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Deep-inelastic scattering and jets

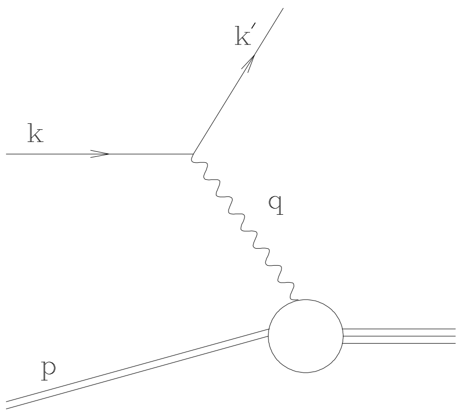
- Deep inelastic scattering
- Factorization
- Renormalization group equation
- Scaling violation
- Small x - Double asymptotic behaviour
- High Energy Behaviour
- Impact Factor
- Small x DIS in the proton rest frame
- Aligned jet model

Talk available at

`http://theory.fnal.gov/people/ellis/Talks/`

Deep inelastic scattering

- Consider lepton-proton scattering via exchange of virtual photon:



- Standard variables are:

$$
\begin{aligned}
x &= \frac{-q^2}{2p \cdot q} = \frac{Q^2}{2M(E-E')} \\
y &= \frac{q \cdot p}{k \cdot p} = 1 - \frac{E'}{E}
\end{aligned}
$$

where $Q^2 = -q^2 > 0$, $M^2 = p^2$ and energies refer to target rest frame.

- Elastic scattering has $(p+q)^2 = M^2$, i.e. $x = 1$. Hence deep inelastic scattering (DIS) means $Q^2 \gg M^2$ and $x < 1$.

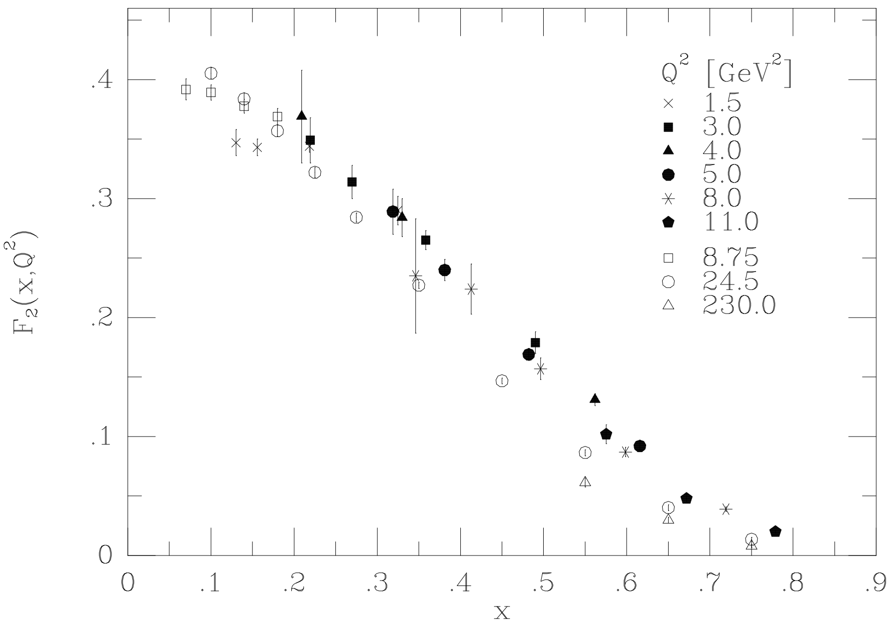
- Structure functions $F_i(x,Q^2)$ parametrise target structure as 'seen' by virtual photon. Defined in terms of cross section

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2}{Q^2xy}\Big[\Big(\frac{1+(1-y)^2}{2}\Big)2xF_1 \\ +(1-y)(F_2-2xF_1)-(M/2E)xyF_2\Big] .$$

- Bjorken limit is Q^2, $p\cdot q \to \infty$ with x fixed. In this limit structure functions obey approximate Bjorken scaling law, i.e. depend only on dimensionless variable x:

$$\begin{aligned} F_i(x,Q^2) &\longrightarrow F_i(x) \\ \sigma_i(x,Q^2) &= \frac{4\pi^2\alpha}{Q^2}F_i(x,Q^2) \sim 1/Q^2 \end{aligned}$$

- Scaling behaviour implies that the cross section falls like $1/Q^2$.



- Figure shows F_2 structure function for proton target. Although Q^2 varies by two orders of magnitude, in first approximation data lie on universal curve.

- Bjorken scaling implies that virtual photon is scattered by *pointlike constituents* (partons) — otherwise structure functions would depend on ratio Q/Q_0, with $1/Q_0$ a length scale characterizing size of constituents.

Naive Parton Model

- Parton model of DIS is formulated in a frame where target proton is moving very fast — *infinite momentum frame*.
  - Suppose that, in this frame, photon scatters from pointlike quark with fraction ξ of proton's momentum. Since $(\xi p + q)^2 = m_q^2 \ll Q^2$, we must have $\xi = Q^2/2p \cdot q = x$.
  - In terms of Mandelstam variables $\hat{s}, \hat{t}, \hat{u}$, spin-averaged matrix element squared for massless $eq \to eq$ scattering is

$$\overline{\sum}|\mathcal{M}|^2 = 2e_q^2 e^4 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$

    where $\overline{\sum}$ denotes average (sum) over initial (final) colours and spins.
  - In terms of DIS variables, $\hat{t} = -Q^2$, $\hat{u} = \hat{s}(y-1)$ and $\hat{s} = Q^2/xy$. Differential cross section is then

$$\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}[1 + (1-y)^2]\frac{1}{2}e_q^2 \delta(x - \xi).$$

  - From structure function definition (neglecting M)

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left\{[1 + (1-y)^2]F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right\}.$$

- Hence structure function for scattering from parton with momentum fraction ξ is

$$\hat{F}_2 = x e_q^2 \delta(x-\xi) = 2x\hat{F}_1 \; .$$

- Suppose probability that quark q carries momentum fraction between ξ and $\xi + d\xi$ is $q(\xi)\, d\xi$. Then

$$\begin{aligned} F_2(x) &= \sum_q \int_0^1 d\xi \; q(\xi) \; x e_q^2 \delta(x-\xi) \\ &= \sum_q e_q^2 x q(x) = 2x F_1(x) \; . \end{aligned}$$

- Relationship $F_2 = 2xF_1$ (Callan-Gross relation) follows from spin-$\frac{1}{2}$ property of quarks ($F_1 = 0$ for spin-0).

- Proton consists of three valence quarks (uud), which carry its electric charge and baryon number, and infinite sea of light $q\bar{q}$ pairs. Probed at scale Q, sea contains all quark flavours with $m_q \ll Q$. Thus at $Q \sim 1$ GeV expect

$$F_2^{em}(x) \simeq \frac{4}{9} x[u(x)+\bar{u}(x)] + \frac{1}{9} x[d(x)+\bar{d}(x)+s(x)+\bar{s}(x)]$$

where

$$
\begin{aligned}
u(x) &= u_V(x) + \bar{u}(x) \\
d(x) &= d_V(x) + \bar{d}(x) \\
s(x) &= \bar{s}(x)
\end{aligned}
$$

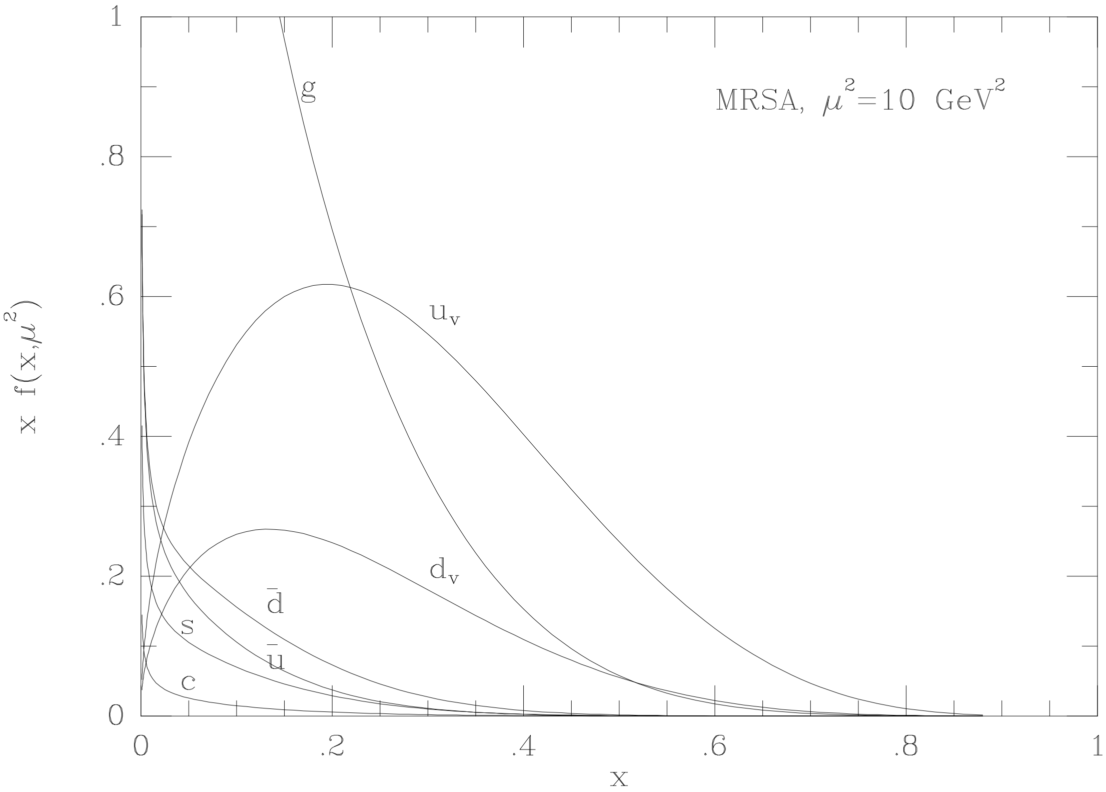
with sum rules

$$\int_0^1 dx\; u_V(x) = 2\,, \quad \int_0^1 dx\; d_V(x) = 1\,.$$

- Experimentally one finds

$$\sum_q \int_0^1 dx\; x[q(x) + \bar{q}(x)] \simeq 0.5\,.$$

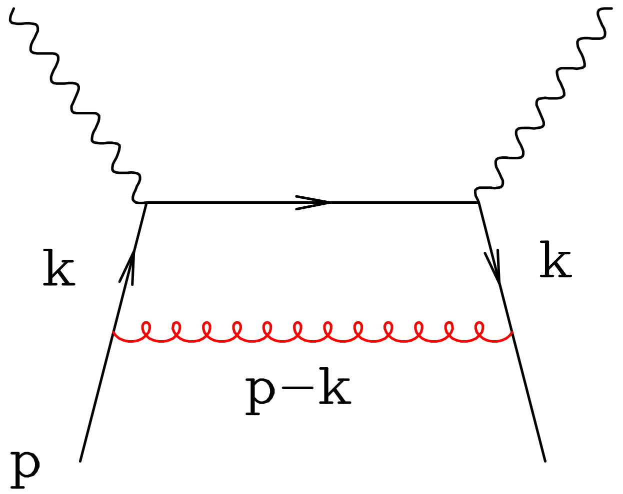
Thus quarks only carry about 50% of proton's momentum. Rest is carried by *gluons*. Although not directly measured in DIS, gluons participate in other hard scattering processes such as large-p_T jet and prompt photon production.



