag{35}$$

Whereas the contributions in the first two lines of Eq. (34) do contain contributions to the anomalous dimension up to $O(a_s^2)$ the third line contributes only in three–loop order. The former terms are *scale–invariant* and are in agreement with the known fixed order results. Eq. (34) therefore makes a prediction on the small x contributions of the yet unknown gluon anomalous dimension in three–loop and higher order, which will be tested in the future. Besides the scale–dependent term $(\beta_0/3)\chi_0(\gamma)\log(Q^2/\mu^2)$ also the second addend depends on the choice of scales, since it is not invariant against the interchange of q_1^2 and q_2^2, cf. [16]. The third addend $6\zeta_3$, being numerically large, contains contributions of the gluonic contribution to the trajectory function $\omega(q_1^2)$. The result given in Ref. [32] was confirmed in a different calculation by Ref. [33]. A departing value was reported in [34].

Numerical results on the impact of the leading and next–to–leading anomalous dimensions and coefficient functions were provided in a series of detailed studies, see e.g. [22,3,25] and references therein. The matrix formalism for the solution of the all order evolution equations, extending a first approach in Ref. [35] to all orders, both for hadronic and photon structure functions, is described in Ref. [3] in detail. The quarkonic contributions lead to a strong enhancement of both $F_2(x,Q^2)$ and $F_L(x,Q^2)$ at small x during the evolution. However, already simple choices for the yet unknown less singular contributions diminish these effects sizably so that a final conclusion cannot be drawn at present. In the case of the resummed gluon anomalous dimension the NLO contributions are found to be extremely large and negative. The large rise due to the LO BFKL term is already canceled to the level of the fixed order contributions by the purely quarkonic contribution to γ_{gg}^{NL}, see Fig. 5. Adding also the gluonic contribution leads to negative values for the resummed splitting function already for

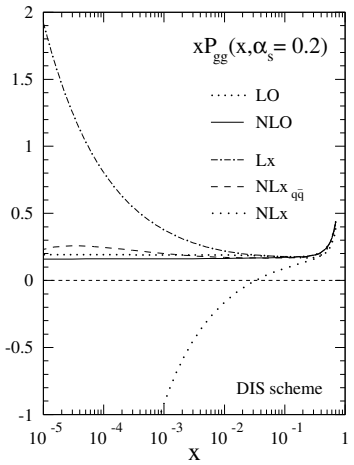

Fig. 5. Different contributions to the resummed splitting function $xP_{gg}(x,\alpha_s)$ in the DIS–scheme (overlayed), Ref. [25].

$\alpha_s = 0.2$ and $x \simeq 0.01$ which has to be regarded as unphysical. The LO and NLO resummed contributions to the gluon anomalous dimension seem to represent the first terms of a diverging series, which might be eventually resummed. This can, however, only be achieved reliably if several more less singular series are calculated *completely*, but not at the present stage.

9 An Exactly Soluble Model

The effect of potential subleading contributions to the LO anomalous dimension was estimated in the previous sections in the case of QCD. In ϕ^3 theory in $D = 6$ dimensions these terms can be determined in explicit form. ϕ^3_6 theory is rather similar to QCD (gluo–dynamics) due to the triple boson interaction and being an asymptotic free field theory. The leading order resummed anomalous dimension can be calculated for *all* values of x solving the Bethe–Salpeter equation [36]

$$T(p,q) = \frac{2^{2-D}}{\pi^{D/2}\Gamma((D-2)/2)} \frac{\lambda_D^2}{(p-q)^2} + \frac{\lambda_D^2}{(2\pi)^D} \int d^D k \frac{T(k,p)}{(q-k)^2 \left(k^2\right)^2} \; , \qquad (36)$$

with $q^2, p^2 < 0$, q the momentum transfer and p^2 a hadronic mass scale. For $D = 6$ the quantity $q.p\; T(p,q)$ is scale invariant and one may expand Eq. (36) into partial waves with

$$p.q\; T_N(p,q) = \left(\frac{q^2}{p^2}\right)^{-(N+1)/2} \left(\frac{q^2}{p^2}\right)^{-\gamma_L(N,a_s)/2} \; , \qquad (37)$$

where $a_s = \lambda_6^2/(4\pi)^3 = \mathsf{const}$. The anomalous dimension $\gamma_L(N,a_s)$ is given by

$$\gamma_L(N,a_s) = \sqrt{(N+2)^2 + 1 - 2\sqrt{(N+2)^2 + 4a_s}} - (N+1) \; . \qquad (38)$$

