DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 67/26
August 1967

Equal Time Commutators of Renormalized Current Operators in Perturbation Theory without Cutoff

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Abstract

Equal time commutators of finite, renormalized current densities are considered in the first two orders of perturbation theory for the models of photon-nucleon and meson-nucleon interaction. After giving a definition of "equal times" for distributions, we calculate without the use of a cutoff, the commutators of vector currents in electrodynamics for the cases where a higher symmetry exists, and where it is broken. Extra terms are found for the j_o, j_o commutator as well as Schwinger terms. Divergences are obtained in this model for the j_k, j_k commutator and for the j_o, j_o commutator in the meson model. Discrepancies arising from the use of a cutoff are discussed and the properties of the extra terms are examined.

I. Introduction

We study here the equal time commutators of non integrated current densities formed in zeroth and first orders of perturbation theory of the interaction of photons with nucleons via Yukawa coupling with the object of obtaining the most important information concerning these quantities rather than in any way attempting a complete treatment of all possible types of currents which we may construct from Dirac matrices. Thus, when the arguments begin to depend on the type of current studied, we will restrict ourselves to the discussion of vector and axial vector currents, these being the only ones that seem to have been of use up to the present. Since simply setting times equal leads to divergent and ambiguous integrals, we are forced to use a limiting process to consider the equal time quantities. This process is neither new nor strange and has already been used by most authors attempting to make rigorous deductions from the assumption of canonical relations. After a precise definition of the equal time limit, we compute the zeroth order commutator (where the fields used are ordinary free fields) and discuss the divergence occurring in the vacuum expectation value. The first order commutator splits naturally into two sets of terms, the first of which are extremely well behaved and which lead precisely to the canonically expected right hand side. The other terms, which in first order are proportional to the Bose field and which contain all the difficulties, will be the origin of any possible non canonical terms. Their limits will be computed and examined as to the most interesting properties.

The most important question is whether the equal time commutator exists, i.e. whether these quantities converge as the equal time limit is taken. If it exists, the next question is whether it vanishes or whether it contributes a non canonical term to the commutator. In the case where the limit exists and does not vanish, it will be of interest to check whether it vanishes after one or two spatial integrations, and, since we are dealing with quantum electrodynamics, whether it is gauge invariant.

Although we have already specified the problem essentially by choosing the perurbational approach to a specific interaction, the calculation will differ from the others in several essential respects. Several authors[1,4] have done the calculation with unrenormalized current operators in the commutator as well as on the right hand side of the equation, keeping apparently divergent terms in check with a cutoff and considering the canonical commutation relations verified when the divergent terms on both sides of the equation can be made equal. The spirit of this work differs from the previous ones in that the currents are renormalized from the beginning, and the apparently divergent terms obtained from the commutator are analyzed more closely and evaluated in the cases where they are finite. A perhaps less serious difference is that we will not introduce any more singularities into the already quite troublesome equal time limit by time ordering the commutator before taking the limit. In [1], for instance, it appears necessary to throw away several polynomials in momentum space on the grounds that they will lead to δ -functions in the time difference variable, and that equal times will be approached through a sequence of unequal times making these contributions vanish identically. In the most important cases which we study, however, the equal time limit will not depend on whether the times are allowed to be equal during the process or not. For a more precise formulation of these considerations, see section 5. As a result of not time ordering, the Feynman formalism does not prove most advantageous for the computation, but this is a matter of method and not of principle.

Our approach differs greatly in another respect from that of Brandt[6] who has calculated the non canonical terms in all orders under the assumption that the renormalized fields satisfy modified canonical commutation relations. Since these commutators can be calculated from perturbation theory as well as those of currents, this statement will be considered here not as assumption, but as a hypothesis capable of being tested.

