Probing Extra Dimensions with Gravitational Waves

S. Solodukhin (IUB)

S. S. & A. Barvinsky,
Nucl. Phys. B675 (03), 159

Newton Potential

$$V(r) = \int_{-\infty}^{+\infty} dx^4 \, D(\vec{x}, x^4) = \frac{1}{2\pi^2} \sum_{n=-\infty}^{+\infty} \frac{1}{r^2 + 4n^2L^2}, \quad r^2 = \vec{x}$$

The sum is taken ~~explicitly~~,

$$V(r) = \frac{1}{4\pi r} \left(1 + \frac{2}{e^{\pi r/L} - 1} \right)$$

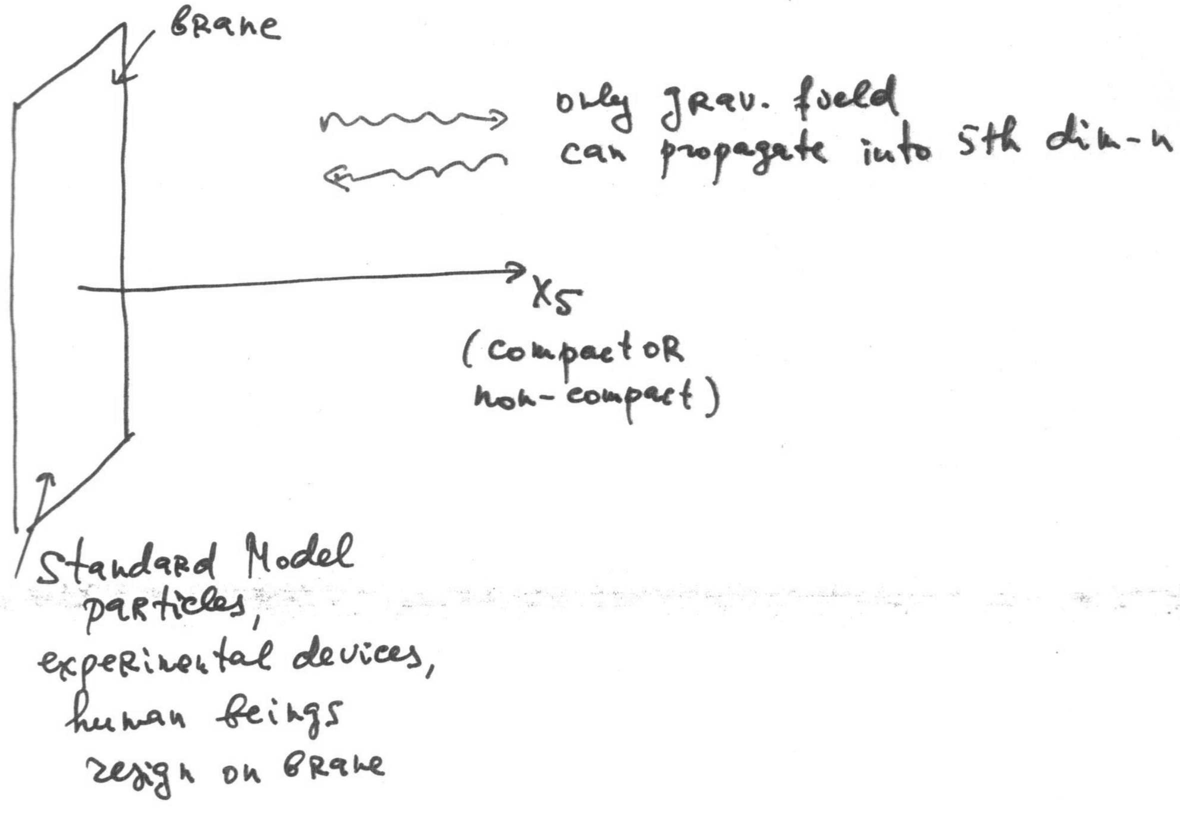
$$r \ll L, \quad V(r) \simeq \frac{1}{r^2} \quad \text{(5-dim-al potential)}$$

$$r \gg L, \quad V(r) \simeq \frac{1}{4\pi r} \left(1 + 2 e^{-\pi r/L} \right)$$

– 4-dim-al law + correction

- Old Kaluza-Klein idea:
  a way to unify GR and Gauge Fields

- Modern Trend: Brane Paradigm
  to solve hierarchy and vacuum energy problems



How to detect Extra Dimension(s)?

