Results from simulations employing lattice chiral symmetry

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Chiral symmetry and the lattice

- Massless fermions:

$$S_F = a^4 \sum_{x,y} \overline{\psi}(x)\, D(x-y)\psi(y)$$

(a) $D(x-y)$ is local
(b) $\widetilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$
(c) $\widetilde{D}(p)$ invertible for $p \neq 0$
(d) $\gamma_5 D + D\gamma_5 = 0$

$\rightarrow$ Nielsen & Ninomiya: (a)–(d) do not hold simultaneously

$\rightarrow$ either left with doublers or chiral symmetry broken explicitly

- Staggered (Kogut-Susskind) fermions
  - U(1) subgroup of SU(N_f)$\times$SU(N_f) unbroken
  - degeneracy not fully lifted

- Wilson fermions
  - degeneracy fully lifted; chiral symmetry broken
  - leading cutoff effects of order a

- Exact chiral symmetry at non-zero a:
  Replace "(d)" by

$$\boxed{\gamma_5 D + D\gamma_5 = aD\gamma_5 D}$$

Ginsparg-Wilson relation (1982)

- Implies an exact global symmetry (Lüscher 1998)

$$\delta\psi = \underbrace{\gamma_5\,(1-aD)}_{\hat{\gamma}_5}\,\psi, \qquad \delta\overline{\psi} = \overline{\psi}\gamma_5$$

  Associated Ward Identities imply correct anomaly in flavour-singlet case

  Action can be split into left- and right-handed parts

- Realisations of Ginsparg-Wilson relation:
  - Domain Wall Fermions (Kaplan, Shamir & Furman)
  - Overlap (Narayanan & Neuberger, Neuberger)
  - Fixed point action
    (Hasenfratz, Niedermayer, Laliena, Bietenholz, Wiese)
  - "Chirally improved" fermions
    (Gattringer, Gattringer, Lang & Hip)

Chiral symmetry breaking may be tolerated in many applications . . .

- In which situations is lattice chiral symmetry necessary?
- Are simulations employing chirally symmetric fermions competitive with Wilson/staggered quarks?

Outline:

I. Properties & technical issues

II. Results:

- Mass of the strange quark
- Quark condensate
- Kaon weak matrix elements

III. Summary

I. Properties & technical issues

Advantages:

+ Separation of chiral and continuum limits

+ Leading discretisation errors of order a^2

+ Mixing patterns of local composite operators like in the continuum, e.g.

$$O^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d]\,[\bar{s}\gamma_\mu(1-\gamma_5)d]$$

[Wilson fermions: mixing with opposite/mixed chiralities occur]

+ Exact index theorem on the lattice

Disadvantages:

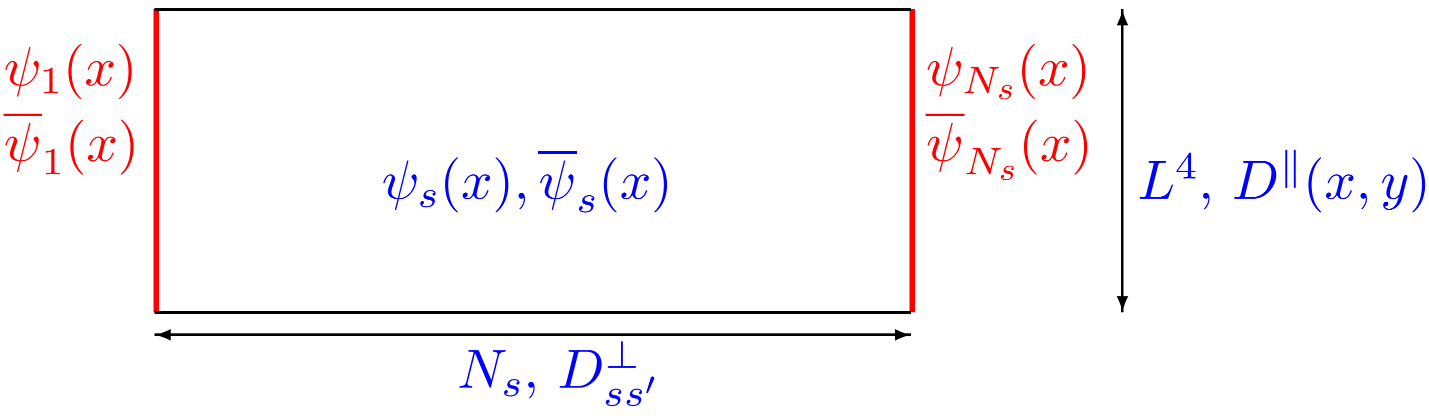
– Expensive to simulate

– "Lattice technology" more difficult to implement (e.g. non-perturbative renormalisation of local operators)