Note that $\gamma_L(N, a_s)$ possesses *no* poles but only branch cuts for $N \in \mathbf{C}$. The anomalous dimension γ_L covers all conformal contributions in leading order. If one expands this quantity it yields in first order in a_s the *complete* leading order anomalous dimension $\gamma_{SS}^{(0)}(N)$, up to an eventual term due to 4–momentum conservation which is easily imposed,

$$\gamma_{SS}^{(0)}(N) = -\frac{2}{(N+1)(N+2)} + \frac{1}{6} \; . \tag{39}$$

Furthermore, all the fixed–order leading poles at $N = -1$ are resummed in this representation. This has been verified by an explicit calculation up to 3–loop order [37]. The complete NLO fixed order anomalous dimension reads [37]

$$\begin{aligned}\gamma_{SS}^{(1)}(N) = &-\frac{1}{6}\frac{22 + 111N + 211N^2 + 138N^3 + 28N^4}{(N+1)^3(N+2)^3} + \frac{5}{3}\frac{S_1(N)}{(N+1)(N+2)} \\ &-\frac{1}{2}\left[1 + (-1)^N\right]\frac{2}{(N+1)^2(N+2)^2} + \frac{13}{216} \; .\end{aligned} \tag{40}$$

One may now derive from Eq. (38) the small–x resummed anomalous dimension, covering the fixed–order leading pole contributions only

$$\gamma_L^{N\to -1}(N, a_s) = (N+1)\left[\sqrt{1 - \frac{4a_s}{(N+1)^2}} - 1\right] \; , \tag{41}$$

which again contains *no* poles for all $N \in \mathbf{C}$. This quantity corresponds to the LO BFKL anomalous dimension in QCD.

In deriving (41) one obtains as well the respective resummed subleading terms. As was shown in Ref. [37] the weight coefficient of these terms are of alternating sign with growing coefficients, which indicates already that the resummation of the leading pole terms ($N = -1$) does not yield the dominant contribution. This is expected, since neither $\gamma_L(N, a_s)$ nor $\gamma_L^{N\to -1}(N, a_s)$ have a pole singularity – as is also the case in the LO BFKL resummation, where a similar behaviour might be expected. Fig. 6 shows the behaviour of the respective splitting functions after the Mellin transform to x–space, normalized to the leading order splitting function $P_0(x) = 2x(1-x)$. $P_L^{x\to 0}(x, a_s)$ is nowhere dominant and departs encreasingly from the complete solution $P_L(x, a_s)$ as $x \to 0$.

10 Conclusions

As in the case of the fixed order calculations the renormalization group equation, through which the factorization of the mass singularities is described, implies the evolution equations for the parton densities including the resummation of the small x terms. Due to the Mellin convolution between the respective evolution kernels and the extended input distributions the detailed knowledge of the kernels at medium x is as important. This is particularly the case for input densities

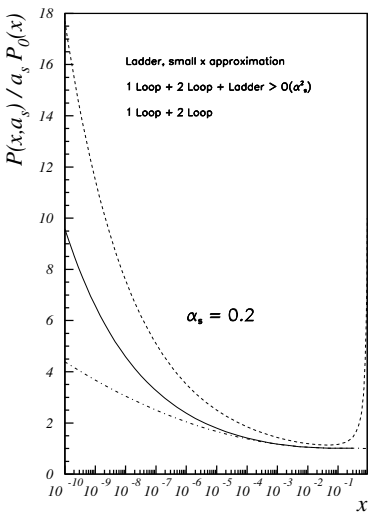

Fig. 6. Fixed-order and resummed splitting functions $P(x, a_s)$, normalized to $a_s P_{SS}^{(0)}(x)$, for $a_s = 0.2$. Dash-dotted line : $P = a_s P_{SS}^{(0)} + a_s^2 P_{SS}^{(1)}$; solid line : $P = P_L$; dashed line : $P = P_L^{x \to 0}$, Ref. [37].

with a large rise towards small x, as the gluon distribution. Less singular contributions to the evolution kernels turn out to have a sizable impact onto the scaling violations. In the example of ϕ_6^3 theory these contributions were calculated for the leading order resummation and turn out to be even more important than the leading pole terms ($N = -1$). The reason for this behaviour is that the resummed anomalous dimension, as also the resummed ($N = -1$)–fixed–order pole contribution, possess *no poles* anymore. This is also the case for the leading order resummed BFKL anomalous dimension and the known resummed NLO contributions.

In a quantitative description of the scaling violation of structure functions the conservation laws as Fermion number conservation in the non–singlet case and energy–momentum conservation have to be obeyed. These integral relations imply strong relations between the small x and medium x contributions also for the resummed evolution kernels. A study of the known fixed–order results in leading and next–to–leading order shows furthermore that the evolution kernels, if approximated in a leading pole representation, require to take into account at least four orders which is likely to be the case for the small x resummed terms as well. The conformal part of the known terms of the small x resummations behaves stable but is not necessarily dominant.

An important future check of the small x resummed calculations is their prediction of the leading and next–to–leading order small x contributions to the 3–loop anomalous dimensions, which are yet unknown. The intimate interplay between small and medium x effects requires to continue consistent calculations

of the anomalous dimensions and coefficient functions to even higher order and to compare these results with the scaling violations measured by experiment.

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