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THE POINCARE-INVARIANT PHASE SPACE WITH FULL KINEMATICS

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THE POINCARE-INVARIANT PHASE SPACE

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by

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Abstract:

A Poincaré-invariant s-particle phase space integral with full kinematics, $\Omega_s(W^2, L^2)$, is formulated as a generalization of the non-relativistic phase space integral at constant energy, momentum, angular momentum, and center of mass. For large particle numbers s the integral $\Omega_s(W^2, L^2)$ is evaluated covariantly with the central limit theorem; the summation

$$\sum_{s=3}^{\infty} \frac{1}{s!} \Omega_s(W^2, L^2)$$

is then carried out for high energies ($m/W \to 0$).

I. Introduction

For a statistical model of elementary particle reactions the fundamental problem is the formulation of a relativistic quantum-mechanical s-particle phase space integral with full kinematics, its evaluation, and the evaluation of such integrals summed over all s [1]. In a previous paper [2] we discussed - in order to gain some insight into the essential features of such integrals - the classical non-relativistic phase space with all integrals of motion of the Galilei group as well as its covariant evaluation by means of the central limit theorem of statistics. In the present paper we shall extend this treatment to the relativistic case; that is, we shall formulate a Poincaré-invariant classical phase space integral $\Omega_s(W^2,L^2)$ at constant total CMS energy W^2 and spin L^2 for a system of s identical scalar particles. For large s this integral will then be evaluated covariantly using the central limit theorem, and finally the sum $\sum_s \frac{1}{s!}\Omega_s(W^2, L^2)$ will be calculated in the high energy limit.

Such a treatment is still a simplification of the problem initially outlined, since - apart from the overall symmetrization factor $1/s!$ in the sum over s - quantum effects are neglected. However, we believe that (in the case of scalar Bose particles) for large particle numbers and high energies such effects do not play the decisive role. This is supported by the fact for large s the non-relativistic classical and quantum-mechanical phase space integrals have very similar features[3]. Thus a classical relativistic model should give quite a good indication of the results of the above mentioned kinematically complete statistical approach.

Before proceeding to the relativistic phase space let us briefly recall the non-relativistic form[2].

For a system of s identical non-relativistic particles (mass m) we denote the kinematic quantities corresponding to total kinetic energy, linear momentum, angular momentum, and center of mass coordinate with T, $\vec{P}$, $\vec{M}$, $\vec{X} = \vec{N}/sm$, respectively. The Galilei invariants for the total CMS energy E and spin $\vec{L}$ are then

$$E = T - \vec{P}^2/2sm \tag{1.1}$$

$$\vec{L}^2 = \left[\vec{M} - \vec{N}\times\vec{P}/sm\right]^2$$

As invariant phase space integral we now set

$$\Omega_s(E,\vec{L}^2) = \int\cdots\int \prod_{i=1}^{s}\left\{d^3p_i\, d^3x_i\right\} e^{-\frac{1}{2sR^2}\sum_{i,j=1}^{s}(\vec{x}_i-\vec{x}_j)^2} \tag{1.2}$$

$$\times\ \delta\left(\sum \frac{\vec{p}_i^{\,2}}{2m} - T\right)\delta^{(3)}\left(\sum \vec{p}_i - \vec{P}\right)\delta^{(3)}\left(\sum \vec{x}_i\times\vec{p}_i - \vec{M}\right)\delta^{(3)}\left(\sum m\vec{x}_i - \vec{N}\right)$$

where the exponential coordinate space restriction provides an "interaction volume". Since

$$\frac{1}{2sR^2}\sum_{i,j=1}^{s}(\vec{x}_i-\vec{x}_j)^2 = \frac{1}{R^2}\sum_{i=1}^{s}(\vec{x}_i - \vec{N}/sm)^2 \tag{1.3}$$

we perform the translation

$$\vec{y}_i = \vec{x}_i - \vec{N}/sm \tag{1.4}$$

to obtain a form quite suitable for relativistic generalization:

