

ORBIFOLDS & BRANES

DESY

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1. Orbifolds

Suppose we are given some closed
string theory describing strings propagating
on manifold \mathcal{M} . Let Γ be a group of
symmetries acting on \mathcal{M} . We want to construct
(closed) string theory on

$$\mathcal{M} / \Gamma \quad \text{orbifold}$$

In the following we shall mainly consider the case where Γ is abelian.

The simplest (interesting) example is

$$\mathcal{M}/\Gamma = S^1/\mathbb{Z}_2$$

where \mathbb{Z}_2 acts either by

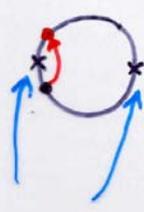
$$x \mapsto x + \pi R$$



half-shift

or by

$$x \mapsto -x$$



inversion

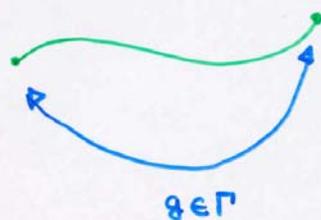
2 fixed points

The main idea of the orbifold construction is that string theory on \mathcal{M}/Γ can be described in terms of the original string theory on \mathcal{M} . This construction proceeds in two steps:

- ① Untwisted sector: restrict space of states of original theory to those that are **invariant under Γ** . (Geometrically these states describe closed strings on the covering space \mathcal{M} that are invariant under Γ and therefore describe closed strings on \mathcal{M}/Γ .)

② Twisted sectors: there are additional

closed strings in \mathcal{M}/Γ that do not
come from closed strings in \mathcal{M} :



This string is open in \mathcal{M} , but closed
in \mathcal{M}/Γ since the two endpoints
are related to one another by the
action of $g \in \Gamma$.

→ Different twisted sectors labelled by
group elements $g \in \Gamma$.

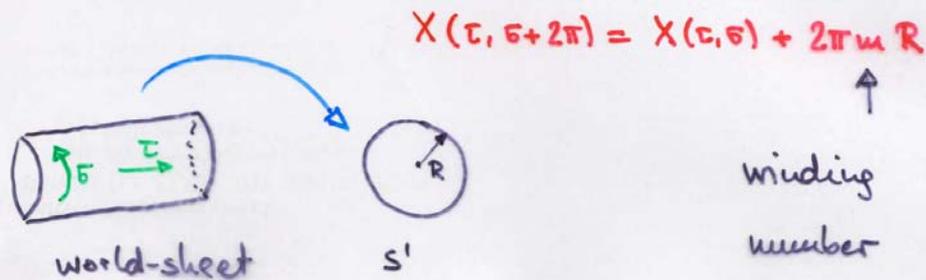
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How can one construct **twisted sectors** in terms of original theory on \mathcal{M} ? — use **modular invariance** of world-sheet (conformal field theory) description. Let us explain this at the hand of the above examples.

We begin with the theory of a single free boson with target space S^1 of radius R :

$$S = \int d^2\sigma \partial_\alpha X \partial^{\alpha} X$$

where



(6)

The equations of motion imply that

$$X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma)$$

and the periodicity constraints imply that

$$X_L(\tau + \sigma) = \frac{1}{2}x + \frac{1}{2}P_L(\tau + \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau + \sigma)}$$

$$X_R(\tau - \sigma) = \frac{1}{2}x + \frac{1}{2}P_R(\tau - \sigma) + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-in(\tau - \sigma)}$$

with

$$(P_L, P_R) = \left(\frac{\hbar}{2R} + mR, \frac{\hbar}{2R} - mR \right)$$

Canonical commutation relations for X further give

$$[\alpha_m, \alpha_n] = m \delta_{m, -n}$$

$$[\tilde{\alpha}_m, \tilde{\alpha}_n] = m \delta_{m, -n}$$

$$[\alpha_m, \tilde{\alpha}_n] = 0.$$

(7)