A third difference is the absence of a cutoff or of any sort of regularization and the correponding increase in the difficulty of the calculation. Hamprecht[5] pointed out, shortly after the appearance of their preprint, that the extra terms of Johnson and Low could be modified and even made to vanish by manipulations with the cutoff that had to be introduced to make them finite. Although this cutoff is not necessary if renormalized currents are used, it simplifies the calculation considerably and leads to temptingly simple and easily interpretable results (see section 6 for this calculation). However this process clearly involves an at best questionable interchange of limits under circumstances in which caution is most urgently recommended as indicated by the uncontrollable behavior of the several divergent terms which should at least partially cancel each other, and as confirmed by the variety of answers obtainable by varying the cutoff. Since the integrals involved are all finite provided renormalized currents are inserted into the commutator, the question of whether they have equal time limits is well posed mathematically, and the limits can be obtained without any further assuptions; this is done in the next two sections for the rather easy case of free fields (zeroth order) and for the first order commutator in quantum electrodynamics of the not necessarily conserved isotopic spin currents. In the following two sections, the properties of the extra terms are discussed and the results compared with those of the authors who have taken one or more of the above mentioned shortcuts in the calculation. Section VII reviews some of the differences which the choice of cutoff can make in the extra terms. Some considerations relevant to the results although not directly connected with the calculation as well as some of the more tedious aspects of the computation of the traces of Dirac matrices are postponed to appendices.

II. Preliminaries

The Yukawa-type coupling is given by the interaction Lagrangian $\mathcal{L}_I = G_{\alpha\beta\gamma}\bar{\psi}^\alpha A^\beta \psi^\gamma = \bar{\psi}\varphi\psi$ where we define $\varphi_{\alpha\gamma} = G_{\alpha\beta\gamma} A^\beta$ and use matrix notation to avoid indices. ψ and A are spinor and tensor fields degenerate along representation spaces of some internal symmetry group. The field equations (without indices) are

$$(\Box + m^2) A = j \qquad A(x) = \int \Delta_R(x-y)\, j(y)$$

$$(\gamma\partial - m)\psi = f \qquad \psi(x) = \int S_R(x-y)\, f(y)$$

where $j_\beta = \bar{\psi}^\alpha G_{\alpha\beta\gamma} \psi^\gamma$ and $f_\alpha = G_{\alpha\beta\gamma} A^\beta \psi^\gamma$ i.e. $f = \varphi\psi$

The following table should clarify the notation:

Interaction	Representation Space ψ	Representation Space A	φ
electron-photon	1 dim	1dim	$e\gamma^\mu A_\mu$
nucleon-photon	2 dim isospin space	1 dim	$e\frac{1+\tau_3}{2}\gamma^\mu A_\mu$
quark-meson 1	3 dim SU_3	1 dim	$\rho_s\phi_s + \rho_{ps}\phi_{ps}\gamma_5$
octet-octet	8 dim SU_3	8 dim SU_3	$(a f_{\alpha\beta\gamma} + b g_{\alpha\beta\gamma})\phi^\beta$

If λ is a matrix in the representation (generalized isospin-) space of ψ and $\gamma_?$ is a Dirac matrix, then the current constructed from the spinors corresponding to the Kronecker product $\Lambda = \lambda \otimes \gamma_?$ is designated and defined as follows:

$$j_\Lambda(x) = \bar{\psi}(x)\Lambda\psi(x)$$

where it is understood that all renormalization steps necessary to produce finite operators in Quantum electrodynamics are also to be taken here. Up to second order, this amounts to subtracting the vacuum expectation value and multiplies of φ, and performing the usual charge renormalization*).

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*) As is done in [2].

