Status of the small-x resummation programme

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Plan of the talk

- **Introduction**
- **Representation of scattering amplitudes in the BFKL approach**
- **Gluon Reggeization**
  - **Gluon Regge trajectory**
  - **Reggeon-parton vertices**
- **Colour singlet channel: BFKL Pomeron**
  - **BFKL kernel**
  - **Impact factors**
- **Colour octet channel: bootstrap of the gluon Reggeization**
  - **"Soft" bootstrap conditions**
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- **Summary**

Introduction

First of all, I have to say that I can not review all works related with the small-x resummation. Therefore I apologize from the beginning that I'll not mention some papers; evidently it does not mean that I consider them as not important. I'll touch mainly those subjects which are familiar to me.

The denotation x came from deep inelastic scattering (DIS), where it was introduced by Bjorken as

$$x = \frac{Q^2}{2pq}\ , \ \ Q^2 = -q^2\ ,$$

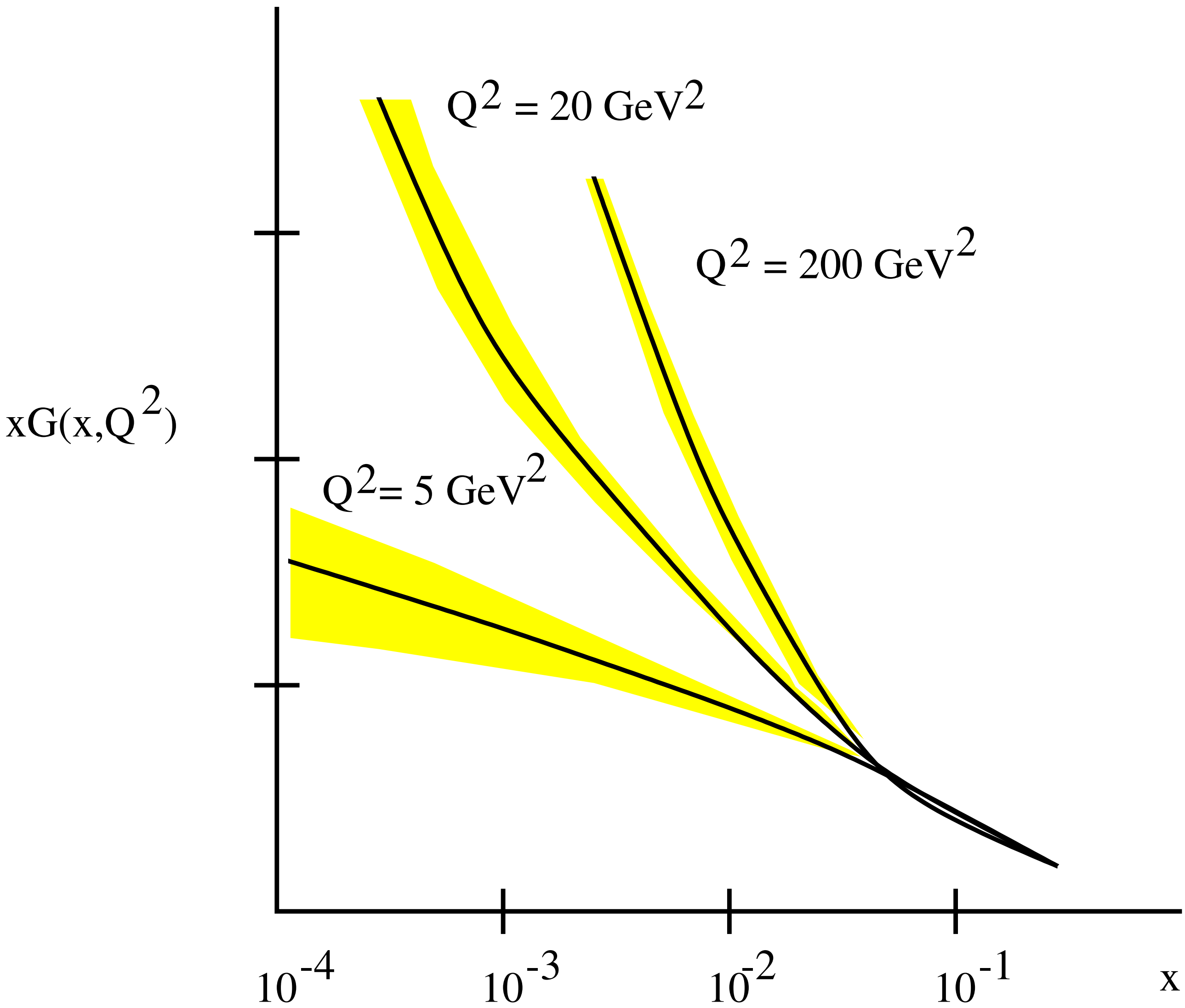
where p and q are the nucleon (N) and virtual photon (γ^*) momenta correspondingly. But now the notion "Small-x Physics" is used for all processes where typical virtuality Q^2 are small compared to squared c.m.s. energy s, so that

$$x = \frac{Q^2}{s}\ .$$

Such definition includes as well soft hadronic interactions, which evidently can not be described in perturbative QCD.

I'll talk about processes with "hard" scale Q^2, which justifies an application of perturbative QCD. In turn, these processes can be divided on two classes: processes with one hard scale and with two (at least) scales. The simplest representative of the first class is $\gamma^*\gamma^*$ scattering with virtualities of both photons of the same order. The total cross section of this process was measured recently at LEP. A typical example of the second kind processes is DIS, which is under investigation more than three decades. In the last decade a comprehensive and careful study of this process in the small-x region was performed at HERA.

Cross sections of DIS are expressed in terms of distributions of partons (quarks and gluons) in a target (let's talk about proton, for definiteness). Since at large energies $\sqrt{s}$ only gluon exchanges in t-channel give non-vanishing contributions to the cross sections, at small $x = Q^2/s$ the gluon distribution $xG(x, Q^2)$ dominates. The striking result of HERA was the rapid growth of the gluon distribution at small x (with the exponent increasing with Q^2).



$Q^2 = 20\ GeV^2$
$Q^2 = 200\ GeV^2$
$xG(x,Q^2)$
$Q^2 = 5\ GeV^2$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
x

Standard QCD evolution (DGLAP) equations
V.V. Gribov and L.N. Lipatov, 1972;
L.N. Lipatov, 1975;
Yu.L. Dokshitzer, 1977;
G. Altarelli and Parisi, 1977,
which are basically renorm group equations, describe an evolution of parton distributions with Q^2. At small x the DGLAP equation for the gluon distribution may be written as

$$\frac{\partial xG(x,Q^2)}{\partial \ln Q^2} = \bar{\alpha}_S(Q^2)\int_x^1 \frac{dx'}{x'}x'G(x',Q^2) \ ,$$

where

$$\bar{\alpha}_S \equiv \frac{\alpha_S N_c}{\pi} \ ,$$

$N_c = 3$ **is the number of colours in QCD,**

$$\alpha_S(Q^2) = \frac{4\pi}{b\ln\frac{Q^2}{\Lambda^2_{QCD}}} \ , \quad b = \frac{11}{3}N_c - \frac{2}{3}n_f \ .$$

At small x the equation gives:

$$xG(x,Q^2) \sim \exp\sqrt{\frac{16N_c}{b}\ln\left(\frac{\ln(Q^2/\Lambda^2_{QCD})}{\ln(Q_0^2/\Lambda^2_{QCD})}\right)\ln\frac{1}{x}} \ ,$$

i.e. $xG(x,Q^2)$ rising with with $1/x$ (but the growth is more weak than power).
The DGLAP equation performs resummation of terms most important in each order of perturbation theory at large $\ln Q^2$ (at fixed $\ln(1/x)$):

$(\alpha_S \ln Q^2)^n$ **at leading order (LO$_Q$)**

$\alpha_S(\alpha_S \ln Q^2)^n$ **at next-to-leading order (NLO$_Q$)**

and so on. Therefore, formally it can not be used in the region of so small x that $\alpha_s \ln(1/x) \sim 1$ at any fixed NNN...NLO$_Q$ order.
There is another evolution equation, so called BFKL (Balitskii-Fadin-Kuraev-Lipatov) equation,
V. F., E.A. Kuraev, L.N.Lipatov, 1975,
E.A. Kuraev, L.N. Lipatov, V.F., 1976,
Ya.Ya. Balitskii, L.N. Lipatov, 1978,

which describes an evolution of the unintegrated gluon distribution $\mathcal{F}(x, \vec{k}^2)$ not in $\ln Q^2$, but in $\ln(1/x)$. The equation can be written symbolically as

$$\frac{\partial \mathcal{F}}{\partial \ln(1/x)} = \mathcal{K} \otimes \mathcal{F},$$

where $\mathcal{K}$ is called BFKL kernel and $\otimes$ means convolution not over fractions of longitudinal momenta as in the DGLAP equation, but over transverse momenta. The BFKL equation resums the terms

$(\alpha_S \ln(1/x))^n$ at leading order (**LO**$_x$),

$\alpha_S(\alpha_S \ln(1/x))^n$) at next-to-leading order (**NLO**$_x$).

In the leading logarithmic approximation (**LLA**), when only the leading terms $(\alpha_S \ln(1/x))^n$ are resummed, it predicts the power growth of cross sections with energy:

$$\sigma \sim \left(\frac{1}{x}\right)^{\omega_P},$$

where the Pomeron intercept (with subtracted 1)

$$\omega_P = 4N_c \frac{\alpha_s}{\pi} \ln 2$$

$$\omega_P \simeq 0.4 \quad for \quad \alpha_s = 0.15 \;.$$

The BFKL equation became famous just due to this prediction, since the rapid growth of the $\gamma^* p$ cross sections was discovered at HERA.