- Figure shows typical set of parton distributions extracted from fits to DIS data, at $Q^2 = 10$ GeV^2.

Gluonic corrections

- To go beyond the naive parton model, investigate gluonic corrections to DIS.



- Sudakov decomposition $k = zp + \beta n + \mathbf{k}$ where $n.n = p.p = n.\mathbf{k} = p.\mathbf{k} = 0$.
- Phase space times propagators, $((p-k)^2 = 0 \to k^2 = -\frac{\mathbf{k}^2}{(1-z)})$

$$PS = \frac{1}{(2\pi)^3} \int \frac{d^n k}{k^4} \, \delta((p-k)^2) = \frac{1}{8\pi^2} \int dz \, \frac{d^{n-2}\mathbf{k}}{\pi \mathbf{k}^4} \, (1-z)$$

- Quadratic divergence is reduced to a logarithmic divergence by helicity conservation at the gluon emission vertex. Similarly other real diagrams have no divergence.

- Lower part of diagram gives $(\gamma.p = \hat{p})$

$$C_F g^2(\mu)^{2\epsilon} \sum_{pol} \left[\hat{k}\hat{\varepsilon}\hat{p}\hat{\varepsilon}\hat{k}\right]_{ij} \times PS = A\ \hat{p}_{ij} + B\ \hat{n}_{ij} + C\ \hat{\mathbf{k}}_{ij}$$

$$\begin{aligned}
A \times PS &= C_F(\mu)^{2\epsilon}\frac{\alpha_s}{2\pi}\int\ dz\ \int^{\sim Q^2} \frac{d^{n-2}\mathbf{k}}{\pi\mathbf{k}^2}\,\frac{1+z^2}{1-z} \\
B \times PS &= O(1)\ \text{non-singular} \\
C \times PS &\to 0\ \text{after angular averaging}
\end{aligned}$$

Using the relation $(n = 4 - 2\epsilon)$

$$\int_0^{Q^2} \frac{d^{n-2}\mathbf{k}}{\mathbf{k}^2} = -\frac{1}{\epsilon}\,\frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}(Q^2)^{-\epsilon}$$

We find the radiative correction to the lower part of the diagram

$$\Gamma(z, \alpha_s, \epsilon) = \delta(1-z) - \frac{1}{\epsilon} C_F \frac{\alpha_s}{2\pi}\,\frac{1+z^2}{(1-z)} + \text{Finite terms}$$

Note that we did not need to specify the upper part of the diagram.

Factorization

- Generalization of the lowest order example above.

- Factorize physical cross section into short-distance part calculable in perturbation theory and universal long distance part to be measured in one process and used in another.

- Factorization implies that the DIS structure functions can be written to all orders in perturbation theory as,

$$F_i\left(x, \frac{Q^2}{\mu^2}, \alpha_S, \frac{1}{\epsilon}\right) = \int_x^1 \frac{dz}{z}\, C_i\left(\frac{x}{z}, \frac{Q^2}{\mu^2}, \alpha_S\right) \Gamma\left(z, \alpha_S, \frac{1}{\epsilon}\right)$$

- C_i is the short distance deep inelastic scattering cross-section (coefficient function) corresponding to the ith structure function. C_i contains no mass singularities.

- Γ is a process independent function depending only on the type of the incoming parton leg. It contains all mass singularities order by order. In the dimensionally regularised scheme these appear as poles in ϵ. In general the function Γ is a matrix in the space of quarks and gluons.

- Consider the diagonal quark-quark term. Rewrite in a form which makes the convolution structure manifest,

$$F_i\left(x,\frac{Q^2}{\mu^2},\alpha_S,\frac{1}{\epsilon}\right)=\int_0^1 dy\int_0^1 dz\,\delta(x-yz)$$
$$C_i\left(y,\frac{Q^2}{\mu^2},\alpha_S\right)\Gamma_{qq}\left(z,\alpha_S,\frac{1}{\epsilon}\right).$$

Since convolutions of this form occur frequently we introduce a symbolic notation,

$$C(x)=\int_0^1 dy\int_0^1 dz\,\delta(x-yz)A(y)B(z)\equiv A\otimes B.$$

We define the moments of any function by the equation,

$$F\left(j\right)=\int_0^1 dz\ \ z^{j-1}F\left(z\right).$$

In moment space the convolutions reduce to a product of moments. Factorization takes the form

$$F\left(j,\frac{Q^2}{\mu^2},\alpha_S,\frac{1}{\epsilon}\right)=C\left(j,\frac{Q^2}{\mu^2},\alpha_S\right)\Gamma_{qq}\left(j,\alpha_S,\frac{1}{\epsilon}\right).$$

In the following we shall often leave unspecified whether we are working in moment space or in the space of longitudinal momentum fractions.

Proof of Factorization

- Factorization in DIS is a consequence of the operator product expansion.
- Factorization in processes with *two* incoming hadron requires an independent proof performed in perturbation theory.
- Theorems of Perturbative QCD (inluding factorization) have been reviewed by Collins and Soper.
- Soft Collinear effective field theory approach may simplify proofs of factorization, see Bauer et al. Such proofs may lead into insight into power corrections.
- In this lecture no proof will be given.

Factorization of the cross section

- Why does the factorization property hold and when it should fail?
- For a heuristic argument consider the simplest hard process involving two hadrons

$$H_1(P_1) + H_2(P_2) \to V + X.$$

- Do the partons in hadron H_1, through the influence of their colour fields, change the distribution of partons in hadron H_2 before the vector boson is produced? Soft gluons which are emitted long before the collision are potentially troublesome.
- A simple model from classical electrodynamics. The vector potential due to an electromagnetic current density J is given by

$$A^\mu(t,\vec{x}) = \int dt' d\vec{x}' \, \frac{J^\mu(t',\vec{x}')}{|\vec{x}-\vec{x}'|} \, \delta(t' + |\vec{x}-\vec{x}'| - t) \; ,$$

where the delta function provides the retarded behaviour required by causality.

Consider a particle with charge e travelling in the positive z direction with constant velocity β. The non-zero components of the current density are

$$\begin{aligned} J^t(t', \vec{x}') &= e\delta(\vec{x}' - \vec{r}(t')) \;, \\ J^z(t', \vec{x}') &= e\beta\delta(\vec{x}' - \vec{r}(t')), \;\; \vec{r}(t') = \beta t' \hat{z}, \end{aligned}$$

$\hat{z}$ is a unit vector in the z direction. At an observation point (the supposed position of hadron H_2) described by coordinates x, y and z, the vector potential (after performing the integrations using the current density given above) is

$$\begin{aligned} A^t(t, \vec{x}) &= \frac{e\gamma}{\sqrt{[x^2 + y^2 + \gamma^2(\beta t - z)^2]}} \\ A^x(t, \vec{x}) &= 0 \\ A^y(t, \vec{x}) &= 0 \\ A^z(t, \vec{x}) &= \frac{e\gamma\beta}{\sqrt{[x^2 + y^2 + \gamma^2(\beta t - z)^2]}}, \end{aligned}$$

where $\gamma^2 = 1/(1 - \beta^2)$. Target hadron H_2 is at rest near the origin, so that $\gamma \approx s/m^2$.

- Note that for large γ and fixed non-zero $(\beta t - z)$ some components of the potential tend to a constant independent of γ, suggesting that there will be non-zero fields which are not in coincidence with the arrival of the particle, even at high energy.
- However at large γ the potential is a pure gauge piece, $A^\mu = \partial^\mu \chi$ where χ is a scalar function
- Covariant formulation using the vector potential A has large fields which have no effect.
- For example, the electric field along the z direction is

$$E^z(t,\vec{x}) = F^{tz} \equiv \frac{\partial A^z}{\partial t} + \frac{\partial A^t}{\partial z} = \frac{e\gamma(\beta t - z)}{[x^2 + y^2 + \gamma^2(\beta t - z)^2]^{\frac{3}{2}}}.$$

The leading terms in γ cancel and the field strengths are of order $1/\gamma^2$ and hence of order m^4/s^2. The model suggests the force experienced by a charge in the hadron H_2, at any fixed time before the arrival of the quark, decreases as m^4/s^2.

Renormalization group behaviour

- In performing the factorization and renormalization we were obliged to introduce a parameter μ to keep the coupling constant dimensionless in d dimensions. μ is an arbitrary parameter and consequently, with fixed bare parameters, no physical result can depend on it. This leads to the property known as renormalization group invariance.