$$\Omega_s(E,\vec{L}^2) = \int\cdots\int \prod_{i=1}^{s}\left\{d^3p_i\, d^3y_i\, e^{-\vec{y}_i^{\,2}/R^2}\right\} \times \tag{1.5}$$

$$\times\ \delta\left(\sum \frac{\vec{p}_i^{\,2}}{2m} - T\right)\delta^{(3)}\left(\sum \vec{p}_i - \vec{P}\right)\delta^{(3)}\left(\sum \vec{y}_i\times\vec{p}_i - \vec{L}\right)\delta^{(3)}\left(\sum m\vec{y}_i\right)$$

Here the translational invariance is explicitly clear, as only quantities invariant under translations enter. In section II we shall formulate and discuss the relativistic extension of this integral; the evaluation for large s and for the sum over all s will be then given in sections III and IV, respectively.

II. The Poincaré-Invariant Phase Space Integral

The position and momentum coordinates of a single particle transform under the Poincaré group as

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu \qquad \mu = 0,1,2,3 \tag{2.1}$$
$$p'_\mu = \Lambda_\mu{}^\nu p_\nu$$

where $\Lambda_{\mu\nu}$ denotes a homogeneous Lorentz transformation and a_μ a space-time translation [4]. The generalized angular momentum tensor $m_{\mu\nu}$ is defined as

$$m_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \tag{2.2}$$

Its space-space components (m_{ik};i,k=1,2,3) are the usual angular momentum, its space-time components (m_{ko};k=1,2,3) a position coordinate.

The ten kinematic quantities (corresponding to the ten infinitesimal generators of the Poincaré group) of an s particle system now form the four-vector of total momentum and the antisymmetric four-by-four tensor of (generalized) angular momentum:

$$P_\mu = \sum_{i=1}^{s} p^i_\mu \; ; \quad M_{\mu\nu} = \sum_{i=1}^{s} m^i_{\mu\nu} \tag{2.3}$$

The group invariants for the total CMS energy and spin are

$$W^2 = P_\mu P^\mu \tag{2.4}$$

$$L^2 = - \Gamma_\mu \Gamma^\mu \quad ; \quad \Gamma_\mu = \frac{1}{2W} \varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$$

where the metric $g_{oo}=-g_{11}=-g_{22}=-g_{33}=1$ is used. We now introduce the four-vector

$$K_\mu = \frac{1}{W} M_{\mu\nu} P^\nu \tag{2.5}$$

which in the CMS (always denoted by a superscript o) becomes

$$\overset{o}{K}_\mu = (0, M_{ko}) = (0, \sum_i (\vec{x}_i p_{io} - x_{io} \vec{p}_i)) \tag{2.6}$$

and thus represents a covariant generalization of the center of mass coordinate. With its help we can define, as generalization of the non-relativistic spin vector $\vec{L}$, the antisymmetric spin tensor $L_{\mu\nu}$

$$L_{\mu\nu} = M_{\mu\nu} - [K_\mu P_\nu - K_\nu P_\mu] \; ; \; \frac{1}{2} L_{\mu\nu} L^{\mu\nu} = L^2 ; \; L_{\mu\nu} P^\nu = 0 \tag{2.7}$$

which is also translation invariant, as can easily be verified from (2.1, 2.2, 2.3).

A natural extension of coordinate space integration ($d^3 \vec{x}$) is given by integration over space-like hypersurfaces [6] $d\sigma_\mu = (d^3\vec{x}, dx_o\, dx_2\, dx_3, dx_o\, dx_1\, dx_3, dx_o\, dx_1\, dx_2)$. For the momentum space the invariant measure becomes the usual $d^4p\, \delta(p^2-m^2)\, \theta(p_o)$. Noting finally that

$$m_{\mu\nu} m^{\mu\nu} = -2 m^2 \vec{x}^2 \tag{2.8}$$

for a single particle at rest, we choose as cut-off function

$$\exp\left\{\frac{1}{2m^2R^2}\, m_{\mu\nu} m^{\mu\nu}\right\} \tag{2.9}$$

The further properties of such a coordinate space restriction will be discussed later on.

