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Derivation of Sum Rules for the Isovector Nucleon Form Factor from a Divergence Condition.

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Abstract

Sum rules for the isovector nucleon form factor are derived by starting from a divergence condition describing how conservation of the isovector-vector current of hadrons is broken by electromagnetism in lowest order.

I. INTRODUCTION

Recently, Veltman gave divergence conditions[1] which describe how the conservation of the isovector current of hadrons is broken by electromagnetic and weak interactions in lowest order. It was stressed that the Cabibbo-Radicati relation may be obtained by using the divergence condition as a starting point instead of the algebra of the isovector current. The Adler-Weisberger relation was also derived.

In this note we start from the divergence condition which describes the violation of the hadron current by electromagnetism and give a derivation of four Compton scattering sum rules. These were previously derived by Gourdin[2] from current algebra by using Fubini's method[3].

In particular, in Section 2, essentially following Veltmann, we derive the general conditions which certain matrix elements have to fulfil on account of the divergence condition. In Section 3, the sum rules are given by assuming dispersion relations for the invariant amplitudes and using the results of Section 2. Section 3 thus closely follows the procedure used in the algebra of currents approach[2]. Some conclusions are drawn in Section 4.

2. CONDITION FOR MATRIX ELEMENTS.

Let J^i_λ be the isovector part of the hadron vector current. Introducing the projections on the following base vectors in isospace

$$\vec{e}^{(t)} = \begin{cases} -\frac{1}{\sqrt{2}}(\vec{e}_1+i\vec{e}_2) & \text{for } t = 1 \\ \vec{e}_3 & \text{for } t = 0 \\ \frac{1}{\sqrt{2}}(\vec{e}_1-i\vec{e}_2) & \text{for } t = -1 \end{cases} \tag{1}$$

from minimality of the electromagnetic interaction we have in first order of the electromagnetic coupling[1]

$$\begin{aligned} \partial^\lambda J^{(\pm)}_\lambda &= ie\, A^\lambda J^{(\pm)}_\lambda \\ \partial^\lambda J^{(o)}_\lambda &= 0. \end{aligned} \tag{2}$$

In this expression A^λ is a free electromagnetic field. Let us now consider the following amplitude, where the initial state contains a proton P (p_1) and the final state contains a neutron N (p_2) and a photon γ (q_2):

$$T^-_\mu \equiv \int d^4x e^{iq_1x} <N,\gamma|\ J^-_\mu(x)|P>,$$

$$q_1 \equiv q_2+p_2-p_1. \qquad (3)$$

Using translation invariance we have

$$q_1^\mu T^-_\mu = i \int d^4x\ e^{iq_1x} <N,\gamma|\partial^\lambda J^-_\lambda(x)|P> \qquad (4)$$

Introducing the divergence condition (2) and simplifying the resulting expression, we have in first order of electromagnetism

$$q_1^\mu\ T^-_\mu = -\frac{ee^\lambda}{\sqrt{2q_2^o}}(2\pi)^4\delta\big(q_1+p_1-q_2-p_2\big)<N|J^-_\lambda(o)|P>. \qquad (5)$$

We now define the amplitude $T^{o-}_{\mu\nu}$ by

$$e\frac{e^\mu}{\sqrt{2q_2^o}}(2\pi)^4\delta\big(p_2+q_2-p_1-q_1\big)\,T^{\,o-}_{\mu\nu} = T^-_\nu \qquad (6)$$

and from (5) we then have

$$T^{o-}_{\mu\nu}q_1^\nu = -<N|J^-_\mu\ (o)|\ P> . \qquad (7)$$

The same argument holds if instead of the outgoing γ we have a lepton pair e, e' interacting in one-photon exchange (q_2^2 spacelike)

$$T^-_\mu = \int d^4x e^{iq_1x} <N,e'|\ J^-_\mu(x)|e,P> .$$

In this case A^λ in (2) is replaced by the lepton current operator and in lowest order of the electromagnetic interaction we again arrive at (7).

The matrix element of the isovector current appearing on the righthand side of (7) is given by

$$<p_2|J^{(t)}_\mu(o)|p_1> = \bar u(p_2)\tfrac{1}{2}\tau^{(t)}\Big[(i\gamma_\mu-\tfrac{1}{M}P_\mu)(F_1^V+F_2^V)+\frac{P_\mu}{M}F_1^V\Big]u(p_1) \qquad (8)$$

where $\tau^{(t)} = \vec{\tau}\vec{e}^{(t)}$, $\vec{\tau} = (\tau^1, \tau^2, \tau^3)$ and τ^i are the Pauli matrices. M is the nucleon mass and $P^\mu = \frac{1}{2}(p_1^\mu + p_2^\mu)$, $t = -(p_1-p_2)^2$ and $F_1^V(t)$ and $F_2^V(t)$ are the isovector nucleon form factors.