- Standard way:
  short-distance physics (corrections to Newton's law)

  high-energy experiments (colliders, cosmic rays)

- In this talk:
  large scale, astronomical type experiments
  involving grav. waves

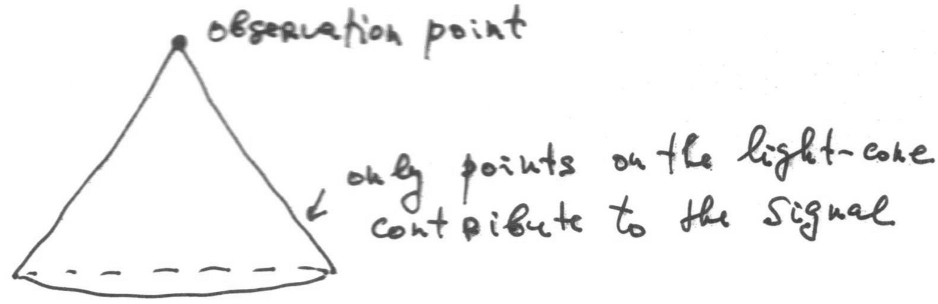
Why waves?

- corrections to static (Newton's) potential
  are exponentially small, $e^{-r/L}$

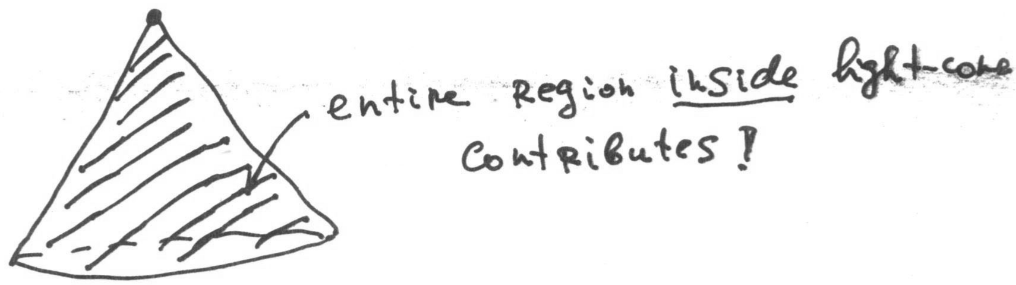
- corrections to purely 4-dim-al law
  of wave propagation have a power law

