Non-perturbative Methods in QCD apart from Lattice Simulations

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1. Introduction

- **Quantum Chromodynamics** (**QCD**) – the quantum gauge field theory of quarks and gluons –
  - assumed to be **the** fundamental theory of **strong interactions**
  - yet **no** generally accepted and uniformly applicable **analytic tool** to handle its **non-perturbative aspects** such as
    * confinement
    * hadron spectroscopy
    * chiral symmetry breaking
    * soft scattering processes

    starting from **first principles:**

  Expectation values of observables as analytic continuation of continuum limit of **lattice-regularized Euclidean functional integral**

- Quite a few aspects of non-perturbative **QCD** can be understood qualitatively as well as quantitatively with the help of **nonperturbative methods** based on expansions complementary to ordinary perturbation theory or model assumptions inspired from **QCD**

⇒ Review (some of) these non-perturbative, non-lattice methods and point out their
  - foundations
  - successes
  - shortcomings
  - mutual connections

- **Content:**

2. Generalities

- **QCD** in terms of **Euclidean functional integral:**
  - Expectation values of observables:

$$
\begin{aligned}
\langle \mathcal{O}[A,\psi,\overline{\psi}]\rangle &= \frac{1}{Z}\int [dA][d\psi][d\overline{\psi}]\mathcal{O}[A,\psi,\overline{\psi}]\mathrm{e}^{-S[A,\psi,\overline{\psi}]} \\
Z &= \int [dA][d\psi][d\overline{\psi}]\,\mathrm{e}^{-S[A,\psi,\overline{\psi}]}
\end{aligned}
$$

  - Physical observables: analytical continuation to **Minkowski** space-time ($x_4 \to \mathrm{i}\, x_0$); Z: vacuum-to-vacuum amplitude
  - **Non-perturbative definition:** continuum limit of **lattice-regularized model** (Lecture $\to$ Karl Jansen; Plenary $\to$ Sinya Aoki, Edwin Laermann, Hartmut Wittig, . . .)
  - Analytical treatment: **ordinary perturbation theory**

- Due to **asymptotic freedom**,

$$\alpha_s(Q) \equiv \frac{g_s^2(Q)}{4\pi} = \frac{2\pi}{\beta_0 \ln(Q/\Lambda)} \left[1 + \mathcal{O}\left(\frac{\ln\left[\ln(Q/\Lambda)\right]}{\ln(Q/\Lambda)}\right)\right];$$

$$\beta_0 = 11 - \frac{2}{3} n_f \,,$$

ordinary perturbation theory applicable at **large momentum transfers** $Q \gg \Lambda \sim 200$ **MeV** $\Leftrightarrow$ **small distances**

- **In low-energy domain,** $Q \lesssim \Lambda \sim 200$ **MeV** $\Leftrightarrow$ **at long distances**, growing of coupling and associated confinement of quarks and gluons make difficult thorough analysis of **QCD** dynamics in terms of these fundamental constituents

3. Chiral perturbation theory

- **At very low energies** – below resonance region, $E \lesssim M_\rho \approx 770$ MeV – **great simplification of strong-interaction dynamics:**

  Hadronic spectrum only contains an **octet of very light pseudoscalar particles (π, K, η)** whose interaction can be understood with **global symmetry considerations**

$\Rightarrow$ **Chiral perturbation theory** [Weinberg '79; Gasser,Leutwyler '84;...]

- Powerful theoretical framework to systematically analyse low-energy implications of the **QCD** symmetries
- Based on two key ingredients:
  - **Chiral symmetry properties of QCD**
  - **Concept of effective field theories**

Chiral symmetry properties of QCD

- In absence of quark masses, QCD Lagrangian $[q = \text{column}(\underbrace{u, d, s, \dots}_{n_f})]$

$$\mathcal{L}^0_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + i\bar{q}_L \gamma^\mu D_\mu q_L + i\bar{q}_R \gamma^\mu D_\mu q_R$$

invariant under independent **global**

$$\boldsymbol{G \equiv SU(n_f)_L \otimes SU(n_f)_R \otimes U(1)_{L+R} \otimes U(1)_{L-R}}$$

transformations

$$\begin{aligned}
SU(n_f)_{L,R} \quad &: \quad q_{L,R} \xrightarrow{G} g_{L,R}\, q_{L,R} \\
U(1)_{L+R} \equiv U(1)_V \quad &: \quad q_{L,R} \xrightarrow{G} \mathrm{e}^{\mathrm{i}\theta}\, q_{L,R} \\
U(1)_{L-R} \equiv U(1)_A \quad &: \quad q_L \xrightarrow{G} \mathrm{e}^{-\mathrm{i}\theta}\, q_L, \qquad q_R \xrightarrow{G} \mathrm{e}^{\mathrm{i}\theta}\, q_R
\end{aligned}$$

Chiral symmetry properties of QCD

- **Chiral symmetry $SU(n_f)_L \otimes SU(n_f)_R$** should be approximately good in light quark sector (u,d,s), but **not seen in hadronic spectrum:**
  – Hadrons can be classified in $SU(3)_V \equiv SU(3)_{L+R}$ representations, but degenerate multiplets with opposite parity do not exist
- Octet of pseudoscalar mesons (π^+, π^-, π^0, η, K^+, K^-, K^0, and $\bar{K}^0$) much lighter than all other hadronic states

$\Rightarrow$ **Ground state of the theory not symmetric under the chiral group:**

$$SU(3)_L \otimes SU(3)_R \stackrel{\text{SCSB}}{\longrightarrow} SU(3)_{L+R}$$

and, according to Goldstone's theorem, an **octet $(3^2 - 1)$ of pseudoscalar massless bosons appears** in the theory; their small masses generated by quark-mass matrix

- **Quark condensate** $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle = \langle 0|\bar{s}s|0\rangle \neq 0$ natural order parameter of **spontaneous chiral symmetry breaking (SCSB)**

Effective chiral Lagrangian

- **Mass gap** separating pseudoscalar octet from rest of hadronic spectrum $\Rightarrow$ can build **effective field theory** containing only Goldstone fields, which may be parametrized by

$$U(\phi) \;=\; \exp\left\{ \mathrm{i}\sqrt{2}\Phi/f \right\}$$

$$\Phi \equiv \frac{\vec{\lambda}}{\sqrt{2}}\vec{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

$U(\phi)$ transforms linearly under chiral group, $g_R\, U(\phi)\, g_L^\dagger$, but induced transformation of Goldstone fields ϕ highly non-linear

Effective chiral Lagrangian

- Low–energy effective Lagrangian realization of **QCD** for the light–quark sector $(u,\, d,\, s)$:
  - Most general $\mathcal{L}_{\rm eff}$ involving $U(\phi)$ consistent with chiral symmetry
  - Organized in terms of increasing powers of momentum or, equivalently, in terms of an increasing number of derivatives $\mathcal{L}_{\rm eff}(U) = \sum_{n\geq 1} \mathcal{L}_{2n}$.
- To lowest order, effective chiral Lagrangian uniquely given by

$$\boldsymbol{\mathcal{L}_2 = \frac{f^2}{4}\,\mathrm{tr}\left[\partial_\mu U^\dagger \partial^\mu U\right]}$$

- Expanding $U(\phi)$ in $\Phi \Rightarrow$ all interactions predicted in terms of single f:

$$\mathcal{L}_2 = \frac{1}{2}\,\mathrm{tr}\left[\partial_\mu \Phi \partial^\mu \Phi\right] + \frac{1}{12 f^2}\,\mathrm{tr}\left[\left(\Phi \overleftrightarrow{\partial}_\mu \Phi\right)\left(\Phi \overleftrightarrow{\partial}^\mu \Phi\right)\right] + \mathcal{O}(\Phi^6/f^4)$$

e.g. $T(\pi^+\pi^0 \to \pi^+\pi^0) = t/f^2$

Effective chiral Lagrangian

- Introduce couplings to **external classical fields**

$$\mathcal{L}_{\rm QCD} = \mathcal{L}^0_{\rm QCD} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q$$

which enable to
– compute the effective realization of Green's functions of quark currents

$$Z\,[v,a,s,p] \quad = \quad \int \mathcal{D}q\,\mathcal{D}\bar{q}\,\mathcal{D}G_\mu\,{\rm e}^{{\rm i}\int d^4x\,\mathcal{L}_{\rm QCD}} \equiv \int \mathcal{D}U\,{\rm e}^{{\rm i}\int d^4x\,\mathcal{L}_{\rm eff}}$$

– incorporate the electromagnetic and semileptonic weak interactions
– take into account explicit χ-symmetry breaking through quark masses

$$\begin{aligned} r_\mu &\equiv v_\mu + a_\mu = e\mathcal{Q}A_\mu + \ldots \\ \ell_\mu &\equiv v_\mu - a_\mu = e\mathcal{Q}A_\mu + \frac{e}{\sqrt{2}\sin\theta_W}(W^\dagger_\mu T_+ + {\rm h.c.}) + \ldots \\ s &= \mathcal{M} + \ldots = {\rm diag}(m_u, m_d, m_s) + \ldots \end{aligned}$$