For the derivation of sum rules we furthermore need equations for the absorptive part $t_{\mu\nu}^{t't}$ of $T_{\mu\nu}^{t't}$. For the absorptive part we have

$$t_{\mu\nu}^{t't} = (-1)^{t'+1} \frac{1}{2} \int d^4x e^{iq_1x} \langle N| \left[J_\mu^{(-t')}(o), J_\nu^{(t)}(x) \right] |P\rangle \qquad (9)$$

From translation invariance, as may be seen in Appendix 1, we obtain

$$t_{\mu\nu}^{t't} q_1^\nu = + i(-1)^{t'+1} \frac{1}{2} \int d^4x e^{iq_1x} \langle N| \left[J_\mu^{(-t')}(o), \partial^\nu J_\nu^{(t)}(x) \right] |P\rangle \qquad (10)$$

and thus from (2) we have in first order of electromagnetism

$$t_{\mu\nu}^{t't} q_1^\nu = 0. \qquad (11)$$

3. DERIVATION OF SUM RULES.

The discussion of this section now will closely follow the dispersion theoretic method introduced by Fubini[3] for the derivation of sum rules from current algebras and used by Gourdin[2] for the derivation of Compton scattering sum rules. $T_{\mu\nu}^{t't}$ and $t_{\mu\nu}^{t't}$ are written in terms of 32 scalar amplitudes in the form given by Gourdin[2]. For easy reference the expansion is written down in Appendix 2. From equations (7) and (8), we now deduce (among others) the following equations

$$- \nu A^{o-} + q_1^2 B_1^{o-} + q_1 \cdot q_2 B_2^{o-} = - \frac{\sqrt{2}}{2M} F_1^V(t)$$

$$- \nu \bar{B}_4^{o-} + q_1^2 \bar{C}_4^{o-} + q_1 \cdot q_2 \bar{C}_3^{o-} + \frac{1}{4M} E_4^{o-} + \frac{1}{2} \bar{E}_4^{o-} = 0$$

$$- \nu D_1^{o-} + q_1^2 E_1^{o-} + q_1 \cdot q_2 E_2^{o-} = - \frac{\sqrt{2}}{2} (F_1^V(t) + F_2^V(t)) \qquad (12)$$

$$- \nu \bar{D}_1^{o-} + q_1^2 \bar{E}_1^{o-} + q_1 \cdot q_2 \bar{E}_2^{o-} = 0$$

where the amplitudes $A^{o-}, \bar{B}_4^{o-}, ..$ are functions of $\nu = - q_1 \cdot P = - q_2 \cdot P, t$, q_1^2 and q_2^2. For the absorptive amplitudes we have corresponding equations with zero on the righthand side, for instance

$$- \nu a^{o-} + q_1^2 b_1^{o-} + q_1 \cdot q_2 b_2^{o-} = 0. \tag{13}$$

We now assume unsubtracted dispersion relations for all the scalar amplitudes in (12). For example, we have

$$A^{o-}\left(\nu, t, q_1^2, q_2^2\right) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{a^{o-}(\nu')}{\nu' - \nu} \, d\nu' \; . \tag{14}$$

Substitution of the dispersion relations into (12), using Equations (13) and the crossing relations (A12) of Appendix 2, finally yields

$$\begin{aligned}
\frac{1}{\pi} \int_0^{\infty} \left(a^{o-}(\nu') + a^{-o}(\nu')\right) d\nu' &= - \frac{\sqrt{2}}{2M} F_1^V(t) \\
\frac{1}{\pi} \int_0^{\infty} \left(\bar{b}_4^{o-}(\nu') + \bar{b}_4^{-o}(\nu')\right) d\nu' &= 0 \\
\frac{1}{\pi} \int_0^{\infty} \left(d_1^{o-}(\nu') + d_1^{-o}(\nu')\right) d\nu' &= \left(-\frac{\sqrt{2}}{2\pi}\right)\left(F_1^V(t) + F_2^V(t)\right) \\
\frac{1}{\pi} \int_0^{\infty} \left(\bar{d}_1^{o-}(\nu') + \bar{d}_1^{-o}(\nu')\right) d\nu' &= 0 \; .
\end{aligned} \tag{15}$$