As a result of the above considerations we now define as our relativistic phase space integral with the full kinematics of the Poincaré group

$$\Omega_s(W^2, L^2) = \int \cdots \int \prod_{i=1}^{s} \left\{ d^4p_i\, \delta(p_i^2-m^2)\, \Theta(p_{i0})\, 2p_i^\mu \cdot \right. \tag{2.10a}$$

$$\left. d\sigma_\mu^i\, e^{m_{\mu\nu}^i m_i^{\mu\nu}/2m^2R^2} \right\} \delta^{(4)}(\sum p_i - P)\, \delta^{(6)}(\sum m_{\mu\nu}^i - L_{\mu\nu})$$

The translation

$$x_\mu \rightarrow x_\mu + K_\mu/W^2$$

then yields as the relativistic analogue of (1.2/1.3):

$$\Omega_s(W^2, L^2) = \int \cdots \int \prod_{i=1}^{s} \left\{ d^4p_i\, \delta(p_i^2-m^2)\, \Theta(p_{i0})\, 2p_i^\mu\, d\sigma_\mu^i \right\} \cdot \tag{2.10b}$$

$$\exp\left\{\frac{1}{2m^2R^2} \sum_{i=1}^{s} \left[(x_\mu^i - \frac{K_\mu}{W^2}) p_\nu^i - (x_\nu^i - \frac{K_\nu}{W^2}) p_\mu^i \right]^2 \right\} \cdot$$

$$\delta^{(4)}(\sum_i p_i - P)\, \delta^{(6)}(\sum m_{\mu\nu}^i - M_{\mu\nu})$$

Equation (2.10) is an invariant definition only if $\Omega_s(W^2, L^2)$ does not depend on the particular choice of space-like hypersurfaces, for which a necessary and sufficient condition is

$$\frac{\partial}{\partial x_\mu} I_\mu(x) = 0 \tag{2.11}$$

where $I_\mu(x)$ denotes the integrand of $\int d\sigma^\mu$. Since

$$p_\alpha \frac{\partial}{\partial x_\alpha} m_{\mu\nu} = p_\alpha \left[\delta_\mu^\alpha p_\nu - \delta_\nu^\alpha p_\mu \right] = 0 \qquad (2.12)$$

this condition is indeed fulfilled, and hence we may, whenever it is convenient, choose in particular the set of surfaces $x_{io}=0$, i.e. replace $p^\mu d\sigma_\mu$ by $p_o d^3\vec{x}$. The integral (2.10) is form-invariant under the homogenous Lorentz group and is a function only of translation invariant quantities - hence its invariance under the full Poincaré group is assured.

Returning now to the coordinate space restriction we have with hypersurfaces $\{ x_{io}=0;\ i=1, \dots ,s \}$

$$m_{\mu\nu} m^{\mu\nu} = -2 \left[p_o^2 \vec{x}^2 - (\vec{x} \times \vec{p})^2 \right] \qquad (2.13)$$

Setting p = (p,o,o) this gives

$$- \frac{m_{\mu\nu} m^{\mu\nu}}{2 m^2 R^2} = \frac{x_1^2}{R^2 (1 - \vec{p}^2/p_o^2)} + \frac{x_2^2}{R^2} + \frac{x_3^2}{R^2} \qquad (2.14)$$

i.e. the interaction volume experiences a Lorentz contraction in the direction of flight, as had been proposed originally by FERMI[6]. Here, however, it follows quite naturally from the invariant form of the phase space integral.