Effective chiral Lagrangian

- Lagrangian $\mathcal{L}_{\rm QCD}$ invariant under following set of **local** $SU(3)_L \otimes SU(3)_R$ transformations:

$$
\begin{array}{lll}
q_L \longrightarrow g_L\, q_L\,, & q_R \longrightarrow g_R\, q_R\,, & s+ip \longrightarrow g_R\,(s+ip)\,g_L^\dagger\,, \\
\ell_\mu \longrightarrow g_L\,\ell_\mu\,g_L^\dagger + i g_L \partial_\mu g_L^\dagger\,, & & r_\mu \longrightarrow g_R\,r_\mu\,g_R^\dagger + i g_R \partial_\mu g_R^\dagger
\end{array}
$$

- At lowest order, the most general effective Lagrangian consistent with Lorentz invariance and (local) chiral symmetry [Gasser,Leutwyler '85]

$$
\mathcal{L}_2 = \frac{f^2}{4}\,{\rm tr}\left[D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U\right]
$$

where $D_\mu U = \partial_\mu U - i r_\mu U + i U \ell_\mu$ and $\chi = 2 B_0\,(s+ip)$

Effective chiral Lagrangian

- Meaning of the constants f and B_0?
  - Taking functional derivatives with respect to r_μ and ℓ_μ $\Rightarrow$ chiral currents $\Rightarrow$ to lowest order, **f equals the pion decay constant, $f_\pi = 92.4$ MeV**, defined as

$$\langle 0|(J^\mu_A)^{12}|\pi^+\rangle = \langle 0|(J^\mu_R - J^\mu_L)^{12}|\pi^+\rangle \equiv \mathrm{i}\sqrt{2} f_\pi p^\mu$$

  - Taking functional derivatives with respect to the scalar $s - \mathrm{i}p$ and pseudoscalar $s + \mathrm{i}p$ sources $\Rightarrow$ to lowest order, **B_0 is related to the quark condensate**,

$$B_0 = -\langle \bar{q} q\rangle / f^2$$

- $\mathcal{L}_2$ encodes all the **current–algebra** results obtained in the sixties, e.g.

$$f_\pi^2 M_\pi^2 = -\frac{1}{2}(m_u + m_d)\,\langle 0|\bar{u}u + \bar{d}d|0\rangle\,; \qquad \frac{M^2_{\pi^\pm}}{m_u + m_d} = \frac{M^2_{K^0}}{m_s + m_d} = \frac{M^2_{K^\pm}}{m_s + m_u}$$

Effective chiral Lagrangian

- At next-to-leading order in momenta, $\mathcal{O}(p^4)$, the computation of the generating functional $Z[v,a,s,p]$ involves three different ingredients:
  - The most general effective chiral Lagrangian of $\mathcal{O}(p^4)$, $\mathcal{L}_4$, to be considered at tree level.
  - One–loop graphs associated with the lowest–order Lagrangian $\mathcal{L}_2$.
  - The Wess–Zumino–Witten functional to account for the chiral anomalies associated with the non-abelian external sources r_μ and ℓ_μ

Effective chiral Lagrangian

- At $\mathcal{O}(p^4)$, the most general Lagrangian, invariant under parity, charge conjugation and the local chiral transformations [Gasser,Leutwyler '85]

$$
\begin{aligned}
\mathcal{L}_4 \quad &= L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle \\
&+ L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi^\dagger U\rangle \\
&+ L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle + L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2 \\
&+ L_7 \langle U^\dagger \chi - \chi^\dagger U\rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\
&- i L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle \\
&+ H_1 \langle F_{R\mu\nu} F_R^{\mu\nu} + F_{L\mu\nu} F_L^{\mu\nu}\rangle + H_2 \langle \chi^\dagger \chi\rangle \,.
\end{aligned}
$$

i	$L_i^r(M_\rho) \times 10^3$	Source
1	0.4 ± 0.3	K_{e4}, $\pi\pi \longrightarrow \pi\pi$
2	1.4 ± 0.3	K_{e4}, $\pi\pi \longrightarrow \pi\pi$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \longrightarrow \pi\pi$
4	-0.3 ± 0.5	Zweig rule
5	1.4 ± 0.5	$F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule
7	-0.4 ± 0.2	Gell-Mann–Okubo, L_5, L_8
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m}) : (m_d - m_u)$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$
10	-5.5 ± 0.7	$\pi \longrightarrow e\nu\gamma$

[Pich '98]

Recent developments in χPT and further reading

- Recent developments include:
  - Effective Lagrangian $\mathcal{L}_6$ at $\mathcal{O}(p^6)$ [Bijnens,Colangelo,Ecker '99;...]
  - Test of basic **assumption of SCSB** and determination of parameters from $\mathcal{L}_{2/4}$ through **QCD** lattice simulations (Lecture $\longrightarrow$ Karl Jansen; Plenary $\longrightarrow$ Hartmut Wittig)
  - ...

$\Rightarrow$ Review, *e.g.*: A. Pich, hep-ph/9806303

$U(1)_A$ problem

- So far discussed how the global **chiral flavour symmetries** as well as the **baryon number conserving symmetry**

$$SU(n_f)_L \otimes SU(n_f)_R \otimes U(1)_V \otimes U(1)_A$$

of the massless QCD Lagrangian

$$\mathcal{L}^0_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} + i\bar{q}_L \gamma^\mu D_\mu q_L + i\bar{q}_R \gamma^\mu D_\mu q_R$$

are realized in nature

- What about the **axial symmetry**?

$U(1)_A$ problem

- If $U(1)_A$ unbroken in vacuum $\Rightarrow$ in chiral limit all massless hadrons would have a massless partner of opposite parity $\Rightarrow$ not realized in nature.

- If $U(1)_A$ spontaneously broken, then there should be an $I = 0$ pseudoscalar Goldstone boson, whose perturbed state should have about the same mass as the pion. Only candidate, $\eta'(958)$, too heavy: $M_{\eta'} \gg M_\eta$.

- Solution involves **$U(1)_A$ anomaly and topological (instanton) effects**

4. Instantons

- **$U(1)_A$ is not a symmetry of the theory at the quantum level:**

  The $U(1)_A$ axial current is not conserved due to an **axial anomaly:**

  [Adler '69; Bell,Jackiw '69]

$$\partial_\mu \left(\bar{q}\gamma^\mu\gamma_5 q\right) = 2\, n_f\, \nu \qquad ; \qquad \nu \equiv \frac{\alpha_s}{16\pi}\, \epsilon^{\mu\nu\rho\sigma}\, G^a_{\mu\nu} G^a_{\rho\sigma} \equiv \frac{\alpha_s}{8\pi} G^a_{\mu\nu} \tilde{G}^{a\,\mu\nu}$$

- ν so-called **topological charge density**. Can be written as total divergence

$$\nu = \partial^\mu K_\mu(x) \;\; ; \;\; K_\mu = \frac{\alpha_s}{4\pi} \epsilon_{\mu\alpha\beta\gamma} A^\alpha_a \left(\partial^\beta A^\gamma_a + \frac{1}{3} g f_{abc} A^\beta_b A^\gamma_c \right)$$

K_μ so-called **Chern-Simons current**

$\Rightarrow$ For sufficiently rapidly vanishing fields can write for integral of anomaly over space-time, $\int \mathrm{d}^4x\,\partial_\mu\left(\bar q\gamma^\mu\gamma_5 q\right) = 2\,n_f \int \mathrm{d}^4x\,\nu \equiv 2\,n_f \int \mathrm{d}^4x\,\partial_\mu K^\mu$,

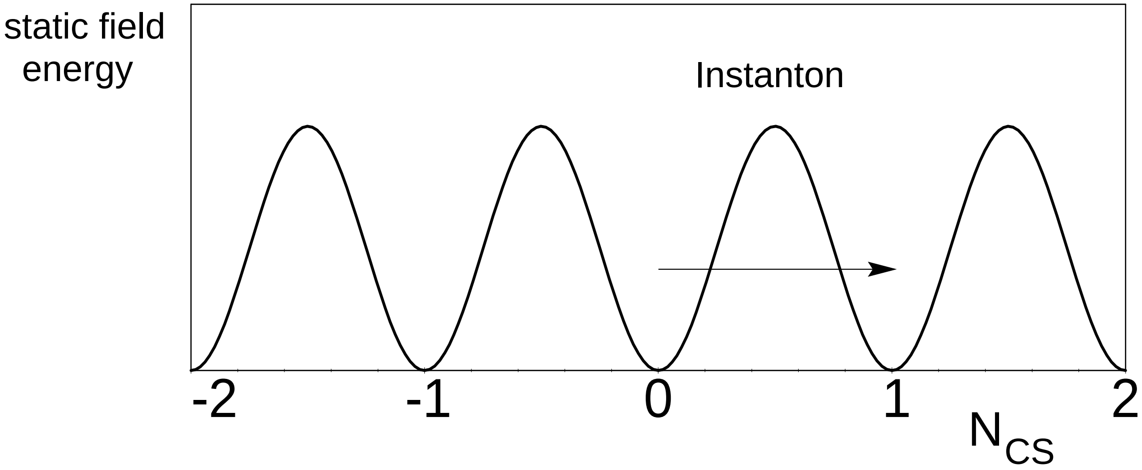
$$\underbrace{\int_{-\infty}^{+\infty} \mathrm{d}t\,\partial_t \int \mathrm{d}^3x\,\bar q\gamma^0\gamma_5 q}_{Q_5(t=\infty)-Q_5(t=-\infty)} = 2\,n_f \underbrace{\int_{-\infty}^{+\infty} \mathrm{d}t\,\partial_t \int \mathrm{d}^3x\,K^0}_{N_{\mathrm{CS}}(t=\infty)-N_{\mathrm{CS}}(t=-\infty)}$$

in short

$$\boldsymbol{\triangle Q_5 = 2\,n_f\,\triangle N_{\mathrm{CS}}}$$

i.e. in the background of gauge fields which evolve in time such that their Chern-Simons number $N_{\mathrm{CS}} = \int \mathrm{d}^3x\,K^0$ changes by $\triangle N_{\mathrm{CS}}$ $\Rightarrow$ the fermionic $U(1)_A$ charge Q_5 changes by $2\,n_f\,\triangle N_{\mathrm{CS}}$