- Define the total logarithmic derivative with respect to μ,

$$\mathcal{D} = \mu \frac{d}{d\mu} \equiv \mu \frac{\partial}{\partial \mu} + \tilde{\beta}(g,\epsilon) \frac{\partial}{\partial g}$$

  where $\tilde{\beta}(g,\epsilon)$ is the logarithmic derivative of the renormalised coupling at fixed bare parameters,

$$\tilde{\beta}(g,\epsilon) = \mu \frac{\partial}{\partial \mu} g \bigg|_{g_0,\epsilon} .$$

  In the MS scheme the relationship between the bare and renormalised parameters is

$$g = \mu^{-\epsilon} Z_g^{-1} g_0 .$$

By construction, Z_g contains only pole terms and therefore has a Laurent expansion in ϵ.

$$Z_g = 1 + \sum_{i=1}^{\infty} \frac{Z_g^i}{\epsilon^i}$$

The quantities Z_g^i have perturbative expansions in the renormalised coupling. We therefore find that $\tilde{\beta}$ satisfies the equation,

$$\left[\tilde{\beta}\left(g,\epsilon\right) + \epsilon g + g\frac{\partial}{\partial \ln \mu}\right] Z_g = 0$$

In the minimal subtraction scheme Z_g has no explicit μ dependence. The μ-dependence enters only through the implicit dependence of the renormalised coupling. So the μ derivative can be related to a derivative with respect to g,

$$\mu\frac{\partial}{\partial\mu}Z_g\left(g\left(g_0,\mu,\epsilon\right),\epsilon\right) = \tilde{\beta}\left(g,\epsilon\right)\frac{\partial}{\partial g}Z_g,$$

may be written as

$$\left[\tilde{\beta}\left(g,\epsilon\right)\frac{\partial}{\partial g}g + \epsilon g\right] Z_g = 0.$$

Now $\beta(g,\epsilon)$ is finite for vanishing ϵ, so that, it is clear that $\tilde{\beta}(g,\epsilon)$ is at most linear in ϵ. Comparing the coefficient of the linear term in ϵ we find,

$$\tilde{\beta}(g,\epsilon) = -\epsilon g + \tilde{\beta}(g).$$

Substituting for $\tilde{\beta}$,

$$\left[\tilde{\beta}(g)\frac{\partial}{\partial g}g - \epsilon g^2 \frac{\partial}{\partial g}\right] Z_g = 0.$$

By further comparing the coefficients of the term of order ϵ^0 we obtain,

$$\tilde{\beta}(g) = g^2 \frac{\partial}{\partial g} Z_g^{(1)}.$$

We therefore conclude that, in the MS renormalization scheme the beta function is determined by the simple poles in Z_g to all orders in g.

In lowest order we have that,

$$Z_g^{(1)} = -\frac{g^2}{16\pi^2}\left[\frac{11N_c - 4n_f T_R}{6}\right] + \ldots$$

as given in Table 2. The result for the beta function is,

$$\tilde{\beta}(g) = \frac{-g^3}{16\pi^2}\left(\frac{11N_c - 4n_f T_R}{3}\right) + O(g^5)$$

- Operate with the renormalization group operator on the LHS of the factorization equation

$$\mathcal{D}\ln F = [\mathcal{D}\ln C + \mathcal{D}\ln\Gamma_{qq}] = 0.$$

- We now introduce a renormalization group quantity called the anomalous dimension γ_{qq},

$$\gamma_{qq} = \frac{1}{2}\mathcal{D}\ln\Gamma_{qq} \equiv \frac{1}{2}\tilde{\beta}(g,\epsilon)\frac{\partial}{\partial g}\ln\Gamma_{qq}$$

and obtain a renormalization group equation for short distance cross-section C.

$$\mathcal{D}C + 2\gamma_{qq}C = 0.$$

Since γ_{qq} appears in an equation involving C and $\mathcal{D}$ which are both finite in the limit $\epsilon \to 0$, γ_{qq} is also finite in that limit. In the minimal subtraction factorization scheme Γ_{qq} has a Laurent expansion.

$$\Gamma_{qq} = 1 + \sum_{i=1}^{\infty}\frac{\Gamma_{qq}^{(i)}}{\epsilon^i}.$$

Using arguments similar to those used for the β function we can show that the anomalous dimension is determined by the simple pole in Γ_{qq}.

$$\gamma_{qq} = -\frac{1}{2} g \frac{\partial}{\partial g} \Gamma_{qq}^{(1)}.$$

Solution of renormalization group equation

- The renormalization group equation for C, the short distance cross-section,

$$\left[\mu\frac{\partial}{\partial\mu}+\tilde{\beta}(g)\frac{\partial}{\partial g}+2\gamma_{qq}\right]C\left(\ln\frac{Q^2}{\mu^2},\alpha_S\right)=0.$$

It is convenient to work in terms of the variables $\alpha_S = g^2/4\pi$ and $t = \ln Q^2/\mu^2$. We have therefore defined the function $\beta(\alpha_\mathrm{S}) = g\tilde{\beta}(g)/(4\pi)$. In terms of these variables the renormalization group equation is,

$$\left[-\frac{d}{dt}+\beta(\alpha_S)\frac{d}{d\alpha_S}+\gamma_{qq}(\alpha_S)\right]C\left(t,\alpha_S\right)=0$$

where the expansion of the renormalization group functions are defined by,

$$\begin{aligned}\beta(\alpha_S) &= -b\alpha_S^2\left(1+b'\alpha_S\cdots\right), \qquad b=\left(\frac{11N_c-4n_fT_R}{12\pi}\right),\\ \gamma_{qq}(\alpha_S) &= -c\alpha_S+c'\alpha_S^2\cdots\end{aligned}$$

b and b' are independent of the renormalization scheme.

The solution to the renormalization group equation is,

$$C(t,\alpha_S) = C\left(0,\alpha\left(t\right)\right) \exp \int_{\alpha(0)}^{\alpha(t)} dx \frac{\gamma_{qq}(x)}{\beta(x)}.$$

A leading logarithmic approximation to this solution can be obtained by including the first term in the perturbative expansion of the β function,

$$t = -\int_{\alpha(0)}^{\alpha(t)} \frac{dx}{bx^2}.$$

which yields the expression,

$$\alpha(t) = \frac{\alpha(0)}{1+\alpha(0)bt}.$$

In the limit of large t we may write this as,

$$\alpha(t) \sim \frac{1}{b \ln Q^2/\Lambda^2} + O(1/\ln^2(Q^2)).$$

Λ is a fundamental parameter of the theory, which sets the scale for the strength of the strong interactions. However it depends on the renormalization scheme. At LO Λ is undefined, since a rescaling in Λ is of the same order as terms which have been dropped.

Restoring the dependence on the moment number j, the solution for the structure function in perturbation theory may be written,

$$F_i\left(\frac{Q^2}{\mu^2}, j, \alpha_S, \frac{1}{\epsilon}\right) = C\left(1, j, \alpha(t)\right)$$
$$\exp \int_{\alpha(0)}^{\alpha(t)} dx \frac{\gamma_{qq}(x)}{\beta(x)}\, \Gamma_{qq}\left(j, \alpha_S, \frac{1}{\epsilon}\right)$$

Altarelli-Parisi equation

The final result for the hadronic structure function is given by the convolution of the perturbative result with a 'bare' distribution of quarks inside a hadron, q_B. It is convenient to define a 'dressed' parton distribution function,

$$q(j,t) = \Gamma_{qq}\left(j, \alpha_S, \frac{1}{\epsilon}\right) q_B\left(j, \frac{1}{\epsilon}\right) \exp \int_{\alpha(0)}^{\alpha(t)} dx \frac{\gamma_{qq}(x)}{\beta(x)}$$

The physics of the low momentum region is certainly non-perturbative. A necessary assumption in order for the whole parton picture to make sense is that the singularities in Γ_{qq} are cancelled by singularities present in the bare quark distribution function. The 'dressed' quark distribution function is therefore a finite function. Its behaviour under changes of the scale t is,

$$\frac{dq(j,t)}{dt} = \gamma_{qq}\left(j, \alpha(t)\right) q\left(j,t\right).$$

P_{qq} is the anti-Mellin transform of γ_{qq}

$$\gamma_{qq}\left(j, \alpha_S\right) = \frac{\alpha_S}{2\pi} \int_0^1 dx x^{j-1} P_{qq}(x, \alpha_S).$$

Splitting Functions

$$\begin{aligned}
\hat{P}_{qq}(z) &= C_F \frac{1+z^2}{1-z} \\
\hat{P}_{qg}(z) &= T_R((1-z)^2+z^2) \\
\hat{P}_{gq}(z) &= C_F \frac{1+(1-z)^2}{z} \\
\hat{P}_{gg}(z) &= C_A \Big[\frac{1}{z}+\frac{1}{(1-z)}-2+z(1-z)\Big]
\end{aligned}$$

- The $\hat{P}$ are technically the unregularized splitting functions
- The full functions have virtual contributions which contribute only at $z=1$. They can be fixed by
- Conservation of quark-number

$$\int_0^1 dz\, P_{qq}(z) = 0$$

- Conservation of parton momentum

$$\int_0^1 dz\; z\Big[P_{qq}(z) + P_{gq}(z)\Big] = 0$$
$$\int_0^1 dz\; z\Big[P_{qg}(z) + P_{gg}(z)\Big] = 0$$

Introduce plus-prescription with definition

$$\int_0^1 dx\; f(x)\, g(x)_+ \;=\; \int_0^1 dx\; [f(x) - f(1)]\, g(x)\;.$$

Using this we can define regularized splitting function

$$P(z) = \hat{P}(z)_+\;,$$

The plus-prescription applies only to P_{qq} and P_{gg} parts giving

$$\begin{aligned} P_{qq}(z) &= \hat{P}_{qq}(z)_+ = C_F\left(\frac{1+z^2}{1-z}\right)_+ \\ P_{qg}(z) &= \hat{P}_{qg}(z) = T_R\,[z^2 + (1-z)^2] \end{aligned}$$

- P_{qq} and P_{gg} can be written in the common forms

$$
\begin{aligned}
P_{qq}(z) &= C_F\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right] \\
P_{gg}(z) &= 2C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right] \\
&\quad +\frac{1}{6}(11C_A-4N_fT_R)\,\delta(1-z)\ .
\end{aligned}
$$