Using (2.4) and (2.7) we now write

$$\delta^{(6)}\Big(\sum_i m_{\mu\nu}^i - L_{\mu\nu} \Big) = \delta\Big(\frac{P^\nu}{W} \sum_i m_{\mu\nu}^i \Big) \delta\Big(\frac{\varepsilon^{\lambda\mu\nu\rho} P_\mu \sum_i m_{\nu\rho}^i}{2W} - \Gamma^\lambda \Big) \qquad (2.15)$$

From this we see that in the non-relativistic limit where

$$\Gamma^\lambda \to (0, \vec{M} - \vec{N} \times \vec{P}/sm) \; ; \; P^\nu/W \to \delta_o^\nu \qquad (2.16)$$

$$m_{\mu\nu} m^{\mu\nu} / 2m^2 R^2 \quad \to \quad - \vec{x}^2 / R^2$$

our form (2.10) gives the original non-relativistic form (1.2).

With the help of (2.15) the phase space at constant energy, i.e. summed over all angular momentum, is found to be

$$\Omega_s(W^2) = \int \cdots \int \prod_{i=1}^{s} \{ d^4p_i \, \delta(p_i^2 - m^2)\, \Theta(p_{i0})\, 2p_i^\mu \times \qquad (2.17)$$

$$\times\, d\sigma_\mu^i \, e^{m_{\mu\nu}^i m_i^{\mu\nu}/2m^2R^2} \} \, \delta^{(4)}(\Sigma p_i - P)\, \delta(\tfrac{P^\nu}{W} \Sigma m_{\mu\nu}^i)$$

In the case of s=2, all integrals may be carried out immediately; the result is

$$\Omega_2(W^2, L^2) = \Omega_2(W^2)\, F_2(W^2, L^2) \qquad (2.18)$$

where

$$\Omega_2(W^2) = \left[\left(\tfrac{\pi}{2}\right)^{3/2} R^3 \left(\tfrac{2m}{W}\right) \frac{8\pi K}{W^2} \right] ; \quad K = \tfrac{1}{2}[W^2 - 4m^2]^{\frac{1}{2}} \qquad (2.19)$$

is the two particle phase space at constant energy and

$$F_2(W^2, L^2) = \frac{e^{-L^2/2K^2R^2}}{(2\pi L)(2K^2R^2)} \quad ; \quad \int d^3L \, F_2(W^2, L^2) = 1 \qquad (2.20)$$

the normalized probability distribution of the angular momentum.

Had we used the (non-invariant) cut-off $\exp\{-\vec{x}^2/R^2\}$ we would have obtained the usual FERMI phase space

$$\Omega_2^{FERMI}(W^2) = \left[\left(\tfrac{\pi}{2}\right)^{3/2} R^3 \, \frac{8\pi K}{W^2} \right] \qquad (2.21)$$

Thus we see from (2.19) that the invariant form (2.10) does indeed lead to a Lorentz contraction of the interaction volume, R^3 (2m/W), which moreover is in the direction of flight of the two particles, since the "radius" R occurring in the angular momentum distribution is uncontracted.

III. The Phase Space Integral for Large s.

Since the application of the central limit theorem in evaluating $\Omega_s(W^2, L^2)$ for large s differs very little from the non-relativistic case, we sketch the calculation only briefly. In contrast to the non-relativistic case, however, the procedure here is covariant under the full group in every step.

We define as single particle generating function

$$\varphi(\xi,\eta) = \int \frac{d^3p}{2p_0} e^{-\xi_\mu p^\mu} 2p_0 d^3x \, e^{-\frac{1}{2}\eta_{\mu\nu} m^{\mu\nu} + \frac{m_{\mu\nu} m^{\mu\nu}}{2m^2R^2}} \qquad (3.1)$$

where ξ_μ and $\eta_{\mu\nu}$ are the contravariants to P_μ and $L_{\mu\nu}$. The function $\varphi(\xi,\eta)$ exists for all ξ with $\{ \mathrm{Re}\, \xi_0 > 0, (\mathrm{Re}\,\xi_0)^2 > (\mathrm{Re}\,\vec{\xi})^2 \}$ and all pure imaginary $\eta_{\mu\nu}$. With its help $\Omega_s(W^2, L^2)$ can be written