Puzzling Property of wave propagation

d=4



d=5



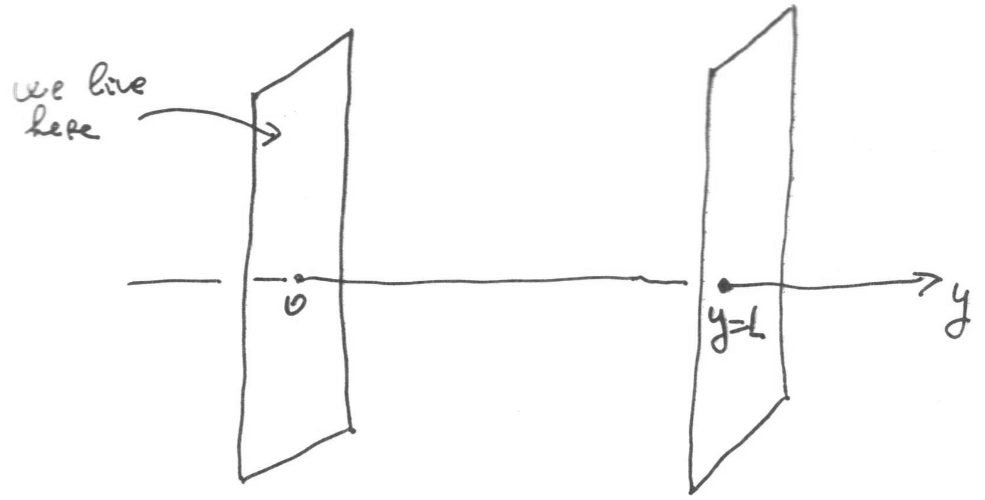
Mathematically: Retarded potential G_R

d=4 $\qquad G_R \sim \delta(t-r)$

d=5 $\qquad G_R \sim \theta(t-r)$

Interesting question: how 4-dim-al law of propagation emerges if it is built of 5-dim-al law?

Set-up



Neumann b.c.: $\partial_y \phi\big|_{y=0} = \partial_y \phi\big|_{y=L} = 0$

Scalar type field eq-n.: $\left(-\partial_0^2 + \partial_y^2 + \partial_{\vec{x}}^2\right)\phi = 0$

ϕ – transverse-traceless part of metric perturbation

– ignore tensor structure

Normalized modes in y-direction:

$$\varphi_n(y) = \frac{1}{\sqrt{L}}, \quad n=0$$

$$\varphi_n(y) = \sqrt{\frac{2}{L}} \cdot \cos(m_n \cdot y), \quad m_n = \frac{\pi n}{L}, \quad n>0$$

Green's Function (Euclidean version)

$$-\nabla_5^2 \, D_5(X, X') = \delta(X, X')$$

$$D_5(X, X') = \frac{1}{(2\pi)^4} \int d^4p \sum_{n=0}^{\infty} e^{ip(x-x')} \frac{\varphi_n(y)\,\varphi_n(y')}{p^2 + m_n^2}$$

$p^2 = p_4^2 + \vec{p}^{\,2}$

KK masses

Propagation along brane ($y=0$)

$$D(x, x') \equiv D_5(X, X')\Big|_{y=y'=0}$$

$$D(x, x') = \frac{1}{(2\pi)^4 L} \int d^4p \sum_{n=-\infty}^{\infty} \frac{e^{ip(x-x')}}{p^2 + m_n^2} = \frac{1}{L} \sum_{n=-\infty}^{+\infty} D(m_n^2 | x-x')$$

$$D(m_n^2 | x-x') = \frac{m}{4\pi^2} \frac{K_1(m|x-x'|)}{|x-x'|}, \quad |x-x'| = \sqrt{(x-x')^2}$$

brane-to-brane propagation
is mediated by set of massive KK fields

Momentum Space Representation

$$\mathcal{D}(x,x') = \frac{1}{(2\pi)^4 L}\int d^4p\, \mathcal{D}(p)\, e^{ip(x-x')}$$

non-trivial propagator

$$\mathcal{D}(p) \equiv \sum_{n=-\infty}^{+\infty} \frac{1}{p^2+\left(\frac{n\pi}{L}\right)^2} = \frac{L}{p}\cdot\frac{\cosh(L\cdot p)}{\sinh(L\cdot p)}\,,\quad p\equiv\sqrt{p^2}$$

In Lorentzian signature $(p \to ip)$

$$\mathcal{D}_M(p) = \frac{L}{p}\,\frac{\cos(Lp)}{\sin(Lp)}\,,\quad p=\sqrt{-p^2}$$

Poles: $L\cdot p = \pi\cdot n \to$ correspond to KK modes

Zeros: $L\cdot p = \frac{\pi}{2}(2n+1)$

(guarantee positivity of residues of the propagator at its poles – non-ghost nature of KK modes)

$p \gg 1/L$: $\mathcal{D}(p) \sim \frac{1}{p}$ (truly 5-dim-al propagator as it is seen from lower-dim-al subspace)