- Classical gauge fields with zero energy have **integer** $N_{\rm CS}$.
- There are topological obstructions to deform such zero energy fields differing in their Chern-Simons numbers smoothly into each other $\Rightarrow$ separated by **energy barrier** [Jackiw,Rebbi 176; Callan,Dashen,Gross '76]



- **Instanton** describes $\triangle N_{\rm CS} = 1$ **tunneling transition**. Associated with anomalous violation of axial charge conservation, $\triangle Q_5 = 2\,n_f$. ['t Hooft '76]

$\Rightarrow$ No reason to expect $U(1)_A$ to be a symmetry $\Rightarrow$ no $U(1)_A$ problem.

Instanton-perturbation theory

- Generalized saddle-points in the Euclidean functional integral formulation of **QCD**,

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dA][d\psi][d\overline{\psi}]\, \mathcal{O}[A,\psi,\overline{\psi}]\, \mathrm{e}^{-S[A,\psi,\overline{\psi}]}\,,$$

$$Z = \int [dA][d\psi][d\overline{\psi}]\, \mathrm{e}^{-S[A,\psi,\overline{\psi}]}\,.$$

- Perturbation theory:
  - **perturbative QCD:** expansion about trivial vacuum solution, i.e. vanishing gluon field and vanishing quark fields and thus vanishing Euclidean action, $S = 0$.
  - **instanton-perturbation theory:** generalized saddle-point expansion of the Euclidean functional integral about non-trivial minima of the Euclidean action with $S \neq 0$.

Instanton-perturbation theory

- Non-trivial minima ($\Leftrightarrow$ solutions) have **integer** **Pontryagin index (topological charge)**

$$Q \equiv \int d^4x\, \nu(x) \equiv \triangle N_{\rm CS} = \pm 1, \pm 2, \ldots,$$

and their **action** is a multiple of $2\pi/\alpha_s$,

$$S \equiv \int d^4x\, \frac{1}{2} {\rm tr}(G_{\mu\nu} G_{\mu\nu}) = \frac{2\,\pi}{\alpha_s}\, |Q| = \frac{2\,\pi}{\alpha_s} \cdot (1, 2, \ldots).$$

Instanton-perturbation theory

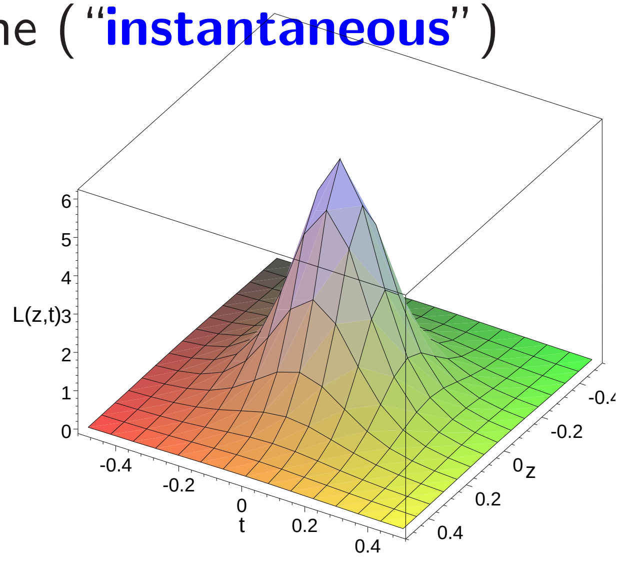
$\Rightarrow$ **Dominant** saddle-point for $\alpha_s \ll 1$: **Instanton $(Q = 1)$:** [Belavin *et al.* '75]

$$A_\mu^{(I)}(x;\rho,U,x_0) = -\frac{\mathrm{i}}{g}\,\frac{\rho^2}{(x-x_0)^2}\,U\,\frac{\sigma_\mu\left(\overline{x}-\overline{x_0}\right)-\left(x_\mu - x_{0\mu}\right)}{(x-x_0)^2+\rho^2}\,U^\dagger$$

- **size ρ, color orientation U, position x_0**
- localized in Euclidean space and time ("**instantaneous**")

$$\mathcal{L}\left(A_\mu^{(I)}(x;\rho,U,0)\right) = \frac{12}{\pi\alpha_s}\cdot\frac{\rho^4}{(x^2+\rho^2)^4}$$

$$\Rightarrow \quad S\left[A_\mu^{(I)}\right] = \frac{2\,\pi}{\alpha_s}$$



Instanton-perturbation theory

- **Instanton**-contribution to vacuum-to-vacuum amplitude Z:

$$\frac{1}{Z^{(0)}}\frac{dZ^{(I)}}{d^4x} = \int_0^\infty d\rho\, D_m(\rho) \int dU$$

- **Size distribution** $D_m(\rho)$ known in instanton-perturbation theory

['t Hooft '76; Bernard '79]

$$\alpha_s(\mu_r)\ln(\rho\,\mu_r) \ll 1, \qquad \rho\, m_i(\mu_r) \ll 1,$$

in 2-loop renormalization-group invariant form, [Morris *et al.* '85]

$$\frac{dn_I}{d^4x\, d\rho} = D_m(\rho) = D(\rho)\prod_{i=1}^{n_f}(\rho\, m_i(\mu_r))\,(\rho\,\mu_r)^{n_f\,\gamma_0\frac{\alpha_s(\mu_r)}{4\pi}}$$

Instanton-perturbation theory

- **Reduced size distribution $D(\rho)$:**

$$D(\rho) = \frac{d}{\rho^5}\left(\frac{2\pi}{\alpha_s(\mu_r)}\right)^{2\,N_c} \mathrm{e}^{-\frac{2\pi}{\alpha_s(\mu_r)}}\,(\rho\,\mu_r)^{\beta_0+(\beta_1-4\,N_c\beta_0)\frac{\alpha_s(\mu_r)}{4\pi}}$$

- Clearly non-perturbative, $\propto \mathrm{e}^{-\frac{2\pi}{\alpha_s(\mu_r)}}$
- Power-law behaviour of (reduced) size distribution,

$$D(\rho) \sim \rho^{\beta_0-5+\mathcal{O}(\alpha_s)}; \qquad \beta_0 = \frac{11}{3}\,N_c - \frac{2}{3}\,n_f$$

  - dominant contribution to ρ integral generically originates from large ρ
  - spoils the applicability of instanton-perturbation theory, $\alpha_s(1/\rho) \ll 1$

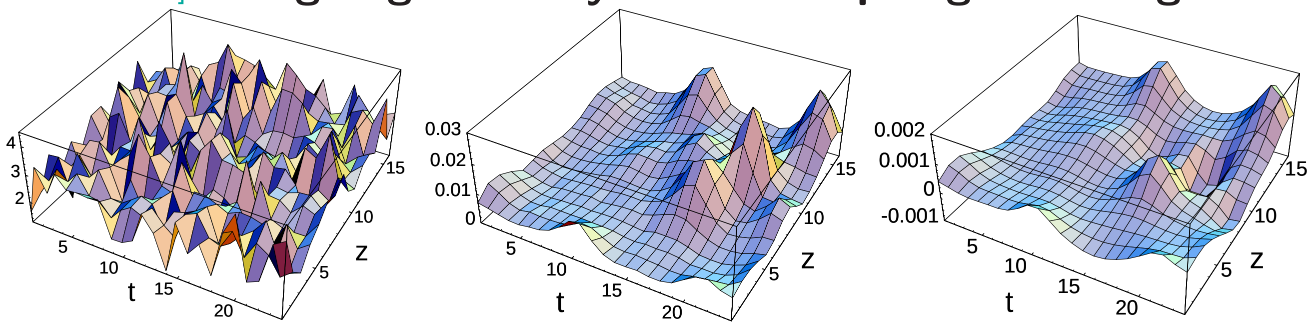
Instanton-perturbation theory

- Size distribution basic building block of **instanton**-perturbation theory:
  - ∘ appears in generic instanton contributions to Green's functions