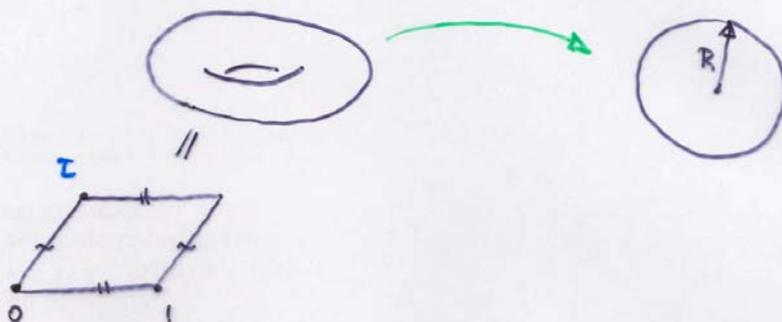
Space of states of the theory, \mathcal{H} , is spanned by

$$\alpha_{-n_1} \dots \alpha_{-n_r} \tilde{\alpha}_{-m_1} \dots \tilde{\alpha}_{-m_s} |(P_L, P_R)\rangle \quad n_i, m_j > 0$$

The torus vacuum amplitude (partition function) is then

$$\begin{aligned} Z &= \text{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{1}{24}} \bar{q}^{\tilde{L}_0 - \frac{1}{24}} \right) \\ &= \frac{1}{\eta(\tau) \bar{\eta}(\bar{\tau})} \sum_{m, n} q^{\frac{1}{2} \left(\frac{n}{2R} + mR \right)^2} \bar{q}^{\frac{1}{2} \left(\frac{n}{2R} - mR \right)^2} \end{aligned}$$

Here $q = e^{-2\pi\tau}$ with τ the modular parameter of world-sheet torus



Using the **Poisson resummation formula** it is easy to see that Z is **modular invariant**, i.e. the same if

$$\tau \mapsto \frac{a\tau+b}{c\tau+d} \quad \begin{array}{l} ad-bc=1 \\ a, b, c, d \in \mathbb{Z} \end{array}$$

Modular group of torus $SL(2, \mathbb{Z})/\mathbb{Z}_2$.

Now we want to analyse orbifold.

Let $g \in \mathbb{Z}_2$ be the non-trivial generator of our orbifold group, i.e.

$$\begin{array}{l} g_1: X \mapsto +X + \pi R \\ \text{or} \\ g_2: X \mapsto -X \end{array}$$

This induces action of g on space of states \mathcal{H} ;

for example

$$g_1 \alpha_n g_1^{-1} = \alpha_n \quad g_1 \tilde{\alpha}_n g_1^{-1} = \tilde{\alpha}_n \quad g_1 |(n, m)\rangle = (-1)^m |(n, m)\rangle$$

$$g_2 \alpha_n g_2^{-1} = -\alpha_n \quad g_2 \tilde{\alpha}_n g_2^{-1} = -\tilde{\alpha}_n \quad g_2 |(n, m)\rangle = |(-n, -m)\rangle.$$

[Strictly speaking, above geometrical action only fixes action on \mathcal{H} up to some signs; actual action of Γ on \mathcal{H} is however part of 'definition of orbifold action'.]

The **projector** onto g -invariant states is simply

$$P = \frac{1}{2} (1 + g)$$

Thus contribution of **untwisted sector** of orbifold theory to **torus vacuum amplitude** is

$$Z_U = \text{Tr}_{\mathcal{H}} \left(\frac{1}{2} (1 + g) q^{L_0 - \frac{1}{24}} \bar{q}^{\bar{L}_0 - \frac{1}{24}} \right) = \frac{1}{2} Z_0 + \frac{1}{2} Z_g.$$

Here Z_0 is original (modular invariant) partition function, and Z_g is

$$Z_g = \text{Tr}_H \left(g q^{L_0 - \frac{c}{24}} \bar{g} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right)$$

$$= \text{Tr}_H \left[\begin{array}{|c|} \hline \square \\ \hline \end{array} \right]$$

Z_g is not modular invariant by itself, and

therefore $Z_U = \frac{1}{2} (Z_0 + Z_g)$ is not either.