Scheme Dependence

- The physical cross section is made up of a coefficient function $\mathbf{C}(t)$ and parton distribution functions $\mathbf{\Gamma}(t)$, (matrices in the space of quarks and gluons)

$$F(t) = \mathbf{C}(t)\ \mathbf{\Gamma}(t)$$

where $\mathbf{\Gamma}(t)$ satisfies DGLAP (renormalization group) eqn

$$\frac{d}{d\ln Q^2}\mathbf{\Gamma}(t) = \mathbf{P}\ \mathbf{\Gamma}(t)$$

These have perturbative expansions

$$\begin{aligned} \mathbf{P} &= \frac{\alpha_s}{2\pi}\Big[\mathbf{P}^{(0)} + \frac{\alpha_s}{2\pi}\mathbf{P}^{(1)} + \ldots\Big] \\ \mathbf{C} &= \mathbf{C}^{(0)} + \frac{\alpha_s}{2\pi}\mathbf{C}^{(1)} + \ldots \end{aligned}$$

- Change the factorization scheme, $(\mathbf{Z} = \mathbf{1} + \frac{\alpha_s}{2\pi}\mathbf{Z}^{(1)} + \ldots)$

$$\mathbf{C}(t) \to \hat{\mathbf{C}}(t) = \mathbf{C}(t)\mathbf{Z}^{-1}(t),\ \ \mathbf{\Gamma}(t) \to \hat{\mathbf{\Gamma}}(t) = \mathbf{Z}(t)\mathbf{\Gamma}(t)$$

Modifies $\mathbf{C}^{(1)}, \mathbf{P}^{(1)}$ – both needed for consistency

$$\begin{aligned}\hat{\mathbf{C}}^{(1)} &= \mathbf{C}^{(1)} - \mathbf{C}^{(1)}\mathbf{Z}^{(1)} \\ \hat{\mathbf{P}}^{(1)} &= \mathbf{P}^{(1)} - [\mathbf{Z}^{(1)}, \mathbf{P}^{(1)}] - \frac{b_0}{2}\mathbf{Z}^{(1)}\end{aligned}$$

- In n-th order a consistent prediction for DIS requires the coefficient function calculated in order α_s^n and the anomalous dimension calculated in order α_s^{n+1}

Scaling violation

- Bjorken scaling is not exact. Structure functions decrease at large x and grow at small x with increasing Q^2. This is due to Q^2 dependence of parton distributions, considered earlier. In present notation, they satisfy DGLAP evolution equations of form

$$t\frac{\partial}{\partial t}q(x,t) = \frac{\alpha_S(t)}{2\pi}\int_x^1 \frac{dz}{z}P(z)q\left(\frac{x}{z},t\right) \equiv \frac{\alpha_S(t)}{2\pi}P\otimes q$$

where P is $q \to qg$ splitting function.

- Taking into account other types of parton branching that can occur in addition to $q \to qg$, we obtain coupled evolution equations

$$\begin{aligned}
t\frac{\partial q_i}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{qq}\otimes q_i + P_{qg}\otimes g\right] \\
t\frac{\partial \bar{q}_i}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{qq}\otimes \bar{q}_i + P_{qg}\otimes g\right] \\
t\frac{\partial g}{\partial t} &= \frac{\alpha_S(t)}{2\pi}\left[P_{gq}\otimes \sum(q_i+\bar{q}_i) + P_{gg}\otimes g\right] .
\end{aligned}$$

  - ❖ Lowest-order splitting functions derived. More generally they are power series in α_S, same for jet fragmentation (timelike branching) and deep inelastic scattering (spacelike branching) in leading order, but differing in higher orders.

- For the present, concentrate on larger x values ($x \gtrsim 0.01$), where PT expansion converges better.

- Recall solution of evolution equations for flavour non-singlet combinations V , e.g. $q_i - \bar{q}_i$ or $q_i - q_j$. Mixing with gluons drops out and

$$t\frac{\partial}{\partial t}V(x,t) = \frac{\alpha_S(t)}{2\pi}P_{qq} \otimes V \ .$$

Taking moments (Mellin transform)

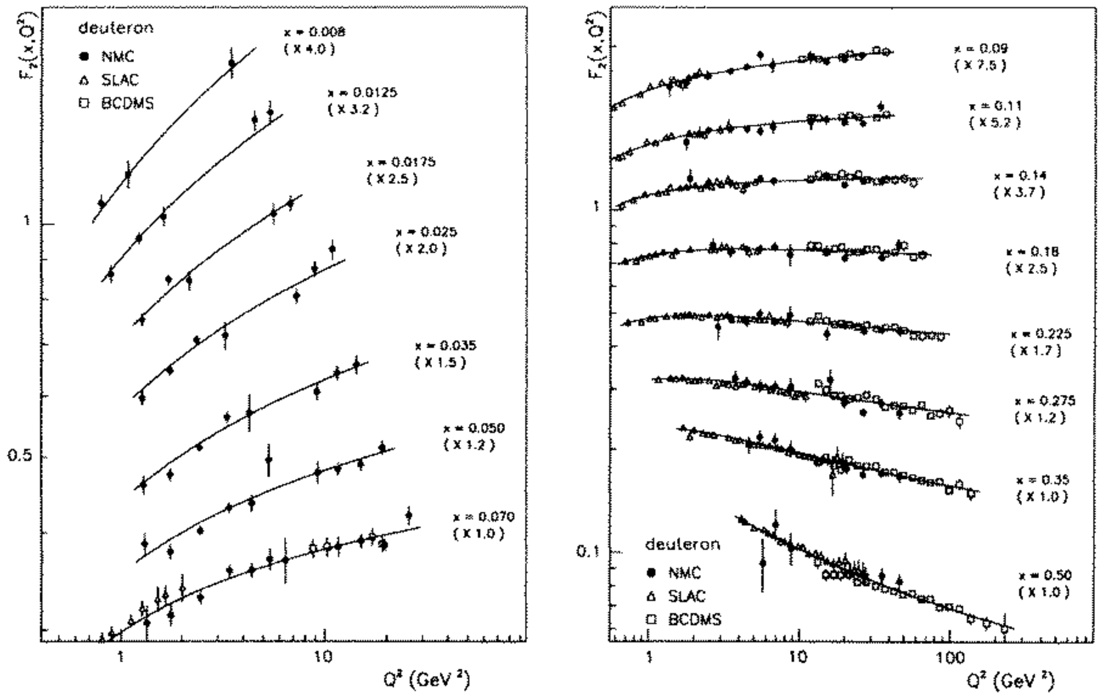
$$\begin{aligned} \tilde{V}(j,t) &= \int_0^1 dx\; x^{j-1}\, V(x,t) \\ t\frac{\partial}{\partial t}\tilde{V}(j,t) &= \frac{\alpha_S(t)}{2\pi}\gamma_{qq}^{(0)}(j)\; \tilde{V}(j,t) \end{aligned}$$

where $\gamma_{qq}^{(0)}(j)$ is Mellin transform of $P_{qq}^{(0)}$. Solution is

$$\tilde{V}(j,t) = \tilde{V}(j,0)\left(\frac{\alpha_S(0)}{\alpha_S(t)}\right)^{d_{qq}(j)}$$

where $d_{qq}(j) = \gamma_{qq}^{(0)}(j)/2\pi b$.

- Now $d_{qq}(1) = 0$ and $d_{qq}(j) < 0$ for $j \geq 2$. Thus as t increases V *decreases* at large x and *increases* at small x. Physically, this is due to increase in the phase space for gluon emission by quarks as t increases, leading to loss of momentum. This is clearly visible in data:



- For flavour-singlet combination, define

$$\Sigma = \sum_i (q_i + \bar{q}_i) \, .$$

Then we obtain

$$t\frac{\partial \Sigma}{\partial t} = \frac{\alpha_S(t)}{2\pi} \left[P_{qq} \otimes \Sigma + 2N_f P_{qg} \otimes g \right]$$

$$t\frac{\partial g}{\partial t} = \frac{\alpha_S(t)}{2\pi}\left[P_{gq}\otimes\Sigma + P_{gg}\otimes g\right] \ .$$

- Thus flavour-singlet quark distribution Σ mixes with gluon distribution g: evolution equation for moments has matrix form