Realisations of Ginsparg-Wilson fermions

Domain Wall Fermions (Shamir & Furman 1993)

- Introduce discrete extra dimension:



$$D^{\mathrm{DWF}}_{ss'}(x,y) = D^{\parallel}(x,y)\delta_{ss'} + \delta(x-y)D^{\perp}_{ss'}$$

$D^{\parallel}(x,y)$: Wilson-Dirac operator with mass $-M$.
$D^{\perp}_{ss'}$: contains Dirac mass m.

- For $m = 0,\ N_s \to \infty$:
  - no fermion doublers
  - chiral modes are trapped in 4D domain walls

- In practice: work at finite N_s

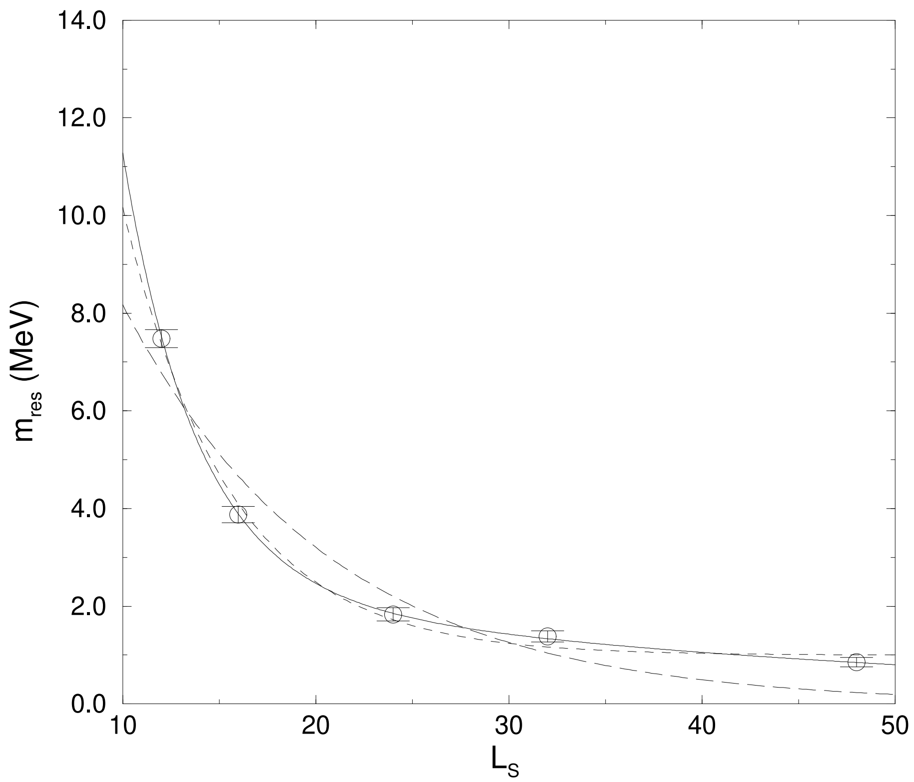
$\longrightarrow$ decoupling of chiral modes not exact:

$$\partial_\mu A_\mu(x) = 2mP(x) + 2J_5(x)$$

Anomalous term in Ward Identity gives rise to residual quark mass at vanishing m_π^2 :

$\longrightarrow$ measure of residual chiral symmetry breaking

(RBC Collab., hep-lat/0007038)



- consistent with exponential fall-off;
  accelerate convergence using different gauge action;
  Residual quark mass:

$$m_{\mathrm{res}} \lesssim 1 - 3\,\mathrm{MeV}$$

- Can realise (almost exact) chiral at the expense of simulating 5D theory

Overlap Fermions

- Reinterpretation of domain wall proposal:
  Overlap (Neuberger's) operator (Neuberger 1998)

  $$aD_{\rm ov} = (1+s)\left[1 - A(A^\dagger A)^{-1/2}\right], \quad A = 1+s-aD_{\rm W}$$