  $\Rightarrow$ important to know the region of validity of the perturbative result

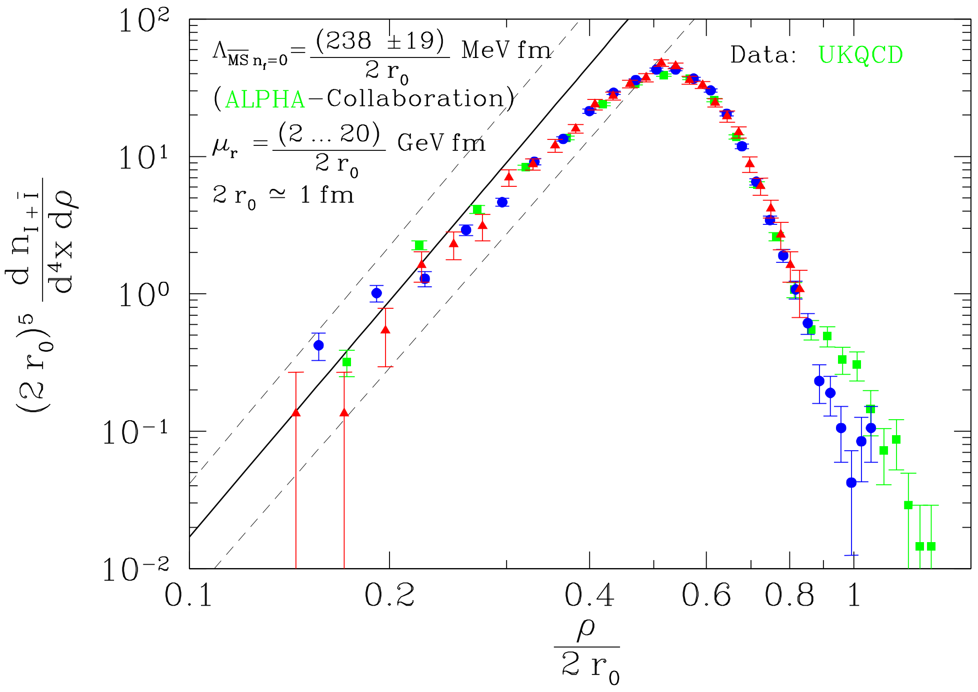
- Crucial information from lattice investigations on the topological structure of the QCD vacuum

[Chu *et. al* '94] **Lagrange Density** **Topological Charge Density**



Instanton-perturbation theory

- Confronting recent lattice data $(n_f = 0)$ [Smith,Teper (UKQCD) '98] on size distribution with perturbative result [AR,F. Schrempp '99]:



$\Rightarrow$ **Instanton-perturbation theory reliable for $\rho\,\Lambda \lesssim 0.4$**

Instanton-perturbation theory

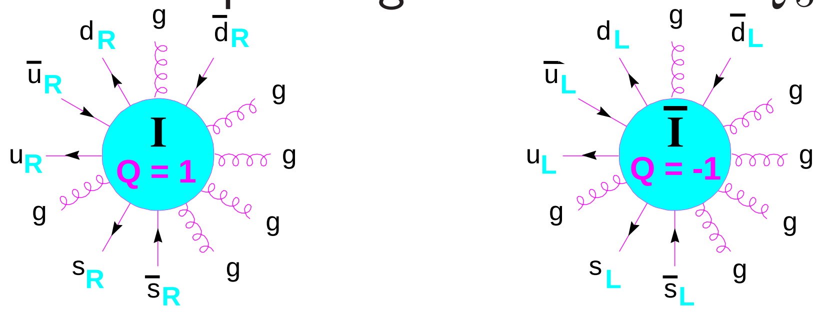
- Suppression of instanton contribution to the vacuum-to-vacuum amplitude for small quark masses $\rho m_i \ll 1$:

$\Leftarrow$ **Axial anomaly**: [Adler '69; Bell, Jackiw '69]

Any gauge field fluctuation with topological charge Q must be accompanied by a corresponding change in axial charge, $\triangle Q_5 = 2\, n_f\, Q$

$\Rightarrow$ pure vacuum-to-vacuum transitions vanish in chiral limit

$\Rightarrow$ Green's functions corresponding to anomalous Q_5 violation: ['t Hooft '76]



  * main contribution due to instantons
  * do not suffer from any mass suppression

Instanton-perturbation theory

- Simplest **example** of one light flavour ($n_f = 1$): Fermionic two-point function

$$\langle\psi(x_1)\overline{\psi}(x_2)\rangle^{(I)} \simeq \int d^4x \int_0^\infty d\rho\, D(\rho) \int dU\, (\rho\, m)\, S^{(I)}(x_1, x_2; x, \rho, U)$$

Quark propagator in the I-background $S^{(I)}$,

$$S^{(I)}(x_1, x_2; \ldots) = \frac{\kappa_0(x_1; \ldots)\, \kappa_0^\dagger(x_2; \ldots)}{m} + \sum_{n\neq 0} \frac{\kappa_n(x_1; \ldots)\, \kappa_n^\dagger(x_2; \ldots)}{m + i\lambda_n},$$

in terms of the spectrum of the Dirac operator in the I-background, which has exactly one right-handed zero mode κ_0, ['t Hooft '76]

$$-\mathrm{i}\gamma_\mu D_\mu^{(I)} \kappa_n = \lambda_n \kappa_n; \quad \text{with } \lambda_0 = 0 \text{ and } \lambda_n \neq 0 \text{ for } n \neq 0,$$

for $m \to 0$ only the zero mode contribution survives,

$$\langle\psi(x_1)\overline{\psi}(x_2)\rangle^{(I)} \simeq \int d^4x \int_0^\infty d\rho\, D(\rho) \int dU\, \rho\, \kappa_0(x_1; x, \rho, U)\, \kappa_0^\dagger(x_2; x, \rho, U)$$

Instanton-perturbation theory

- First principle calculations of instanton contributions only possible for quantities to which large size instantons do not contribute:
  - **Short-distance coefficient functions** in the operator product expansion of two-point functions. [Andrei,Gross '78;...;Novikov *et al.* '80;...; Balitsky,Braun '93]
    Problem:
    - ∗ No physically relevant two-point function known which receives contribution solely from instantons.
    - $\longrightarrow$ Instanton contribution typically hidden beyond large perturbative background.
  - **Unique possibility:**
    Anomalous $\triangle Q_5 = \pm 2\, n_f$ processes in deep-inelastic scattering at large momentum transfer [Moch,AR,F.Schrempp '97]
    - $\longrightarrow$ In chiral limit, receive contributions only from (anti-)instantons.

Recent developments in instantons and further reading

- Instantons in high energy scattering (Plenary $\longrightarrow$ Hans-Günther Dosch):
  - Theory and phenomenology of **small-size** **QCD**-instanton induced processes in deep-inelastic scattering at **HERA**

    [AR,F.Schrempp '94–'01]
  - Possible role of **larger size instantons** (beyond instanton perturbation theory) in **diffraction scattering** in **QCD** $\Rightarrow$ needs modelling (**instanton liquid**) or information from lattice

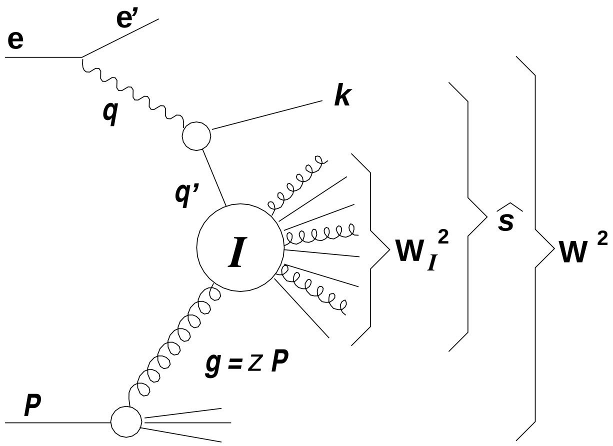
    [Shuryak *et al.* '00; Levin *et al.* '01; Schrempp '01;...]
- Instantons in $\mathcal{N} = 2$ supersymmetric gauge theories [Seiberg,Witten '94;...]

$\Rightarrow$ No review which covers all of above . . .