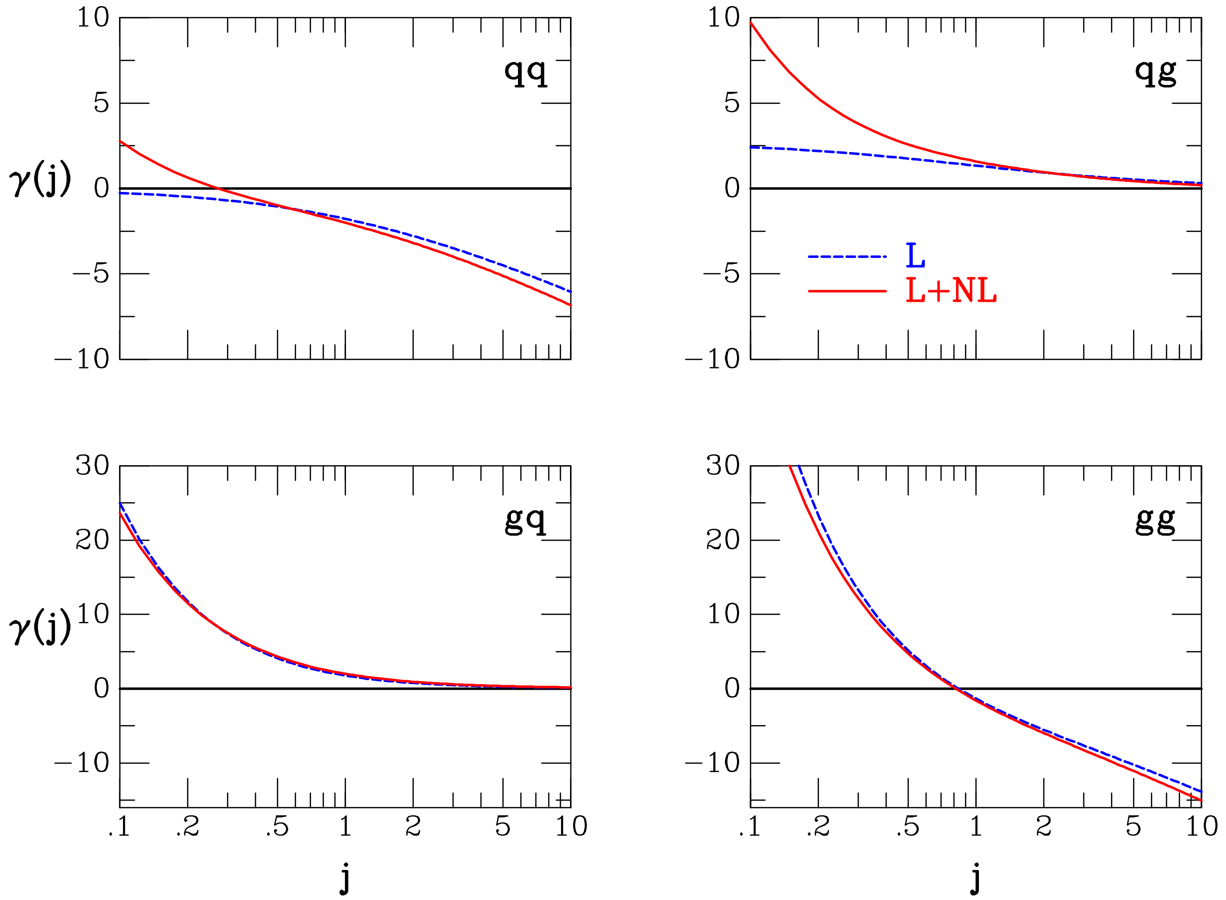
$$t\frac{\partial}{\partial t}\left(\begin{array}{c}\tilde{\Sigma}\\ \tilde{g}\end{array}\right) = \left(\begin{array}{cc}\gamma_{qq} & 2N_f\gamma_{qg}\\ \gamma_{gq} & \gamma_{gg}\end{array}\right)\left(\begin{array}{c}\tilde{\Sigma}\\ \tilde{g}\end{array}\right)$$

- Singlet anomalous dimension matrix has two real eigenvalues $\gamma_\pm$ given by

$$\gamma_\pm = \tfrac{1}{2}[\gamma_{gg}+\gamma_{qq}\pm\sqrt{(\gamma_{gg}-\gamma_{qq})^2+8N_f\gamma_{gq}\gamma_{qg}}] \ .$$

- Expressing $\tilde{\Sigma}$ and $\tilde{g}$ as linear combinations of eigenvectors $\tilde{\Sigma}_+$ and $\tilde{\Sigma}_-$, we find they evolve as superpositions of terms of above form with $\gamma_\pm$ in place of γ_{qq}.

Form of anomalous dimension matrix as a function of j, $(\alpha_s/(2\pi) = 1/30, n_f = 5)$.



NB

- Rapid growth at small j in gq and gg elements at lowest order
- Change in small j in qq and qg at NLO

Large x

The large-x limit is dominated by the diagonal terms of the splitting function matrix:

$$
\begin{aligned}
P_{qq} &\rightarrow \frac{2C_F}{(1-x)_+}\left(1+\frac{\alpha_S}{2\pi}\kappa\right) \\
P_{gg} &\rightarrow \frac{2C_A}{(1-x)_+}\left(1+\frac{\alpha_S}{2\pi}\kappa\right) \quad ,
\end{aligned}
$$

where $\kappa = C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right) - T_f\frac{10}{9}$.

- This gives $\ln j$ behaviour at large j in qq and gg elements

$$
\begin{aligned}
&\int_0^1 dx\, \frac{x^{j-1}}{[1-x]_+} \equiv \int_0^1 dx\, \frac{x^{j-1}-1}{[1-x]} \\
\equiv &\quad -\sum_{n=1}^{j-1}\int_0^1 dx\, x^{n-1} = -\sum_{n=1}^{j-1}\frac{1}{n}
\end{aligned}
$$

Next-to-leading correction to the splitting functions at large j is a universal factor associated with soft emission. By redefining

$$
\alpha^{MC} = \alpha^{\overline{MS}}\left(1+\frac{\alpha_S}{2\pi}\kappa\right) \quad ,
$$

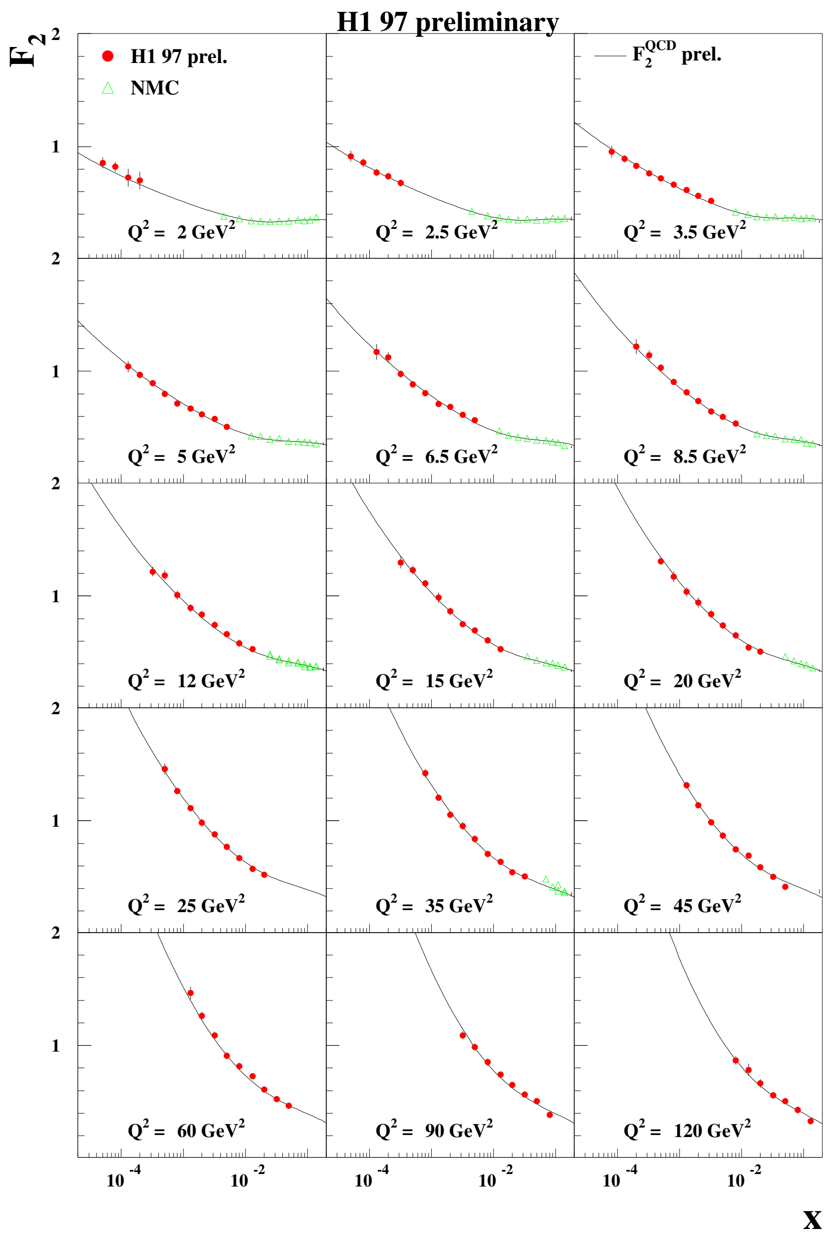
the major two loop effects at large x can be included, e.g. in a branching Monte Carlo progam.

Small x

- At small x, corresponding to $j \to 1$,

$$\gamma_{+} \to \gamma_{gg} \to \infty \; , \quad \gamma_{-} \to \gamma_{qq} \to 0 \; ,$$

Therefore we expect structure functions to grow rapidly at small x, which is as observed:



- Higher-order corrections also become large in this region:

$$\begin{aligned}
\gamma_{qq}^{(1)}(j) &\to \frac{40 C_F N_f T_R}{9(j-1)} \\
\gamma_{qg}^{(1)}(j) &\to \frac{40 C_A T_R}{9(j-1)} \\
\gamma_{gq}^{(1)}(j) &\to \frac{9 C_F C_A - 40 C_F N_f T_R}{9(j-1)} \\
\gamma_{gg}^{(1)}(j) &\to \frac{(12 C_F - 46 C_A) N_f T_R}{9(j-1)} .
\end{aligned}$$

- Thus we find

$$\begin{aligned}
\gamma_+ &\to \frac{2C_A}{j-1}\frac{\alpha_S}{2\pi}\left[1+\frac{(26C_F-23C_A)N_f}{18C_A}\frac{\alpha_S}{2\pi}+\dots\right] \\
&= \frac{2C_A}{j-1}\frac{\alpha_S}{2\pi}\left[1-0.64N_f\frac{\alpha_S}{2\pi}+\dots\right]
\end{aligned}$$

where neglected terms are either non-singular at $j=1$ or higher-order in α_S. Thus NLO corrections is relatively small.

- In general one finds (BFKL) that for small x ($j \to 1$)

$$\gamma_+ \to \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{\gamma^{(n,m)}}{(j-1)^m} \left(\frac{\alpha_S}{2\pi}\right)^n$$

where it happens that $\gamma^{(2,2)}$ (and $\gamma^{(3,3)}$) are zero. This is much less singular than the timelike (jet fragmentation) case.

  - This is probably why significant deviations from NLO QCD have not yet been seen in DIS at small x, whereas they are obvious in jet fragmentation.
  - Crucial difference is coherence (angular ordering), which suppresses soft gluon emission in low-x fragmentation, but does not suppress low-x spacelike branching in DIS.

Small x: Asymptotic limits

- In the asymptotic limit where $x \to 0$ and $t \to \infty$ it is possible to determine the behaviour of the distributions directly from the GLAP equations.

- only the most singular (leading $\ln(1/x)$) x-dependent contributions are retained (double-leading-logarithm approximation, DLLA).

