RESUMMED PERTURBATIVE EVOLUTION AT HIGH ENERGY

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We present a small x resummation for the GLAP anomalous dimension and its corresponding dual BFKL kernel, which includes all the available perturbative information and nonperturbative constraints.

An accurate perturbative determination of the hard partonic cross-sections and of the anomalous dimensions which govern parton evolution is necessary for the precise extraction of parton densities. As is well known, such high-order perturbative calculations display classes of terms containing large logarithms, which ultimately signal the breakdown of perturbation theory. Because these terms are scale–dependent and in general non universal, lack of their inclusion can lead to significant distortion of the parton densities in some kinematical regions, thereby leading to loss of accuracy if parton distributions extracted from deep-inelastic scattering (DIS) or the Drell-Yan (DY) processes are used at the LHC.

Logarithmic enhancement of higher order perturbative contribution may take place when more than one large scale ratio is present. In DIS and DY

this happens in the two opposite limits when the centre-of-mass energy of the partonic collision is close to the threshold for the production of the final state, or much higher than the characteristic scale of the process. These correspond respectively to the small x and large x kinematical regions, where $0 \leq x \leq 1$ is defined in terms of the invariant mass of the non-leptonic final state $\frac{(1-x)Q^2}{x}$. The corresponding perturbative contributions are respectively enhanced by powers of $\ln \frac{1}{x}$ and $\ln(1-x)$, or, equivalently, in the space of Mellin moments, by powers of $\frac{1}{N}$ and $\ln N$, where $N \to 0$ moments dominate as $x \to 0$ while $N \to \infty$ moments dominate as $x \to 1$. Here we will be concerned only with the small x (and thus small N) region.

In the DGLAP evolution equation one resums collinear logarithms first, resulting at LO and NLO in a good description of many data sets, and in particular the HERA F_2 data at perturbative Q^2 and values of x as low as 10^{-4} [1–3]. However in the singlet (gluonic) channel the fixed order splitting functions contain small x logarithms also, which at NNLO [4] begin to destabilise the perturbative expansion, so that a further resummation is needed if the evolution is to be reliable at small x. Small x logarithms may be resummed by using the BFKL equation. However the fixed order kernels of this equation, currently known to NLO [5] are perturbatively unstable for all realistic values of α_s, and are thus by themselves useless for reliable calculations [6–8]. This is because they contain collinear logarithms, which must be resummed even at LO if reliable predictions are to be obtained [9]. This collinear resummation of the BFKL kernel restores longitudinal momentum conservation [10], and leads ultimately to a stable expansion.

Two approaches to the simultaneous resummation of these two classes of logs have recently reached the stage where their phenomenological application can be envisaged. In the duality (ABF) approach [10–16] one concentrates on the problem of obtaining an improved anomalous dimension (splitting function) for DIS which reduces to the ordinary perturbative result at large N (large x), while including resummed BFKL corrections at small N (small x), determined through the BFKL kernel. The CCSS approach [17,18,20,21] is built up within the BFKL framework, by improving the whole hierarchy of subleading kernels in the collinear region consistently with the renormalization group. The BFKL quation is then solved and a perturbative splitting function extracted numerically.

Here we will briefly review the theoretical underpinnings of the duality approach, and then compare phenomenological results obtained in both approaches.

1. High Energy Duality

In the ABF approach one constructs an improved anomalous dimension (splitting function) for DIS which reduces to the ordinary perturbative result at large N (large x) given by:

$$\gamma(N,\alpha_s) = \alpha_s\gamma_0(N) \ + \ \alpha_s^2\gamma_1(N) \ + \ \alpha_s^3\gamma_2(N) \ \ \ldots . \tag{1}$$

while including resummed BFKL corrections at small N (small x) which are determined by the aforementioned BFKL kernel $\chi(M,\alpha_s)$:

$$\chi(M,\alpha_s) = \alpha_s\chi_0(M) \ + \ \alpha_s^2\chi_1(M) \ + \ \ldots , \tag{2}$$

which is the Mellin transform of the angular averaged kernel K with respect to $t = \ln\frac{k^2}{k_0^2}$. The main theoretical tool which enables this construction is the duality relation between the kernels χ and γ

$$\chi(\gamma(N,\alpha_s),\alpha_s) = N, \tag{3}$$

which is a consequence of the fact that the solutions of the BFKL and DGLAP equations coincide at leading twist [8, 10, 19]. Further improvements are obtained exploiting the symmetry under gluon interchange of the BFKL gluon-gluon kernel and through the inclusion of running coupling effects.