QCD-Instantons at HERA

[AR,F.Schrempp '94–'01]

- **Kinematics:**



Deep-inelastic scattering variables:

$S = (e + P)^2$
$Q^2 = -q^2 = -(e - e')^2$
$x_{\rm Bj} = Q^2 / \ (2P \cdot q)$
$y_{\rm Bj} = Q^2 / \ (S\, x_{\rm Bj})$
$W^2 = (q + P)^2 = Q^2(1/x_{\rm Bj} - 1)$
$\hat{s} = (q + g)^2$
$z = x_{\rm Bj}\ (1 + \hat{s}/Q^2)$

Variables of instanton-subprocess:

$Q'^2 = -q'^2 = -(q - k)^2$
$x' = Q'^2 / \ (2\ g \cdot q')$
$W_I^2 = (q' + g)^2 = Q'^2(1/x' - 1)$

- "Fiducial" kinematical region from lattice constraints: [AR,F.Schrempp '99;'01]

$$\left(\rho^*\Lambda_{\overline{\rm MS}}^{(0)} \lesssim 0.4,\ \frac{R^*}{\rho^*} \gtrsim 1.0\right) \Rightarrow \left(Q'/\Lambda_{\overline{\rm MS}}^{(n_f)} \gtrsim 30.8,\ x' \gtrsim 0.35\right)$$

QCD-Instantons at HERA

- Dedicated search by **H1**

  [H1 collab. '02]

- Based on MC QCDINS 2.0

  [AR,F.Schrempp '00]

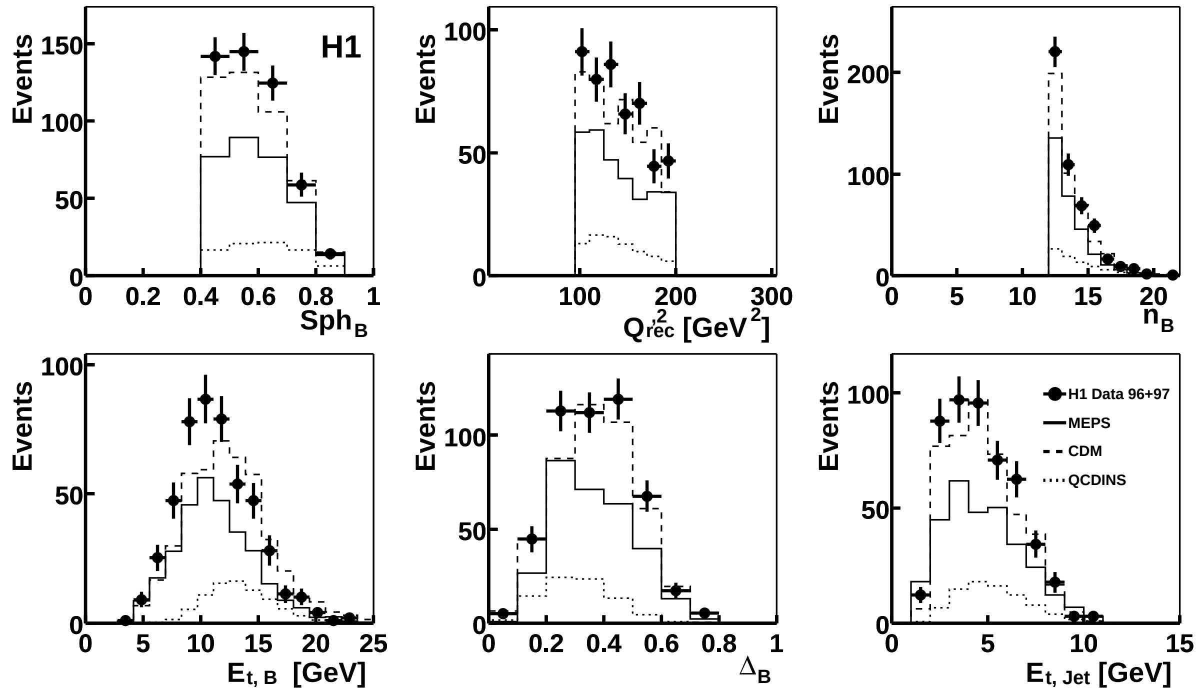
  compared to MEPS and CDM

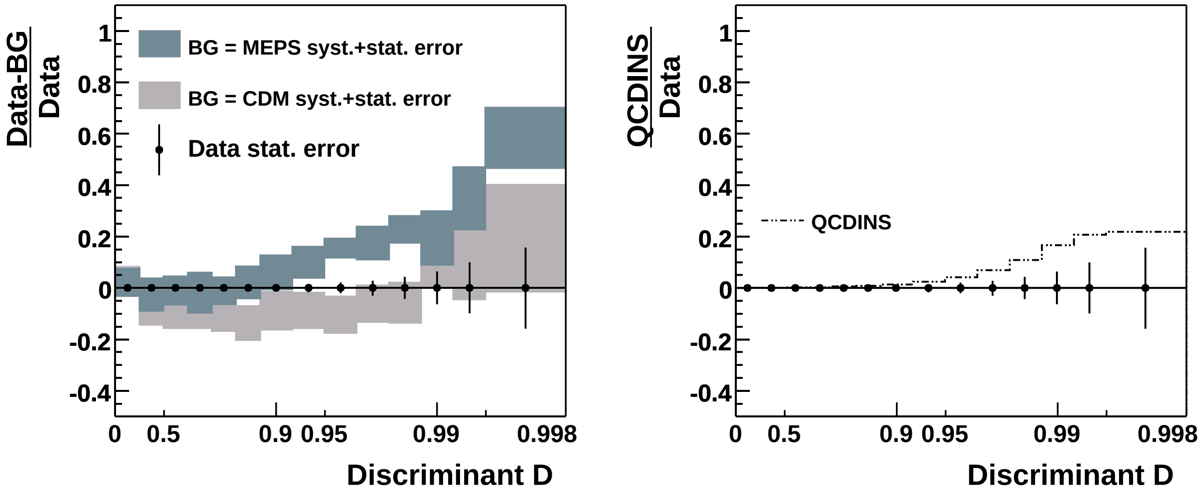
  – Excess with I-like topology, compatible with I-signal

  – Statistical significant in comparison to MEPS

  – Uncertainties in background simulations?

$\Rightarrow$ Upper limit on σ

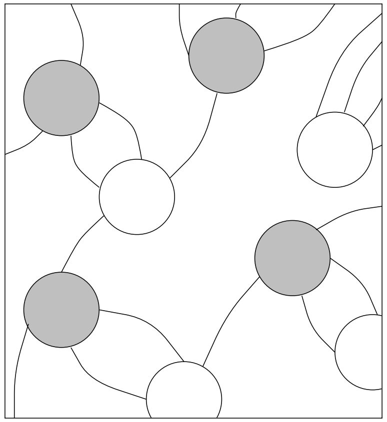




Beyond instanton-perturbation theory

- Instanton contributions to quantities which receive also contribution from larger size instantons have to be **modelled:**

$\Leftarrow$ **Instanton liquid model** [Shuryak '82; Diakonov,Petrov '86;...]



- Mainly characterized by two phenomenological parameters:

$$\langle\rho\rangle \approx 1/3 \text{ fm}\,, \qquad \frac{\mathrm{d}n}{\mathrm{d}^4x} \approx 1 \text{ fm}^{-4}$$

- **Phenomenologically** quite successful: T. Schäfer, E. Shuryak, hep-ph/9610451

5. $1/N_c$ expansion

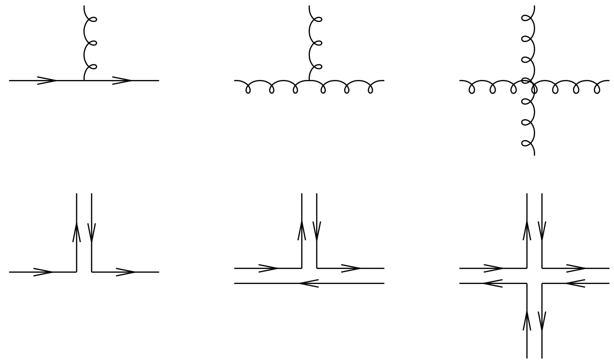
- Many features of **QCD**
  - Zweig's rule
  - Light pseudoscalar **nonet** (π, K, η, η')
  - $M_{\eta'} \gg M_\eta$

  can be **understood** by studying a gauge theory based on **$SU(N_c)$ in the limit $N_c \to \infty$ with $\alpha_s N_c$ and n_f kept fixed**. ['t Hooft '74;...;Witten '79;...]

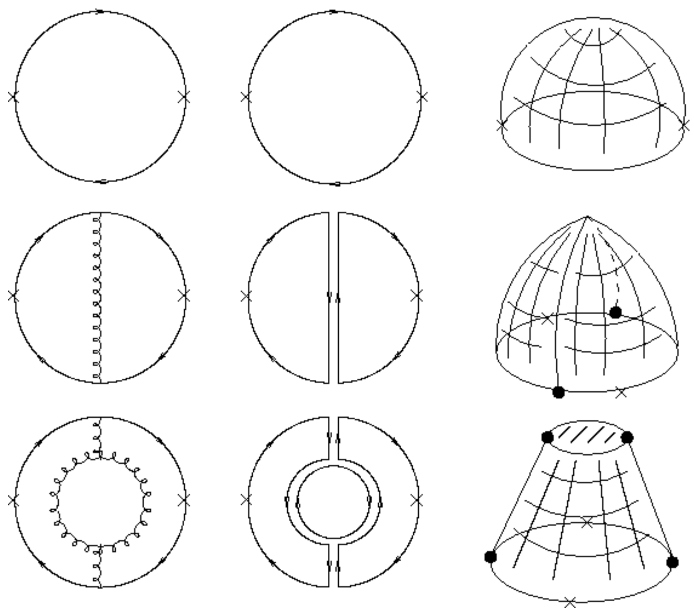
- Why useful limit?
  - $SU(N_c)$ gauge theory simplifies in this limit: planar diagrams dominant
  - $1/N_c$ expansion equivalent to a semi-classical expansion for an effective theory of color singlet mesons and baryons

Planar dominance

Double-line notation $\Rightarrow$ ['t Hooft '74]



[Lebed '98]



Large N_c power counting for $N_c \to \infty$, $\alpha_s N_c$ and n_f fixed:

- **Planar diagrams** dominate.
- Each non-planarity costs a suppression factor of $1/N_c^2$.
- Each internal quark loop costs a factor of $1/N_c$.