$p \ll 1/L$: $\mathcal{D}(p) \sim \frac{1}{p^2}$ – 4-dim-al law

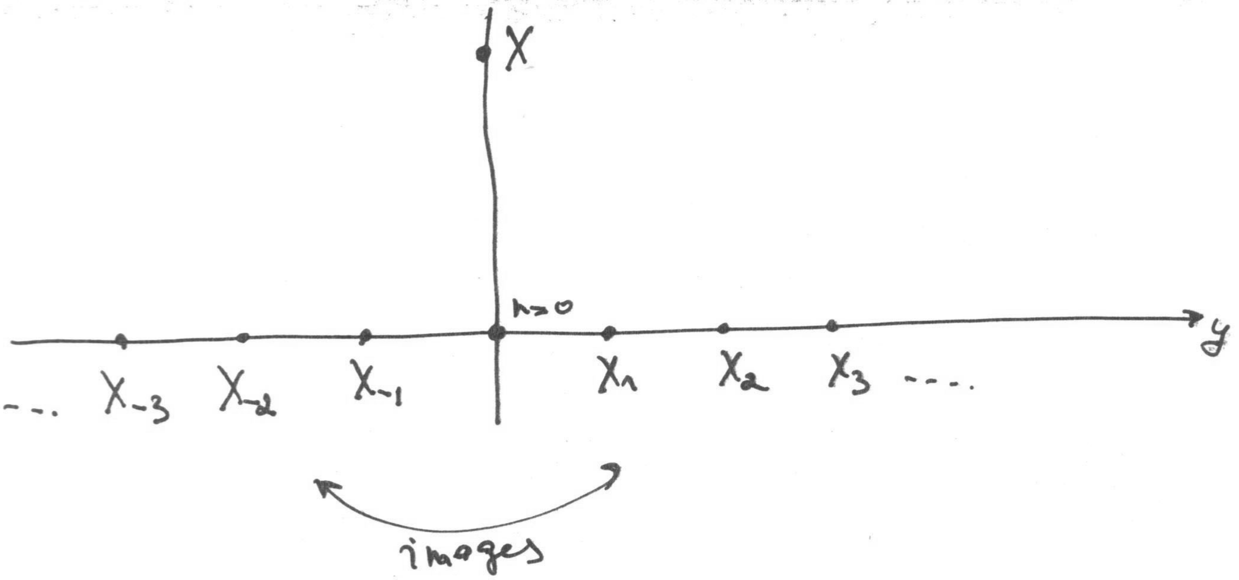
Sum over images ~~representation~~

perform integration $\int d^4p$

$$D(x,x') = \frac{1}{4\pi^2} \sum_{n=-\infty}^{+\infty} \frac{1}{\left[(x-x')^2 + 4n^2L^2 \right]^{3/2}}$$

$$= 2 \sum_{n=-\infty}^{+\infty} D_5 (X - X_n) \Big|_{X=(x,0)}$$

$$X_n = (0, y = 2nL), \quad (X - X_n)^2 = x^2 + 4n^2L^2$$



Large interval limit: $(x-x')^2 \gg L^2$

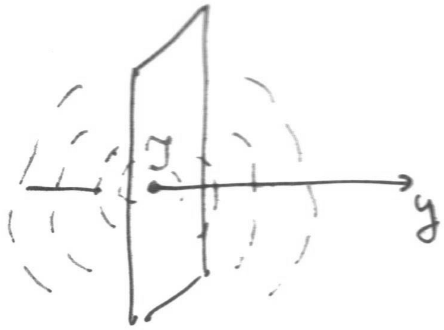
replace sum by integral, $\sum_n = \frac{1}{2L} \int dy$, $y = 2L \cdot n$

$$D(x,x') \simeq \frac{1}{8\pi^2 L} \int_{-\infty}^{+\infty} dy \frac{1}{\left((x-x')^2 + y^2\right)^{3/2}} = \frac{1}{4\pi^2 L} \frac{1}{(x-x')^2} \equiv D_4(x,x')$$

4-dim-al law recovered for large intervals as a result of superposition of contributions of large number of images