  The $x \to 0$ limit of the parton distributions is controlled by the behaviour of the anomalous dimensions $\gamma(j)$ near $j = 1$. At leading order, the anomalous dimensions $\gamma_{gq}^{(0)}$ and $\gamma_{gg}^{(0)}$ are singular at $j = 1$ and it is these that dominate the evolution of the distributions at small x.

- Consider the evolution with gluons only:

$$t\frac{\partial}{\partial t}g(j,t) = \frac{\alpha(t)}{2\pi}\gamma_{gg}^{(0)}(j)g(j,t)$$

  where

$$\gamma_{gg}^{(0)}(j) \approx \frac{2N}{j-1}\,.$$

- The solution moments of the gluon distribution is

$$g(j,t) = g(j,t_0)\exp\left(\frac{\xi}{j-1}\right),$$

where ξ is defined by

$$\xi = \frac{N}{\pi}\int_{t_0}^{t}\frac{dt'}{t'}\alpha_S(t') = \frac{N}{\pi b}\ln\left(\frac{\ln t/\Lambda^2}{\ln t_0/\Lambda^2}\right) .$$

- the inverse Mellin transform is given by

$$\begin{aligned} xg(x,t) &= \frac{1}{2\pi i}\int dj\; x^{-(j-1)} g(j,t) \\ &\equiv \frac{1}{2\pi i}\int dj\; g(j,t_0)\exp\Big[f(j)\Big], \end{aligned}$$

where the exponent f is

$$f(j) = (j-1)Y + \frac{\xi}{j-1},$$

with $Y \equiv \ln(1/x)$. In the limit in which both Y and ξ become asymptotically large, we can estimate this integral by expanding about the saddle point of the exponential,

$$f(j) = 2\sqrt{\xi Y} + O(j - j_0)^2, \qquad j_0 = 1 + \sqrt{\frac{\xi}{Y}} \, ,$$

which gives, for the asymptotic solution,

$$xg(x,t) = g(j_0, t_0) \exp(2\sqrt{\xi Y}) \; .$$

Expressed in terms of the original variables, this becomes

$$g(x, \mu^2) \sim \frac{1}{x} \exp \sqrt{\frac{4N}{\pi b} \ln \frac{\ln \mu^2/\Lambda^2}{\ln \mu_0^2/\Lambda^2} \ln \frac{1}{x}}, \quad N = 3$$

$b = \frac{(33-2n_f)}{12\pi}$

- The dependence on the starting distribution enters via the j_0th moment of g. Therefore at fixed ξ/Y the initial information enters only as an overall factor.

Caveats

- An important point to note is that the above derivation breaks down if the starting distribution is too singular as $x \to 0$. The contour in the complex j-plane must be located to the right of all the singularities of $g(j, t_0)$.

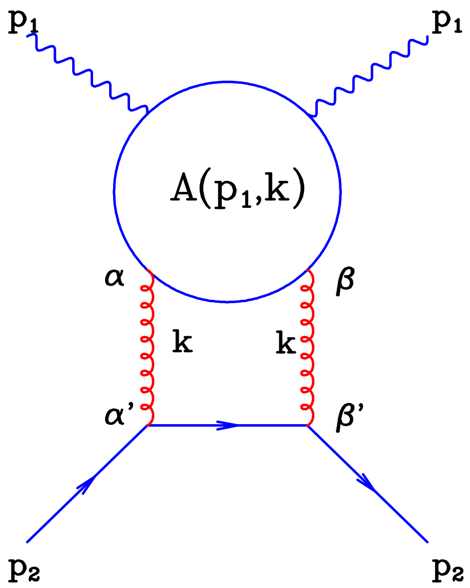
- If $xg(x, t_0) \sim x^{-\lambda}$ as $x \to 0$, then $g(j, t_0)$ has a pole at $j = 1 + \lambda$. If the saddle point j_0, is to the left of this, $i.e.\ \lambda > j_0 - 1$, then the asymptotic behaviour depends on λ.

- Physically, if the starting gluon distribution is too steep then there are not enough faster moving gluons to emit the slower gluons which build up the perturbative rise at small x.

High energy behaviour

- The high energy behaviour of scattering cross sections is controlled by the spin exchanged in the t-channel, $\sigma = \nu^{J-1}$ where $\nu = p_1 \cdot p_2$.

- Consider the diagram with spin-one gluon exchange.



- We can perform a Sudakov decomposition of the gluon polarization tensors

$$g^{\alpha\alpha'} = \frac{p_2^\alpha p_1^{\alpha'}}{\nu} + \frac{p_1^\alpha p_2^{\alpha'}}{\nu} + \text{transverse terms}$$

The first term is the dominant polarization at high energy.

- If $k = z_1 p_1 + z_2 p_2 + \mathbf{k}$ and $\nu = p_1 \cdot p_2$ the limitation of the recoil mass $(p_2 - k)^2 = 0$ implies that $z_1 = \frac{\mathbf{k}^2}{(1-z_2)\nu}$. z_1 can be neglected at high energy for the calculation of the upper parts, $\mathbf{k}^2 \ll \nu$

- Define impact factor $\Phi = A_{\alpha\beta} \frac{p_2^\alpha p_2^\beta}{\nu^2}$

- Using the Ward Identity $k_\alpha A^{\alpha\beta} = 0$ and the special momenta $k = z_2 p_2 + \mathbf{k}$, we find that

$$\frac{-1}{k^2}\Phi(\mathbf{k}) = \frac{1}{z_2^2 \nu^2} A_{\alpha\beta} \frac{\mathbf{k}^\alpha \mathbf{k}^\beta}{\mathbf{k}^2} \to \frac{1}{z_2^2 \nu^2} A_{\alpha\beta} \frac{1}{2} \sum \varepsilon_T^\alpha \varepsilon_T^\beta$$

- The impact factor tends to the on-shell spin-averaged transverse gluon amplitude squared as $\mathbf{k} \to 0$.

- The impact factor controls the high energy behaviour of a physical cross section and hence is gauge invariant.

Unintegrated Gluon distribution

- We can write a factorized form for the high-energy behaviour of gluon induced processes, using the unintegrated gluon distribution $\mathcal{F}(x, \mathbf{k})$. The longitudinal and transverse structure functions are given as a convolution of the impact factor and the unintegrated gluon distribution.

$$
\begin{aligned}
F_2(x, Q^2) &= \frac{Q^2}{4\pi^2\alpha} \int \frac{d^2\mathbf{k}}{\mathbf{k}^4} \frac{\mathcal{F}(x, \mathbf{k})}{4\pi} \left[\Phi_T(\mathbf{k}) + \Phi_L(\mathbf{k})\right] \\
F_L(x, Q^2) &= \frac{Q^2}{4\pi^2\alpha} \int \frac{d^2\mathbf{k}}{\mathbf{k}^4} \frac{\mathcal{F}(x, \mathbf{k})}{4\pi} \Phi_L(\mathbf{k})
\end{aligned}
$$

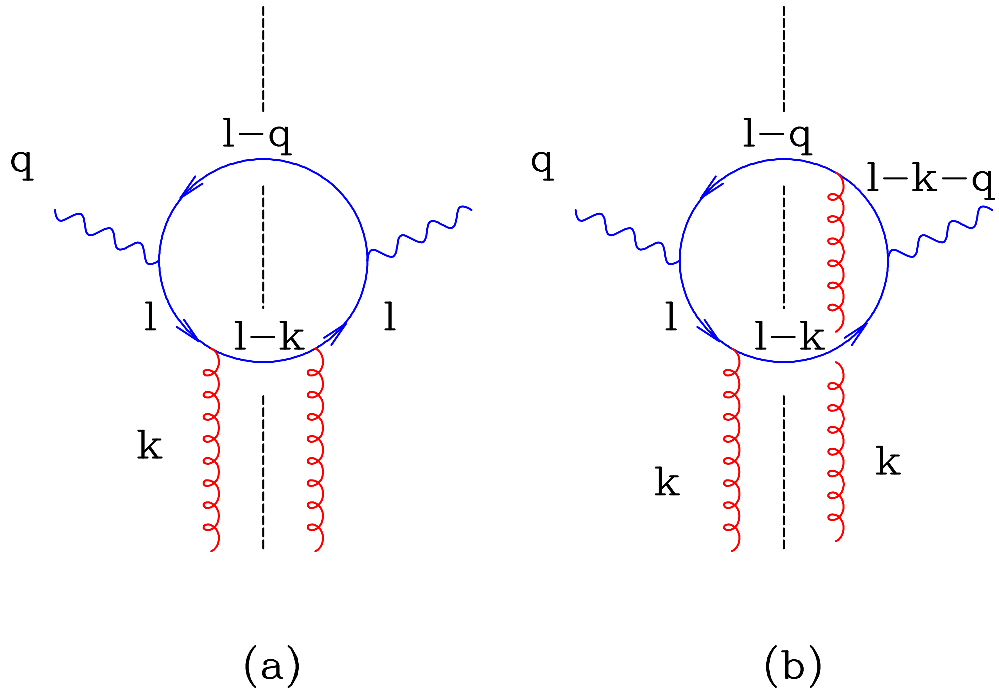
- After integrating $\mathcal{F}(x, \mathbf{k})$ over the transverse momentum we recover the normal gluon distribution

$$
G(x, Q^2) = \int \frac{d^2\mathbf{k}}{\pi \mathbf{k}^2} \Theta(Q^2 - \mathbf{k}^2) \mathcal{F}(x, \mathbf{k})
$$

- For $\mathbf{k}^2 << Q^2$ we recover the normal DGLAP factorization results.

Impact Factors

- Simple perturbative calculation for the impact factor



- Sudakov decomposition of momenta, $p_1.p_2 = \nu$.

$$
\begin{aligned}
q &= p_1 - x p_2 \\
k &= p_2 + \mathbf{k} \\
l &= z p_1 + \lambda p_2 + \mathbf{l}
\end{aligned}
$$

- q is primarily in the plus direction, (along p_1)

- At high energy the component of k along p_1 can be ignored.