Expressing the amplitudes in terms of amplitudes belonging to total isospins $\frac{1}{2}$ and $\frac{3}{2}$ as defined by (A13) and extracting Born terms[2)] gives the following sum rules for virtual Compton scattering which were first written down by Gourdin:

$$\begin{aligned}
&\left[F_1^V(q^2)\right]^2 + \frac{q_1 \cdot q_2}{4M^2} \left[F_2^V(q^2)\right]^2 + \frac{M}{\pi} \int_{(M+\mu)^2}^{\infty} \rho_2(s,t,q^2) ds = F_1^V(t) \\
&\frac{q^4}{4Mt(4M^2Q^2+q^2)} \left[F_1^V(q^2) + F_2^V(q^2)\right]^2 + \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \rho_7(s,t,q^2) ds = 0
\end{aligned} \tag{16}$$

$$\frac{1}{2} F_1^V(q^2) \left[F_1^V(q^2) + F_2^V(q^2)\right] + \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \rho_9(s,t,q^2)ds = F_1^V(t) + F_2^V(t)$$

$$\frac{MQ^2}{4M^2Q^2+q^4} \left[F_1^V(q^2) - \frac{q^4}{4M^2Q^2} F_2^V(q^2)\right] \left[F_1^V(q^2)+F_2^V(q^2)\right] +$$

$$+ \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \rho_{10}(s,t,q^2)ds = 0$$

By specialising $q^2 = 0$ we obtain sum rules for real Compton scattering. Finally, the sum rule for electroproduction given by Kramer and Meetz and others[5] and the Cabibbo-Radicati relation[6] may be derived from (16). We refer to 2) for details.

4. CONCLUSIONS.

In this rather straightforward investigation we have shown that four Compton scattering sum rules previously obtained from the algebra of the isovector electromagnetic current may also be obtained from the divergence condition (2). This condition is intuitionally rather evident and also follows from well established principles: Conservation of the isovector hadron current as long as electromagnetic and weak interactions are neglected and breaking of this conservation law by electromagnetism according to the principle of minimal electromagnetic interaction[1]. The assumption of unsubtracted dispersion relations is common to this and the derivation resting on the algebra of currents. But it should be emphasized that the assumption of unsubtracted dispersion relations in this work is made for a T-matrix amplitude, whereas it is made for the retarded commutator in the algebra of currents approach. By using what he calls the Adler method, Gourdin derived a few more sum rules for Compton scattering. In his derivation the expression for $\frac{\partial}{\partial x_\rho} J_\mu^t$ plays a part. Thus it may well be that these sum rules are not obtainable from the divergence condition only.

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Appendix 1.

To show relation (11), let us start by defining[4)]

$$A_{\mu\nu}(y) \equiv \frac{1}{2}\int d^4x e^{i(q_1+p)(x+y)} \langle p_2|\left[J_\mu(-x),\, J_\nu(y)\right]|p_1\rangle \quad \text{(A1)}$$

where

$$p \equiv p_1 - p_2 \quad \text{(A2)}$$

and where the isospin indices of the currents have been dropped, as they are unimportant in this context. From translation invariance we have

$$A_{\mu\nu}(y) = \frac{1}{2} e^{ipy}\int d^4x\; e^{iq_1(x+y)} \langle p_2|\left[J_\mu(o), J_\nu(x+y)\right]|\,p_1\rangle \quad \text{(A3)}$$

Comparison with (9) yields

$$A_{\mu\nu}(o) = t_{\mu\nu} \quad \text{(A4)}$$

From (A3), by changing the variable of integration we find

$$A_{\mu\nu}(y) = e^{ipy}t_{\mu\nu} \quad \text{(A5)}$$

By differentiation we finally get from (A1) and (A5)

$$i(q_1^\nu+p^\nu)A_{\mu\nu}(y) + \frac{1}{2}\int d^4x e^{i(q_1+p)(x+y)} \langle p_2|\left[J_\mu(-x), \partial^\nu J_\nu(x)\right]|\,p_1\rangle = \quad \text{(A6)}$$

$$= ip^\nu e^{ipy}t_{\mu\nu}$$

Using translation invariance again and specialising y=0 we arrive at (10).

Appendix 2.