Therefore BFKL is usually associated with the evolution equation for the unintegrated gluon distribution. But the region of applicability of the BFKL approach is much wider. The evolution equation for the unintegrated gluon distribution appears in this approach as a particular result for the imaginary part of the forward scattering amplitude ($t = 0$ and vacuum quantum numbers in the t-cahnnel). But the approach gives a description of scattering amplitudes at any fixed momentum transfer $\sqrt{-t}$ in the limit of large center of mass energy $\sqrt{s}$ (**Regge limit**).

I have to add that the approach was developed, and is more suitable, for the description of processes with only one hard scale, such as $\gamma^*\gamma^*$ scattering with both photon virtualities of the same order, where the DGLAP evolution is absent.

Evidently the power growth violate the Froissart bound

$$\sigma_{tot} < const(\ln s)^2$$

so that the BFKL resummation is not applicable at asymptotically large energies. Note that calculation of radiative corrections in the BFKL approach (at any fixed nex-to-next-...order) does not help. To go to the asymptotics one must change the approach and talk about screening, saturation, unitarisation, etc.. I don't touch this subject at all and will talk about the next-to-leading approximation (**NLA**) in the BFKL approach, where, besides the leading $(\alpha_S \ln s)^n$ terms, the subleading terms $\alpha_S(\alpha_S \ln s)^n$ are also resummed.

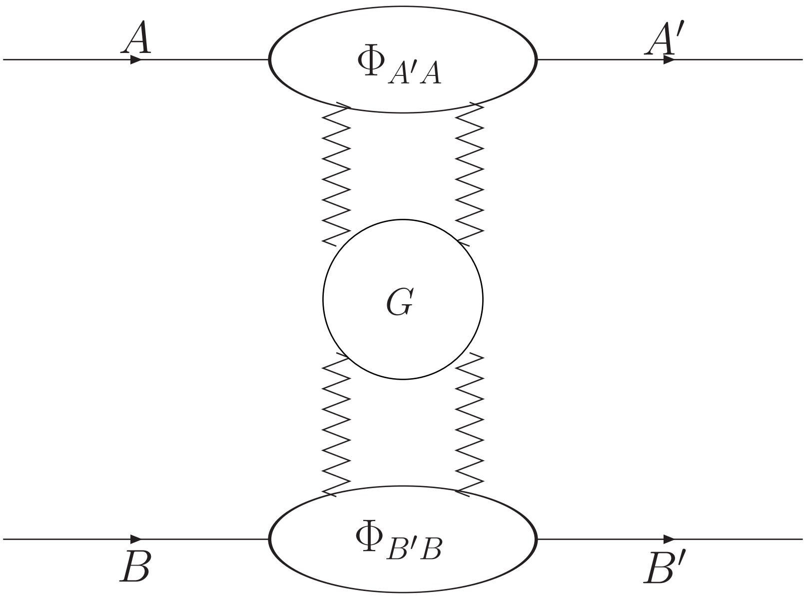
Although the radiative corrections to the BFKL equation at any fixed order can not restore unitarity at high energy, they are very important, since in the leading order neither scale of s nor argument of the running coupling constant α_s are fixed. Therefore corrections are necessary to fix the Pomeron intercept and normalization of cross sections.

Scattering amplitudes in the BFKL approach

The scattering amplitude for the process

$$A + B \longrightarrow A' + B'$$

at large center of mass energy $\sqrt{s}$ **and fixed momentum transfer** $\sqrt{-t}$, $s \gg |t|$, **can be represented by the picture**



and may be symbolically written as the convolution

$$\Phi_{A'A} \otimes G \otimes \Phi_{B'B} \ ,$$

where the impact factors $\Phi_{A'A}$ **and** $\Phi_{B'B}$ **describe the transitions** $A \to A'$ **and** $B \to B'$ **due to scattering on the Reggeized gluons, while** G **is the Green's function for the two interacting Reggeized gluons. All dependence on properties of particles** $A, \ A'$ $(B, \ B')$ **is contained in the impact factors** $\Phi_{A'A}$ $(\Phi_{B'B})$**, which are energy independent, so that dependence on energy is determined by the universal (process independent) Green's function** G**. This representation is valid both in the leading logarithmic approximation (LLA), when only the leading terms** $(\alpha_S \ln s)^n$ **are resummed, and in the next-to-leading approximation (NLA), when the** $\alpha_S(\alpha_S \ln s)^n$ **terms are also resummed, not only for forward scattering with** $t = 0$**, but for the non-forward case as well.**

In the case of the forward scattering ($A' = A, \ \ B' = B \ \ , q = 0$) with the help of the optical theorem we obtain:

$$\sigma_{AB}(s) = \int \frac{d^2 q_A}{2\pi \vec{q}_A^{\,2}} \int \frac{d^2 q_B}{2\pi \vec{q}_B^{\,2}} \Phi_A(\vec{q}_A) G(s;\ \vec{q}_A, -\vec{q}_B) \Phi_B(\vec{q}_B),$$

where the vector sign is used for vector components transverse to the initial momenta p_A, p_B. The unintegrated gluon distribution is given in the BFKL approach by the convolution of the Green function with the impact factor of the target.

The Pomeron, which determines the high energy behavior of cross sections, appear in this approach as compound states of two Reggeized gluons.

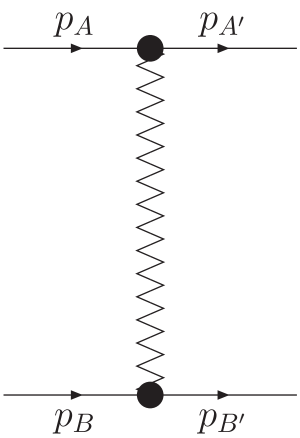
It is clear from this that the foundation of the BFKL approach is given by the remarkable property of QCD - the gluon Reggeization. It appeared as the assumption (based on the three-loop calculations) used in the derivation of the BFKL equation,
V. F, E.A. Kuraev, L.N. Lipatov, 1975
but later it was proved in the LLA
Ya.Ya. Balitskii, V. F., L.N. Lipatov, 1979 .
The key to the proof is given by the bootstrap requirement.

Whereas two Reggeized gluons in the color singlet state create the BFKL Pomeron, in the antisymmetric color octet state they must reproduce the Reggeon itself. This is the requirement of self-consistency of the BFKL approach, since it is based on the gluon Reggeization. Fulfillment of all bootstrap conditions gives the basis on which the proof of Reggeization was constructed in the LLA. An analogous proof can be constructed in the next-to-leading approximation as well.

The gluon Reggeization

In the following I use the term "gluon Reggeization" in some restricted meaning, which is necessary (and sufficient) for the BFKL approach. Remind that I discuss the NLA, so that all relations must be considered with this accuracy.

For the elastic scattering $A + B \to A' + B'$ in Regge kinematical region: $s \simeq -u \to \infty$, t fixed (i.e. not growing with s) it means that the amplitude with the gluon quantum numbers in the t-channel and negative signature (*i.e.* odd under $s \leftrightarrow u$ exchange) has the Regge form



$$(\mathcal{A}_8^-)_{AB}^{A'B'} = \Gamma_{A'A}^c \left[\left(\frac{-s}{-t} \right)^{j(t)} - \left(\frac{s}{-t} \right)^{j(t)} \right] \Gamma_{B'B}^c \ ;$$

$j(t) = 1 + \omega(t)$ – Reggeized gluon trajectory , $j(0) = 1$

$\Gamma_{A'A}^c = g\langle A'|T^c|A\rangle \Gamma_{A'A}$ -Reggeon-parton-parton vertices;

T^c -colour group generators in fundamental (quarks) or adjoint (gluons)representation.

With this assumption the vertices $\Gamma_{A'A}^{(0)}$ and the gluon Regge trajectory $\omega^{(1)}$ can be easily calculated in the leading order (LO). It is sufficient to calculate corresponding amplitudes in the Born approximation to find the vertices and in the one-loop approximation to find the trajectory. Of course, the vertices and the trajectory obtained from different processes must coincide, that gives a check of the assumption. The result is:

$$\Gamma_{A'A}^{(0)} = \delta_{\lambda_{A'}\lambda_A} \ , \ \ \omega^{(1)}(t) = \frac{g^2 t}{(2\pi)^{D-1}} \frac{N}{2} \int \frac{d^{D-2}k_\perp}{k_\perp^2 (q-k)_\perp^2} = -\bar{g}^2 (\vec{q}^{\ 2})^\epsilon \frac{\Gamma^2(\epsilon)}{\Gamma(2\epsilon)} \ ,$$

where Γ is the Euler gamma-function and

$$\bar{g}^2 = \frac{g^2 N \Gamma(1-\epsilon)}{(4\pi)^{D/2}} \ , \qquad t = q^2 \simeq q_\perp^2 \ ;$$

The space-time dimension $D = 4 + 2\epsilon \neq 4$ to regularize infrared divergencies.

Of course, both the calculations and the results are not so simple in the (NLO). In particular, s-channel helicity is not conserved in this approximation, so that

$$\Gamma^{(1)}_{A'A} = \delta_{\lambda_{A'}\lambda_A}\Gamma^{(+)}_A(t) + \delta_{\lambda_{A'},-\lambda_A}\Gamma^{(-)}_A(t) \ .$$

The vertices $\Gamma^{(\pm)}_A$ for quarks and gluons were calculated at arbitrary D.