Once the current operators are obtained and their commutators are constructed in the various orders, it is necessary to have a procedure for passing to equal times. In order that this treatment retain as wide a range of applicability as possible, the following procedure, which seems to make the mildest distribution-theoretical requirements and still produce operator functions of x and y, will be employed. Discussion of this topic is postponed to a later section. The commutator will be smeared by the function $f(\vec{\xi},\vec{X})\, g(X_0)\, \phi(\xi_0)$ $[\xi = x-y\,,\ X = \frac{1}{2}(x+y)]$ where f and g remain arbitrary and ϕ will be made to approach a δ-function, i.e. $\tilde{\phi} \rightarrow 1$. This limiting procedure will be illustrated for the case of the free field current commutator which, formed according to the contraction rules for Wick products from $j_\Lambda(x) = i:\bar{\psi}(x)\Lambda\psi(x):\ ,\quad j_\Sigma(y) = i:\bar{\psi}(y)\Sigma\psi(y):$ is

$$i:\bar{\psi}(x)\Lambda S(x-y)\Sigma\psi(y): - i:\bar{\psi}(y)\Sigma S(y-x)\Lambda\psi(x): - \sum_{\pm} \pm S_p\left(\Lambda S^{(\pm)}(x-y)\,\Sigma\, S^{(\mp)}(y-x)\right)$$

When the first term is smeared with $f(\vec{\xi},\vec{X})\, g(X_0)\, \phi(\xi_0)$, the result is

$$i\int \tilde{f}\left(\tfrac{1}{2}(\vec{p}+\vec{q})+\vec{r},\ \vec{p}+\vec{q}\right)\tilde{g}(p_0+q_0)\,\tilde{\phi}\left(\tfrac{1}{2}(p_0+q_0)+r_0\right):\tilde{\bar{\psi}}(p)\Lambda(i\gamma r-m)\Sigma\tilde{\psi}(q):\,d\Omega^{(\varepsilon)}(r)\,d^4p,q$$

where the tilde's represent Fourier transforms and $d\Omega^{(\varepsilon)}(r) = \varepsilon(r_0)\,\delta(r^2+m^2)\,d^4r$ The limit $\tilde{\phi} \rightarrow 1$ can be performed with no trouble, and from the anti-symmetry of the measure, all terms in $i\gamma r - m$ drop out except $i\gamma^0 r_0$. The integral then becomes

$$i^2\int \tilde{f}(\vec{r},\vec{p}+\vec{q})\,d^3\vec{r}\ g(p_0+q_0):\tilde{\bar{\psi}}(p)\Lambda\gamma_0\Sigma\tilde{\psi}(q):\,d^4p,q$$

The $\vec{r}$-integration of $\tilde{f}$ is the same as smearing f with $\delta(\vec{\xi})$ and with this done, we recognize [3] the defining expression for

$$\int d^4X,\xi\ \ i\delta(\vec{\xi})\ f(\vec{\xi},\vec{X})\,g(X_0)\ i:\bar{\psi}\Lambda\gamma_0\Sigma\psi:(X)$$

With the notation $[\Lambda,\gamma_0,\Sigma] = \Lambda\gamma_0\Sigma - \Sigma\gamma_0\Lambda$ one has the rigorous result

$$[j_\Lambda(x), j_\Sigma(y)]^{(0)} - \langle[\cdot\cdot]\rangle_0 \xrightarrow[\xi_0\to 0]{} i\,\delta(\vec{\xi})\ j_{[\Lambda,\gamma_0,\Sigma]}(X)$$

III. First Order

The renormalized current is, in first order

$$j^{(1)}_{\Lambda}(x) = \int d^4y \left(:\bar\psi(x)\Lambda S_R(x-y)\varphi(y)\psi(y): + :\bar\psi(y)\varphi(y)S_A(y-x)\Lambda\psi(x): \right)$$

and the commutator, in first order, is

$$[j_\Lambda(x), j_\Sigma(y)]^{(1)} = [j^{(0)}_\Lambda(x), j^{(1)}_\Sigma(y)] + [j^{(1)}_\Lambda(x), j^{(0)}_\Sigma(y)]$$

where each term on the right can be obtained from the other by changing the sign and exchanging (x,Λ) with (y,Σ). A calculation of the first gives (where the z-integration is understood)