(in mathematics known as descent method)

Signal Propagation



$$\nabla_5^2 \Phi(X) = -G_5 \cdot J(X) \quad \text{(Einstein equation)}$$

5d Newton constant

Source is on the brane, $J(x,y) = f(t)\,\delta^{(3)}(\vec{x})\,\delta(y)$

$$\Phi(t,\vec{x}) = G_5 \int_{-\infty}^{+\infty} dt' \cdot D_R(t-t', \vec{x})\, f(t')$$

Finite duration T signal: $f(t) = \theta(t) - \theta(t-T)$

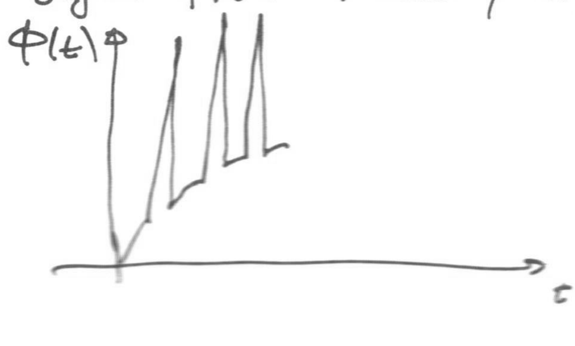
$$\Phi(t,\vec{x}) = \frac{G_5}{16\pi^2 L^3} \cdot t\,\theta(t-\tau)\, I(\alpha,\beta|[\alpha]) - (t \to t-T)$$

$$I(\alpha,\beta|[\alpha]) = \sum_{n=-m}^{m} \frac{1}{(\beta^2+n^2)\sqrt{\alpha^2-n^2}}, \quad m = [\alpha]$$

integer part of α

$$\alpha = \frac{\sqrt{t^2-\tau^2}}{2L}; \quad \beta = \frac{\tau}{2L}$$

- Signal is <u>not</u> continuous!
  there appears a singular spike every time when signal from a new pair of images arrives



Retarded Potentials

The Wick rotation $x_E \equiv (x^4, \vec{x}) = (i x^0, \vec{x})$

$$x_E^2 \equiv (x^4)^2 + \vec{x}^2 = -(x^0)^2 + \vec{x}^2 + i\varepsilon = x^2 + i\varepsilon$$

Feynman propagator: $D_F(x) = i D_E(x_E)\Big|_{x_E = (i x^0, \vec{x})}$

Retarded propagator: $D_R(x) = 2\,\theta(x^0)\cdot \mathrm{Re}\, D_F(x)$

Example: $d=4$, $m^2=0$, $D_F(x) = \frac{1}{4\pi^2}\frac{i}{x^2+i\varepsilon}$

$$D_R(x) = \frac{\theta(x^0)}{2\pi^2}\,\mathrm{Re}\,\frac{i}{x^2+i\varepsilon} = \frac{\theta(t)\cdot\delta(r-t)}{4\pi\cdot r}, \quad t = x^0,\ r = |\vec{x}|$$