V. F., L.N. Lipatov, 1993; V. F., R. Fiore, 1992;
V. F., R. Fiore, A. Quartarolo; 1994; V. F., R. Fiore, M.I. Kotsky, 1995

The two-loop contribution $\omega^{(2)}(t)$ to the gluon trajectory was obtained at arbitrary D in terms of integrals over transverse momenta.

V. F., 1995; V. F., R. Fiore, M.I. Kotsky, 1995; V. F., R. Fiore, A. Quartarolo, 1996; V. F., R. Fiore, M.I. Kotsky, 1996

The integral can be expressed in terms of elementary functions only for $\epsilon \to 0$:

$$\omega^{(2)}(t) \simeq \left(\frac{\bar{g}^2\left(\vec{q}^{\,2}\right)^{\epsilon}}{\epsilon}\right)^2 \left[\frac{11}{3} + \left(2\psi'(1) - \frac{67}{9}\right)\epsilon + \left(\frac{404}{27} + \psi''(1) - \frac{22}{3}\psi'(1)\right)\epsilon^2\right] \ ,$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$.

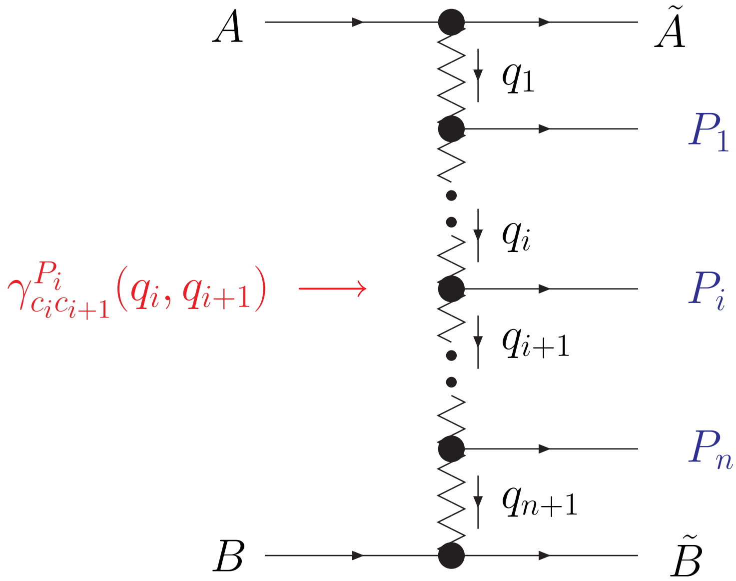
V. F., M. Kotsky, 1996; J. Bluemlein, V. Ravindran, W.L. van Neerven, 1998.

Recently the gluon Regge trajectory was re-evaluated
V. Del Duca, E.W.N. Glover, 2001
in a completely independent way, by taking the high energy limit of the two-loop amplitudes for parton-parton scattering amplitudes with gluon exchanges in the t-channel. The validity of the gluon Reggeization at NLO was confirmed and full agreement with previous results was found.

Much stronger is the asumption about amplitudes for production of arbitary number of partons in multi-Regge kinematics (MRK), where all prodiced partons are strongly ordered in rapidity space. The production amplitudes have complicated analytical structure, which is not simple even in the MRK. Fortunately, this structure is not essential in the NLA. Only the real parts of the production amplitudes are important. It is supposed that they have a simple factorized form

$$\Re\,\mathcal{A}^{\tilde{A}\tilde{B}+n}_{AB} = 2s\,\Gamma^{c_1}_{\tilde{A}A}\left(\prod_{i=1}^{n}\gamma^{P_i}_{c_ic_{i+1}}(q_i,q_{i+1})\left(\frac{s_i}{s_R}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_R}\right)^{\omega(t_{n+1})}\Gamma^{c_{n+1}}_{\tilde{B}B} \ ,$$

$\gamma^{P_i}_{c_ic_{i+1}}(q_i,q_{i+1})$ - (non-local) effective vertex for production of the parton P_i in the Reggeon-Reggeon collisions.



LLA: $\{P_i\}$ states of a single gluon

$$\gamma^{G_i}_{c_i c_{i+1}}(q_i, q_{i+1}) = gT^{d_i}_{c_i c_{i+1}} e^*_\mu(k_i) C^\mu(q_i, q_{i+1})$$

$$C(q_{i+1}, q_i) = -q_i - q_{i+1} + \left(\frac{q_i^2}{(k_i\, p_1)} + 2\frac{(k_i\, p_2)}{(p_1\, p_2)}\right) p_1 - \left(\frac{q_{i+1}^2}{(k_i\, p_2)} + 2\frac{(k_i\, p_1)}{(p_1\, p_2)}\right) p_2$$

L.N. Lipatov, 1976; V. F., E.A. Kuraev, L.N. Lipatov, 1975

The calculation of the one loop corrections to the vertex $\gamma^{G_i}_{c_i c_{i+1}}$ has a long history

V. F., L.N. Lipatov (1993)
V. F., R. Fiore, A. Quartarolo (1994)
V. F., R. Fiore, M.I. Kotsky (1996)
V. F., R. Fiore, A. Papa (2001).
Now they are known now also at arbitrary D.

In the NLA the quasi multi-Regge kinematics (QMRK), where instead of one of produces in the MRK partons there is a couple of partons with limited invariant mass, becomes also important. Reggeon vertices in this kinematics (they are necessary only in the Born approximation) are also known

- $\Gamma^{c\,(\text{1-loop})}_{P'P}$

V. F., R. Fiore (1992) V. F., L.N. Lipatov (1993)
V. F., R. Fiore, A. Quartarolo (1994)
V. F., R. Fiore, M.I. Kotsky (1995)

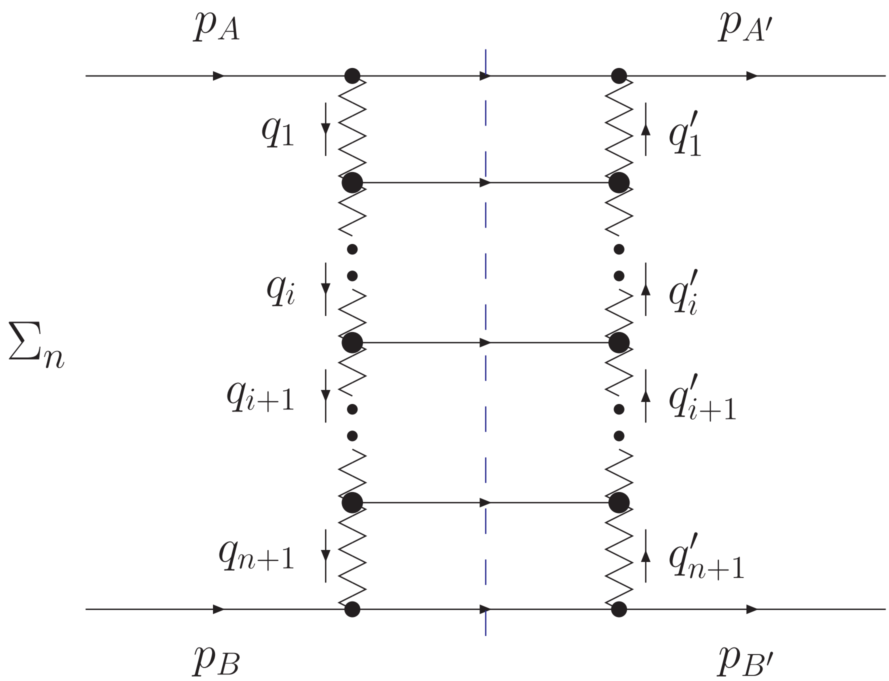
- $\gamma^{Q\overline{Q}\,(\mathrm{Born})}_{c_i c_{i+1}}$

V. F., R. Fiore, A. Flachi, M.I. Kotsky (1997)
S. Catani, M. Ciafaloni, F. Hautmann (1990)
G. Camici, M. Ciafaloni (1996)

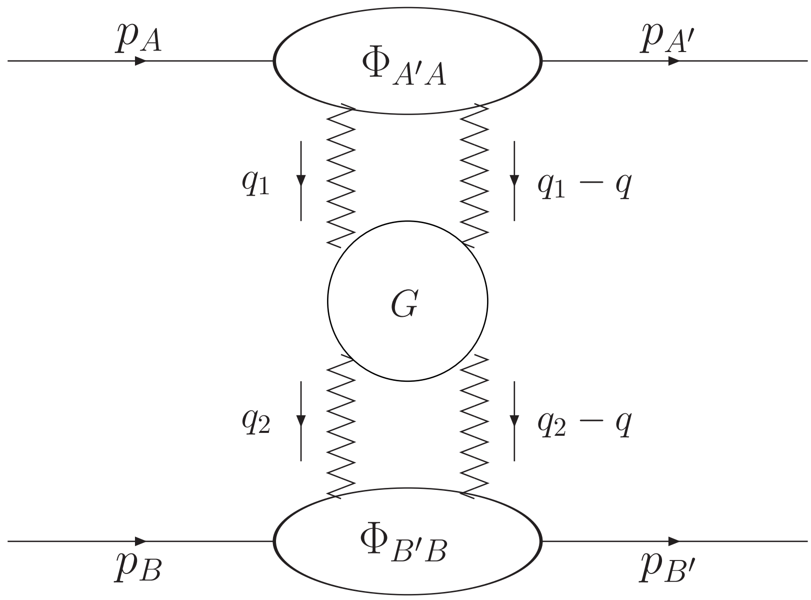
- $\gamma^{GG(\mathrm{Born})}_{c_i c_{i+1}}$