In order to obtain consistent orbifold theory,

we need to add twisted sectors so that

$$Z_{\text{orbifold}} = Z_U + Z_T$$

is modular invariant. \uparrow contribution from twisted sectors

Advantage of this point of view: (at least)

partition function of **twisted sectors** can

be **determined** from knowledge of \mathcal{H} and

the Γ -action on it!

Pictorially speaking, under the action of

$$S: \tau \mapsto -1/\tau$$

$$1 \begin{array}{|c|} \hline \square \\ \hline g \\ \hline \end{array} \mapsto g \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline \end{array} \cong g\text{-twisted sector}$$

and

$$T: \tau \mapsto \tau + 1$$

$$g \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline \end{array} \mapsto g \begin{array}{|c|} \hline \square \\ \hline g \\ \hline \end{array} \cong g\text{-twisted sector} \\ \text{with insertion of } g.$$

Thus the modular invariant extension of Z_0 is simply

$$Z_{\text{orbifold}} = \frac{1}{2} \sum_{i \in \{1, 2\}} \text{Tr} \rho_i$$

For a general (abelian) orbifold group Γ the obvious generalisation is

$$Z_{\text{orb}} = \frac{1}{|\Gamma|} \sum_{g, h \in \Gamma} \text{Tr} \rho_h$$

$$= \sum_{g \in \Gamma} \text{Tr}_{\mathcal{H}_g} \left(\mathcal{P} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right)$$

g -twisted
Sector

$$\mathcal{P} = \frac{1}{|\Gamma|} \sum_{h \in \Gamma} h$$

projector onto

Γ -invariant

states

Thus orbifold theory consists of Γ -invariant states of \mathcal{H} , together with the Γ -invariant states of each twisted sector \mathcal{H}_g , $g \in \Gamma$. The partition function of \mathcal{H}_g can be obtained as

$$Z_g = \text{Tr}_{\mathcal{H}_g} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right)$$

$$= S \left[\text{Tr}_{\mathcal{H}} \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} \right) \right]$$

↑
 $\tau \mapsto -1/\tau$.

How does this work in practice? — Consider as an example the two theories

$$S^1/\mathbb{Z}_2$$

If $\mathbb{Z}_2 = \text{half-shift}$

$$1 \square_g = \frac{1}{\eta(q) \eta(\bar{q})} \sum_{n,m} (-1)^n q^{\frac{1}{2} \left(\frac{n}{2R} + mR\right)^2} \bar{q}^{\frac{1}{2} \left(\frac{n}{2R} - mR\right)^2}$$



S-transformation $\tau \mapsto -1/\tau$

$$g \square_1 = \frac{1}{\eta(q) \eta(\bar{q})} \sum_{n,m} q^{\frac{1}{2} \left(\frac{n}{2R} + mR\right)^2} \bar{q}^{\frac{1}{2} \left(\frac{n}{2R} - mR\right)^2}$$

m : half-odd integer

Both untwisted & g -twisted sector must be

Γ -invariant: n must be even! Thus total

orbifold theory is

$$\text{Zorbifold} = \frac{1}{\eta(q) \eta(\bar{q})} \sum_{\substack{n \in 2\mathbb{Z} \\ m \in \frac{1}{2}\mathbb{Z}}} q^{\frac{1}{2} \left(\frac{n}{2R} + mR\right)^2} \bar{q}^{\frac{1}{2} \left(\frac{n}{2R} - mR\right)^2}$$

\rightarrow theory at $R' = \frac{1}{2}R!$

If $\mathbb{Z}_2 = \text{inversion}$, on the other hand, we have

$$1 \square_g = \frac{1}{q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1+q^n)} \quad \frac{1}{\bar{q}^{\frac{1}{24}} \prod_{n=1}^{\infty} (1+\bar{q}^n)}$$

since then only sector with $(u,n) = (0,0)$ contributes.