$\Rightarrow$ Expansion selects the **topology** of Feynman diagrams rather than their order.

$\Rightarrow$ Resembles perturbative expansion in string theory in terms of g_S

Mesons at large N_c

Only singularities in n-point function of local quark bilinears $J = \bar{q}\,\Gamma q$

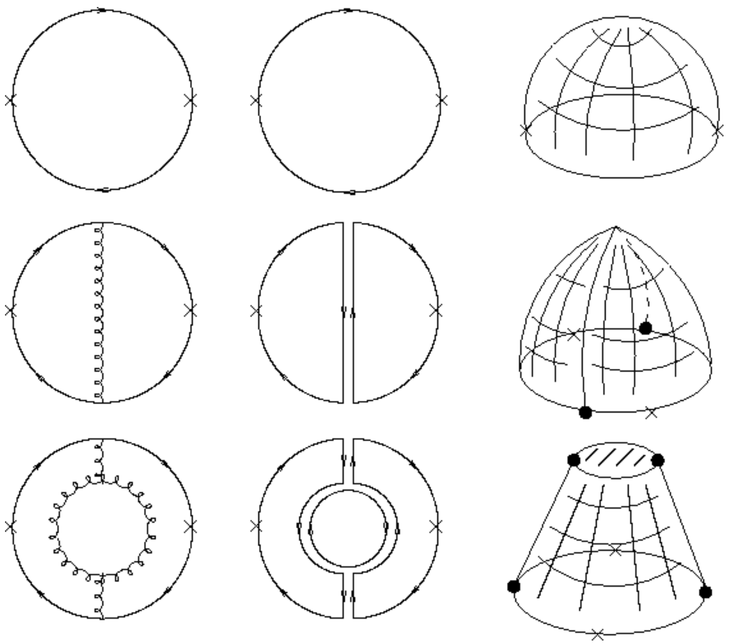
$$\langle T\,(J_1 \cdots J_n)\rangle \sim O(N_C)\,.$$

are one-meson poles; e.g.

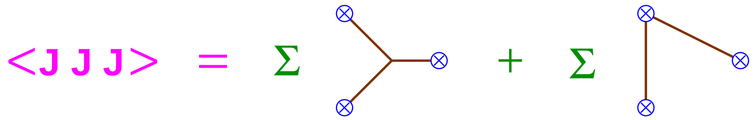
$$\langle J(k)\,J(-k)\rangle = \sum_n \frac{f_n^2}{k^2 - M_n^2}\,.$$

- $f_n = \langle 0|J|n\rangle \sim O(\sqrt{N_C}\,)$ and $M_n \sim O(1)$.
- Infinite number of meson states, since $\langle J(k)\,J(-k)\rangle \sim \log\,k^2$ for large k^2

Weakly interacting theory of mesons, glueballs (and baryons) with coupling constant $1/\sqrt{N_c}$ [Witten '79]



[Lebed '98]



[Pich '02]

Chiral perturbation theory and large N_c

- To leading order in $1/N_C$, the **$U(1)_A$ anomaly is absent.**

$\Rightarrow$ Massless **QCD** has then larger $U(n_f)_L \otimes U(n_f)_R$ chiral symmetry

- At $N_c \to \infty$, symmetry group breaks down spontaneously to $U(n_f)_V$

[Coleman,Witten '80]

$\Rightarrow$ **$n_f^2 = 3^2$ Goldstone bosons (π, K, η, η')**; conveniently collected in

$$\widetilde{U}(\phi) \equiv \exp\left\{ \mathrm{i}\sqrt{2}\,\widetilde{\Phi}/f \right\}, \qquad \widetilde{\Phi} \equiv \eta_1\, I_3/\sqrt{3} + \vec{\lambda}\,\vec{\phi}/\sqrt{2}\,.$$

$\Rightarrow$ To lowest order in chiral expansion, interactions described by

$$\mathcal{L}_2 = \frac{f^2}{4}\,\mathrm{tr}\left[D_\mu \widetilde{U}^\dagger D^\mu \widetilde{U} + \widetilde{U}^\dagger \chi + \chi^\dagger \widetilde{U} \right]$$

Chiral perturbation theory and large N_c

- Large N_c counting rules:
  - Leading-order chiral Lagrangian $\mathcal{L}_2 \sim \mathcal{O}(N_c)$ involving single trace, with $f^2 \sim \mathcal{O}(N_C)$, $\widetilde{U} \sim \chi \sim \mathcal{O}(1) \Rightarrow 1/N_c$ expansion equivalent to semi-classical expansion
  - Next-to-leading-order chiral Lagrangian $\mathcal{L}_4$: single trace terms $\mathcal{O}(N_c)$, two trace terms $\mathcal{O}(1)$

L_i	Value	Order
$2L_1 - L_2$	-0.6 ± 0.5	1
L_4	-0.3 ± 0.5	1
L_6	-0.2 ± 0.3	1
L_7	-0.4 ± 0.2	1
L_2	1.4 ± 0.3	N_c
L_3	-3.5 ± 1.1	N_c
L_5	1.4 ± 0.5	N_c
L_8	0.9 ± 0.3	N_c
L_9	6.9 ± 0.7	N_c
L_{10}	-5.5 ± 0.7	N_c

$U(1)_A$ problem and large N_c

- To leading order in $1/N_C$, the **$U(1)_A$ anomaly is absent.**
  $\Rightarrow$ Study η' as pseudo-Goldstone boson, with $1/N_c$ as $U(1)_A$ symmetry breaking parameter [Witten '79; Veneziano '79]

- To lowest non-trivial order in $1/N_C$, the chiral symmetry breaking effect induced by the $U(1)_A$ anomaly can be taken into account in the effective low-energy theory through the term [Rosenzweig *et al.* '80; Di Vechia,Veneziano '80; Witten '01]

$$\mathcal{L}_{U(1)_A} = -\frac{f^2}{4}\frac{a}{N_C}\left\{\theta - \frac{\mathrm{i}}{2}\,\mathrm{tr}\left[\ln(\widetilde{U}) - \ln(\widetilde{U}^\dagger)\right]\right\}^2 ,$$

  which breaks $U(3)_L \otimes U(3)_R$ but preserves $SU(3)_L \otimes SU(3)_R \otimes U(1)_V$;
  $a \sim \theta \sim \mathcal{O}(1)$ in large-N_c counting rules

$\Rightarrow$ η_1 field becomes massive even in the chiral limit:

$$M^2_{\eta_1} = 3\,a/N_C + O(\mathcal{M}) .$$

$U(1)_A$ problem and large N_c

$\Rightarrow$ Masses of η' and η,

$$M^2_{\eta'/\eta} = M_K^2 + \frac{3}{2}\frac{a}{N_c} \pm \frac{1}{2}\sqrt{\left(2M_K^2 - 2M_\pi^2 - \frac{a}{N_c}\right)^2 + 8\frac{a^2}{N_c^2}}$$

- Parameter a related to topological susceptibility of pure $SU(N_c)$ Yang-Mills theory,

$$\chi_{\rm t}\,|_{\rm YM} \equiv -{\rm i}\int {\rm d}^4x\,\langle T\,\nu(x)\nu(0)\rangle|_{\rm YM} = \frac{1}{2}\frac{a}{N_c}f^2$$

$$\Rightarrow \frac{6\,\chi_{\rm t}\,|_{\rm YM}}{f^2} \simeq M_{\eta'}^2 + M_\eta^2 - 2M_K^2$$

Phenomenologically successful for $\chi_{\rm t}\,|_{\rm YM} \simeq (180\ {\rm MeV})^4$; order of magnitude confirmed by lattice calculations [Teper '88;...]

Recent developments in large N_c and further reading

- Recent developments include:
  - Possible to build a consistent combined expansion in powers of momenta, quark masses and $1/N_C$, by counting the relative magnitude of these parameters as: [Leutwyler '96]

$$\mathcal{O}(p^2) \sim \mathcal{O}(\mathcal{M}) \sim \mathcal{O}(1/N_C)\,.$$

  - Information on large N_c gauge theories utilizing the AdS/CFT $\Leftrightarrow$ gravity/gauge theory correspondence [Maldacena '98;...]
    - $*$ Expectation value of Wilson loop
    - $*$ Glueball spectrum

    Existing computations correspond to strong 't Hooft coupling, $N_c \longrightarrow \infty$ with $\alpha_s N_c \longrightarrow \infty \Rightarrow$ phase transition?
  - . . .