Using Lorentz invariance and P-invariance $T^{t't}_{\rho\sigma}$ may be shown to depend on 32 scalar amplitudes. We use the following expansion[2)]

$$T^{t't}_{\rho\sigma} = \bar{u}(p_2)\, M^{t't}_{\rho\sigma}\, u(p_1)$$

$$
\begin{aligned}
M^{t't}_{\rho\sigma} = {} & P_\rho P_\sigma (A^{t't} + [\not{q}_1, \not{q}_2] \bar{A}^{t't}) + \\
& + P_\rho q_{1\sigma} (B_1^{t't} + [\not{q}_1, \not{q}_2] \bar{B}_1^{t't}) \\
& + P_\rho q_{2\sigma} (B_2^{t't} + [\not{q}_1, \not{q}_2] \bar{B}_2^{t't}) \\
& + q_{1\rho} P_\sigma (B_3^{t't} + [\not{q}_1, \not{q}_2] \bar{B}_3^{t't}) \\
& + q_{2\rho} P_\sigma (B_4^{t't} + [\not{q}_1, \not{q}_2] \bar{B}_4^{t't}) \\
& + q_{1\rho} q_{2\sigma} (C_1^{t't} + [\not{q}_1, \not{q}_2] \bar{C}_1^{t't}) \\
& + q_{1\rho} q_{1\sigma} (C_2^{t't} + [\not{q}_1, \not{q}_2] \bar{C}_2^{t't}) \\
& + q_{2\rho} q_{2\sigma} (C_3^{t't} + [\not{q}_1, \not{q}_2] \bar{C}_3^{t't}) \\
& + q_{2\rho} q_{1\sigma} (C_4^{t't} + \not{q}_1, \not{q}_2 \quad \bar{C}_4^{t't}) \\
& + g_{\rho\sigma} \quad (C_5^{t't} + [\not{q}_1, \not{q}_2] \bar{C}_5^{t't}) \\
& + (i\gamma_\rho - \frac{P_\rho}{M}) P_\sigma D_1^{t't} + [\gamma_\rho, \not{Q}] P_\sigma \bar{D}_1^{t't} \\
& + P_\rho (i\gamma_\sigma - \frac{P_\sigma}{M}) D_2^{t't} + P_\rho [\gamma_\sigma, \not{Q}] \bar{D}_2^{t't} \\
& + (i\gamma_\rho - \frac{P_\rho}{M}) q_{1\sigma} E_1^{t't} + [\gamma_\rho, \not{Q}] q_{1\sigma} \bar{E}_1^{t't} \\
& + (i\gamma_\rho - \frac{P_\rho}{M}) q_{2\sigma} E_2^{t't} + [\gamma_\rho, \not{Q}] q_{2\sigma} E_2^{t't} \\
& + q_{1\rho} (i\gamma_\sigma - \frac{P_\sigma}{M}) E_3^{t't} + q_{1\rho} [\gamma_\sigma, \not{Q}] \bar{E}_3^{t't} \\
& + q_{2\rho} (i\gamma_\sigma - \frac{P_\sigma}{M}) E_4^{t't} + q_{2\rho} [\gamma_\sigma, \not{Q}] \bar{E}_4^{t't} .
\end{aligned}
$$

In this expression $P^\mu = \frac{1}{2}(p_1^\mu+p_2^\mu)$, $Q^\mu = \frac{1}{2}(q_1^\mu+q_2^\mu)$ and the scalar functions depend on 4 independent variables chosen to be ν, t, q_1^2 and q_2^2, where

$$\nu = - q_1 \cdot P = - q_2 \cdot P = - Q \cdot P \tag{A8}$$

$$t = - (p_1-p_2)^2 = - (q_1-q_2)^2$$

For the absorptive part $t_{\rho\sigma}^{t't}$ we write

$$t_{\rho\sigma}^{t't} = \bar{u}(p_2)\, m_{\rho\sigma}^{t't}\, u(p_1) \tag{A9}$$

and the expansion follows from (A7) by substituting corresponding small letters for the capital letters describing scalar amplitudes.
Finally, for the case $q_1^2 = q_2^2 \equiv q^2$, by using gauge invariance and (11) and T-invariance, the number of amplitudes may be reduced to 12:

$$m_{\rho\sigma}^{t't} = \sum_{r=1}^{12} h_r^{t't} I_{\rho\sigma}^{(r)} \quad \text{where} \tag{A10}$$

