DESY Theory Workshop 2004

Recent developments in second order perturbation theory

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Motivation?

Currently popular models of structure formation:

- Standard inflation models ("single" field, starts adiabatic and stays adiabatic)
- Curvaton models (multi-field, starts non-adiabatic or isocurvature, then becomes adiabatic)

$\Rightarrow$ Different mechanism, but similar observational signatures, i.e. nearly scale invariant spectrum of adiabatic fluctuations with very small amount of gravitational waves, *at linear order*

But: Very distinctive predictions at *second order*

Second order things we might observe

- Everybody's favourite, the non-linearity parameter $f_{\rm NL}$, defined as

$$\Phi = \Phi_{(1)} + f_{\rm NL}\Phi_{(1)}{}^2$$

  where $\Phi^{(1)}$ is the Bardeen potential at first order

  see e.g. review by Bartolo, Komatsu, Matarrese and Riotto (2004)

- Other examples: Gravitational collapse is non-linear
  - Primordial black hole formation

    Green, Liddle, M., and Sasaki (2004)

  - Galaxy formation

But how does it work?

- Splitting a tensor into homogeneous background and inhomogeneous perturbations, e.g. the metric tensor

$$g_{\mu\nu}(\eta, x^i) = {g_{\mu\nu}}^{(0)}(\eta) + \delta g^{(1)}_{\mu\nu}(\eta, x^i) + \frac{1}{2}\delta g^{(2)}_{\mu\nu}(\eta, x^i) + \dots$$

- Problem: under coordinate transformation

$$\widetilde{x^\mu} = x^\mu + \xi_1^\mu + \frac{1}{2}\left(\xi^\mu_{1,\nu}{\xi_1}^\nu + \xi_2^\mu\right)$$

perturbations transform as

$$\begin{aligned} \widetilde{\delta g^{(1)}_{\mu\nu}} &= \delta g^{(1)}_{\mu\nu} + \pounds_{\xi_1} g^{(0)}_{\mu\nu} \\ \widetilde{\delta g^{(2)}_{\mu\nu}} &= \delta g^{(2)}_{\mu\nu} + \pounds_{\xi_2} g^{(0)}_{\mu\nu} + \pounds^2_{\xi_1} g^{(0)}_{\mu\nu} + 2\pounds_{\xi_1}\delta g^{(1)}_{\mu\nu} \end{aligned}$$

Bruni, Matarrese, Mollerach and Sonego (1997)

Solution:

⇒ Construct gauge-invariant variables.

Quick first order recap:

- Curvature perturbation transforms as

$$\widetilde{\psi_1} = \psi_1 - \frac{a'}{a}\delta\eta_1$$

- density perturbation transforms as

$$\widetilde{\delta\rho_1} = \delta\rho_1 + \rho_0'\delta\eta_1$$

  where $a = a(\eta)$ is the scale factor, coordinate systems are related by $\widetilde{x}^\mu = x^\mu + \delta x^\mu$, in particular $\widetilde{\eta} = \eta + \delta\eta_1$

- Combine and get curvature perturbation on uniform density hypersurfaces

$$\zeta_1 = -\psi_1 - \frac{a'}{a}\frac{\delta\rho_1}{\rho_0'}$$

Bardeen (1980,1988)

Second order gauge-invariant variables

- Similar to first order case, but slightly more complicated!
- To ease presentation: large scales only, no need to specify "threading".
- Second order coordinate transformation simplifies to:
  $\widetilde{\eta} = \eta + \delta\eta_1 + \frac{1}{2}(\delta\eta_1'\delta\eta_1 + \delta\eta_2)$

Curvature perturbation and density perturbation change as

$$\widetilde{\psi_2} = \psi_2 - \delta\eta_1 \left[\mathcal{H}\delta{\eta_1}' + \left(\mathcal{H}' + 2\mathcal{H}^2\right)\delta\eta_1 - 2\psi_1' - 4\mathcal{H}\psi_1\right] - \mathcal{H}\delta\eta_2$$

and

$$\widetilde{\delta\rho_2} = \delta\rho_2 + \rho_0'\delta\eta_2 + \delta\eta_1\left[\rho_0''\delta\eta_1 + \rho_0'\delta{\eta_1}' + 2\delta{\rho_1}'\right]$$

where $\mathcal{H} = \frac{a'}{a}$.

