On signature change in cosmology

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Abstract

A classical model of gravitation is studied in which a self interacting scalar field is coupled to gravity with the metric undergoing a continuous signature transition. We obtain dual signature changing classical solutions for the Einstein field equations. These dual classical solutions correspond to the same quantum cosmology. Based on this correspondence, it is hoped to look for a scenario for quantum creation of the Lorentzian universe.

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1 Introduction

The initial idea of signature change is due to Hartle and Hawking [1] which makes it possible to have both Euclidean and Lorentzian regions in quantum gravity. It was then shown that signature change may happen even in classical general relativity . From a classical point of view, the signature change may prevent the occurrence of singularities in general relativity, such as the big-bang, which may be replaced by a compact Euclidean domain prior to the birth of time in the Lorentzian domain, the so-called no-boundary proposal. Alternatively, the classical signature change scenario may be an effective classical description of the quantum tunneling approach for the creation of the Lorentzian universe.

2 Dual classical solutions

Consider the Einstein-Hilbert action

$$I = \int \sqrt{|g|}\,\left[\frac{1}{16\pi G}\mathcal{R} + \mathcal{L}_M\right] d^4x, \tag{1}$$

where $\mathcal{L}_M = \frac{1}{2}\partial_0\phi\partial^0\phi - U(\phi)$ is the real scalar field Lagrangian and ϕ is assumed to be a homogeneous field which depends merely on the time parameter. We take the chart $\{\beta, x^1, x^2, x^3\}$ and parametrize the metric as [2]

$$g = -\beta d\beta \otimes d\beta + \frac{R^2(\beta)}{[1+\frac{k}{4}r^2]^2}\sum_i dx^i \otimes dx^i, \tag{2}$$

where β is the lapse function with the hypersurface of signature change at $\beta = 0$. By the transformations [2]

$$X = R^{3/2}\cosh(\alpha\phi), \tag{3}$$

$$Y = R^{3/2}\sinh(\alpha\phi), \tag{4}$$

where $-\infty < \phi < +\infty,\ 0 \leq R < \infty$ and $\alpha^2 = \frac{3}{8}$, the corresponding effective Lagrangian is obtained

$$\mathcal{L} = \frac{1}{2}\alpha^{-2}\left\{-\dot{X}^2 + \dot{Y}^2 + \frac{9k}{4}(X^2 - Y^2)^{1/3} - 2\alpha^2(X^2 - Y^2)U(\phi(X,Y))\right\}, \tag{5}$$

Concentrating on cosmologies with $k = 0$, we obtain

$$\mathcal{L} = \frac{1}{2}\alpha^{-2}\left\{-\dot{X}^2 + \dot{Y}^2) - (X^2 - Y^2)U(\phi(X,Y))\right\}, \tag{6}$$

$$\mathcal{H} = \frac{1}{2}\alpha^{-2}\left\{-\dot{X}^2 + \dot{Y}^2) + (X^2 - Y^2)U(\phi(X,Y))\right\}. \tag{7}$$

Now, we take the potential as

$$U(\phi) = \lambda + \frac{1}{2\alpha^2}m^2\sinh^2(\alpha\phi) + \frac{1}{2\alpha^2}b\sinh(2\alpha\phi), \tag{8}$$

where λ, m^2 and b are the bare cosmological constant, positive mass square and coupling constant respectively. Variation of the action with respect to the dynamical variables X and Y gives the dynamical equations

$$\ddot{\xi} = M\xi, \tag{9}$$

where $\xi = \begin{pmatrix} X \\ Y \end{pmatrix}$. These equations are subjected to the zero energy condition

$$\dot{\xi}^T J \dot{\xi} = \xi^T J M \xi, \tag{10}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and M is the matrix of potential's parameters. One may then define the normal modes $\xi = S\alpha$ where $\alpha = \begin{pmatrix} u \\ v \end{pmatrix}$ and

$$S = \begin{pmatrix} -\frac{m^2-\sqrt{m^4-4b^2}}{2b} & -\frac{m^2+\sqrt{m^4-4b^2}}{2b} \\ 1 & 1 \end{pmatrix}. \tag{11}$$