$$\Omega_s(W^2, L^2) = e^{\alpha_\mu P^\mu} [\varphi(\alpha,0)]^s \, U_s^\alpha(W^2, L^2) \qquad (3.2)$$

where α is real and

$$U_s^\alpha(W^2, L^2) = \int \dots \int \prod_{i=1}^{s} \Big\{ \frac{d^3p_i}{2p_{i0}} e^{-\alpha_\mu p_i^\mu} 2p_{i0} d^3x_i \, \varphi^{-1}(\alpha,0) \times \qquad (3.3)$$

$$\times \, e^{m^i_{\mu\nu} m_i^{\mu\nu}/2m^2R^2} \Big\} \, \delta^{(4)}(\sum p_i - P) \, \delta^{(6)}(\sum m^i_{\mu\nu} - L_{\mu\nu})$$

The function $\varphi(\alpha,0)$ is found to be

$$\varphi(\alpha,0) = (2\pi^{3/2} m R^3)\left(\frac{2\pi m}{\alpha} K_1(m\alpha)\right) ; \quad \alpha = \sqrt{\alpha^2} \qquad (3.4)$$

i.e., up to a constant factor it is $\Delta^{+}(-i\alpha)$. Since $U_s^{\alpha}(W^2, L^2)$ is positive-definite and normalized

$$\int d^4P \, d^6 L_{\mu\nu} \, \mathcal{U}_s^{\alpha}(W^2, L^2) = 1 \tag{3.5}$$

we can apply the central limit theorem.

The characteristic function of U_s^{α} is

$$\Psi_s^{\alpha}(\beta, \gamma) = \int d^4P \, d^6 L_{\mu\nu} \, e^{i\beta_\mu P^\mu + \frac{i}{2}\gamma_{\mu\nu}L^{\mu\nu}} \, \mathcal{U}_s^{\alpha}(W^2, L^2) \tag{3.6}$$

$$= \left[\varphi(\alpha - i\beta, -i\gamma) / \varphi(\alpha, 0) \right]^s$$

where from (3.1)

$$\varphi(\alpha - i\beta, -i\gamma) = 2\pi^{3/2} m R^3 \int \frac{d^3p}{2p_0} \, e^{-(\alpha_\mu - i\beta_\mu)p^\mu + \frac{R^2}{4}\gamma_{\mu\nu}p^\nu \gamma^{\mu\sigma} p_\sigma} \tag{3.7}$$

with $\gamma_{\mu\nu}p^\nu \gamma^{\mu\sigma} p_\sigma$ negative-definite. Combining P_μ, $L_{\mu\nu}$ and β_μ, $\gamma_{\mu\nu}$ to A_k and ξ_k, $k = 1, \ldots, 10$, respectively, we have for the first moments

$$\bar{A}_k(\alpha) = \int d^{10}A \; A_k \, \mathcal{U}_s^{\alpha}(A) = \left[\frac{\partial \Psi_s^{\alpha}}{\partial i \xi_k} \right]_{\xi = 0} \tag{3.8}$$

and for the ten-by-ten dispersion matrix

$$B_{k\ell}(\alpha) = \int d^{10}A \, (A_k - \bar{A}_k)(A_\ell - \bar{A}_\ell) \, \mathcal{U}_s^{\alpha}(A) = \left[\frac{\partial^2 \Psi_s^{\alpha}}{\partial i \xi_k \, \partial i \xi_\ell} \right]_{\xi = 0} \tag{3.9}$$

The central limit theorem then yields as asymptotic form for large s the Gaussian

$$\mathcal{U}_s^{\alpha}(A) \rightarrow (2\pi)^{-5} (\det B_{k\ell})^{-1/2} \, e^{-\frac{1}{2}(B^{-1})_{k\ell}(A_k - \bar{A}_k)(A_\ell - \bar{A}_\ell)} \tag{3.10}$$

Correction terms of higher orders in (1/s) can be calculated for (3.10) in the usual fashion [7)8)].