  $D_{\rm W}$: massless Wilson-Dirac operator

- $D_{\rm ov}$ satisfies Ginsparg-Wilson relation

- Can show that $D_{\rm ov}$ is local:

  $$||D_{\rm ov}(x,y)\,\phi(y)|| \propto {\rm e}^{-\gamma|x-y|}$$

  (Hernández, Lüscher, Jansen 1998)

- Construct numerical approximation of $1/\sqrt{A^\dagger A}$ for every appplication of $D_{\rm ov}$:

  – Chebyshev polynomials:

  $$P_{n,\epsilon}(x) \approx 1/\sqrt{x}, \quad x \in [\epsilon, 1]$$

  $n = {\rm O}(200)$ in typical simulations

  – Rational polynomial approximation; continued fraction
  – MinMax polynomials

Fixed point (FP) action

(Hasenfratz, Niedermayer, Laliena, Bietenholz, Wiese,...)

- Constructed from renormalisation group properties:
  - add irrelevant operators to the lattice Dirac operator
  - fix their coefficients via FP equation
  - exact FP Dirac operator satisfies Ginsparg-Wilson relation

- In practice: use approximate FP Dirac operator D_{FP}
  $\longrightarrow$ residual chiral symmetry breaking

- Can replace D_{W} by D_{FP} in the definition of the overlap operator;
  $\longrightarrow$ reduce degree n in polynomial approximation?
  net gain?

Chirally improved fermions

(Gattringer; Gattringer, Lang & Hip)

- Irrelevant couplings in D_{CI} fixed through Ginsparg-Wilson relation

- In practice: use approximate solution

- Similar properties as FP action

II. Results

The mass of the strange quark

$$F_{\rm K} m_{\rm K}^2 = (m_{\rm s} + \widehat{m}) \underbrace{\langle 0 \, |\overline{s}\gamma_5 d| \, K\rangle}_{G_{\rm K}}$$

$$\frac{(m_{\rm s} + \widehat{m})_{\rm R}}{Q} = Z_{\rm R} \left(\frac{F_{\rm PS} Q}{G_{\rm PS}}\right)^{\rm bare}_{\rm lat} \Bigg|_{m_{\rm PS}=m_{\rm K}} \left(\frac{m_{\rm K}}{Q}\right)^2_{\rm exp} + O(a^p)$$

Q : lattice scale $(= m_\rho, m_{K^*}, F_\pi, \ldots)$

$Z_{\rm R}$: renormalisation factor

Condition $m_{\rm PS} = m_{\rm K}$ fixes bare strange quark mass

$\longrightarrow$ compute renormalised matrix elements of A_μ and P

Typical results in continuum limit (quenched approx.!):