By using duality, one can construct a more balanced expansion for both γ and χ, the "double leading" (DL) expansion, where the information from χ is used to include in γ all powers of α_s/N and, conversely, γ is used to improve χ by all powers of α_s/M. A great advantage of the DL expansion is that it resums the collinear poles of χ at $M = 0$, enabling the imposition of the physical requirement of momentum conservation $\gamma(1,\alpha_s) = 0$, whence, by duality:

$$\chi(0,\alpha_s) = 1. \tag{4}$$

This procedure eliminates in a model independent way the alternating sign poles $+1/M, -1/M^2, \ldots..$ that appear in χ_0, $\chi_1, \ldots$. These poles make the perturbative expansion of χ unreliable even in the central region of M: e.g., $\alpha_s\chi_0$ has a minimum at $M = 1/2$, while, at realistic values of α_s, $\alpha_s\chi_0 + \alpha_s^2\chi_1$ has a maximum.

At this stage, while the poles at $M = 0$ are eliminated, those at $M = 1$ remain, so that the DL expansion is still not finite near $M = 1$. The resummation of the $M = 1$ poles can be accomplished by exploiting the collinear-anticollinear symmetry, as in the CCSS approach [17, 18, 21]. In Mellin space, this symmetry implies that at the fixed-coupling level the

kernel χ for evolution in $\ln \frac{s}{kk_0}$ must satisfy $\chi(M) = \chi(1-M)$ order by order in perturbation theory. This symmetry is however broken by the DIS choice of variables $\ln \frac{1}{x} = \ln \frac{s}{Q^2}$ and by the running of the coupling. In the fixed coupling limit the kernel $\chi_{\rm DIS}$, dual to the DIS anomalous dimension, is related to the symmetric one χ_σ through the implicit equation [5]

$$\chi_{\rm DIS}(M + 1/2\chi_\sigma(M)) = \chi_\sigma(M). \tag{5}$$

Hence, the $M = 1$ poles can be resummed by performing the double-leading resummation of $M = 0$ poles of $\chi_{\rm DIS}$, determining the associated χ_σ through eq. (5), then symmetrizing it, and finally going back to DIS variables by using eq. (5) again in reverse. Using the momentum conservation eq. (4) and eq. (5), it is easy to show that $\chi_\sigma(M)$ is an entire function of M, with $\chi_\sigma(-1/2) = \chi_\sigma(3/2) = 1$, and thus necessarily has a minimum at $M = 1/2$. Through this procedure one obtains order by order from the DL expansion a symmetrized DL kernel $\chi_{\rm DIS}$, and its corresponding dual anomalous dimension γ. The kernel $\chi_{\rm DIS}$ has to all orders a minimum and satisfies a momentum conservation constraint $\chi_{\rm DIS}(0) = \chi_{\rm DIS}(2) = 1$.

The final ingredient of the ABF approach is a treatment of the running coupling corrections to the resummed terms. Indeed, their inclusion in the resummed anomalous dimension greatly softens the asymptotic behavior near $x = 0$. Hence, the dramatic rise of structure functions at small x, which characterized resummations based on leading–order BFKL evolution and is ruled out phenomenologically, is replaced by a much milder rise. This requires a running coupling generalization of the duality equation (3), which is possible noting that in M space the running coupling $\alpha_s(t)$ becomes a differential operator, since $t \to d/dM$. Hence, the BFKL evolution equation for double moments $G(N, M)$, which is an algebraic equation at fixed coupling, becomes a differential equation in M for running coupling. In the ABF approach, one solves this differential equation analytically when the kernel is replaced by its quadratic approximation near the minimum. The solution is expressed in terms of an Airy function if the kernel is linear in α_s [12], for example in the case of $\alpha_s\chi_0$, or of a Bateman function in the more general case of a nonlinear dependence on α_s [16] as is the case for the DL kernels. The final result for the improved anomalous dimension is given in terms of the DL expansion plus the "Airy" or "Bateman" anomalous dimension, with the terms already included in the DL expansion subtracted away.