It remains to fix α ; we follow the standard procedure of KHINCHIN[7] in choosing the (unique) solution of

$$\bar{P}_\mu(\alpha) = P_\mu \tag{3.11}$$

and denote the thus determined value with $\bar{\alpha}$.
For the first moments we find

$$\bar{P}_\mu(\alpha) = -\frac{s\alpha_\mu}{\alpha\,\varphi(\alpha)}\frac{d\varphi}{d\alpha} \equiv s\,Q(\alpha)\,\alpha_\mu \, ; \quad \bar{L}_{\mu\nu}(\alpha) = 0 \tag{3.12}$$

The dispersion matrix B_{kl} (α) is the direct sum of the usual[8] four-by-four momentum dispersion matrix

$$B_{\mu\nu}(\alpha) = s\left[m^2 + 4Q - \alpha^2 Q^2\right]\frac{\alpha_\mu \alpha_\nu}{\alpha^2} - s\,Q\,g_{\mu\nu} \tag{3.13}$$

and the six-by-six angular momentum dispersion matrix

$$B_{\mu\nu,\rho\sigma}(\alpha) = -\frac{sR^2}{2}\left\{\frac{(m^2+4Q)}{\alpha^2}\left[g_{\mu\rho}\alpha_\nu\alpha_\sigma + g_{\nu\sigma}\alpha_\mu\alpha_\rho + g_{\mu\sigma}\alpha_\nu\alpha_\rho + g_{\nu\rho}\alpha_\mu\alpha_\sigma\right] - 2Q\left[g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right]\right\} \tag{3.14}$$

A straight-forward calculation now gives, upon resubstitution of (3.10) into (3.2), as asymptotic phase space for large s

$$\Omega_s^{as}(W^2, L^2) = \Omega_s^{as}(W^2)\; F_s(W^2, L^2) \tag{3.15}$$

where

$$\Omega_s^{as}(W^2) = \left\{ e^{\bar{\alpha}_\mu P^\mu}\left(\frac{s}{2\pi}\right)^{7/2}\left[\varphi(\bar{\alpha},0)\right]^s \times \left[\bar{Q}^3\left(s^2m^2 + 4\bar{Q} - \frac{\bar{\alpha}^2\bar{Q}^2}{s^2}\right)R^6\left(\bar{Q} + s^2 m^2/2\right)\right]^{-1/2}\right\} \tag{3.16}$$

is the asymptotic form of the phase space at constant energy (2.17), and

$$F_s(W^2, L^2) = s^{3/2}\left[2\pi R^2 \bar{Q}\right]^{-3/2} e^{-sL^2/2R^2\bar{Q}} \tag{3.17}$$

is the normalized angular momentum probability distribution. We have here used the notation

$$\bar{Q}(\bar{\alpha}) \equiv s^2 Q(\bar{\alpha}) = \frac{sW}{\bar{\alpha}} \tag{3.18}$$

Finally $\bar{\alpha}$ (W) is determined as solution of (3.11):

$$-\frac{s}{\varphi(\bar{\alpha},0)} \frac{d\varphi(\bar{\alpha},0)}{d\bar{\alpha}} = W \tag{3.19}$$

with $\varphi(\bar{\alpha},0)$ given by (3.4). Equation (3.19), which is identical to the determining equation for $\bar{\alpha}$ in the momentum space evaluation via central limit theorem[8)], in general has to be solved by numerical calculations. We refer to 8), where these calculations are performed for a wide range of energies.