$$m_{\rm s}^{\overline{\rm MS}}(2\,{\rm GeV}) = \ 97 \pm 4\,{\rm MeV},$$

ALPHA/UKQCD 1999
Wilson fermions

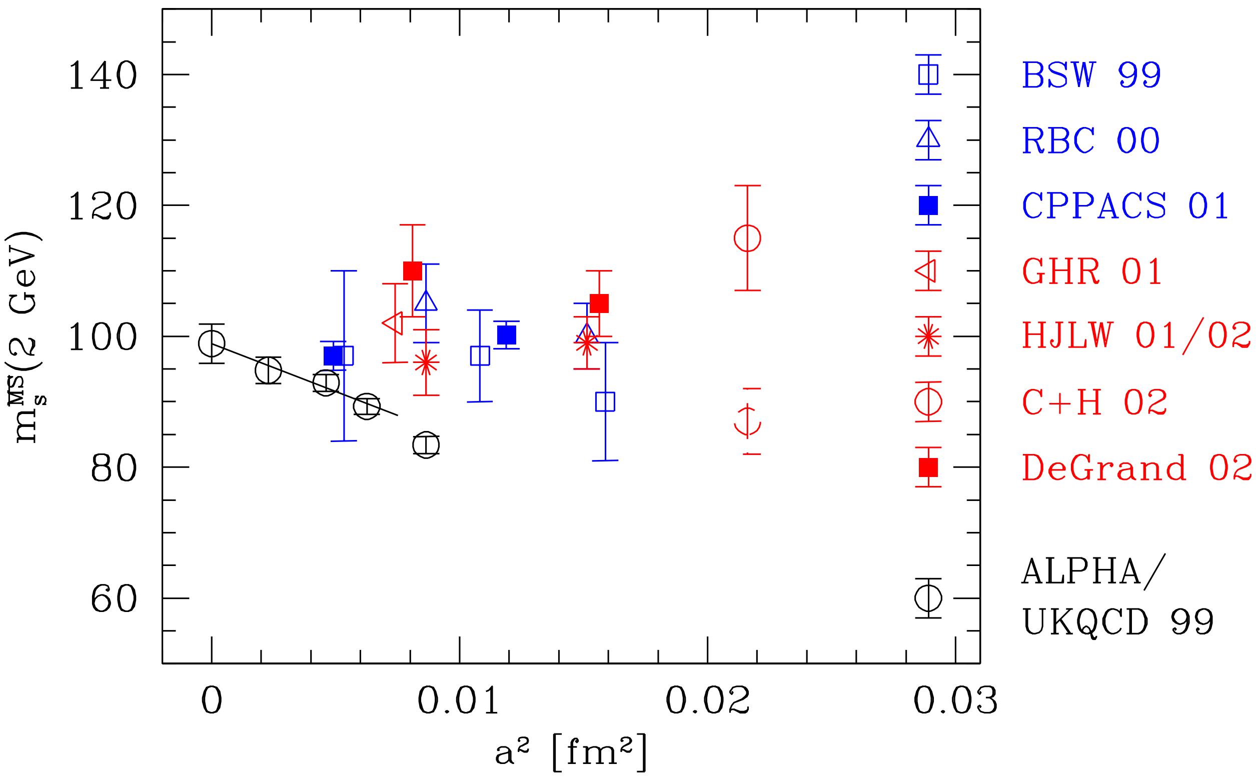
$$m_{\rm s}^{\overline{\rm MS}}(2\,{\rm GeV}) = 106 \pm 7\,{\rm MeV},$$

JLQCD 1999
KS fermions

$\longrightarrow$ Discretisation & renormalisation effects controlled at 5% level

Collab.	year	action	a [fm]	Ren.	m_s [MeV]
RBC	2002	DWF	≥ 0.099	RI	~ 130
CP-PACS	2001	DWF	≥ 0.066	PT	99(2)(6)
RBC	2000	DWF	0.093 0.123	RI	105(6)(21) 100(5)(20)
BSW	1999	DWF	≥ 0.073	PT	96(26)
HJLW	2002	Ovlp	0.093 0.123	SF	96(5)(7) 99(4)(7)
C+H	2002	Ovlp	0.147	PT	115(8)
DeGrand	2002	Ovlp	0.090 0.125	PT	110(7) 105(7)
GHR	2001	Ovlp	0.086	RI	102(6)(18)

($\overline{\mathrm{MS}}$-scheme at $\mu = 2\,\mathrm{GeV}$)



The quark condensate

- Spontaneous chiral symmetry breaking

$$\langle\overline{\psi}\psi\rangle \neq 0$$

- Chiral Perturbation Theory: identify $\langle\overline{\psi}\psi\rangle$ with the low-energy constant $\Sigma \equiv B_0 F_0^2$:

$$\mathcal{L}_{\mathsf{eff}}^{(2)} = \tfrac{1}{4}F_0^2\,\mathrm{Tr}\,\left(\partial_\mu U^\dagger \partial^\mu U\right) + \tfrac{1}{2}B_0 F_0^2\,\mathrm{Tr}\,\left(\mathcal{M}(U+U^\dagger)\right)$$

Gell-Mann–Oakes–Renner:

$$\lim_{m\to 0}\frac{F_{\mathrm{PS}}^2 m_{\mathrm{PS}}^2}{2m} = \Sigma$$

- Basis for lattice determinations:

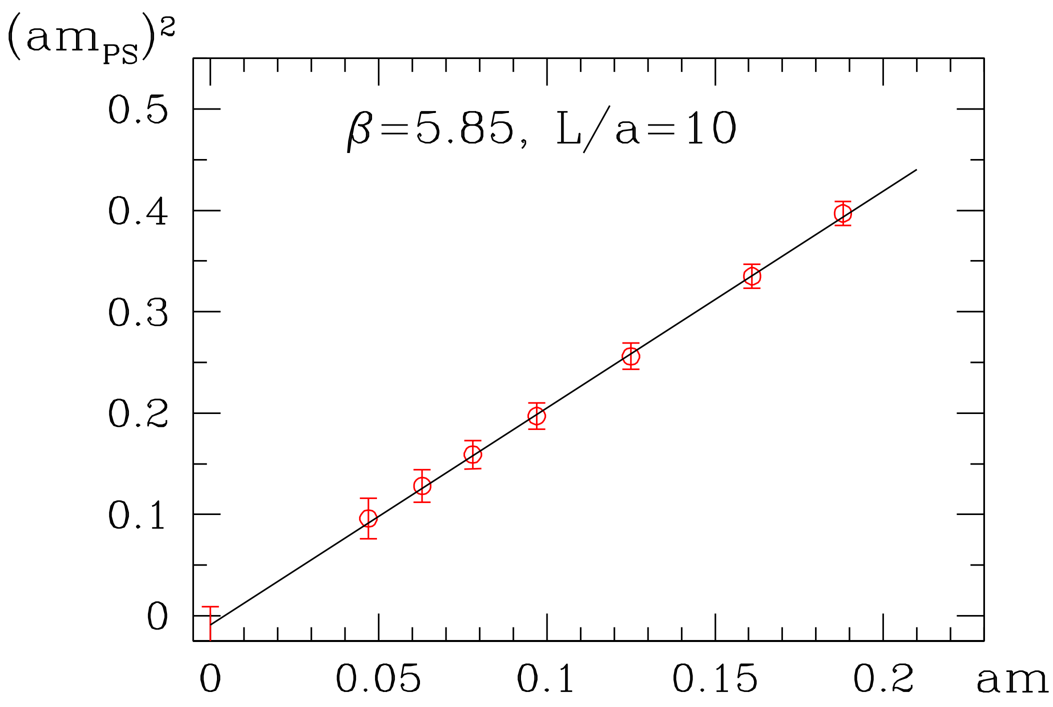
$$\frac{m_{\mathrm{PS}}^2}{2m} = \frac{\Sigma}{F_0^2}\left\{1 + \text{chiral logs} + \mathrm{O}(m)\right\}$$

$\rightarrow$ Map out dependence of m_{PS}^2 on m; compare with ChPT;

not justified for non-zero a if chiral symmetry broken explicitly

Hernández, Jansen, Lellouch, HW, JHEP 0107 (2001) 018

- Overlap fermions



- LO behaviour well satisfied in the studied mass range;
  $\longrightarrow$ Determine Σ/F_0^2 from slope

- Including NLO contributions does not change Σ/F_0^2 significantly;

$\longrightarrow$ strong cancellation between chiral log's and analytic terms at NLO

Alternative approach:

- Consider theory in finite volume: $m_\pi L \ll 1$
- For $F_\pi L \gg 1$ theory is described by usual chiral Lagrangian
  reorganise chiral expansion $\longrightarrow$ "ϵ-expansion"
- Quark condensate for fixed topology ν:
  (Damgaard, Osborn, Toublan, Verbaarschot 1999)

$$\frac{\Sigma_\nu}{\Sigma'} = z\Big(I_\nu(z)K_\nu(z) + I_{\nu+1}(z)K_{\nu-1}(z)\Big) + \frac{\nu}{z}$$