For example, at leading DL order, i.e. only using $\gamma_0(N)$ and $\chi_0(M)$,

the improved anomalous dimension is

$$\gamma_I^{NL}(\alpha_s, N) = \left[\alpha_s\gamma_0(N) + \alpha_s^2\gamma_1(N) + \gamma_s(\frac{\alpha_s}{N}) - \frac{n_c\alpha_s}{\pi N}\right] + \gamma_A(c_0, \alpha_s, N) - \frac{1}{2} + \sqrt{\frac{2}{\kappa_0\alpha_s}[N - \alpha_s c_0]}. \quad (6)$$

The terms within square brackets give the LO DL approximation, i.e. they contain the fixed–coupling information from γ_0 and (through γ_s) from χ_0. The "Airy" anomalous dimension $\gamma_A(c_0, \alpha_s, N)$ contains the running coupling resummation, i.e. it is the exact solution of the running coupling BFKL equation which corresponds to a quadratic approximation to χ_0 near $M = 1/2$. The last two terms subtract the contributions to $\gamma_A(c_0, \alpha_s, N)$ which are already included in γ_s and γ_0. In the limit $\alpha_s \to 0$ with N fixed, $\gamma_I(\alpha_s, N)$ reduces to $\alpha_s\gamma_0(N) + O(\alpha_s^2)$. For $\alpha_s \to 0$ with α_s/N fixed, $\gamma_I(\alpha_s, N)$ reduces to $\gamma_s(\frac{\alpha_s}{N}) + O(\alpha_s^2/N)$, i.e. the leading term of the small x expansion. Thus the Airy term is subleading in both limits. However, if $N \to 0$ at fixed α_s, the Airy term replaces the leading singularity of the DL anomalous dimension, which is a square root branch cut, with a simple pole, located on the real axis at rather smaller N, thereby softening the small x behaviour. The quadratic approximation is sufficient to give the correct asymptotic behaviour up to terms which are of subleading order in comparison to those included in the DL expression in eq. (6).

The running coupling resummation procedure can be applied to a symmetrized kernel, which possesses a minimum to all orders, and then extended to next-to-leading order [15, 16]. This entails various technical complications, specifically related to the nonlinear dependence of the symmetrized kernel on α_s, to the need to include interference between running coupling effects and the small x resummation, and to the consistent treatment of next-to-leading log Q^2 terms, in particular those related to the running of the coupling.

2. Results

Even though the basic underlying physical principles of the ABF and CCSS approaches are close, there are technical differences in the construction of the symmetrized DL kernel, in the derivation from it of an anomalous dimension and associated splitting function, and in the inclusion of running coupling effects. Therefore, we will compare results for the resummed fixed-coupling χ kernel (BFKL characteristic function), then the corresponding fixed-coupling splitting functions, and finally the running coupling split-

ting functions which provide the final result in both approaches. In order to assess the phenomenological impact on parton evolution we will finally compare the convolution of the splitting function with a "typical" gluon distribution.