Under the S -transformation $\tau \mapsto -1/\tau$, this

becomes

$$g \square_1 = 2 |q|^{-\frac{1}{24}} \frac{1}{\prod_{n=1}^{\infty} (1-q^{n-\frac{1}{2}}) (1-\bar{q}^{n-\frac{1}{2}})}$$

This suggests that there are 2 twisted sector contributions, one from each fixed point



For each twisted sector component, the Fock space is generated by half-integer moded bosonic generators

$$\alpha_{-n_1+\frac{1}{2}} \cdots \alpha_{-n_r+\frac{1}{2}} \tilde{\alpha}_{-m_1+\frac{1}{2}} \cdots \tilde{\alpha}_{-m_s+\frac{1}{2}} |\mathbb{T}'\rangle \quad n_i, m_j \in \mathbb{N}$$

and the ground state energy is

$$(L_0 - \frac{1}{24}) |\mathbb{T}'\rangle = \frac{1}{48} |\mathbb{T}'\rangle$$

$$L_0 |\mathbb{T}'\rangle = \tilde{L}_0 |\mathbb{T}'\rangle = \frac{1}{16} |\mathbb{T}'\rangle$$

$$(\tilde{L}_0 - \frac{1}{24}) |\mathbb{T}'\rangle = \frac{1}{48} |\mathbb{T}'\rangle$$

This is **in accord** with our previous geometrical analysis: in the twisted sector we should have

$$X(\tau_1, \sigma + 2\pi) = -X(\tau, \sigma) + 2\pi i R$$

(17)

and the most general solution with this boundary condition is

$$X(\tau, \xi) = x_0 + \frac{i}{2} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \frac{1}{n} e^{-in\tau} (x_n e^{-in\xi} + \tilde{x}_n e^{in\xi})$$

where x_0 is one of the two fixed points, i.e.

$$x_0 = 0 \quad \text{or} \quad x_0 = \pi R.$$

Here we have used that

$$e^{-in(\xi + 2\pi)} = -e^{-in\xi} \quad \text{for } n \in \mathbb{Z} + \frac{1}{2}. \quad \downarrow$$

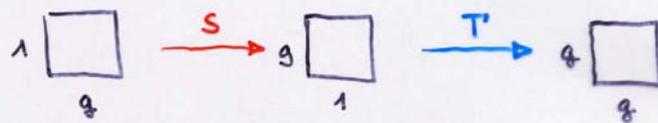
There is an interesting subtlety that arises for the construction of the twisted sectors if Γ is the product of more than one cyclic group.

[The simplest example arises for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$.] As

we have seen, given the action of Γ on \mathcal{H} ,

we can construct the twisted sectors \mathcal{H}_g , one

for each $g \in \Gamma$:



Furthermore, by applying T , we can determine

the action of g on \mathcal{H}_g . However, in order

to construct the actual twisted sector, we

also need to know the action of an arbitrary $h \in \Gamma$ on \mathcal{H}_g (so that we can restrict \mathcal{H}_g to the Γ -invariant states). In general, there is no canonical definition for this action; in fact, suppose one has a consistent definition

$$\Gamma \ni h : \mathcal{H}_g \rightarrow \mathcal{H}_g \quad h : \psi \mapsto h\psi$$

then the action

$$h : \psi \mapsto \epsilon(h, g) h\psi$$

is also consistent provided that the phases $\epsilon(h, g)$ satisfy the following two constraints:

(i) representation property

$$\varepsilon(h_1 h_2, g) = \varepsilon(h_1, g) \varepsilon(h_2, g)$$

(ii) modular invariance

$$\varepsilon(h, g) = \varepsilon(h \begin{pmatrix} a & b \\ c & d \end{pmatrix}, g) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

This ambiguity in the definition of the action of Γ on the twisted sectors is called **discrete torsion**.