$$I_{\rho\sigma}^{(1)} = g_{\rho\sigma} - \frac{1}{q_1 \cdot q_2} q_{1\rho} q_{2\sigma}$$

$$I_{\rho\sigma}^{(2)} = P_{2\rho} P_{1\sigma}$$

$$I_{\rho\sigma}^{(3)} = P_{2\rho} Q_{1\sigma} + Q_{2\rho} P_{1\sigma}$$

$$I_{\rho\sigma}^{(4)} = Q_{2\rho} Q_{1\sigma}$$

$$I_{\rho\sigma}^{(5)} = \left(g_{\rho\sigma} - \frac{1}{q_1 \cdot q_2} q_{1\rho} q_{2\sigma}\right) [\not{q}_1, \not{q}_2]$$

$$I_{\rho\sigma}^{(6)} = P_{2\rho} P_{1\sigma} [\not{q}_1, \not{q}_2]$$

$$I_{\rho\sigma}^{(7)} = \left(P_{2\rho} Q_{1\sigma} + Q_{2\rho} P_{1\sigma}\right) [\not{q}_1, \not{q}_2]$$

$$I_{\rho\sigma}^{(8)} = Q_{2\rho} Q_{1\sigma} [\not{q}_1, \not{q}_2]$$

$$I^{(9)}_{\rho\sigma} = \left(i\gamma_\rho - i\not{Q}\,\frac{q_{1\rho}}{q_1\cdot q_2}\right) P_{1\sigma} + P_{2\rho}\left(i\gamma_\sigma - i\not{Q}\,\frac{q_{2\sigma}}{q_1\cdot q_2}\right) - \frac{2}{M}\, I^{(2)}$$

$$I^{(10)} = \left(\left[\gamma_\rho, \not{Q}\right] + \frac{1}{2}\left[\not{q}_1, \not{q}_2\right] \frac{q_{1\rho}}{q_1\cdot q_2}\right) P_{1\sigma} -$$

$$- P_{2\rho}\left(\left[\gamma_\sigma, \not{Q}\right] - \frac{1}{2} \left[\not{q}_1, \not{q}_2\right] \frac{q_{2\sigma}}{q_1\cdot q_2}\right)$$

$$I^{(11)} = \left(i\gamma_\rho - i\not{Q}\,\frac{q_{1\rho}}{q_1\cdot q_2}\right) Q_{1\sigma} + Q_{2\rho}\left(i\gamma_\sigma - i\not{Q}\,\frac{q_{2\sigma}}{q_1\cdot q_2}\right) - \frac{1}{M}\, I^{(3)}$$

$$I^{(12)} = \left(\left[\gamma_\rho, \not{Q}\right] + \frac{1}{2} \left[\not{q}_1, \not{q}_2\right]\frac{q_{1\rho}}{q_1\cdot q_2}\right) Q_{1\sigma} -$$

$$- Q_{2\rho} \left(\left[\gamma_\sigma, \not{Q}\right] - \frac{1}{2} \left[\not{q}_1, \not{q}_2\right]\frac{q_{2\sigma},}{q_1\cdot q_2}\right)$$

and

$$P_2 = P + \frac{\nu}{q_1\cdot q_2}\, q_1, \qquad P_1 = P + \frac{\nu}{q_2\cdot q_1}\, q_2$$

$$Q_2 = q_2 - \frac{q^2}{q_1\cdot q_2}\, q_1, \qquad Q_1 = q_1 - \frac{q^2}{q_2\cdot q_1}\, q_2$$

In our discussion, we need the following relations between $a^{t't}$, $\bar{a}^{t't}$,... and $h_r^{t't}$

$$a^{t't} = h_2^{t't}, \qquad \bar{b}_4^{t't} = h_7^{t't}$$

$$d_1^{t't} = h_9^{t't}, \qquad \bar{d}_1^{t't} = h_{10}^{t't} \tag{A11}$$

For the amplitudes of (A11) we note the crossing relations

$$h_j^{t',t}\,(-\nu) = -\,(-1)^{t'-t}\, h_j^{-t,-t'} \tag{A12}$$

Finally, introducing the amplitudes $h_j^{(1)}$ and $h_j^{(3)}$ belonging to total isospin 1/2 and 3/2 we have

$$h_j^{o-} + h_j^{-o} = -\frac{2}{3}\sqrt{2}\,\left(h_j^{(1)} - h_j^{(3)}\right) \equiv -\sqrt{2}\,\rho_j \tag{A13}$$

Footnotes and References

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