In fig. 1 we show the solution $\chi_{\rm DIS}$ of eq. (5) for the symmetrized NLO DL kernel. The pure Lx and NLx (BFKL) and next-to-leading $\ln Q^2$ (DGLAP) are also shown. All curves are determined with frozen coupling ($\beta_0 = 0$), and with $n_f = 0$, in order to avoid complications related to the diagonalization of the DGLAP anomalous dimension matrix and to the choice of scheme for the quark parton distribution. Also shown is the corresponding resummed kernel in the RGI CCSS approach. The resummed ABF and CCSS results are very close, the main difference being due to the fact that at small M the ABF result coincides by construction with NLO DGLAP, whereas for very small M (i.e. large x) the CCSS result reduces to LO DGLAP. Because of the underlying symmetry this small difference is also seen in the anticollinear region $M \sim 1$, though the two curves coincide by construction at the momentum conservation points $M = 0$ and $M = 2$. In comparison to DGLAP, the resummed kernels have a minimum, related to the fact that both collinear and anticollinear logs are resummed. In comparison to BFKL, which has a minimum at LO but not NLO, the resummed kernels always have a perturbatively stable minimum, characterized by a lower intercept than leading–order BFKL: specifically, when $\alpha_s = 0.2$, $\lambda \sim 0.25$ instead of $\lambda \sim 0.5$. This corresponds to a softer small x rise of the associated splitting function.

The fixed–coupling resummed splitting functions up to NLO are shown in fig. 2, along with the unresummed DGLAP splitting functions up to NNLO. For $n_f = 0$ the NLO DGLAP splitting function has the property that it vanishes at small x — this makes it relatively straightforward to combine not just LO DGLAP but also NLO DGLAP with the NLLx resummation. In the ABF approach the splitting function is the inverse Mellin transform of the anomalous dimension obtained using duality eq. (3) from the symmetrized DL χ kernel. Hence, the LO and NLO resummed result respectively reproduce all information contained in the LO and NLO χ and γ kernel with the additional constraint of collinear-anticollinear symmetry. Both the ABF LO and NLO results are shown in fig. 2. The CCSS NLx+LO and NLx+NLO curves are also shown for comparison.

In comparison to unresummed results, the resummed splitting functions display the characteristic rise at small x of fixed-coupling leading-order BFKL resummation, though the small x rise is rather milder ($\sim x^{-0.25}$

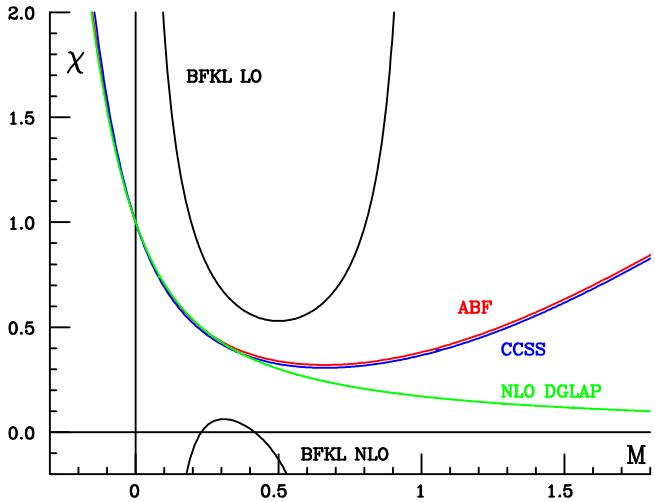

Figure 1. The kernel χ (BFKL characteristic function) for fixed coupling ($\beta_0 = 0$) $\alpha_s = 0.2$ and $n_f = 0$. The BFKL curves are the LO and NLO truncations of eq. (2), the DGLAP curve is the dual eq. (3) of the NLO anomalous dimension eq. (1), while the ABF and CCSS curves are respectively the solution $\chi_{\rm DIS}$ of eq. (5) and the solution for ω of a similar eqn. in the CCSS approach.

instead of $\sim x^{-0.5}$ for $\alpha_s = 0.2$). At large x there is good agreement between the resummed results and the corresponding LO (dashed) or NLO (solid) DGLAP curves. At small x the difference between the ABF LO and CCSS NLx+LO (dashed) curves is mostly due to the different way the symmetrization is implemented, as both approaches contain the same dominant small x terms. This difference is reduced when one compares the CCSS NLx+nLO with ABF NLO, and it might be taken as an estimate of the intrinsic ambiguity of the fixed–coupling resummation procedure. At intermediate x the NLO resummed splitting functions is of a similar order of magnitude as the NLO DGLAP result even down to quite small x, but with a rather different shape, characterized by a dip at $x \sim 10^{-3}$. The unstable small x drop of the NNLO DGLAP result is a consequence of the unresummed $\frac{\alpha_s^3}{N^2}$ double pole in the NNLO anomalous dimension.