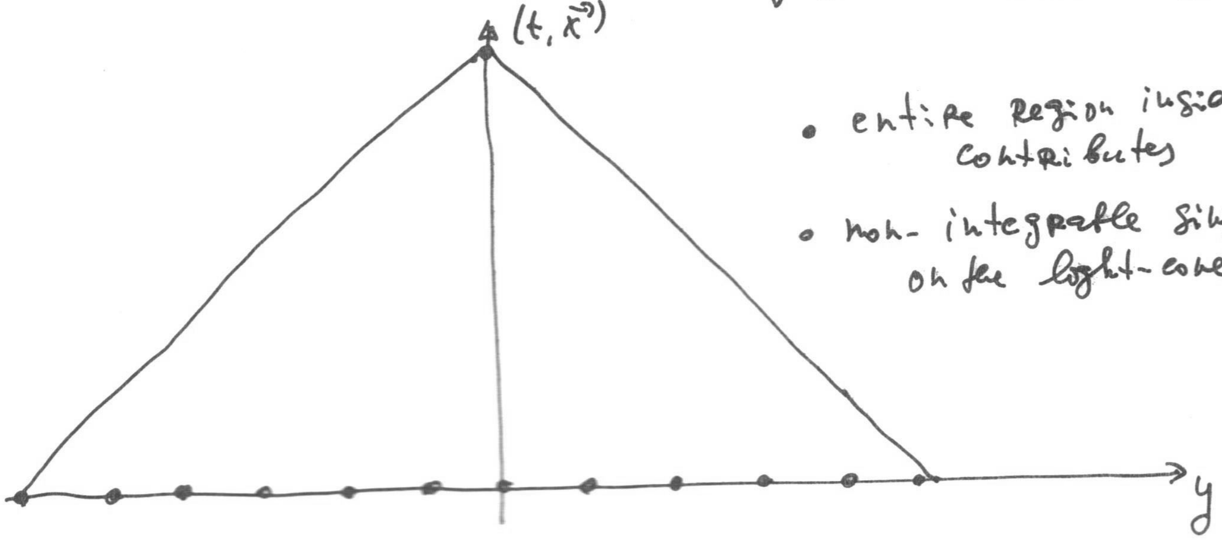
Useful formula: $\frac{1}{x^2+i\varepsilon} - \frac{1}{x^2-i\varepsilon} = -2\pi i\,\delta(x^2)$

$d=5$ (similar but a bit more involved)

$$D_R^{(5)}(r,t) = \frac{1}{4\pi^2}\,\mathrm{Re}\,\frac{i}{(r^2-t^2+i\varepsilon)^{3/2}} = -\frac{1}{4\pi^2}\cdot\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\theta(t-r)}{(t^2-r^2)^{1/2}}\right]$$

Our case: Sum over images

$$D_R(t,r) = -\frac{1}{2\pi^2}\sum_{n=-\infty}^{+\infty}\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\theta\left(t-\sqrt{r^2+4n^2L^2}\right)}{\sqrt{t^2-r^2-4n^2L^2}}\right]$$



- entire region inside LC contributes
- non-integrable singularit. on the light-cone

Further Steps

- replace the sum $\sum_n$ by contour integral, deform contour
- in the limit $\alpha >> 1$ $(t^2 - z^2 >> L^2)$ evaluate the integral
- available devices can not resolve the spike

  – average over period of the spike

  $$\left(\Delta t = 2L \sqrt{1 - \frac{z^2}{t^2}} \rightarrow 2L \right)$$

E.g.

$$< (\alpha - [\alpha])^{-1/2} > \equiv \int_{[\alpha]}^{[\alpha]+1} \frac{d\alpha}{\sqrt{\alpha - [\alpha]}} = 2$$