The zero energy condition then reads as

$$\dot{\alpha}^T \mathcal{J} \alpha = \alpha^T \mathcal{I} \alpha, \tag{12}$$

where $\mathcal{J} = S^T J S$ and $\mathcal{I} = SJMS$. Evaluating this condition at $t = 0$, choosing $\dot{\alpha}(0) = 0$, leads to

$$\alpha^T(0)\mathcal{I}\alpha(0) = 0. \tag{13}$$

The normal mode solutions are then obtained [2]

$$u^{\pm}(t) = 2A^{\pm}\cosh(\sqrt{\lambda_+}t),$$
$$v(t) = 2\cosh(\sqrt{\lambda_-}t),$$

where $A^{\pm}$ are the roots of the zero energy condition. The normal modes $\begin{pmatrix} u \\ v \end{pmatrix}$, with zero energy condition, for $\lambda_+, \lambda_- < 0$ lead to the following classical loci[3]

$$\begin{aligned} v &= 2\cos\left[\frac{1}{r}\cos^{-1}\left(\epsilon\frac{ru}{2}\right)\right], \quad |u| \leq \frac{2}{r} \\ v &= 2\cosh\left[\frac{1}{r}\cosh^{-1}\left(\epsilon\frac{ru}{2}\right)\right], \quad |u| > \frac{2}{r} \end{aligned} \tag{14}$$

where $r = \sqrt{\frac{\lambda_\pm}{\lambda_-}}$, $0 < r < 1$, $\epsilon = \pm 1$ indicates two ways of satisfying the constraint $\mathcal{H} = 0$, and $\lambda_\pm$ are the eigenvalues of the matrix M.

An interesting feature of this model is that one can find a class of transformations on the space of potential's parameters leaving the eigenvalues $\lambda_\pm$ of the matrix M invariant. These transformations can be written as

$$\lambda \to \tilde{\lambda} \equiv \frac{1}{4\alpha^4}\lambda^{-1},$$

$$m^2 \to \tilde{m}^2 \equiv m^2 - \frac{4\alpha^4\lambda^2 - 1}{\alpha^2\lambda}, \tag{15}$$

$$b^2 \to \tilde{b}^2 \equiv b^2 + m^2[(2\alpha^2\lambda)^{-1} - 2\alpha^2\lambda] + [(2\alpha^2\lambda)^{-1} - 2\alpha^2\lambda]^2. \tag{16}$$

It is seen that although the classical loci (14) do not change under (16), the corresponding solutions $R(\beta)$ and $\phi(\beta)$ change, since $X(\beta)$ and $Y(\beta)$ are related to $u(\beta)$ and $v(\beta)$ by the decoupling matrix which changes under (16). Therefore, if we define (16) as *duality* transformations, then we have two sets of solutions for $R(\beta)$ and $\phi(\beta)$ corresponding to dual sets of physical parameters. We interpret the new parameters as dual bare cosmological constant, dual mass square and dual coupling constant, respectively.

In the case of small parameters λ, m^2 and b, the dual transformations map these small values to very large values of the corresponding dual parameters. It then follows that two different classical cosmologies, one with very small bare cosmological and coupling constants and the other with large ones, exhibit the same signature dynamics on the configuration space (u, v).

We have shown that it is possible to find dual classical cosmologies on the (R, ϕ) configuration space corresponding to the same classical cosmology defined on the (u, v) configuration space. On the other hand it is shown that [3] the corresponding Wheeler-Dewitt equation in terms of variables (u, v) has analytic solutions and that the absolute value of these solutions have maxima in the vicinity of classical loci (11) on the (u, v) configuration space which can exhibit a signature transition. It then turns out that for any distinct quantum cosmology in terms of the variables (u, v), we may correspond dual classical solutions (R, ϕ) and $(\tilde{R}, \tilde{\phi})$, admitting signature transition from Euclidean to Lorentzian space-time. It would be interesting if one could associate the solutions (R, ϕ) and $(\tilde{R}, \tilde{\phi})$ to the Lorentzian and Euclidean regions, respectively. Then, the solutions $(\tilde{R}, \tilde{\phi})$ would jump to (R, ϕ) in passing from Euclidean to Lorentzian region. We suggest that these possible jumps may be interpreted as quantum effects describing the quantum creation of the Lorentzian universe.

References

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