The running-coupling resummed splitting functions are displayed in fig. 3. Note that the unresummed curves are the same as in the fixed coupling case since their dependence on α_s is just through a prefactor of α_s^k, whereas in the resummed case there is an interplay between the running of the coupling and the structure of the small-x logs. All the resummed

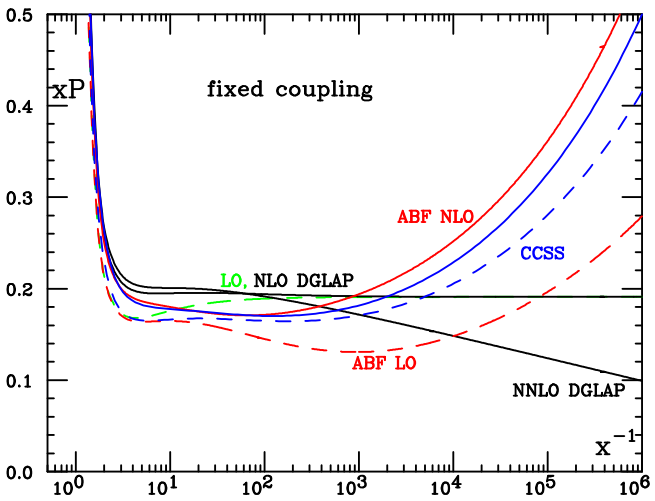

Figure 2. The fixed coupling ($\beta_0 = 0$) $xP_{gg}(x)$ splitting function, evaluated with $\alpha_s = 0.2$ and $n_f = 0$. The dashed curves are LO for DGAP, symmetrized LO DL for ABF, and NLx+LO for CCSS, while the solid curves are NLO and NNLO for DGLAP, symmetrized NLO DL for ABF and NLx+NLO for CCSS.

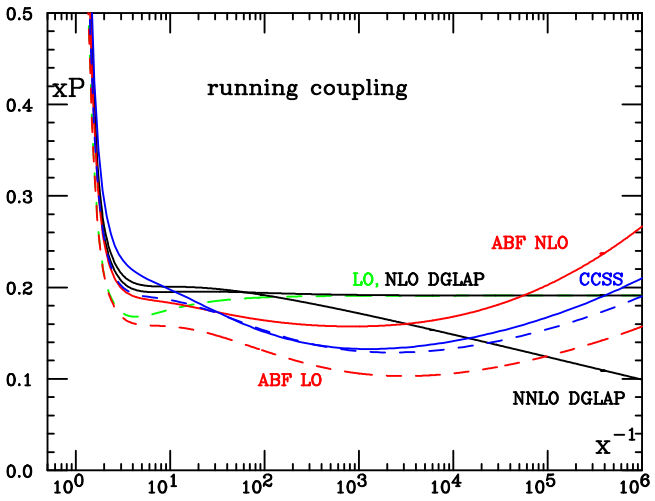

Figure 3. The running coupling $xP_{gg}(x)$ splitting function, evaluated with $\alpha_s = 0.2$ and $n_f = 0$. The various curves correspond to the same cases as in fig. 2.

curves display a considerable softening of the small x behaviour in comparison to their fixed-coupling counterparts, due to the softening of the

leading small x singularity in the running-coupling case [12, 17]. As a consequence, the various resummed results are closer to each other than in the fixed-coupling case, and also closer to the unresummed LO and NLO DGLAP results. The resummed perturbative expansion appears to be stable, subject to moderate theoretical ambiguity, and qualitatively close to NLO DGLAP.