In the extreme-relativistic limit (sm/W→0), equation (3.19) may be solved in closed form to give

$$\bar{\alpha} = \frac{2s}{W} \tag{3.20}$$

This yields

$$\Omega_s^{ER}(W^2) = \left[4\pi^{5/2} mR^3W^2\right]\left(\frac{e}{2s}\right)^{2s}\left(\frac{s}{2\pi}\right)^{4/2} \frac{2^3}{(RW)^3\,W^4} \tag{3.21}$$

$$F_s^{ER}(W^2, L^2) = s^{3/2}\left[\pi R^2 W^2\right]^{-3/2} e^{-sL^2/R^2W^2} \tag{3.22}$$

The exact extreme-relativistic momentum space integral is known to give

$$Q_s^{ER}(W^2) = \int\cdots\int \prod_{i=1}^{s} \left\{ \frac{d^3p_i}{2p_{i0}} \right\} \delta^{(4)}(\Sigma p_i - P) \approx \frac{\pi}{2} \frac{[\pi W^2/2]^{s-2}}{(s-1)!\,(s-2)!} \qquad (3.23)$$

Thus we see that (3.21) as in the non-relativistic case is (apart from factors due to the center of mass restriction) the momentum space integral with Stirling's approximation applied to the Γ -functions.

The dispersion of the angular momentum probability density is

$$\sigma = R^2 W^2/s \qquad (3.24)$$

i.e., the angular momenta are effectively "cut-off" for $L^2 > R^2W^2/s$. For s=2 this cut-off occurs (see 2.20) for $L^2 > 2K^2R^2 = W^2R^2/2$. Thus at fixed energy the phase space becomes more isotropic the larger the particle number s is, in the sense that the main contributions then come from lower angular momenta. For fixed angular momentum, the function $F_s(W^2, L^2)$ has a maximum if

$$W^2 = \frac{2}{3} \frac{sL^2}{R^2} \qquad (3.25)$$

Now we have for a single particle the mean square values

$$\langle \vec{p}^{\,2} \rangle = W/s \qquad (3.26a)$$

$$\langle \vec{x}^{\,2} \rangle = \frac{3}{2} R^2 \qquad (3.26b)$$

$$\langle \sin^2\theta \rangle = 1 \qquad (3.26c)$$

where θ is the angle between $\vec{x}$ and $\vec{p}$. Relation (3.26) is due to the fact that in the extreme-relativistic limit the the interaction volume is contracted to zero in the direction of flight. Thus from (3.25) and (3.26) the probability of obtaining a particular fixed value L_o^2 becomes maximal for an energy such that

$$L_o^2 = s < [\vec{x} \wedge \vec{p}]^2 > \tag{3.27}$$

i.e., if L_0^2 is the sum of the individual mean square angular momenta.

IV. The Sum over s : E - R Limit

To begin with, let us recall the approximations involved in our calculation of $\Omega_s(W^2,L^2)$ for large s. In the result (3.10/15) of the central limit theorem application we have retained only the leading term in s and to obtain the explicit expression (3.20) for $\bar{\alpha}(W)$ we have considered the extreme-relativistic limit ($sm/W \to 0$). Both these approximations (which with increasing particle number and energy converge to the exact form) can easily be removed in a numerical evaluation; the first by calculating higher order terms[8] in (1/s) for (3.1o/15), the second by using the exact solution of (3.19) as e.g. tabulated in ref. 8).

Since our aim here is study the general high energy features of the relativistic phase space we shall however use the extreme-relativistic form (3.21/22) in order to obtain a closed expression for the sum

$$\Omega_R(W^2, L^2) = \sum_{s=3}^{\infty} \frac{1}{s!} \Omega_s(W^2, L^2) \tag{4.1}$$

In the similar case[9] of summing over s the phase space integrals $(mR^3)^s Q_s(W^2)$ at constant energy (see 3.23), it can be seen that the sum for $m \neq 0$ converges to the (exactly soluble) sum for m=0 only in the so-called thermodynamic limit[6], i.e., at quite high energies. For quantitative questions at intermediate energies it would thus be necessary to investigate correction terms arising from the above mentioned approximations.