$\Rightarrow$ Reviews: Aneesh Manohar, hep-ph/9802419; Paolo Di Vecchia, hep-th/9908148

6. Sum rules, power corrections, renormalons, and all that

- **QCD sum rules:** [Shifman,Vainshtein,Zakharov '79;...]

$\Leftrightarrow$ Method of expansion of two-point functions of local currents in terms of vacuum condensates with subsequent matching via dispersion relations

are well rooted in quantum field theory, exploiting heavily basic concepts such as

– Dispersion relations for two-point functions [Källén '52; Lehmann '54]
– Operator product expansion (**OPE**) [Wilson '69;...;Novikov *et al.* '80;...]

and are still widely used in a variety of phenomenological investigations

$\Rightarrow$ Hadronic parameters (masses, couplings) in terms of universal condensates

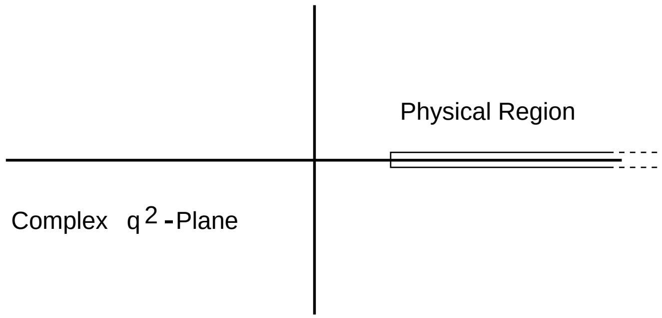
- **Infrared renormalons** intimately related to vacuum condensates; may be exploited also for investigation of power corrections to quantities which have no OPE

Dispersion relations

- Consider two-point functions of local operators

$$\Pi(q^2) = \mathrm{i} \int \mathrm{d}^4x \, e^{\mathrm{i}q\cdot x} \langle 0 | T\, J(x) J(0)^\dagger | 0 \rangle$$

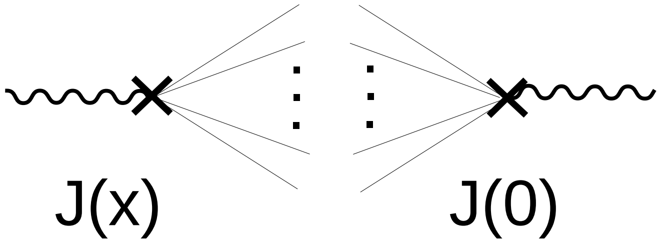
with $J(x) = \bar{q}\,\Gamma\, q,\ \mathrm{tr}\,(G^{\mu\nu}G_{\mu\nu}),\ \ldots$



- Dispersion relation from analyticity

[de Rafael '98]

$$\Pi(q^2) = \int_0^\infty \frac{\mathrm{d}s}{s - q^2 - \mathrm{i}\epsilon} \frac{1}{\pi} \mathrm{Im}\Pi(s) + \sum_{n=0}^{N} \mathrm{a}_n \, (q^2)^n$$



- **Spectral density $\rho(s) \equiv (1/\pi)\mathrm{Im}\Pi(s)$** proportional to **total cross-section**

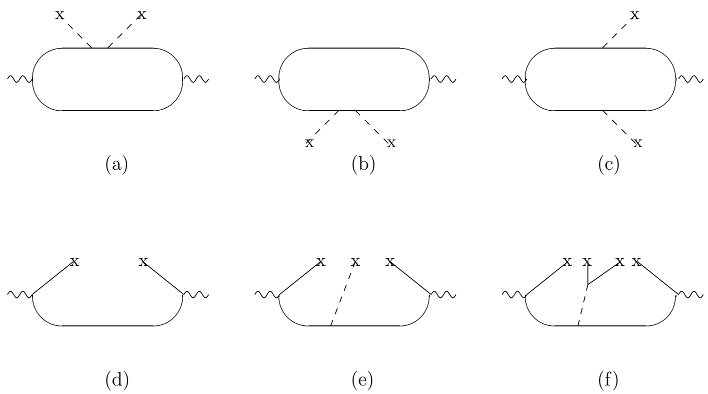
Operator product expansion

- For large space-like momenta, $Q^2 = -q^2$, the operator product expansion for the product of currents $J(x)J(0)$ leads to

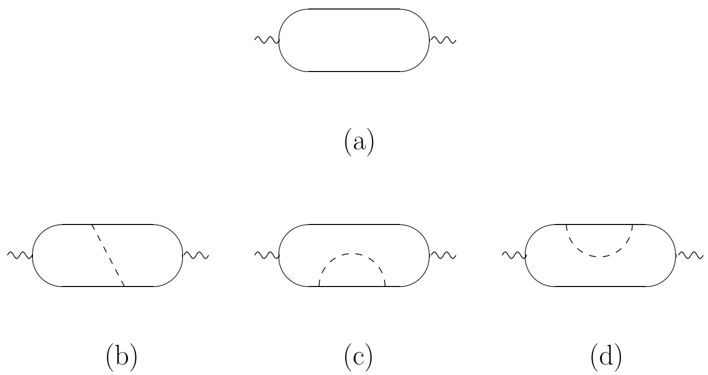
$$\Pi(Q^2) = Q^{\dim J} \sum_d \frac{C_d(Q,\mu)}{Q^d} \langle 0|O_d|0\rangle(\mu)$$

  - Universal matrix elements of local operators O_d, ordered by their dimension, $O_0 = 1$, $O_3 = \bar{q}q$, $O_4 = G^a_{\mu\nu}G^a_{\mu\nu}$, ...
  - Short-distance coefficients, $C_d(Q,\mu)$, calculable for $Q \to \infty$, due to asymptotic freedom of **QCD**, e.g.

$$C_d(Q,\mu) = A_d\,[1 + b_d\alpha_s(Q) + \ldots]$$



[Khodjamirian '02]



Matching

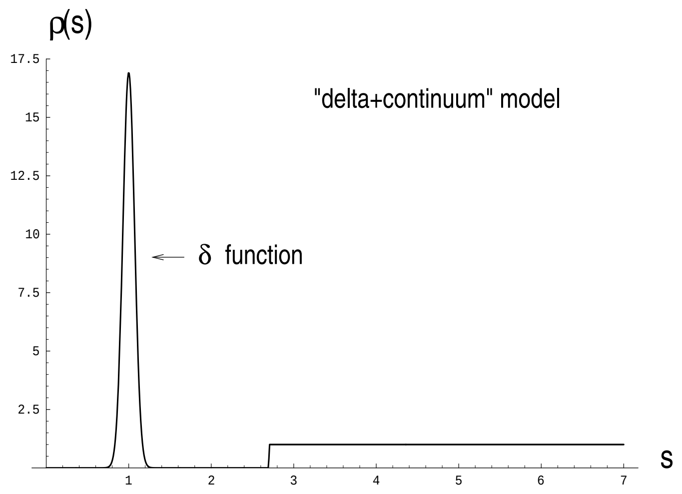
- Equate the dispersion and the operator product expression, $\Rightarrow$ **master equation**

$$\int_0^\infty \frac{\mathrm{d}s}{s+Q^2-\mathrm{i}\epsilon}\rho(s)+\ldots$$

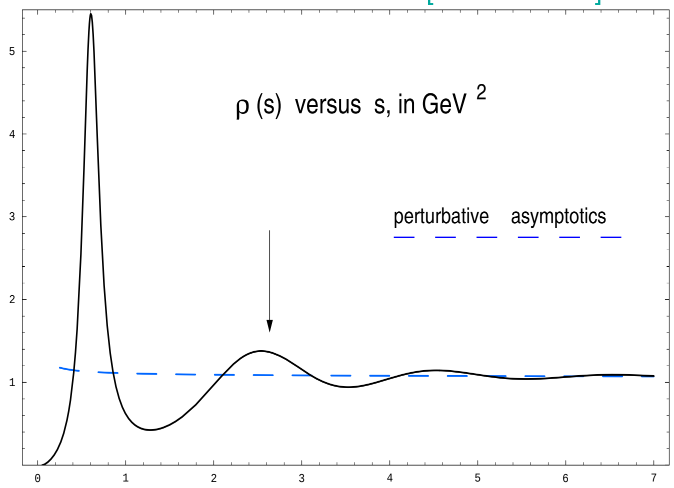
$$\stackrel{Q^2\to\infty}{=} Q^{\dim J}\sum_d \frac{C_d(Q,\mu)}{Q^d}\langle 0|O_d|0\rangle(\mu)$$

- Need experimental and theoretical information about spectral density $\rho(s)$ $\Rightarrow$ **model dependence**

- In deep Euclidean region ($Q\to\infty$) unit operator O_0 dominant, but to approach physical region, need more and more **power corrections** of type $\langle O_d\rangle/Q^d$