Vafa

The possible choices of $\varepsilon(g, h)$ satisfying (i) & (ii) are in one-to-one correspondence with elements in

$$H^2(\Gamma, \mathbb{U}(1)).$$

(21)

This cohomology group consists of the
 2-cycles $c(g, h) \in U(1)$, satisfying the **cocycle**
condition

$$c(g_1, g_2 g_3) c(g_2, g_3) = c(g_1 g_2, g_3) c(g_1, g_2)$$

where we identify cocycles that differ by
 a **coboundary** $c_g \in U(1)$

$$c'(g, h) = \frac{c_g c_h}{c_{gh}} c(g, h).$$

The relation between between discrete torsion
 phases and cohomology elements is given by

$$\varepsilon(g, h) = \frac{c(g, h)}{c(h, g)}.$$

It is not difficult to show that this defines a one-to-one correspondence.

We therefore conclude that if $H^2(\Gamma, \mathbb{U}(1)) \neq 1$ the orbifold theory is not uniquely determined by the action of Γ on \mathcal{H} — in addition we need to specify which Γ -action is chosen on the twisted sectors.

The relation between discrete torsion and $H^2(\Gamma, \mathbb{U}(1))$ is important for the following since the latter is directly related to projective representations of Γ .

Suppose that for each $g \in \Gamma$

$$g(g): V \rightarrow V$$

is a linear map on a vector space V . If g satisfies

$$g(g_1) g(g_2) = c(g_1, g_2) g(g_1 g_2)$$

then g defines a projective representation. Here

$c(g_1, g_2)$ satisfies cycle condition [required

for above action to be associative]. Furthermore,

cocycles that differ by a coboundary correspond

to representations that differ as

$$g'(g) = c_g g(g).$$

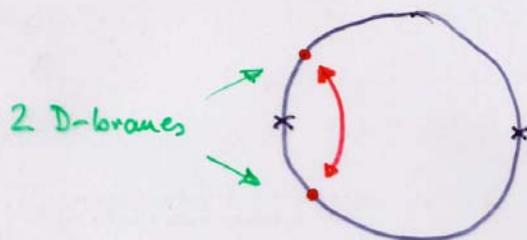
2. D-branes in orbifolds

Up to now we have described **closed strings** on the orbifold \mathcal{M}/Γ (mainly in terms of the original closed string theory on \mathcal{M}). Now we want to consider **open strings**. Equivalently we may analyse **D-branes** (since D-branes are just a geometrical way of describing open string boundary conditions). We shall mainly describe the D-branes of the orbifold theory in terms of the D-branes of the original theory on \mathcal{M} .

For simplicity we shall only consider D-branes in the following that are points on \mathcal{M} . Then the D-branes of the orbifold theory can be described as follows:

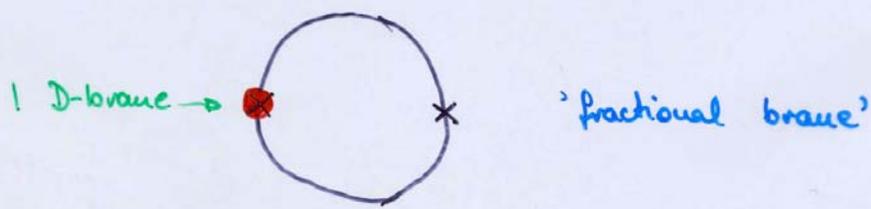
Douglas & Moore

- ① Consider a Γ -invariant configuration of D-branes on \mathcal{M} . In the generic case this will involve $|\Gamma|$ D-branes of the original string theory on \mathcal{M} .



'bulk brane'
possesses moduli
that describes its
position on \mathcal{M} .

However, if action of Γ on \mathcal{M} has
 fixed points, there also exist Γ -invariant
 configurations of D-branes that involve
 fewer branes



These fractional branes are stuck to
 the fixed points because they are too
 few in number to produce a Γ -invariant
 configuration of D-branes at a generic
 point. However, if a suitable number

of fractional D-branes come together,
they can 'pair up' to produce a bulk
brane. (For example, in the above case

2 fractional D-branes are needed to
produce 1 bulk brane — the fractional
D-branes therefore only carry half the
charge & tension of the bulk branes.