V. F., L.N. Lipatov (1996)
V. F., M.I. Kotsky, L.N. Lipatov (1997)

Unitarity relations:



give



Colour decomposition:

$$\mathcal{A}_{AB}^{A'B'} = \sum_{\mathcal{R}} (\mathcal{A}_{\mathcal{R}})_{AB}^{A'B'}$$

$$\mathbf{Im}_s (A_{\mathcal{R}})_{AB}^{A'B'} = \frac{s}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_1}{\vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^2} \int \frac{d^{D-2}q_2}{\vec{q}_2^{\,2}(\vec{q}_2-\vec{q})^2} \sum_{\nu} \Phi_{A'A}^{(\mathcal{R},\nu)}(\vec{q}_1;\vec{q};s_0)$$
$$\times \int_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left[\left(\frac{s}{s_0}\right)^{\omega} G_{\omega}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2,\vec{q}) \right] \Phi_{B'B}^{(\mathcal{R},\nu)}(-\vec{q}_2;-\vec{q};s_0)$$

$\Phi_{P'P}^{(\mathcal{R},\nu)}$ – **impact factors in the t-channel color state $(\mathcal{R},\nu)$**

$G_{\omega}^{(\mathcal{R})}$ – **Mellin transform of the Green's functions for Reggeon-Reggeon scattering in the t-channel color representation $\mathcal{R}$**

The Green's functions obey the equation

$$\omega G_{\omega}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2,\vec{q}) = \vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^{\,2}\delta^{(D-2)}(\vec{q}_1-\vec{q}_2)$$
$$+ \int \frac{d^{D-2}q_r}{\vec{q}_r^{\,2}(\vec{q}_r-\vec{q})^{\,2}} \mathcal{K}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_r;\vec{q})\, G_{\omega}^{(\mathcal{R})}(\vec{q}_r,\vec{q}_2;\vec{q})$$

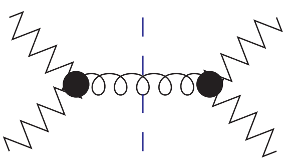
$$\mathcal{K}^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2;\vec{q}) = \left[\omega\left(-\vec{q}_1^{\,2}\right) + \omega\left(-(\vec{q}_1-\vec{q})^{\,2}\right)\right]\delta^{(D-2)}(\vec{q}_1-\vec{q}_2) + \mathcal{K}_r^{(\mathcal{R})}(\vec{q}_1,\vec{q}_2;\vec{q})$$

Kernel: **"virtual" part** **"real" part**

Impact factors and kernel are defined without any ambiguity in terms of the gluon Regge trajectory and vertices of Reggeon interactions

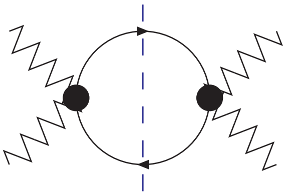
Knowledge of the Reggeon vertices gives a possibility to find the "real" part of the kernel

$$\mathcal{K}_r = \mathcal{K}_{RRG}^{(B)} + \mathcal{K}_{RRG}^{(1)} + \mathcal{K}_{RRQ\overline{Q}}^{(B)} + \mathcal{K}_{RRGG}^{(B)}$$



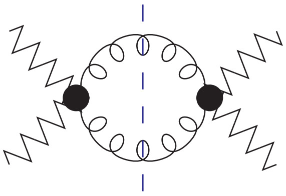
$\mathcal{K}_{RRG}^{(1)}$

V. F., L.N. Lipatov (1993)
V. F., R. Fiore, A. Quartarolo (1994)
V. F., R. Fiore, M.I. Kotsky (1996)



$\mathcal{K}^{(B)}_{RRQ\overline{Q}}$

forward:
V.F., R. Fiore, A. Flachi, M.I. Kotsky (1997)
non-forward:
V.F., R. Fiore, A. Papa, (1999)



$\mathcal{K}^{(B)}_{RRGG}$

− counterterm

forward:
V.F., L.N. Lipatov, M.I. Kotsky (1997)
non-forward, octet:
V.F., D.A. Gorbachev (2000)

- **The NLO kernel is known completely in the forward case ($t = 0$) for the all color representations**
- **In the non-forward case, the $\mathcal{K}^{(B)}_{RRGG}$ contribution is missing for the singlet color representation**
- **The NLA kernel is known completely in the non-forward case for the octet color representation**

Colour singlet channel: BFKL Pomeron

The forward LO kernel averaged over the azimuthal angle between $\vec{q}_1$ and $\vec{q}_2$ is:

$$\int d^{D-2}q_2\; \mathcal{K}(\vec{q}_1,\vec{q}_2) f(\vec{q}_2^{\,2}) = \frac{N_c\alpha_s}{\pi}\int \frac{d\vec{q}_2^{\,2}}{|\vec{q}_2^{\,2}-\vec{q}_1^{\,2}|}\left[f(\vec{q}_2^{\,2}) - 2\frac{\min(\vec{q}_2^{\,2},\vec{q}_1^{\,2})}{(\vec{q}_2^{\,2}+\vec{q}_1^{\,2})} f(\vec{q}_1^{\,2})\right];$$

It is explicitly scale invariant.
Eigenfunctions: $(\vec{q}_2^{\,2})^{\gamma-1}$;
Eigenvalues $\omega(\nu)$ $\frac{N_c\alpha_s}{\pi}\chi_0(\gamma)$;

$$\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)\,,\;\; \psi(\gamma) = \Gamma'(\gamma)/\Gamma(\gamma)\,.$$

Complete set: $\gamma = 1/2 + i\nu,\;\; -\infty < \nu < \infty$;
The maximal eigenvalue at $\nu = 0$ $\chi_0\left(\frac{1}{2}\right) = 4ln2$;
Pomeron intercept: $\omega_P = 4N_c(\alpha_s/\pi)\ln 2$.

Problems of the NLO BFKL

The explicit expression for the NLO kernel obtained by
V.F., L.N. Lipatov (1998)
G. Camici, M. Ciafaloni (1998)
leads to the Pomeron intercept

$$\omega_P = \omega_P^B(1 - 2.4\,\omega_P^B) \qquad\qquad \omega_P^B = 4\ln 2 N\frac{\alpha_s(\vec{q}^{\,2})}{\pi}$$

large correction!

The evident problems related with the NLO BFKL are:
— large size of the scale-invariant correction;

— running coupling.

In fact, only the first problem is really new.

The correction is so large, that for $\alpha_s = 0.15$, where $\omega_P^{(L)} \simeq 0.4$ we obtain $\omega_P \simeq 0.02$.

It means that in the HERA region the naive estimate gives the correction which almost completely cancel the main term.

The large size of the correction leads to such unpleasant phenomena as negative values for the gluon splitting function $P_{gg}(x, \alpha_s)$
R.D. Ball, S. Forte, 1998;
J. Bluemlein, V. Ravindran, W.L. van Neerven, 1998.

There was attempt to overcome the problem of the large and negative correction to the Pomeron intercept using the fact, that the corrected eigenvalue function

$$\omega(q^2, \nu) = \frac{\alpha_s(q^2)\, N_c}{\pi} \left(\chi_0(\gamma) + \frac{\alpha_s(q^2) N_c}{\pi} \chi_1(\gamma) \right) , \gamma = \frac{1}{2} + i\nu$$

completely changes its ν-dependence
D.A. Ross, 1998.

In the LLA the point $\nu = 0$ corresponds to the maximal eigenvalue of the kernel (Pomeron intercept). In the NLO it is true only at very small α_s ($\alpha_s \leq 0.05$). For the values of α_s above 0.5 instead of having a single maximum at $\nu = 0$ the function $\omega(q^2, \nu)$ has a local minimum here. Near $\nu = 0$

$$\omega(q^2, \nu) \simeq \omega_0 + a\nu^2 - b\nu^4$$

where for $\alpha_s = 0.15$

$$\omega_0 = 0.021, \quad a = 4.19, \quad b = 47.4.$$

The maximum of the eigenfunction is at $\nu^2 = \frac{a}{2b}$ and

$$\sigma \sim s^{(\omega_0 + \frac{a^2}{4b})} \simeq s^{0.12}.$$

However, the solution for the Green function obtained by Ross contains oscilations.

So, it is very dezirable to diminish the relative value of the correction. It is possible to do by several ways.

—Choice of an appropriate renormalization scheme and scale setting
S.J. Brodsky, V.S. Fadin, V.T. Kim, L.N. Lipatov, G.B. Pivovarov, 1999.

—Limitation from below on relative rapidities of the produced particles
C.R. Scmidt, 1999.

—Improving of the equation using the renormalization group constraints
G. Salam, 1998;
M. Ciafaloni, D. Colferai, G. Salam, 1999.

Another problem - running coupling. **The function** $\omega(q^2,\nu)$ **contains a piece** $-\left(\frac{N_c\alpha_s(\mu^2)}{\pi}\right)^2$ **which evidently can not be considered as a small correction at all** q^2**. Treating formally, this piece makes questionable the BFKL approach itself, because the spectrum of the BFKL kernel in the NLA becomes unbounded, that makes the use of the complex angular momentum variable impossible N.Armesto, J.bartels, M.A. Brown, 1998.**

The running coupling leads to a non-Regge behaviour of the scattering amplitudes; in particular

$$G(q_1,q_2) =\simeq \frac{1}{2\pi\sqrt{\pi D q_1^2 q_2^2 Y}} exp\left[\omega_P Y - \frac{\left(\ln q_1^2 - \ln q_2^2\right)^2}{4DY} + \frac{D}{3}\left(\alpha_s(\mu^2)\omega_P b\right)^2 Y^3\right];$$

$$Y = \ln\left(\frac{s}{s_0}\right)$$

so that the Regge-BFKL asymptotics is valid only in the limited region of energy

$$Y \le (\alpha_s)^{-5/3}$$

Y.V. Kovchegov, A.H. Mueller, 1998;
E.Levin, 1998.