Finally

Curvature perturbation on uniform density hypersurfaces at second order (large scales)

$$\zeta_2 = -\psi_2 - \frac{\mathcal{H}}{\rho_0'}\delta\rho_2 + 2\frac{\mathcal{H}}{{\rho_0'}^2}\delta{\rho_1}'\delta\rho_1 + 2\frac{\delta\rho_1}{\rho_0'}\left(\psi_1' + 2\mathcal{H}\psi_1\right)$$
$$+\frac{{\delta\rho_1}^2}{{\rho_0'}^2}\left(\mathcal{H}\frac{\rho_0''}{\rho_0'} - \mathcal{H}' - 2\mathcal{H}^2\right)$$

M. and Wands (2003)

- Note: ζ_2 contains terms $\mathcal{O}(\epsilon^2)$ *and* $(\mathcal{O}(\epsilon))^2$
- Above definition can, naturally, be extended to all scales.

But this is only a variable $\Rightarrow$ need evolution equations.

Simple evolution

- Full Einstein equations *very* complicated

  see e.g. review by Noh and Hwang (2003)

- But, to get evolution of curvature perturbation on large scales, only need energy conservation $\nabla_\mu T^{\mu\nu} = 0$,

$$\delta\rho_1{}' + 3\mathcal{H}\,(\delta\rho_1 + \delta P_1) - 3\,(\rho_0 + P_0)\,\psi_1' \simeq 0$$

$$\delta\rho_2{}' + 3\mathcal{H}\,(\delta\rho_2 + \delta P_2) - 3\,(\rho_0 + P_0)\,\psi_2' - 6\psi_1'\,[\delta\rho_1 + \delta P_1 + 2\,(\rho_0 + P_0)\,\psi_1] \simeq 0$$

Sufficient to show

$$\zeta_1' = 0\,, \qquad \zeta_2' = 0$$

on large scales for adiabatic perturbations, i.e. $P = P(\rho)$.

Wands, M., Lyth and Liddle (2000),
M. and Wands (2003)

Details

Rewriting energy conservation in terms of ζ_1 and ζ_2 gives

$$\zeta_1' \simeq -\frac{\mathcal{H}}{(\rho+P)}\widetilde{\delta P_1}|_\rho$$

and

$$\zeta_2' \simeq -\frac{\mathcal{H}}{(\rho+P)}\widetilde{\delta P_2}|_\rho - \frac{2}{\rho_0+P_0}\left[\widetilde{\delta P_1}|_\rho - 2(\rho_0+P_0)\zeta_1\right]\zeta_1'$$

where $\widetilde{\delta P_1}|_\rho = \delta P_{1\rm nad}$, and $\widetilde{\delta P_2}|_\rho = \delta P_{2\rm nad} - 2\frac{\delta\rho_1}{\rho_0'}\delta P_1{}'_{\rm nad}$ and where the non-adiabatic pressures are defined

$$\begin{aligned}\delta P_{1\rm nad} &= \delta P_1 - c_s^2\delta\rho_1\\ \delta P_{2\rm nad} &= \delta P_2 - c_s^2\delta\rho_2 - \frac{dc_s^2}{d\rho_0}\delta\rho_1{}^2\end{aligned}$$

where $c_s^2 \equiv dP/d\rho$.

Conclusions

- Second order gauge-invariant perturbations can be readily constructed without having to use whole set of Einstein's field equations
- principles are similar to first order, albeit slightly more difficult
- variables other than ζ, for example curvature on uniform scalar field hypersurfaces, "$\mathcal{R}$", of course also possible
- 2^{nd} order perturbation theory is different to 1^{st} order squared
- read: K. A. M. and D. Wands, Class. Quantum Grav. **21** No. 11 L65, astro-ph/0307055