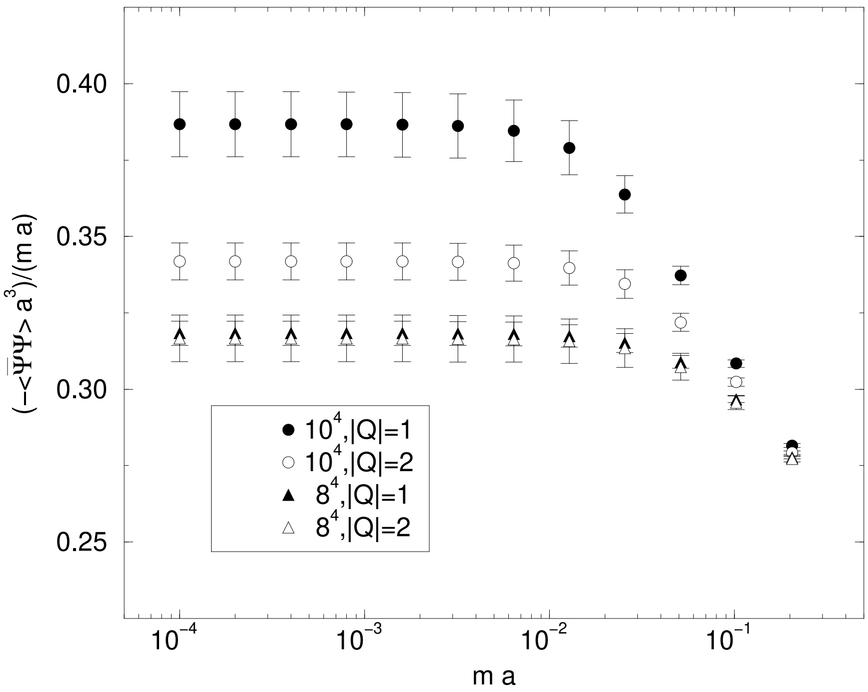
$$z = m\Sigma' V$$

$$\Sigma' = \Sigma\,\{1 + \text{chiral logs} + \ldots\}$$

$\longrightarrow$ Extract Σ via Finite-size scaling (FSS) analysis

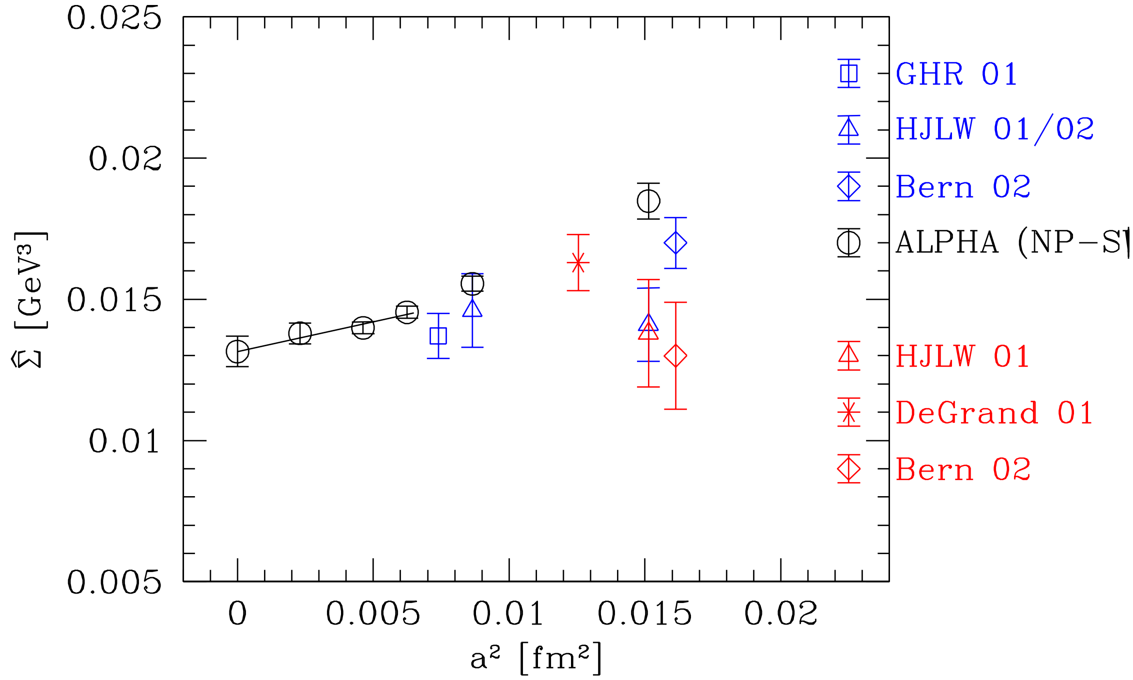
- Consistency of results obtained using GMOR or FSS?

Hasenfratz et al., hep-lat/0205010



- Bare condensate needs to be renormalised;

  Renormalisation factors computed non-perturbatively by
  Hernández, Jansen Lellouch, HW, JHEP 0107 (2001) 018
  Giusti, Hoelbling, Rebbi, Phys Rev D64 (2001) 114508



- Condensate determined via FSS and GMOR agree

  $\longrightarrow$ consistent description of regimes ϵ/p-expansion

- Exact chiral symmetry leads to conceptually clean determination of Σ

- Typical result:

$$\Sigma_{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = (267 \pm 38 \pm 24\,\mathrm{MeV})^3$$

(assume $\Sigma_{N_\mathrm{f}=0} \approx \Sigma_{N_\mathrm{f}=3} \equiv -\left\langle\overline{\psi}\psi\right\rangle$)