- We must perform projection of the polarizations of the virtual photon.

$$
\begin{aligned}
\mathcal{F}_T &\sim -g^{\mu\rho} + \frac{p^\mu q^\rho}{\nu} + \frac{p^\rho q^\mu}{\nu} - \frac{4x^2}{q^2} p^\mu p^\rho \\
\mathcal{F}_L &\sim -\frac{4x^2}{q^2} p^\mu p^\rho, \;\; \mathcal{F}_\Sigma \sim -g^{\mu\rho}
\end{aligned}
$$

- We must also project out the dominant gluon polarization at high energy, (the term proportional to $p_1^\alpha p_1^\beta$), by contracting with $\frac{p_2^\alpha p_2^\beta}{\nu^2}$

- Imposing the mass shell conditions $(l-q)^2 = 0, (l-k)^2 = 0$ we find that

$$
\begin{aligned}
l^2 &= -\left[\frac{\mathbf{l}^2 + \bar{Q}^2}{1-z}\right] \equiv -\frac{D_1}{1-z} \\
(l-q-k)^2 &= -\left[\frac{(\mathbf{l}-\mathbf{k})^2 + \bar{Q}^2}{z}\right] \equiv -\frac{D_2}{z}
\end{aligned}
$$

- These are standard energy denominators. The important detail to note is the dependence on $\bar{Q}^2 = z(1-z)Q^2$.

Impact factor

The result for diagram (b)

$$\Phi^{(b)}_{\Sigma}(\mathbf{k}) = -16e^2g^2\sum_{q=1}^{n_f} e_q^2 \int dz\, d^2\mathbf{l}\,\frac{1}{D_1D_2}$$

$$\Big\{z(1-z)(D_1+D_2-2\bar{Q}^2)+\mathbf{l}\cdot(\mathbf{l}-\mathbf{k})(z^2+(1-z)^2)\Big\}$$

$$\Phi^{(b)}_{L}(\mathbf{k}) = -32e^2g^2\sum_{q=1}^{n_f} e_q^2 \int dz\, d^2\mathbf{l}\Big\{\bar{Q}^2\frac{z(1-z)}{D_1D_2}\Big\}$$

where $D_1 = \mathbf{l}^2 + Q^2z(1-z)$ and $D_2 = (\mathbf{l}-\mathbf{k}_1)^2 + Q^2z(1-z)$.

- No separate calculation needed for diagram (a) the result is given by $$\Phi^{(a)}_{L/T} = -\Phi^{(b)}_{L/T}(0)$$
- So full result is $\Phi_{L/T} = \Phi^{(b)}_{L/T}(0) - \Phi^{(b)}_{L/T}(\mathbf{k})$.
- The transverse cross section can be reconstructed, $\Phi_T = (\Phi_L + \Phi_\Sigma)$

Transverse factorization in impact parameter

We choose to express the impact factors in a mixed momentum- (z) and impact parameter- ($\mathbf{r}$) representation. Using the identity

$$\int \frac{d^2\mathbf{l}}{(\mathbf{l}^2+\bar{\mathbf{Q}}^2)((\mathbf{l}-\mathbf{k})^2+\bar{\mathbf{Q}}^2)} = \int d^2\mathbf{r}\ e^{i\mathbf{k}.\mathbf{r}}\ K_0^2(\bar{Q}r)$$

we may write

$$\begin{aligned}\Phi_L(\mathbf{k}) &= 32\alpha\alpha_s \sum_{q=1}^{n_f} e_q^2 \\ &\quad \int_0^1 dz \int d^2\mathbf{r}\ (1-e^{i\mathbf{k}.\mathbf{r}})\bar{Q}^2\ z(1-z)K_0^2(\bar{Q}r)\end{aligned}$$

where $\bar{Q}^2 = z(1-z)Q^2$.

- We may therefore re-interpret the transverse factorization formula

$$\sigma_{T/L}(x,Q^2) = \int \frac{d^2\mathbf{k}}{\mathbf{k}^4}\frac{\mathcal{F}(x,\mathbf{k})}{4\pi}\left[\Phi_{T/L}(\mathbf{k})\right]$$

Dipole scattering formula

High energy cross section is product of the wave function of the virtual photon to produce a dipole times the cross section for the dipole to interact. The cross section for the scattering of a longitudinal or transverse photon is is

$$\sigma_{L/T}(x,Q^2) = \int dz\; d^2\mathbf{r} |\Psi_{L/T}(z,r)|^2 \sigma(x,r)$$

$$\sigma(x,r) = \frac{4\pi\alpha_S}{3} \int \frac{d^2\mathbf{k}}{\mathbf{k}^4} \mathcal{F}(x,\mathbf{k}) \big(1-\exp(i\mathbf{k}\cdot\mathbf{r})\big)$$

At small r, the cross section $\sigma \sim r^2$, (colour transparency) whereas at large r the cross section tends to a constant. where for longitudinal photons the wave function is,

$$|\Psi_L(z,r^2)|^2 = \frac{6}{\pi^2}\alpha \sum_{q=1}^{n_f} e_q^2 \bar{Q}^2\; z(1-z)\; K_0^2(\bar{Q}r)$$

and for transverse photons we have

$$|\Psi_T(z,r^2)|^2 = \frac{3}{2\pi^2}\alpha \sum_{q=1}^{n_f} e_q^2 (z^2+(1-z)^2)\bar{Q}^2\; K_1^2(\bar{Q}r)$$

Integrals

- The wave function of the photon in the space of z and r is given by the Fourier transform of energy denominators.

$$\int \frac{d^2\mathbf{l_1}}{(\mathbf{l_1}^2+\bar{Q}^2)((\mathbf{l_1}-\mathbf{k})^2+\bar{Q}^2)}$$

$$= \frac{1}{(2\pi)^2}\int d^2\mathbf{l_1}\int d^2\mathbf{l_2}\int d^2\mathbf{r}\frac{\exp(i(\mathbf{l_2}+\mathbf{k}-\mathbf{l_1})\cdot r)}{(\mathbf{l_1}^2+\bar{Q}^2)(\mathbf{l_2}^2+\bar{Q}^2)}$$

$$\int d^2\mathbf{r}\ \exp(i\mathbf{k}\cdot\mathbf{r})\left|\frac{1}{2\pi}\int d^2\mathbf{l}\ \frac{\exp(i\mathbf{l}\cdot\mathbf{r})}{\mathbf{l}^2+\bar{Q}^2}\right|^2$$

$$= \int d^2\mathbf{r}\ \exp(i\mathbf{k}\cdot\mathbf{r})\ K_0^2(\bar{Q}r)$$

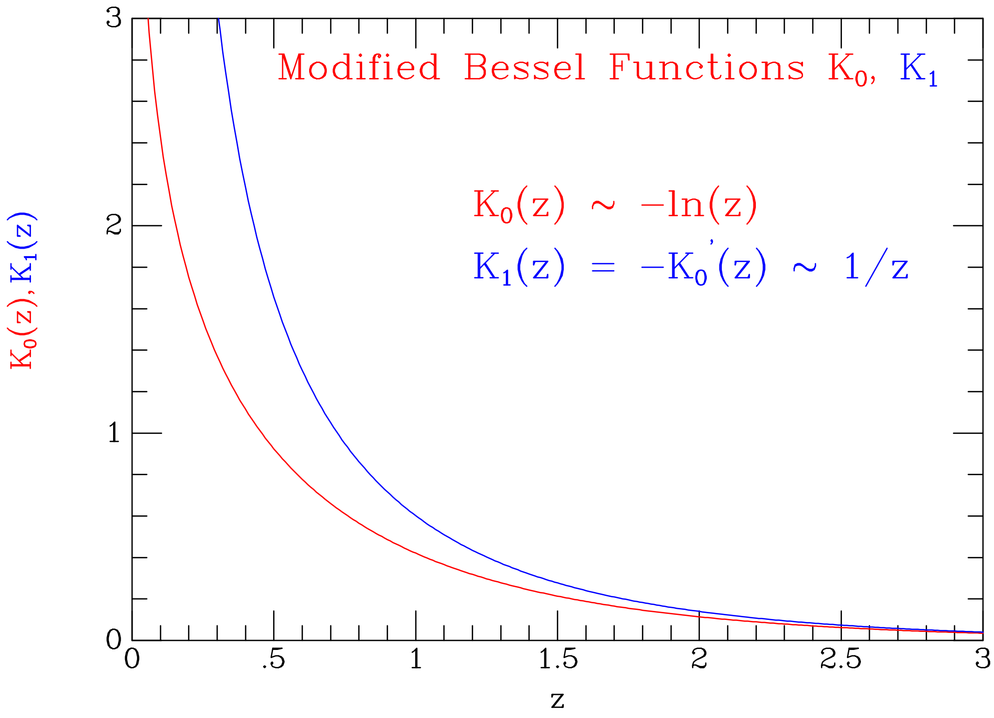
because

$$\frac{1}{\pi}\int_0^\pi d\theta\ e^{i\mathbf{l}\cdot\mathbf{r}} = J_0(lr)$$

and

$$\int_0^\infty dl\ l^{\nu+1}\frac{J_\nu(lr)}{l^2+\bar{Q}^2} = \bar{Q}^\nu\ K_\nu(\bar{Q}r)$$

$$K_1(z) = -\frac{d}{dz}K_0(z)$$



- The modified Bessel functions can be approximated by theta functions

$$K_0(\bar{Q}r) \sim \Theta(1-\bar{Q}r)$$
$$K_1(\bar{Q}r) \sim \frac{1}{\bar{Q}r}\Theta(1-\bar{Q}r)$$

So we can identify two important regions

- ❖ $r \sim 1/Q$ and $z \sim \frac{1}{2}$
- ❖ $r \sim R$ and z or $(1-z) \leq \frac{1}{R^2Q^2}$ for $R > 1/Q$.