The particle-number dependence of the extreme-relativistic phase space (3.21/22)

$$s^5 \left[\frac{e}{2s}\right]^{2s} \left[4\pi^{5/2} m R^3 W^2 e^{-L^2/W^2R^2}\right]^s \tag{4.2}$$

is within the framework of the central limit theorem evaluation equivalent to

$$2\pi \quad s \left[\pi^{5/2} m R^3 W^2 e^{-L^2/W^2R^2}\right]^s \Big/ (s-1)!\,(s-2)! \tag{4.3}$$

since Stirling's formula applied to (4.3) gives (4.2). Thus the sum (4.1) over s becomes

$$\Omega_R^{ER}(W^2, L^2) = \frac{2^{1/2}}{\pi^4 R^6 W^{10}} \sum_{s=3}^{\infty} \frac{\left[\pi^{5/2} m R^3 W^2 e^{-\frac{L^2}{W^2R^2}}\right]^s}{(s-1)!\,(s-2)!\,(s-3)!} \tag{4.4}$$

This can be written

$$\Omega_R^{ER}(W^2, L^2) = \left\{\frac{\pi^{7/2} m^3 R^3}{2^{1/2} W^4} e^{-3\frac{L^2}{W^2R^2}}\right\} \left\{2\sum_{s=0}^{\infty} \frac{\left[\pi^{5/2} m R^3 W^2 e^{-\frac{L^2}{W^2R^2}}\right]^s}{s!\,\Gamma(s+2)\,\Gamma(s+3)}\right\} \tag{4.5}$$

The sum in (4.5) is just the generalized hypergeometric function [10]

$${}_0F_2(2,3;x) = \sum_{s=0}^{\infty} \frac{x^s}{s!} \frac{\Gamma(2)\,\Gamma(3)}{\Gamma(s+2)\,\Gamma(s+3)} \tag{4.6}$$

and the first term on the r.h.s. of (4.5) is the three-particle e-r phase space. Thus we obtain as the e-r limit of the sum over all relativistic s-particle phase space integrals (s ⩾ 3)

$$\Omega_R^{ER}(W^2, L^2) = \Omega_3^{ER}(W^2, L^2)\; {}_0F_2\left(2,3;\pi^{5/2} m R^3 W^2 e^{-\frac{L^2}{W^2R^2}}\right) \tag{4.7}$$

The function ${}_0F_2(2,3;x)$ goes to unity for small x

$${}_0F_2(2,3;x) = 1 + \frac{x}{6} + O(x^2) \tag{4.8}$$

and diverges exponentially for large x

$$ {}_0F_2(2.3;x) = \frac{e^{3x^{1/3}}}{3^{1/2}\pi x^{4/3}}\left[1+\frac{8}{9x^{1/3}}+O(x^{-2/3})\right] \qquad (4.9) $$

Hence the total relativistic phase space as function of W^2 and L^2 exhibits the following behaviour in the limit of high energies:

(a) for $L^2/W^2 \to \infty$ it converges to the three-particle phase space;

(b) for $L^2/W^2 \to 0$ it diverges as more and more particle number components come into play.

Thus we have again the expected behaviour that the more isotropic (small L^2) parts of $\Omega_R^{ER}(W^2,L^2)$ are dominated by large, the more anisotropic (large L^2) parts by small particle numbers.

V. Concluding Remarks

The relativistic classical phase space integral is thus seen to exhibit on one hand many features quite similar to the non-relativistic case, in particular the factorization $\Omega_s(W^2)\ F_s(W^2,L^2)$, while on the other hand the Poincaré-invariant formulation leads quite naturally to an interaction volume Lorentz-contracted in the direction of flight. The functional form (2.10) is moreover that of an uncorrelated assembly of s particles, which, besides being useful for the covariant application of statistical methods, would facilitate an investigation of different types of cut-off functions. The sum over s finally leads to an asymptotic total phase space form expressing the increasing dominance of large particle number systems with increasing isotropy.

The present results will subsequently be applied in the evaluation of a relativistic statistical model with full kinematics.

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