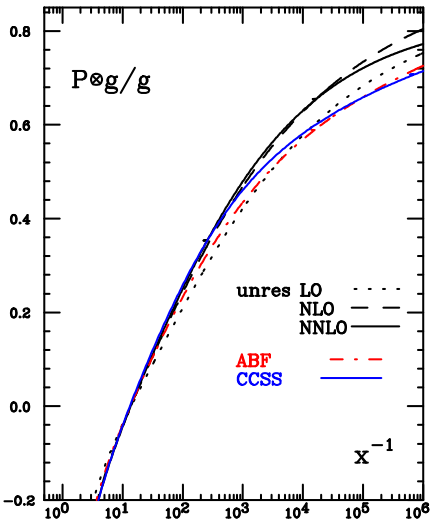

Figure 4. Convolution of resummed and fixed-order P_{gg} splitting functions with a toy gluon distribution, eq. (7), normalised to the gluon distribution itself, with $\alpha_s = 0.2$ and $n_f = 0$. The resummed ABF and CCSS curves are obtained using respectively the ABF NLO and CCSS NLx+NLO splitting function shown in fig. 3.

Finally, to appreciate the impact of resummation it is useful to investigate not only the properties of the splitting function, but also its convolution with a physically reasonable gluon distribution. We take the following toy gluon

$$xg(x) = x^{-0.18}(1-x)^5\,, \tag{7}$$

and show in fig. 4 the result of its convolution with various splitting functions of fig. 3. The differences between resummed and unresummed results, and between the ABF and CCSS resummations are as expected partly washed out by the convolution, even though the difference between the unresummed LO and NLO DGLAP results is clearly visible. In particular, differences between the fixed-order and resummed convolution start to

become significant only for $x \lesssim 10^{-2} - 10^{-3}$, even though resummation effects started to be visible in the splitting functions at somewhat larger x. However it should be clear from this figure why the structure function data from HERA were so well described by LO and NLO GLAP evolution [1–3]: significant deviations from GLAP will only be seen for smaller x and larger Q^2 than is accessible at HERA, but may well be important for many important processes at the LHC.

References

1. R.D. Ball and S. Forte, Phys. Lett. **B335**, 77 (1994).
2. R.D. Ball and S. Forte, Phys. Lett. **B336**, 77 (1994).
3. R.D. Ball and S. Forte, Phys. Lett. **B358**, 365 (1995).
4. S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. **B726**, 317 (2005).
5. V.S. Fadin and L.N. Lipatov, Phys. Lett. **B429**, 127 (1998).
6. R.D. Ball and S. Forte, hep-ph/9805315, 1998.
7. D.A. Ross, Phys. Lett. **B431**, 161 (1998).
8. R.D. Ball and S. Forte, Phys. Lett. **B465**, 271 (1999).
9. G.P. Salam, J.H.E.P. **9807**, 019 (1998).
10. G. Altarelli, R.D. Ball and S. Forte, Nucl. Phys. **B575**, 313 (2000).
11. G. Altarelli, R.D. Ball and S. Forte, Nucl. Phys. **B599**, 383 (2001).
12. G. Altarelli, R.D. Ball and S. Forte, Nucl. Phys. **B621**, 359 (2002).
13. G. Altarelli, R.D. Ball and S. Forte, Nucl. Phys. **B674**, 459 (2003).
14. G. Altarelli, R.D. Ball and S. Forte, hep-ph/0310016, 2003.
15. G. Altarelli, R.D. Ball and S. Forte, Nucl. Phys. Proc. Suppl. **135**, 163 (2004).
16. G. Altarelli, R.D. Ball and S. Forte, hep-ph/0512237, 2005.
17. M. Ciafaloni, D. Colferai and G.P. Salam, Phys. Rev. **D60**, 114036 (1999).
18. M. Ciafaloni, D. Colferai, G.P. Salam and A.M. Stasto, Phys. Lett. **B576**, 143 (2003).
19. R.D. Ball and S. Forte, Phys. Lett. **B405**, 317 (1997).
20. M. Ciafaloni and D. Colferai, hep-ph/0507106, 2005.
21. M. Ciafaloni, D. Colferai, G.P. Salam and A.M. Stasto, Phys. Rev. **D68**, 114003 (2003).