Kinematics of DIS at small x_B

- The conclusions drawn from the derivation can be obtained from kinematics alone.
- Kinematics in the rest frame of the proton, using on-mass shell old fashioned perturbation theory

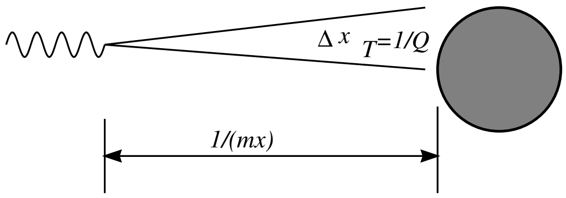
$$
\begin{aligned}
p &= (M, M, \mathbf{0}) \\
q &= (q^+, \frac{Q^2}{2q^+}, \mathbf{0}) \\
k_q &= (zq^+, \frac{\mathbf{k^2}}{2zq^+}, \mathbf{k}) \\
k_{\bar{q}} &= ((1-z)q^+, \frac{\mathbf{k^2}}{2(1-z)q^+}, -\mathbf{k})
\end{aligned}
$$

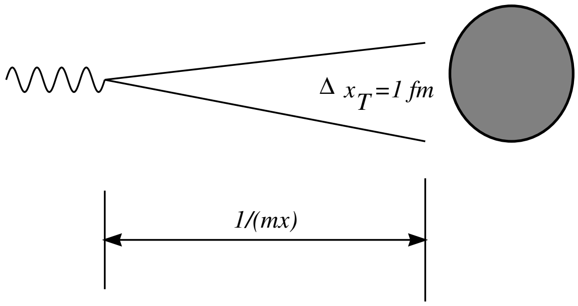
The energy deficit is given by

$$\Delta E = \frac{1}{2}\Big(\frac{\mathbf{k^2}}{2z(1-z)q^+} + \frac{Q^2}{2q^+}\Big) = \frac{Mx_B}{2}\Big(1 + \frac{\mathbf{k^2}}{z(1-z)Q^2}\Big)$$

since $x_B = \frac{Q^2}{2Mq^+}$.

Physical Picture





- The average lifetime of the fluctuation $\Delta y \sim 1/\Delta E \sim 1/(M x_B)$.
- At small x_B, Δy can be much larger than the target size.
- Virtual photon materializes into a $q\bar{q}$ upstream of the target and arrives at the face of nucleus in hadronic guise.
- Dipoles which interact are either small with symmetric longitudinal momenta or large with with asymmetric longitudinal momenta.

Dipole cross section

- For small dipoles ($r \sim 1/Q$)
  - The cross section is perturbative
  - It has the property of colour transparency $\sigma \sim r^2$
  - For $r < 1/\mu$, σ is related to the gluon density

$$\sigma(x,r) = \frac{\pi^2}{3} r^2 \alpha_s x g(x,\mu)$$

  - For $z \sim 1/2, r \sim 1/Q$, physical X-sects are

$$\begin{aligned}\sigma_T(x,Q^2) &= 1 \times \frac{1}{Q^2} = \frac{1}{Q^2} \\ \sigma_L(x,Q^2) &= Q^2 \times \frac{1}{Q^4} = \frac{1}{Q^2}\end{aligned}$$

Dipole cross section, (cont)

- For large dipoles $r \sim R \gg 1/Q,\ z, (1-z) \sim 1/(Q^2 R^2)$
  - ❖ The aligned jet configuration there is a large colour separation. The perturbative calculation of the wave function is not appropriate. (VMD region)
  - ❖ The dipole cross section has the property of saturation

$$\sigma_T(x, Q^2) = \frac{1}{Q^2 R^2} \times R^2 = \frac{1}{Q^2}$$

$$\sigma_L(x, Q^2) = \frac{1}{Q^4 R^6} \times R^4 = \frac{1}{Q^4 R^2}$$

- Bjorken-Gribov paradox
  - ❖ Given that a large dipole interacts like a vector meson, why does it lead to a cross section which scales $\sigma \sim 1/Q^2$?
  - ❖ Answer: Only very asymmetric splittings z or $(1-z) \sim 1/(Q^2 R$ contribute.

KGB model

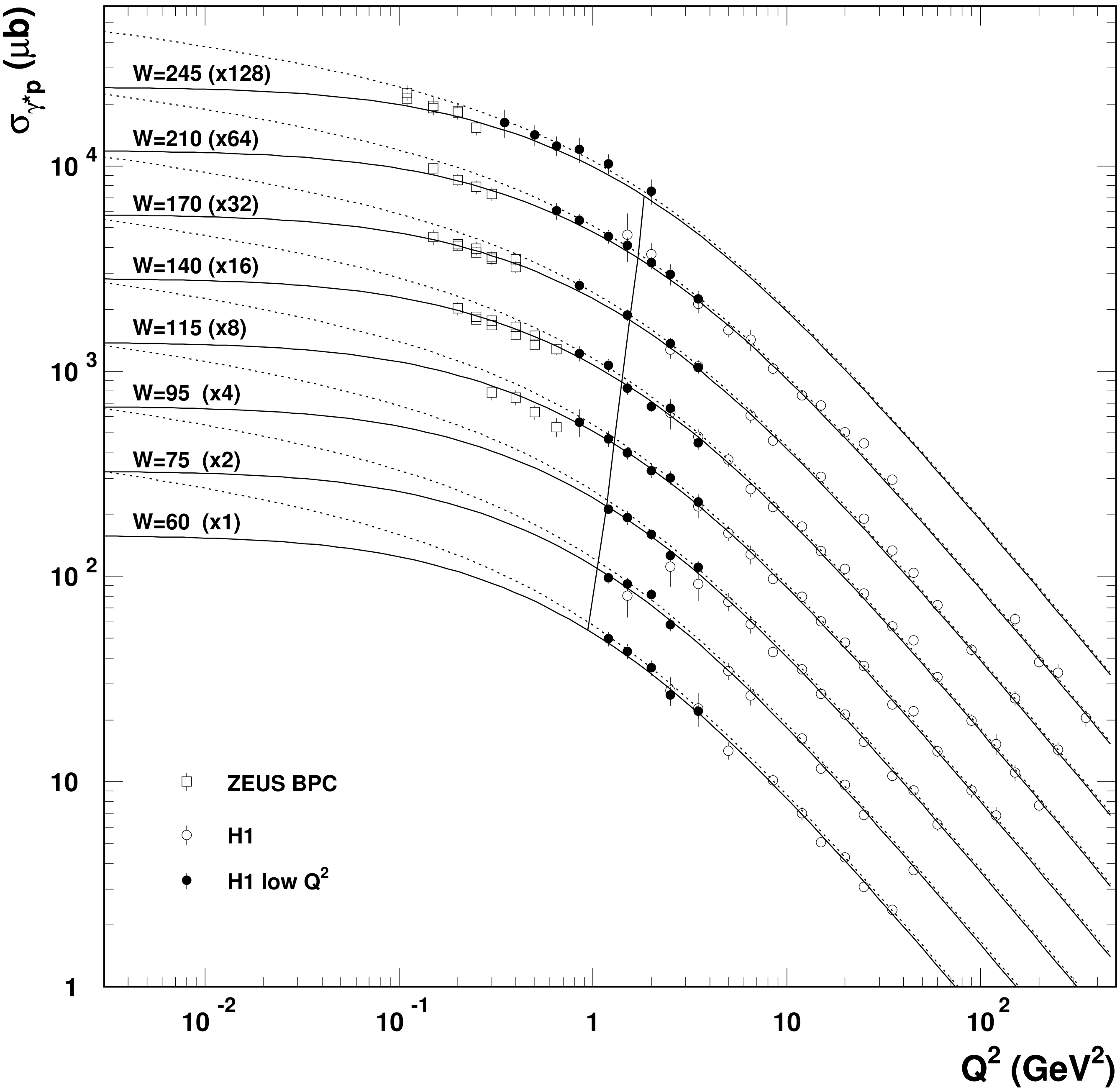
- The KGB model (K. Golec-Biernat-Wusthoff) attempts to describe both regions
  - The dipole cross section is bounded by an energy independent value σ_0 which assures unitarity of F_2,

$$\hat{\sigma}(r,x) = \sigma_0 \left\{1 - \exp\left(-\frac{r^2}{4R_0^2(x)}\right)\right\},$$

where $R_0(x)$, called saturation radius, is given by

$$R_0(x) = 1 \text{ GeV}^{-1} \, (x/x_0)^{\lambda/2} .$$

  - The three parameters, $\sigma_0 = 23$ mb, $\lambda \simeq 0.3$ and $x_0 = 3{\cdot}10^{-4}$, were fitted to all small-x DIS data with $x < 10^{-2}$.
  - For large r, $\hat{\sigma}$ saturates, $\hat{\sigma} \simeq \sigma_0$. The transition between the two regimes is controlled by the saturation radius $R_0(x)$ which increases with x.
  - for decreasing x, the dipole cross section saturates for smaller dipole sizes.



σ_{γ^*p} (μb)
W=245 (x128)
W=210 (x64)
W=170 (x32)
W=140 (x16)
W=115 (x8)
W=95 (x4)
W=75 (x2)
W=60 (x1)
10^4
10^3
10^2
10
1
ZEUS BPC
H1
H1 low Q^2
10^{-2}
10^{-1}
1
10
10^2
Q^2 (GeV^2)

Diffraction

$$\frac{d\,\sigma_T^D}{dt}_{|\,t=0} = \frac{1}{16\,\pi}\int d^2\mathbf{r}\,dz\,|\Psi_{T,L}(\mathbf{r},z)|^2\,\sigma^2(x,\mathbf{r})$$

$$\sim R^2 \times \frac{1}{Q^2 R^2} \times \frac{\bar{Q}^2}{\bar{Q}^2 R^2} \times R^4 \text{ (Large dipoles)}$$

$$\sim r^2 \times 1 \times \frac{\bar{Q}^2}{\bar{Q}^2 r^2} \times r^4 \text{ (Small dipoles)}$$

- Diffractive cross section is dominated by large dipoles.
- Small dipoles suppressed by an extra power of $1/Q^2$ item Diffractive events induced by transverse photon are a scaling phenomenon.

Conclusion on impact factors

- k_T factorization and impact factors have not yet been developed as systematically as a calculational tool as the DGLAP formulation of the theory.
- Nevertheless, the formulation of DIS in the rest frame gives an appealing physical picture.
- Model of dipole interacting with a nucleus at rest gives greater information about the structure of jets, rapidity gaps, in the event.
- Used to provide insight into Diffraction, Shadowing, Vector Meson production . . .

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