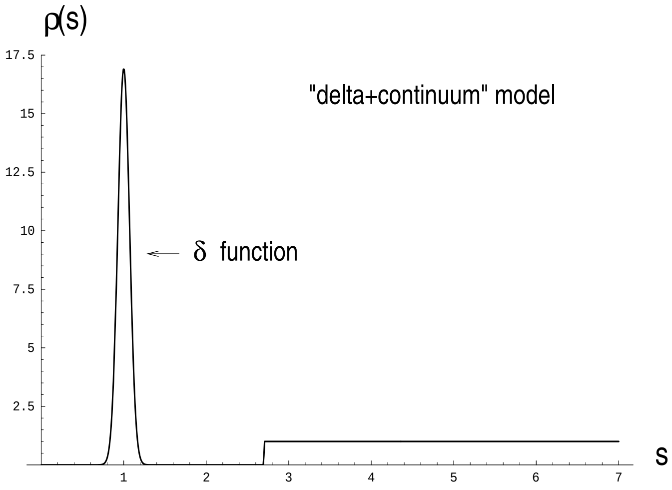


[Shifman '98]



In praxis:

- General steps:
  - Choose J and write dispersion relation
  - Model spectral density $\rho(s)$
  - Employ OPE including all O_d up to $d \leq 6$
  - Calculate C_d as expansion in α_s
  - Employ moment or transform techniques
  - Vary parameters until stability of output

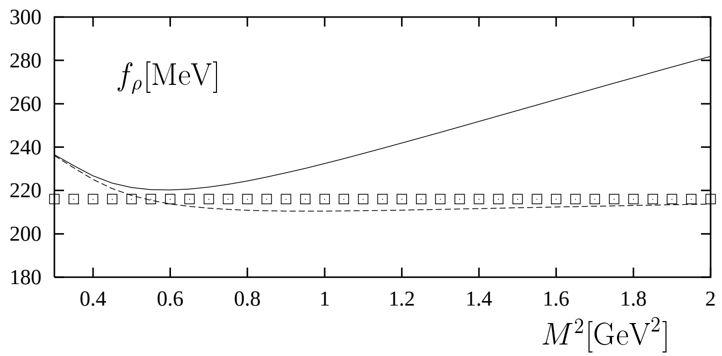


[Shifman '98]

- E.g.: Borel transform of master equation for current with ρ quantum numbers

$$f_\rho^2 e^{-m_\rho^2/M^2} + \{\text{excited,continuum } \rho \text{ states}\}$$

$$= M^2 \left[\frac{1}{4\pi^2} \left(1 + \frac{\alpha_s(M)}{\pi} \right) + \frac{(m_u + m_d)\langle \bar{q} q \rangle}{M^4} + \frac{1}{12} \frac{\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^{a\mu\nu} \rangle}{M^4} - \frac{112\pi}{81} \frac{\alpha_s \langle \bar{q} q \rangle^2}{M^6} \right],$$



[Khodjamirian '02]

Condensates:

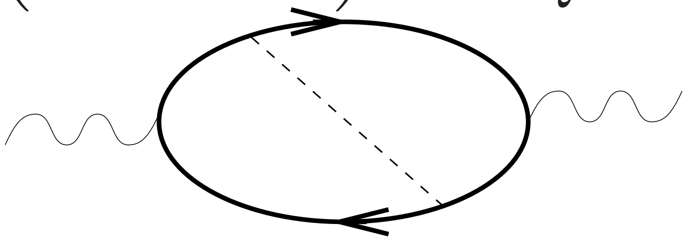
- All condensates, i.e. vacuum expectation values of local operators, should be calculable in principle from lattice gauge theory once the renormalization prescriptions are specified.
- At present, only theoretically determined combinations are the products $(m_u + m_d)\langle \bar{u}u + \bar{d}d\rangle_0 \simeq -2\, M_\pi^2 f_\pi^2$, which follow from lowest-order chiral perturbation theory. Sum rule calculations often adopt the value
$$\langle \bar{q}q\rangle_0(\mu = 1\ \mathrm{GeV}) = -(225 \pm 25\ \mathrm{MeV})^3$$
- Gluon condensate from sum rule analysis of charmonium system,
$$\langle \frac{\alpha_s}{\pi} G^a_{\mu\nu} G^a_{\mu\nu}\rangle_0 = (0.024 \pm .011)\ \mathrm{GeV}^4$$

⇒ Review, *e.g.*, Mikhail Shifman, hep-ph/9802214

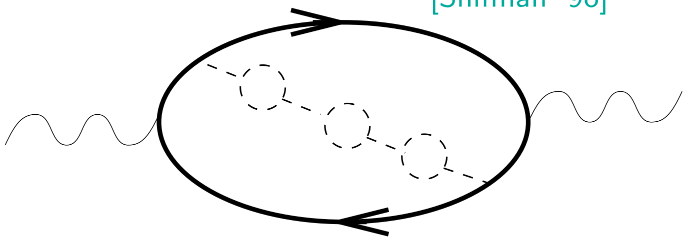
Renormalons and power corrections

- Implicitely assumed that contributions to coefficient functions from soft fluctuations $k^2 < \mu^2$ have been subtracted, e.g. Adler function $D(Q^2) = -(4\pi^2)Q^2(\mathrm{d}\Pi/\mathrm{d}Q^2)$,

$$D(Q^2) = \left[1 + \left(\frac{\alpha_s(Q)}{\pi} - \frac{\alpha_s}{\pi}\frac{\mu^4}{Q^4}\right) + \mathcal{O}(\alpha_s^2)\right] + \left(\frac{2\pi^2}{3} + \mathcal{O}(\alpha_s)\right)\frac{\langle\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^a_{\mu\nu}\rangle(\mu)}{Q^4} + \ldots$$



[Shifman '98]



- Coefficient functions and condensates satisfy renormalization group equation such that physical quantity $D(Q^2)$ independent of μ. In praxi: $Q^2 \gg \mu^2 \gg \Lambda^2$.

- Without infrared cutoff μ, series expansion of coefficient functions diverges factorially: **infrared renormalon** ['t Hooft '77;..; Mueller '85;..;]

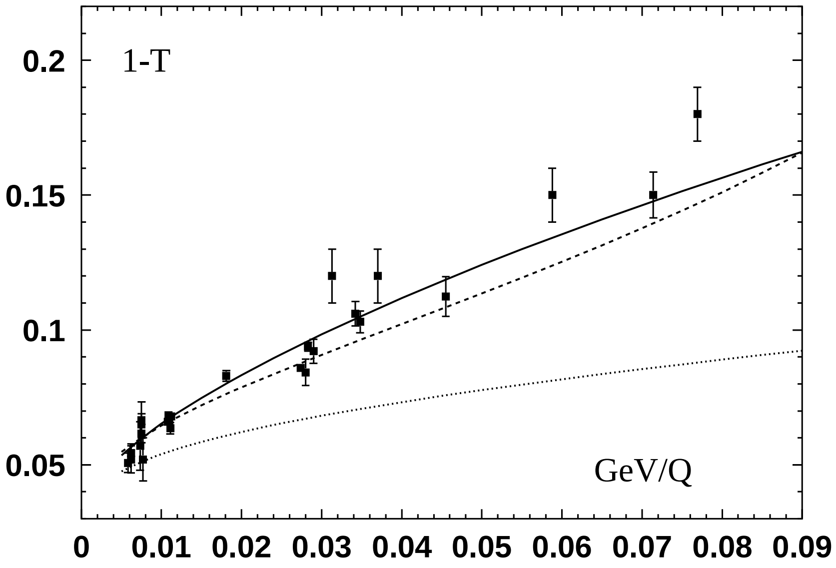
Renormalons and power corrections

- Quite general: Even for short-distance dominated observables, **QCD** perturbation expansion never complete, most importantly due to infrared renormalons.

- Also for infrared safe quantities without an OPE, e.g. hadronic event shape variables or in heavy quark physics, an infrared renormalon analysis, i.e. analysis of the IR cutoff dependence of Feynman amplitudes, can give information on **non-perturbative power corrections**. [Mueller '85; Beneke,Braun '94; Bigi *et al.* '94; Webber '94; Manohar,Wise '95;...]

$\Rightarrow$ Review of rapidly evolving field:
M. Beneke and V. Braun, hep-ph/0010208

From infrared renormalon analysis $\Rightarrow$ Power correction in $\langle 1 - T \rangle$ is $1/Q$; confirmed by experiment:



[Beneke,Braun '00]

7. Summary

- Reviewed powerful **analytical** methods and tools to handle **QCD** beyond the ordinary perturbative treatment:
  - chiral perturbation theory
  - instantons
  - $1/N_c$ expansion
  - sum rules, power corrections, renormalons, and all that
- These tools can be backed up with **numerical** information to be gained with lattice simulations (Lecture → Karl Jansen)
- Left out: Approaches based on **Schwinger-Dyson equations** (truncations → model dependence), **discretized light-cone quantization of QCD (DLQCD)** (not an analytic method), **model of the stochastic vacuum** (Plenary → Hans-Günther Dosch), . . .