In fact, the problem of the running coupling is not new. It was clear that this problem will arise from the begining, much before the calculation of the BLO corrections. Therefore, this problem was considered by many authors:

L.N. Gribov, E.M. Levin, M.G. Ryskin, 1983;
J. Kwiechinski, 1985
L.N. Lipatov, 1986;
J.C. Collins, J. Kwiechinski, 1997;
R. E. Hancock, D.A. Ross, 1992;
L.P.A. Haakman, O. V. Kancheli, J.H. Koch, 1997;
G. Camici, M. Ciafaloni, 1997

both for one-scale ($\gamma^*\gamma^*$) and two-scale (DIS) processes.

Whereas for the one scale processes a general solution of the problem was not found, for the two-scale processes the factorization of the Mellin transformed Green function

$$G_\omega(q_1,q_2) \simeq F_\omega(q_1)\tilde{F}_\omega(q_2)$$

at $q_1 \gg q_2$ was proved, with calculable function $F_\omega(q_1)$. It means that although the Green function itself can not be calculated in the perturbation theory, its dependence from q_1^2 is calculable. The result was confirmed after appearing of the total NLO kernel

R.S. Thorne, 1999,

M. Ciafaloni, D. Colferai, G.P. Salam, 1999, M. Ciafaloni, M.Taiuti, A.Mueller, 2001.

When

$$\ln\frac{q_1^2}{\Lambda_{QCD}^2} \ll Y \ll \left(\ln\frac{q_1^2}{\Lambda_{QCD}^2}\right)^{\frac{5}{3}}$$

the Green function

$$G(q_1,q_2) \sim exp\left[\omega_s(t)Y\right];$$

$\omega_s(t) = \bar{\alpha}_s(t)\chi_m,\ \chi_m$ **- the saddle point of $\chi(\gamma)$; $t=\ln\frac{q_1^2}{\Lambda_{QCD}^2},\ q_1^2 \gg q_2^2$;**

At $t \ll Y \ll t^2$

$$G(q_1,q_2) \sim \frac{1}{\sqrt{4\pi D\omega_s(t)Y}} exp\left[\omega_s(t)Y\left(1+\frac{\Delta t}{2t}\right) - \frac{(\Delta t)^2}{4D\omega_s(t)Y} + \frac{\eta^3}{12}\right];$$

$\Delta t = t - t_0,\ \ \eta = D^{\frac{1}{3}}\omega_s(t)Y t^{\frac{-2}{3}};$

For $Y \gg t^2$

$$G(q_1,q_2) \sim exp\left[\omega_P Y\right]$$

with oscillating prefactor.

Resummation of $(\alpha_s/N)^n$ terms in the anomalous dimension.

T. Jaroszewicz, 1982

$$G_\omega(q_1,q_2) \;=\; \int_{C_+} \frac{d\gamma}{2\pi^2 q_2^2} \frac{exp[\gamma\left(\ln q_1^2 - \ln q_2^2\right)]}{\left(\omega - \frac{N_c\alpha_s}{\pi}\chi^B(\gamma)\right)}.$$

At $q_1^2 \gg q_2^2$:

$$G_\omega(q_1,q_2) \;\sim\; \left(q_1^2\right)^{\gamma_\omega} \;;$$

where γ_ω is determined by the equation:

$$\omega = \frac{N_c\alpha_s}{\pi}\chi^B(\gamma_\omega).$$

It is just the renormgroup behaviour for fixed α_s. γ_ω appears as the resummed anomalous dimension.

$$\gamma_\omega = \frac{\bar\alpha_s}{\omega} + 2.4\left(\frac{\bar\alpha_s}{\omega}\right)^4 + 7\left(\frac{\bar\alpha_s}{\omega}\right)^6 + ...$$

$$\bar\alpha_s = \tfrac{N_c\alpha_s}{\pi}$$

At NLO $\chi(\gamma)$ becomes an operator valued. Nevertheless, the BFKL equation remains fully compatible with the standard factorized perturbative evolution equations for parton distributions.

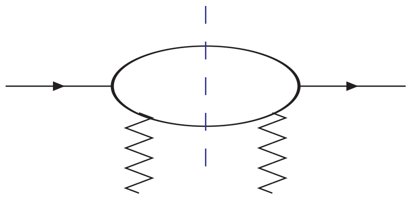
The oscillations at $Y \gg t^2$ do not affect the associated splitting functions, which remain smooth in the small x limit

G. Altarelli, R.D. Ball, S. Forte, 2001

Impact factors

The impact factors are expressed in terms of the effective vertices for Reggeon interactions:

$$\Phi_{P'P} = \sum_{\{f\}} \int \frac{d\kappa}{2\pi} d\rho_f \Gamma^c_{\{f\}A} (\Gamma^{c'}_{\{f\}A'})^*$$



$-$ counterterm

and are known now in the NLO at the parton level (for quarks and gluons)
V. F., R. Fiore, M.I. Kotsky, A. Papa, 2000

for arbitrary colour state in the t-channel and at arbitrary D. In the colour

octet case the impact factors enter in the bootstrap conditions. In the color singlet they are used for the calculation of the cross section of the NLO jet vertex
J. Bartels, D. Colfertai, G.P.Vacca, 2002

which represent one of the building blocs in the production of Mueller-Navalet

jets at hadron-hadron collisions and of forward jets at DIS.

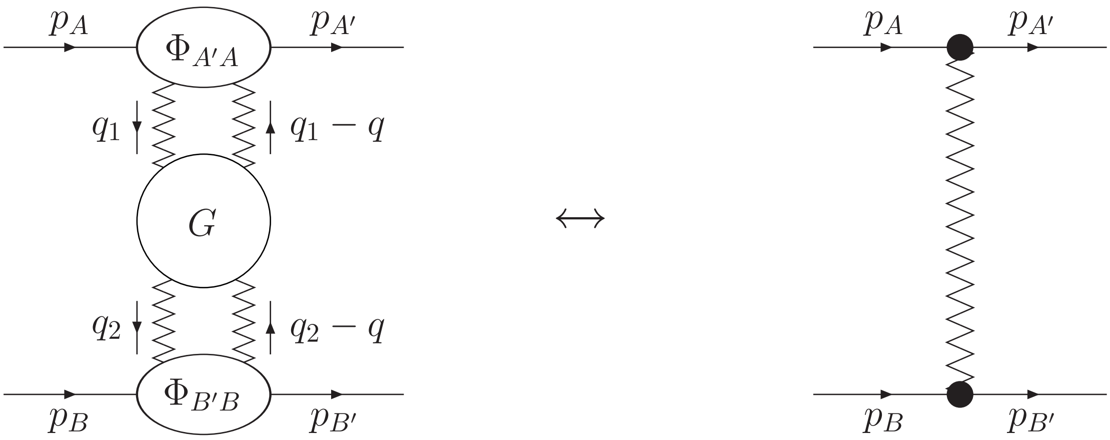
The most interesting are the impact factors of virtual photon. They are not yet calculated, but the work is in progress. Their infrared safety is proved
V. F., A.D. Martin, 1999

and the effective vertices for γ^*-Reggeon interaction are calculated

V.F., M.I. Kotsky, D.Yu. Ivanov, 2000
J. Bartels, S. Gieseke, C.F. Qiao, 2001
J. Bartels, S. Gieseke, A. Kyrieleis, 2001
J. Bartels, D. Colfertai, S. Gieseke, A. Kyrieleis, 2002

Bootstrap of the gluon Reggeization

In the case of Gluon quantum numbers in the t-channel the representation for the elastic scattering process $A+B \longrightarrow A'+B'$ derived from s-channel unitarity, must reproduce the representation with one Reggeized gluon exchange in the t-channel.



$$\mathbf{Im}_s(A_{8^-})_{AB}^{A'B'} \quad = \quad \Gamma_{A'A}^{c\,(\mathrm{Born})}\left(\frac{s}{|t|}\right)^{1+\omega^{(1)}(t)} \pi\omega^{(1)}(t)\Gamma_{A'A}^{c\,(\mathrm{Born})}$$

It leads to the following conditions: **V.F., R. Fiore (1999)**

First bootstrap condition (on the NLA kernel)

$$\frac{g^2Nt}{2\,(2\pi)^{D-1}}\int\frac{d^{D-2}q_1}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}\int\frac{d^{D-2}q_2}{\vec{q}_2^{\,2}\vec{q}_2^{\,\prime\,2}}\mathcal{K}^{(8)(1)}\left(\vec{q}_1,\vec{q}_2;\vec{q}\right)=\omega^{(1)}\left(t\right)\omega^{(2)}\left(t\right)$$

Second bootstrap condition (on the NLA impact factors)

$$\frac{ig\sqrt{N}t}{(2\pi)^{D-1}}\int\frac{d^{D-2}q_1}{\vec{q}_1^{\,2}\vec{q}_1^{\,\prime\,2}}\Phi_{A'A}^{(8,a)(1)}(\vec{q}_1,\vec{q};s_0)=\Gamma_{A'A}^{a(1)}\omega^{(1)}(t)$$
$$+\frac{1}{2}\Gamma_{A'A}^{a(B)}\left[\omega^{(2)}(t)+\left(\omega^{(1)}(t)\right)^2\ln\left(\frac{s_0}{\vec{q}^{\,2}}\right)\right]$$

In the NLA, the check of the bootstrap was very important

- as a (partial) check of correctness of the NLO BFKL calculations
- since a formal proof of the gluon Reggeization to all orders of perturbation theory does not exist in this approximation.