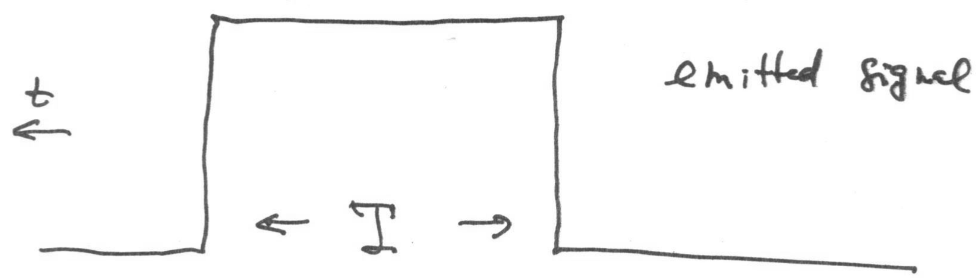
The Result

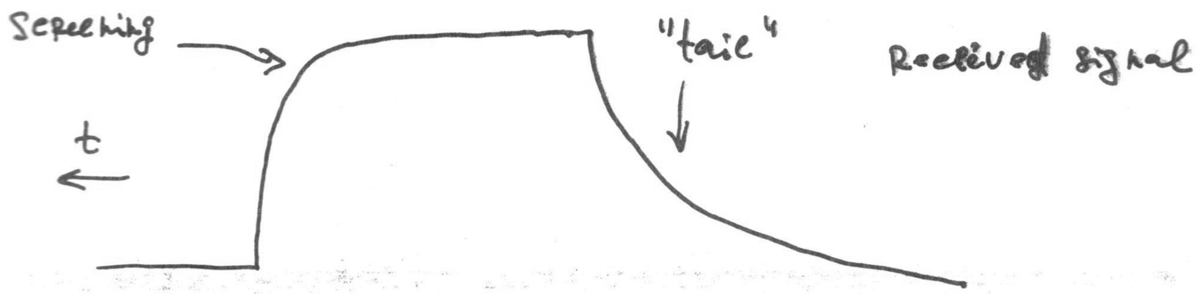
$$\langle \Phi(t, \vec{x}) \rangle = \frac{G_4}{4\pi r}\left(1 + \frac{2}{e^{\pi r/L} - 1}\right)\theta(t-r) -$$

$$- \frac{G_4}{3\pi^2 t}\left(\frac{L^2}{t^2 - r^2}\right)^{1/4}\left(2^{5/2} - 5\right)\cdot\theta(t-r) - (t \to t - T)$$

$G_4 = \frac{G_5}{L}$ - 4d Newton constant

Two effects:





The "screening" effect:

$$\langle \phi(t,\vec{x}) \rangle = \frac{G_4}{4\pi^2}\left(1 - \frac{2}{3\pi}\,\frac{2^{5/2}-5}{\alpha(t)^{1/2}}\right)$$

$$\alpha(t) = \frac{(t^2-\tau^2)^{1/2}}{2L}$$ – number of spikes that have already passed through observer by the time of observation

"Tail":

After entire signal has passed through the observation point our devices still detect some signal

$$\langle \phi(t,\vec{x}) \rangle_{tail} = (2^{5/2}-5)\frac{G_4}{3\pi^2}\left[\frac{L^{1/2}}{(t-T)\left((t-T)^2-\tau^2\right)^{1/4}} - \frac{L^{1/2}}{t\,(t^2-\tau^2)^{1/4}}\right]$$

$t \gg \tau,\ t \gg T$

$$\langle \phi(t,\vec{x}) \rangle_{tail} = (2^{5/2}-5)\frac{G_4}{2\pi^2}\,\frac{L^{1/2}\cdot T}{t^{5/2}}$$

– universal $t^{-5/2}$ behavior

Periodic Signal

$$f(t) = e^{i\omega t}$$

$$\Phi(r,t) = -\frac{G_5}{8\pi i}\, e^{i\omega t}\, \frac{1}{r}\frac{\partial}{\partial r} \sum_{n=-\infty}^{+\infty} H_0^{(2)}\left(\omega\sqrt{r^2+4n^2L^2}\right)$$

Replace $\displaystyle\sum_{n=-\infty}^{+\infty} = \int_{-\infty}^{+\infty} dx$

$$\Phi(r,t) \simeq \Phi_0(r,t) = \frac{G_4}{8\pi r}\, e^{i\omega(t-r)}$$ – purely 4d signal

Correction:

$$\Phi_1(r,t) = \frac{G_4}{8\pi i}\,\frac{L\omega}{r}\, e^{i\omega t}\, H_1^{(2)}(\omega r)$$

$\omega r \gg 1$

$$\Phi_1(r,t) \simeq \sqrt{\frac{2L^2\omega}{\pi r}}\, e^{i\pi/4}\, \Phi_0(r,t)$$

- frequency-dependent amplification of the amplitude $|\Phi_0(r,t) + \Phi_1(r,t)|$

Observable Effects:

- forefront screening
- universal $t^{-5/2}$ tail after backfront
- frequency-dependent amplification of amplitude of periodic signal

$$\Downarrow$$

One day we may discover small extra dimensions by just looking at the sky through "gravitational telescope"