Kaon weak matrix elements

- $K^0 - \overline{K}^0$ mixing:
  Kaon B-parameter B_K : non-perturbative contribution to indirect CP violation:

$$B_K = \frac{\left\langle \overline{K}^0 \left| O^{\Delta S=2} \right| K^0 \right\rangle}{\frac{16}{3} F_{\rm K}^2 m_{\rm K}^2}$$

$$\begin{aligned} O^{\Delta S=2} &= [\bar{s}\gamma_\mu(1-\gamma_5)d]\,[\bar{s}\gamma_\mu(1-\gamma_5)d] \\ &= O^{\Delta S=2}_{\rm VV+AA} - O^{\Delta S=2}_{\rm VA+AV} \end{aligned}$$

Wilson fermions:

- $O^{\Delta S=2}_{\rm VV+AA}$ mixes with other chiralities:

$$O^{\Delta S=2}_{\rm VV+AA;\,R} = Z \left\{ O^{\Delta S=2}_{\rm VV+AA} + \sum_{i=1}^{4} c_i O_i^{\Delta S=2} \right\}^{\rm lat}$$

Staggered fermions:

- Construction of $O^{\Delta S=2}_{\rm VV+AA}$ complicated:
  spin-flavour structure only recovered in continuum limit

- Complicated continuum extrapolation due to large cutoff effects

Lattice QCD with exact chiral symmetry

- No mixing problem;
  Construction of operator with correct chiral properties:

$$\delta\psi = \lambda^a \gamma_5 (1 - aD)\psi, \qquad \delta\overline{\psi} = \overline{\psi}\gamma_5 \lambda^a$$

$\longrightarrow$ 4-quark operators do not form simple multiplets

- Define

$$\tilde{\psi} = (1 - \tfrac{1}{2} aD)\psi \;\Rightarrow\; \delta\tilde{\psi} = \lambda^a \gamma_5 \tilde{\psi}.$$

$\longrightarrow$ 4-quark operators constructed from $\tilde{\psi}$, $\overline{\psi}$ transform like operators in the continuum, e.g.

$$O^{\Delta S=2} = \left[\overline{s}\gamma_\mu (1-\gamma_5)\tilde{d}\right] \left[\overline{s}\gamma_\mu (1-\gamma_5)\tilde{d}\right]$$

Also. . .

QCD with a chirally twisted mass term
(ALPHA Collab.: Frezzotti, Guagnelli, Heitger, Pena, Sint, Vladikas, Weisz, 2000-2002)

- Mixing problem absent even for Wilson fermions:

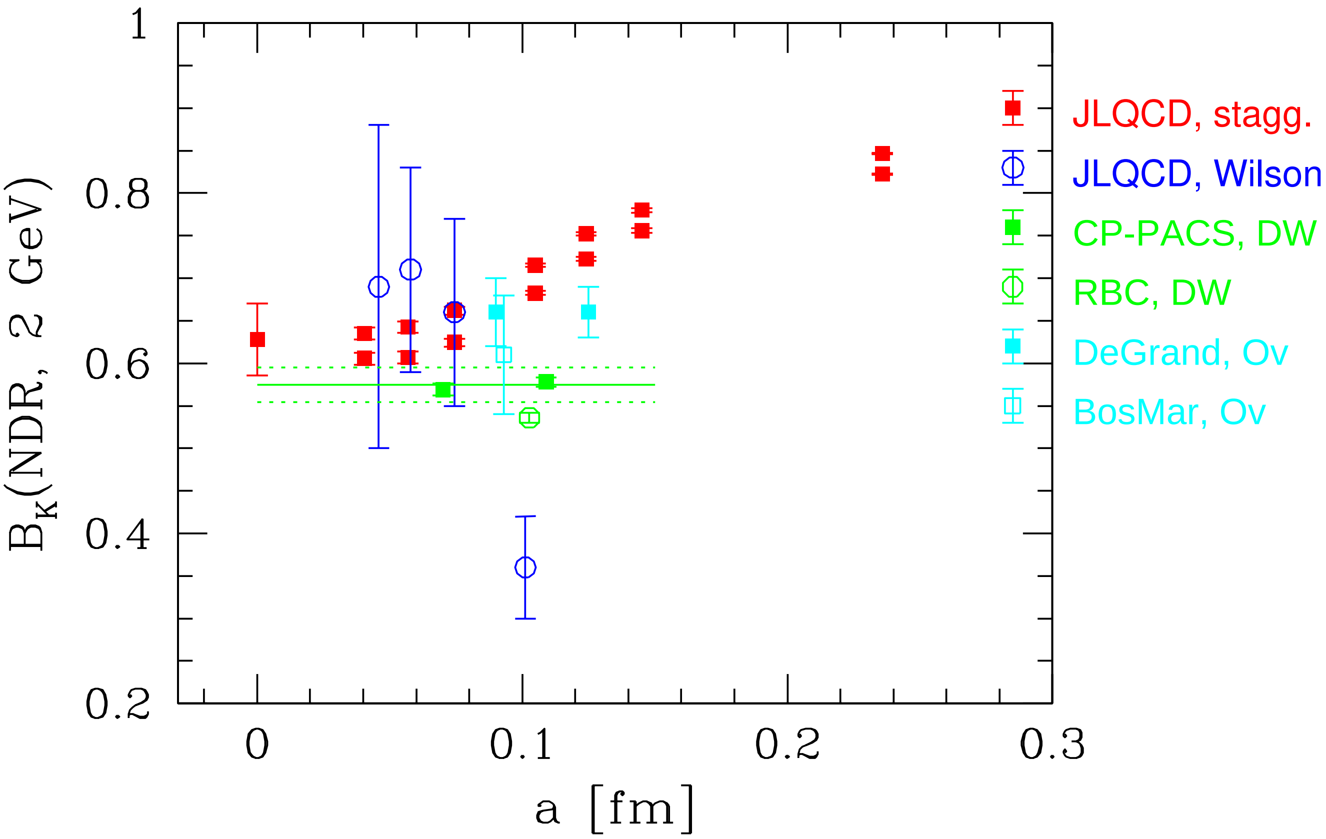
$$\left\langle \overline{K}^0 \left| O^{\Delta S=2}_{\mathrm{VA+AV}} \right| K^0 \right\rangle_{\mathrm{tmQCD}} = i \left\langle \overline{K}^0 \left| O^{\Delta S=2}_{\mathrm{VV+AA}} \right| K^0 \right\rangle_{\mathrm{QCD}}$$

$O^{\Delta S=2}_{\mathrm{VA+AV}}$: renormalised multiplicatively

- Results to be published

Collab.	year	action	a [fm]	Ren.	B_K
JLQCD	1997	KS	0	PT	0.628(42)
APE	1997	W	0.07	RI	0.66(11)
JLQCD	1999	W	"0"	RI/WI	0.68(7)
CP-PACS	2001	DWF	$\gtrsim 0.07$	PT	0.575(6)(19)
RBC	2001	DWF	0.1	RI	0.532(11)
DeGrand	2002	Ovlp	$\gtrsim 0.09$	PT	0.66(4)
BosMar	2002	Ovlp	0.1	RI	0.61(7)

$\overline{\mathrm{MS}}$,NDR-scheme at $\mu = 2\,\mathrm{GeV}$



$K \to \pi\pi$

- A lot of recent activity:
  - numerical & conceptual/analytical
  - lattice actions with and without exact chiral symmetry

  $\longrightarrow$ talk by Chris Sachrajda

- Study $K \to \pi\pi$ via $K \to \pi$ and ChPT

  Lattice chiral symmetry:
  - continuum-like transformation properties of 4-quark operators
  - application of ChPT justified
  - study $K \to \pi$ in finite volume
    (ϵ-expansion – c.f. quark condensate)
    `(Giusti, Hernández, Hoelbling, Jansen, Laine, Lellouch, Lüscher, Weisz, HW -- in progress)`

III. Summary

- Lattice chiral symmetry offers many advantages for QCD simulations:
  - small lattice artefacts (order a^2)
  - simplified construction of operators with correct transformation properties under $\mathrm{SU}(N_\mathrm{f})_L \times \mathrm{SU}(N_\mathrm{f})_R$
  - clean matching of lattice results to ChPT at non-zero lattice spacing

- Results for m_s, B_K are becoming competitive with conventional lattice fermions in the quenched approximation
  - verify current benchmark calculations
  - careful investigations of systematics still required: scaling behaviour, renormalisation effects, . . .
  - simulations with dynamical quarks: conventional discretisations will dominate for some time

- Lattice chiral symmetry promises new ways to tackle non-leptonic decays $(\Delta I = 1/2,\ \epsilon'/\epsilon)$;

  await further results . . .