The first bootstrap condition has been verified

- at arbitrary space-time dimension, for the part concerning the quark contribution to the kernel (in massless QCD)
  V.F., R. Fiore, A. Papa, (1999)

Due to simplicity of this part of the kernel:

$$\mathcal{K}_r^{Q\bar{Q}}\left(\vec{q}_1, \vec{q}_2; \vec{q}\right) = \frac{g^4 n_f N}{24(2\pi)^5} \times$$

$$\left\{ -\frac{1}{\pi^\epsilon} \frac{6}{(4\pi)^{2\epsilon}} \Gamma(-\epsilon) \frac{[\Gamma(2+\epsilon)]^2}{\Gamma(4+2\epsilon)} \left[\frac{(\vec{k}^2)^\epsilon}{\vec{k}^2} (\vec{q}_1^{\,2} \vec{q}_2^{\,\prime\, 2} + \vec{q}_2^{\,2} \vec{q}_1^{\,\prime\, 2}) + \vec{q}^{\,2} \left((\vec{q}^{\,2})^\epsilon - (\vec{q}_1^{\,2})^\epsilon - (\vec{q}_2^{\,2})^\epsilon \right) \right] \right.$$
$$\left. - \frac{(\vec{q}_1^{\,2} \vec{q}_2^{\,\prime\, 2} - \vec{q}_2^{\,2} \vec{q}_1^{\,\prime\, 2})}{\vec{k}^2} \ln\left(\frac{\vec{q}_1^{\,2}}{\vec{q}_2^{\,2}} \right) + \quad (\vec{q}_1 \longleftrightarrow \vec{q}_1^{\,\prime}, \quad \vec{q}_2 \longleftrightarrow \vec{q}_2^{\,\prime}) \right\}$$

it was not a complicated problem.

- in the $D \to 4$ limit, for the part concerning the gluon contribution to the kernel
  V.F., R. Fiore, M.I. Kotsky, (2000)

The second bootstrap condition is process-dependent (it should be checked for every new impact factor which is calculated!).

It has been checked at arbitrary space-time dimension for quark and gluon impact factors in QCD with massive quarks.

V.F., R. Fiore, M.I. Kotsky, A. Papa (2000)

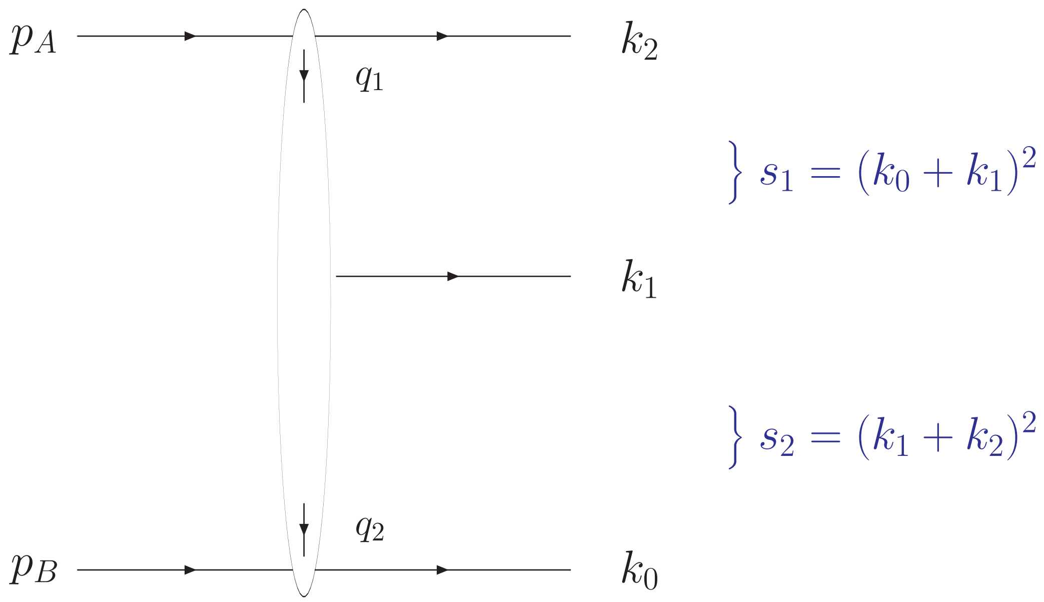
Strong bootstrap

The bootstrap conditions for elastic amplitudes give a possibility to check, but not to prove the hypothesis of the gluon Reggeization. However, it can be proved (so that the BFKL approach can obtain a firm basis), with the help of s-channel unitarity relations for production amplitudes.

Of course, we need to use analytical properties of these amplitudes. Unfortunately, we know not much about them. But, fortunately, we don't need much.

Let us consider the one-particle production amplitude

$$A + B \quad \rightarrow \quad A' + B' + G$$



in MRK. Note that we are interested in only negative signatutes in both t_i- channels, i.e. in the part of the amplitude which is antisymmetric with respect to any of the substitutions $s_i \rightarrow -s_i \;, \;\; i = 1, 2$. Note that due to the relation

$$\frac{s_1 s_2}{s} = \vec{k}_1^{\,2}$$

which is hold in all physical channels, this part is also antisymmetric with respect to $s \rightarrow -s$.

Dependece on $\vec{k}_1^{\,2}$ can be very complicated (the simplest example is $(-\vec{k}_1^{\,2})^\epsilon$; it reproduces correctly real value at all negative s-is). Fortunately, it can be made irrelevant taking appropriate combinations of the discontinuities. .

An important observation is that a discontinuity of a product fg is expressed through the discontinuities of f and g:

$$f_+g_+ - f_-g_- = \frac{1}{2}(f_+ - f_-)(g_+ + g_-) + \frac{1}{2}(f_+ + f_-)(g_+ - g_-) \ .$$

The second important ovservation that in the sum of discontinuities of $F(\vec{k}_1^{\,2})$ on s_1 and s (as well as the sum of the discontinuities on s_2 and s or the difference of the discontinuities on s_1 and s_2) is zero. From these two observations it follows that in such sum (or diference) we can take instead of analytic functions of $\vec{k}_1^{\,2}$ their real parts.

Now, if the variables s_1, s_2 and s does not enter into the combination $s_1s_2/s = \vec{k}^{\,2}$, they can enter only as $\hat{\mathcal{S}}\ln^{n_1}(-s_1)\ln^{n_2}(-s_2)\ln^{n_3}(\pm s)$, with $n_1+n_2+n_3=n$, where n is less or equal the order of perturbation theory and $\hat{\mathcal{S}}$ is the operator of symmetrization with respect to exchanges

$$s_1 \leftrightarrow -s_1 \ , \quad s \leftrightarrow -s \ ,$$

and

$$s_2 \leftrightarrow -s_2 \ , \quad s \leftrightarrow -s \ .$$

Note that the terms containing products of $\ln(-s_i)\ln(s_i)$, where s_i can be s_1, s_2 or s, are forbiden, on the same ground as the terms containing $\ln(-s)\ln(s)$ are forbiden in elastic scattering amplitudes. Since in the NLO we need to keep only the first two leading total powers n, calculating imaginary part of discontinuity in anyone of variables s_1, s_2 or s we can take only real parts of logarithms of other variables. It means that

$$\Re\left[\frac{1}{-2\pi i}disc_{s_i}\left(\hat{\mathcal{S}}\ln^{n_1}(-s_1)\ln^{n_2}(-s_2)\ln^{n_3}(\pm s)\right)\right]$$
$$=\frac{1}{2}\frac{\partial}{\partial\ln(s_i)}\Re\left[\hat{\mathcal{S}}\ln^{n_1}(-s_1)\ln^{n_2}(-s_2)\ln^{n_3}(\pm s)\right] ,$$

where s_i can be s_1, s_2 or s and the partial derivative is taken at fixed $s_j \neq s_i$. Therefore we have, for example

$$\Re\left[\frac{1}{-2\pi i s}\left(disc_{s_1}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\left(\frac{\partial}{\partial\ln(s_1)}+\frac{\partial}{\partial\ln(s)}\right)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'GB'}\right] ,$$

where in the right part the first derivative is taken at fixed s_2 and s and the second at fixed s_1 and s_2. Using that

$$\left(\frac{\partial}{\partial\ln(s_1)}+\frac{\partial}{\partial\ln(s)}\right)f(s_1,s_2,s)=\frac{\partial}{\partial\ln(s_1)}f(s_1,s_2,\frac{s_1,s_2}{\vec{k}_1^{\,2}}) ,$$

we come to

$$\Re\left[\frac{1}{-2\pi i s}\left(disc_{s_1}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\frac{\partial}{\partial\ln(s_1)}\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'GB'}\right] ,$$

where in the right part the amplitude is considered as function of s_1, s_2 and $\vec{k}_1^{\,2}$.