This explains their name.)



2 fractional



1 bulk

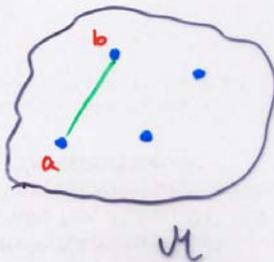
- ② Restrict the open string spectrum to Γ -invariant states.

In general, Γ -invariant configuration of D-branes involves N different branes. (For example, for bulk branes $N=|\Gamma|$.) Thus we have N Chan-Paton indices, and the full open string spectrum is of the form

$$H_{\text{open}} = \bigoplus_{a,b} H_{ab}$$



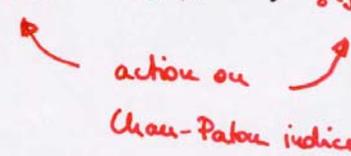
open string from
D-brane a to
D-brane b.



The action of $g \in \Gamma$ on an element

$|\psi, ab\rangle \in H_{ab}$ is of the form

$$g |\psi, ab\rangle = \gamma(g)_{a'a} |U(g)\psi, a'b'\rangle \gamma(g)^{-1}_{b'b'}$$



 action on
 Clebsch-Gordan indices

Usually, $U(g)$ is a conventional representation, i.e.

$$U(g_1) U(g_2) = U(g_1 g_2).$$

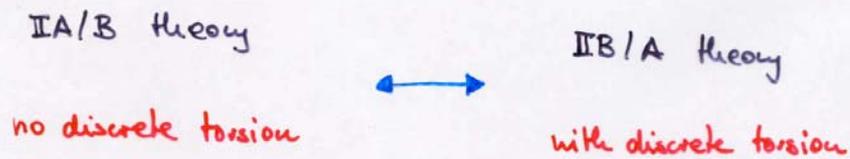
In order to have an actual representation of Γ on H_{ab} , this implies that γ has to be at least a projective representation of Γ

Simon & Polchinski

$$\gamma(g_1)_{a'a'} \gamma(g_2)_{a''a''} = c(g_1, g_2) \gamma(g_1 g_2)_{a'a''}$$

It seems that this statement cannot be true in full generality. As was shown by Vafa & Witten some time ago, for the orbifold $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$

mirror symmetry exchanges

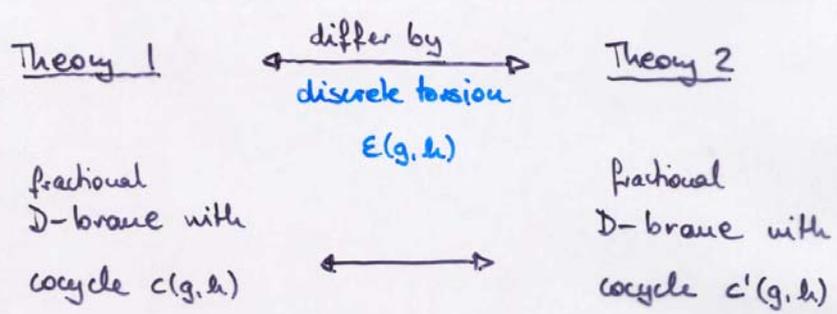


Mirror symmetry does not modify the nature of the group representation on Chan-Paton indices, and thus one would expect that both theories contain branes with a conventional group rep. and branes with a projective group rep. on Chan-Paton indices.

MRG
MRG, Craps

This expectation could also be confirmed by a detailed boundary state analysis of these branes.

The general statement (that is consistent with these findings) seems to be:



$$\frac{c'(g,h)}{c'(h,g)} = E(g,h) \frac{c(g,h)}{c(h,g)}$$

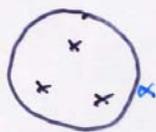
This makes also abstract sense since discrete torsion is only relative concept.

Boundary states

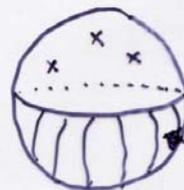
efficient way to construct and describe D-branes.

Basic idea: characterise D-brane in terms of the

closed string states to which it couples



disk diagram with
boundary condition α



sphere
diagram

$$\langle V(z_1, z_1) \dots V(z_n, z_n) \rangle_{\alpha} = \langle V(z_1, z_1) \dots V(z_n, z_n) ||\alpha\rangle$$

It is directly possible to read off from boundary state

the charge, tension, etc of the D-brane.

Boundary state must satisfy 'gluing conditions', e.g.

$$(\alpha_n^i \pm \tilde{\alpha}_{-n}^i) ||\alpha\rangle\rangle = 0 \quad \begin{matrix} + & N \\ - & D \end{matrix}$$

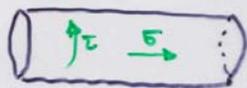
These ensure that correct open string boundary condition is implemented at boundary.

The gluing conditions fix the boundary state up to a few (normalisation) constants; these are then determined by the Cardy condition



closed string
tree diagram

||



open string
one-loop diagram

The boundary states must lie in the **actual closed string space**. For orbifold theories

this means in particular that they must be **invariant under action of orbifold group**.

Because of Cardy condition boundary state

either only involves

untwisted sector \leftrightarrow **bulk brane**

or it involves both

untwisted & twisted sectors \leftrightarrow **fractional brane**



location of
fixed point

Powerful method since it allows one to determine D-brane spectrum from the knowledge of the corresponding closed string theory.

Also this construction does not rely on spacetime supersymmetry. Can use it to construct stable non-BPS D-branes for certain orbifold theories.

Bigman, MR6
See

3. Asymmetric orbifolds

Up to now we have implicitly assumed that the orbifold group Γ has a 'geometrical' action on M . In particular this requires that it treats left- and right-movers symmetrically.

One can also consider **asymmetric orbifolds** for which the orbifold action on left- and right-movers may be different. The analysis of asymmetric orbifolds is essentially the same as for symmetric orbifolds, but there are a number of additional subtleties that need to be considered.

First of all, not every Γ -action on \mathcal{H} defines a consistent orbifold. As we have explained before, given the action of Γ on \mathcal{H} , we can calculate

$$1 \begin{array}{|c|} \hline \square \\ \hline g \\ \hline \end{array} \xrightarrow{s} g \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline \end{array} \xrightarrow{T^n} g \begin{array}{|c|} \hline \square \\ \hline g^n \\ \hline \end{array}$$

Now suppose that $g \in \Gamma$ is of order N , i.e. $g^N = 1$.

Then we must have that

$$T^N \left(g \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline \end{array} \right) = g \begin{array}{|c|} \hline \square \\ \hline 1 \\ \hline \end{array}$$

For a general asymmetric Γ -action on \mathcal{H} this is not automatically the case; consistent orbifolds must however satisfy this constraint (for all $g \in \Gamma$):

Level matching

Narain, Susskind,
Vafa

Secondly, the construction of D-branes we have explained above, certainly works for asymmetric orbifolds, but it does not, in general, produce **all D-branes** of the theory. Not much is known in general about how to construct all the D-branes of an asymmetric orbifold in a **unified manner**. A number of examples have been studied in detail (often using specific properties of the systems in question) but a general prescription is still lacking.

Bouw et al
Gutperle
Graps, MRG, Harvey
MRG, Schiffo-Namchi
:

4. Summary:

- explained how to construct orbifold theory \mathcal{H}/Γ for a given group action Γ on \mathcal{H} .
- in general there is ambiguity in construction described by **discrete torsion**.
- D-branes for orbifolds can be described in terms of Γ -invariant configurations of D-branes of original theory. In addition open string spectrum is restricted to Γ -invariant states.
- Γ -action on open string states requires choice of **(projective) representation or Chan-Paton indices**.
- Asymmetric orbifolds are only consistent if they satisfy **level matching condition**.