The requirement of the Reggeized form of the amplitude in the right part gives us the bootstrap relation:

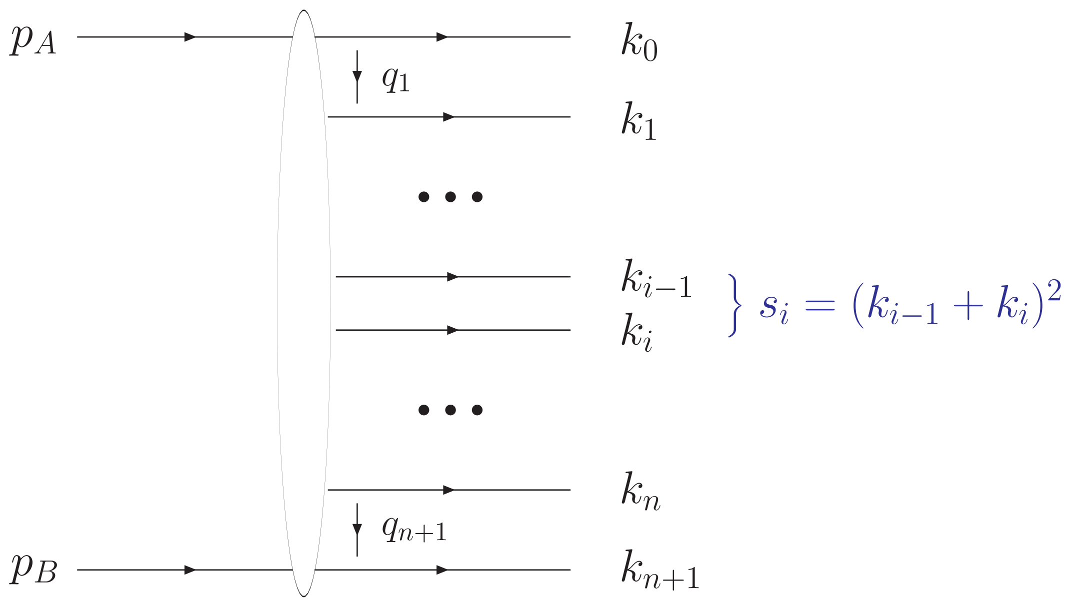
$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_1}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\omega(t_2)\Re\,\mathcal{A}_{AB}^{A'GB'} .$$

In the same way we obtain

$$\Re\left[\frac{1}{-2\pi i}\left(disc_{s_2}+disc_s\right)\mathcal{A}_{AB}^{A'GB'}\right]=\frac{1}{2}\omega(t_2)\Re\,\mathcal{A}_{AB}^{A'GB'} .$$

The bootstrap relations for amplitudes of multi-particle production can be derived in the same way. Denoting $\mathcal{A}_{AB}^{A'B'+n}$ the amplitude of the production of n particles in the MRK in the process

$$A + B \;\rightarrow\; A' + B' + n,$$



we have

$$\Re\left[\frac{1}{-2\pi i s}\left(\sum_{l=k+1}^{n+1} disc_{s_{kl}} - \sum_{l=0}^{k-1} disc_{s_{lk}}\right)\mathcal{A}_{AB}^{A'B'+n}\right]$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial\log s_{k,k+1}} - \frac{\partial}{\partial\log s_{k-1,k}}\right)\Re\left[\frac{1}{s}\mathcal{A}_{AB}^{A'B'+n}(s_{i-1,i})\right] \;,$$

where $s_{ij} = (k_i + k_j)^2$ and in the right part the amplitudes are expressed in terms of $s_{i-1,i}$, which are considered as independent variables.

If the amplitudes have the Reggeized form,

$$\Re\mathcal{A}(s_{i_1,i}) = 2s\,\Gamma_{\tilde{A}A}^{c_1}\left(\prod_{i=1}^{n}\gamma_{c_i c_{i+1}}^{P_i}(q_i,q_{i+1})\left(\frac{s_i}{s_R}\right)^{\omega(t_i)}\frac{1}{t_i}\right)\frac{1}{t_{n+1}}\left(\frac{s_{n+1}}{s_R}\right)^{\omega(t_{n+1})}\Gamma_{\tilde{B}B}^{c_{n+1}} \;.$$

we obtain bootstrap relations.

The bootstrap relations:

$$\Re \left[\frac{1}{-2\pi i} \left(\sum_{l=k+1}^{n+1} disc_{s_{kl}} - \sum_{l=0}^{k-1} disc_{s_{lk}} \right) \mathcal{A}_{AB}^{A'B'+n} \right]$$
$$= \frac{1}{2} \left(\omega(t_{k+1}) - \omega(t_k) \right) \Re \, \mathcal{A}_{AB}^{A'B'+n}$$

are much more restrictive ("strong bootstrap") than the conditions obtained from the requirement of the Reggezation in the elastic amplitudes. Their fulfilment means a proof of the Reggeization in the NLO, since the energy dependence of amplitudes can be calculated order by order in perturbation theory using the equalities

$$\Re \left[\frac{1}{-2\pi i s} \left(\sum_{l=k+1}^{n+1} disc_{s_{kl}} - \sum_{l=0}^{k-1} disc_{s_{lk}} \right) \mathcal{A}_{AB}^{A'B'+n} \right]$$
$$= \frac{1}{2} \left(\frac{\partial}{\partial \log s_{k,k+1}} - \frac{\partial}{\partial \log s_{k-1,k}} \right) \Re \left[\frac{1}{s} \mathcal{A}_{AB}^{A'B'+n}(s_{i-1,i}) \right] ,$$

valid in the NLO.

Indeed, the discontinuities entering in the L.H.S. of these equations in some order in the coupling constant g can be expressed with the help of the unitarity relations through the multi-particle amplitudes in lower orders in g. Fulfillment of the bootstrap relations means that the energy dependence has the Regge form.

The bootstrap relations

$$\Re\left(\sum_{l=k+1}^{n+1}\frac{\Delta_{s_{kl}}\mathcal{A}}{-2\pi i}-\sum_{l=0}^{k-1}\frac{\Delta_{s_{lk}}\mathcal{A}}{-2\pi i}-\right)$$

$$=\frac{1}{2}\left(\omega(t_{k+1})-\omega(t_k)\right)\Re\,\mathcal{A}_{AB}^{A'B'+n}$$

impose connections on the gluon trajectory, colour octet BFKL kernel and vertices of interaction of the Reggeized gluon with quarks an gluons (bootstrap conditions). To formulate these conditions it is convenient to introduce operators in transverse momentum representation. From t-channel point of view we have to consider two interacting Reggeized gluons with "coordinates" in the transverse momentum space $\vec{r}$ and $\vec{q}-\vec{r}$, where $\vec{q}$ is the total transverse momentum in the t-channel. Let us introduce $\hat{\vec{r}}$ as operator of "coordinate" of one of the Reggeized gluons in the transverse momentum space:

$$\hat{\vec{r}}\,|\vec{q}_i\rangle=\vec{q}_i|\vec{q}_i\rangle\ .$$

The total transverse momentum $\vec{q}$ is conserved and is considered as the c-number. It is convenient to use the normalization

$$\langle\vec{q}_1|\vec{q}_2\rangle=\vec{q}_1^{\,2}(\vec{q}_1-\vec{q})^{\,2}\delta^{(D-2)}(\vec{q}_1-\vec{q}_2)\ ,$$

so that

$$\langle A|B\rangle=\langle A|\vec{k}\rangle\langle\vec{k}|B\rangle=\int\frac{d^{D-2}k}{\vec{k}^2(\vec{k}-\vec{q})^2}A(\vec{k})B(\vec{k})\ .$$

The impact factors appear in this formalism as the wave functions of the t-channel states

$$\Phi_{A'A}(\vec{q}_1,\vec{q},s_0)=\langle\vec{q}_1|\Phi_{A'A}\rangle$$

and the BFKL kernel as the operator

$$\mathcal{K}(\vec{q}_2,\vec{q}_1,\vec{q})=\langle\vec{q}_2|\hat{\mathcal{K}}|\vec{q}_1\rangle\ .$$

In these denotations the s-channel discontinuity of the elastic amplitude can be written as

$$\frac{disc_s \mathcal{A}_{AB}^{A'B'}}{-2\pi i} = \frac{-2s}{(2\pi)^{D-1}} \langle A'A| \left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle \ .$$

If the bootstrap relation

$$\frac{-2s}{(2\pi)^{D-1}} \langle A'A| \left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle = \frac{\omega(t)}{2}\, 2s\, \Gamma_{A'A} \left(\frac{s}{s_0}\right)^{\omega(t)} \Gamma_{B'B}$$

were exact, then it should be

$$\left(\hat{\mathcal{K}} - \omega(t)\right) |A'A^a\rangle = 0$$

and

$$|A'A^a\rangle = \Gamma^a_{A'A} |R_\omega\rangle \ ,$$

with an universal (process independent) eigenstate $|R_\omega\rangle$ of the kernel with the eigenvalue $\omega(t)$. In usual denotations it reads

$$\int \frac{d^{(D-2)} q_2}{\vec{q}_2^{\,2} (\vec{q}_2 - \vec{q})^2} \mathcal{K}(\vec{q}_1, \vec{q}_2; \vec{q})\, R_\omega(\vec{q}_2, \vec{q}, s_0) = \omega(t)\, R_\omega(\vec{q}_1, \vec{q}, s_0)$$

and

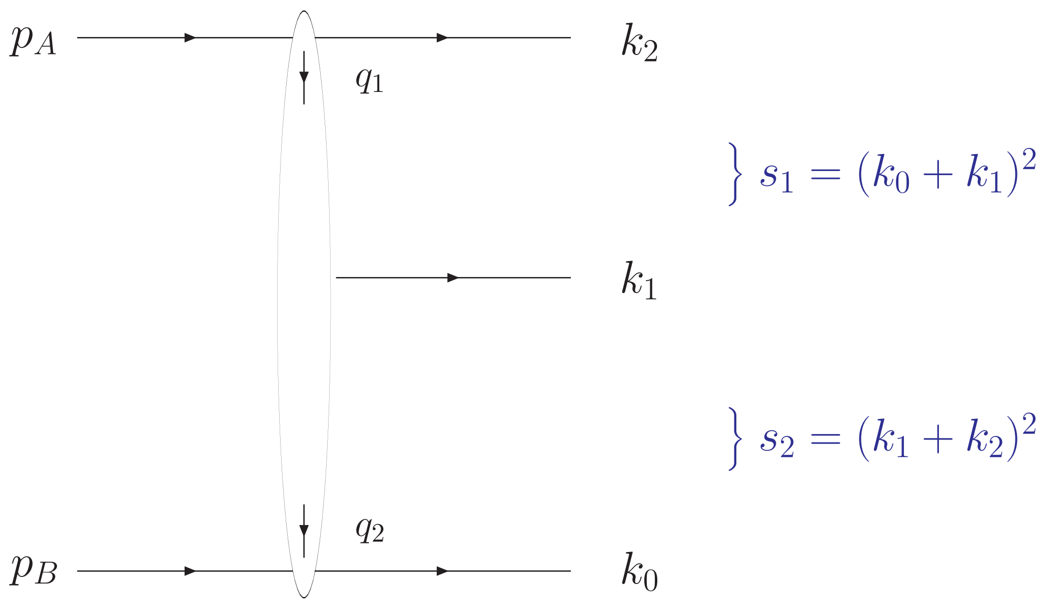
$$\Phi^a_{A'A}(\vec{q}_1, \vec{q}, s_0) = \Gamma^a_{A'A} R_\omega(\vec{q}_1, \vec{q}, s_0)$$

These conditions are known as the strong bootstrap conditions for the kernel and impact factors. They were suggested, without derivation, by M. Braun, G.P. Vacca,1999.

In fact, since we work in the NLO, these conditions can not be derived from the bootstrap requirement for the elastic amplitudes. Nevertheless, they were checked and their fulfillment was shown for the quark and gluon impact factors
M. Braun, G.P. Vacca,1999,
V.F., R. Fiore, M.I. Kotsky, A. Papa, 2000,
for the quark contribution to the kernel
M. Braun, G.P. Vacca,1999,
V.F, R. Fiore, M.I. Kotsky, A. Papa, 2000
and, finaly, quite recently for the gluon part of the kernel
V.F, A. Papa, 2002.

It occurs, that these conditions must be satisfied for fulfillment of the bootstrap relations for inelastic amplitudes. Thus, for the gluon production amplitude

$$A + B \;\rightarrow\; A' + B' + G$$



in the MRK we obtain

$$\frac{disc_{s_2}\mathcal{A}_{AB}^{A'GB'}}{-2\pi i} = \frac{-2s}{(2\pi)^{D-1}}\,\Gamma_{A'A}\frac{1}{t_1}\left(\frac{s}{s_0}\right)^{\omega(t_1)}\;\langle GR|\left(\frac{s}{s_0}\right)^{\hat{\mathcal{K}}}|B'B\rangle\;,$$

where $\langle GR|$ is the t_2-channel state of the produced gluon G and the t_1-channel Reggeized gluon R with the wave function, which is expressed in terms of vertices of interactions of the Reggeized gluon (in fact, this wave function is the impact factor in the case when instead of on mass-shell particle we have the Reggeized gluon).

For the s-channel discontinuity we obtain

$$\frac{disc_s \mathcal{A}_{AB}^{A'GB'}}{-2\pi i} = \frac{-2s}{(2\pi)^{D-1}} \langle A'A| \left(\frac{s_1}{s_0}\right)^{\hat{\mathcal{K}}} \hat{\mathcal{G}} \left(\frac{s_2}{s_0}\right)^{\hat{\mathcal{K}}} |B'B\rangle \ ,$$

where $\hat{\mathcal{G}}$ is the operator of the gluon production with change of total two-Reggeon state momentum from q_1 to q_2. The matrix elements of these operator are also expressed in terms of vertices of interactions of the Reggeized gluon.

The bootstrap relation

$$\Re \left[\frac{1}{-2\pi i} \left(disc_{s_2} + disc_s\right) \mathcal{A}_{AB}^{A'GB'}\right]$$

$$= \frac{\omega(t_2)}{2}\, 2s\, \Gamma^{i}_{A'A} \frac{1}{t_1} \left(\frac{s_1}{s_0}\right)^{\omega(t_1)} \gamma^{G}_{ij}(q_1,q_2)\, \frac{1}{t_2} \left(\frac{s_2}{s_0}\right)^{\omega(t_2)} \Gamma^{j}_{B'B}$$

leads, together with the strong bootstrap conditions for the impact factors and kernel, to new bootstrap condition, which can be written as

$$\frac{-2t_2}{(2\pi)^{D-1}} \left[t_1 \langle R_\omega|\hat{\mathcal{G}}|R_\omega\rangle + \langle RG|R_\omega\rangle\right]$$

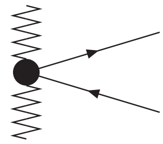
$$= \omega(t_2)\gamma^{G}_{ij}(q_1,q_2)$$

Fulfillment of this bootstrap condition is not checked yet.

Till now we have discussed the bootstrap relations for the scattering amplitudes in the MRK. But in the NLA in the unitarity relations not only multi-Regge, but quasi-multi-Regge kinematics as well does contribute, so that in the derivation of the NLO BFKL the multi-Regge form was assumed for production amplitudes in the QMRK as well.

If rapidities of components of the the produced couple (it can be or gg or $q\bar{q}$ pair) are far away from rapidities of colliding particles, then it is created by two Reggeized gluons, and its production is described by the vertices

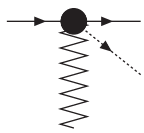
$\gamma_{c_1c_2}^{P_1P_2}(q_1, q_2)$



where q_1, c_1 and $-q_2, c_2$ are momenta and colour indices of the Reggeized gluons.

If the pair is produced in the region of fragmentation of particle A due to interaction of this particle with the Reggeized gluon with momentum q and colour index c, the production is described by the vertices

$\Gamma^c_{\{P_1P_2\},A}(q)$



The produced particles can be gg or $q\bar{q}$ pair if the particle A is a gluon and qg when the particle A is a quark.

Note that because using of multi-Regge kinematics leads to lost of large logarithm, these vertices are needed only in the LO.

The QMRK vertices must satisfy the following bootstrap conditions:

$$-if^{jbb'}\int\frac{d^{D-2}r_\perp}{r_\perp^2(q_2-r)_\perp^2}\Bigg[\sum_G\gamma_{ib}^G(q_1,r)\Gamma^{b'}_{\{P_1P_2\},G}$$

$$+igf^{ia'b}\frac{q_{1\perp}^2}{(q_1-r)_\perp^2}\gamma_{a'b'}^{P_1(k_1)P_2(k_2)}(q_1-r,q_2-r)$$

$$+\sum_{P_1'}\Gamma^{b'}_{P_1,P_1'}\gamma_{ib}^{P_1'(k_1')P_2(k_2)}(q_1,r)$$

$$+\sum_{P_2'}\Gamma^{b'}_{P_2,P_2'}\gamma_{ib}^{P_1(k_1)P_2'(k_2')}(q_1,r)$$

$$+igf^{ia'a}\frac{q_{1\perp}^2}{(q_1-k_1')_\perp^2{k'}_{1\perp}^{2}}$$

$$\times\gamma_{ab}^{P_2(k_2)}(q_1-k_1',r)\gamma_{a'b'}^{P_1(k_1)}(k_1',q_2-r)\Bigg]$$

$$=-g\frac{N_c}{2}\gamma_{ij}^{P_1(k_1)P_2(k_2)}(q_1,q_2)\int\frac{d^{D-2}r_\perp}{r_\perp^2(q_2-r)_\perp^2}\,.$$

(note that the last term in the L.H.S. does contribute only in the case of two-gluon producion)

and

$$\int\frac{d^{D-2}q_{1\perp}}{q_{1\perp}^2(q-q_1)_\perp^2}\frac{if^{cc_1c_1'}}{N_c}\sum_{\{i\}}\Gamma^{c_1}_{\{i\},A}(q_1)\Gamma^{c_1'}_{\{i\},P_1P_2}(q-q_1)$$

$$=\frac{g}{2}\Gamma^c_{P_1P_2,A}(q)\int\frac{d^{D-2}q_{1\perp}}{q_{1\perp}^2(q-q_1)_\perp^2}.$$

These conditions are checked
V. F., M.G. Kozlov, A.V. Reznichenko, 2002
and it is shown that they are satisfied.

Summary

- The forward BFKL kernel is known in the NLO.
- Methods of treating of the NLO corrections are developed.
- The non-forward kernel
  - is known for the colour octet in the the t-channel;
  - for the colour singlet the two-gluon production contribution is not yet calculated.
- The known NLO impact factors are those of the quarks and gluons.
- Work is in progress for the calculation of impact factors of colorless particles.
- The NLO bootstrap conditions are proved to be satisfied
  - for the kernel and the parton's impact factors;
  - for the pair production in the fragmentation region, as well as for the pair production in Reggeon-Reggeon collisions in QMRK.
- The condition for the gluon production in Reggeon-Reggeon collisions is formulated now. Its check is under consideration.
- Fulfilment of all these conditions opens a way to prove the gluon Reggeization, that means the justification of the BFKL approach, in the NLO.