$$-:\bar\psi(x)\Lambda S(x-y)\Sigma S_R(y-z)\varphi(z)\psi(z): + :\bar\psi(y)\Sigma S_R(y-z)\varphi(z)S(z-x)\Lambda\psi(x):$$

$$-:\bar\psi(x)\Lambda S(x-z)\varphi(z)S_A(z-y)\Sigma\psi(y): + :\bar\psi(z)\varphi(z)S_A(z-y)\Sigma S(y-x)\Lambda\psi(x):$$

$$-\sum_{\pm}\pm\left(S_p(\Lambda S^{(\pm)}(x-y)\Sigma S_R(y-z)\varphi(z)S^{(\mp)}(z-x)) + S_p(S^{(\pm)}(x-z)\varphi(z)S_A(z-y)\Sigma S^{(\mp)}(y-x)\Lambda)\right)$$

The treatment used for the free-field currents is applied to the first and fourth terms and their counterparts in the reversed commutator because they contain $S(\xi)$ the result is the expected $i\delta(\vec\xi)\, j^{(1)}_{[\Lambda,\gamma,\Sigma]}(X)$. The second term should be grouped with the counterpart of the third (and vice versa) which together amount to

$$:\bar\psi(y)\Sigma S_R(y-z)\varphi(z)S(z-x)\Lambda\psi(x): + :\bar\psi(y)\Sigma S(y-z)\varphi(z)S_A(z-x)\Lambda\psi(x):$$

and will cancel each other. The formal argument is that since $S_R(x) = -\theta(-x)\ S(x)$ the θ function at equal times may be moved over to the S function changing it to S_A and thus making the first term cancel the second exactly. A rigorous proof which does not interchange limits and deal in cancelation of infinities is given in appendix 1.

IV. Extra Terms in First Order

The terms arising in the commutator after two Wick-contractions are responsible for the extra terms; these will be proportional to the 0th order electromagnetic field and therefore contribute only to matrix elements in which the photon number differs by one. This corresponds, therefore to the calculation carried out by Johnson and Low [1] for mesons. We will restrict ourselves, however, to the case $\Lambda = \gamma_\mu \otimes \lambda \quad \Sigma = \gamma_\nu \otimes \sigma$.

The two-contraction terms are:

$$-i\int \sum_{\pm} \pm \Big(S_p\big(\Lambda S^{(\pm)}(x-y)\,\Sigma\, S_R(y-z)\,\varphi(z)\,S^{(\mp)}(z-x)\big) + S_p\big(S^{(\pm)}(x-z)\,\varphi(z)\,S_A(z-y)\,\Sigma\, S^{(\mp)}(y-x)\,\Lambda\big)\Big)\, d^4z - (x\Lambda \leftrightarrow y\Sigma)$$

In the case that the symmetry is not broken, i.e. the mass matrices are scalars and λ, σ commute with the coupling matrix in $\varphi(x)$, we may conclude from the Furry-theorem-argument:

$$S_p\Big(C^{-1}\big(S^{(\pm)}(x-z)\,\varphi(z)\,S_A(z-y)\,\Sigma\, S^{(\mp)}(y-x)\,\Lambda\big)^T C\Big) = S_p\Big(\Lambda^C S^{(\pm)}(x-y)\,\Sigma^C S_R(y-z)\big(-\varphi(z)\big)\, S^{(\mp)}(z-x)\Big)$$

that the terms in the brackets cancel each other, for in the case which we study, $\Lambda^C = -\gamma^\mu \otimes \lambda^T$, $\Sigma^C = -\gamma^\nu \otimes \sigma^T$ and $S_p(\lambda^T \sigma^T \tau^T) = S_p(\lambda\sigma\tau)$.
Therefore, we consider a natural and simple case of symmetry breaking where $\varphi(z) = \tau\gamma_\mu A^\mu(z)$; $\tau = \frac{1}{2}(1+\tau_3)$, $\Lambda = \tau_+ \otimes \gamma_\mu$, $\Sigma = \tau_- \otimes \gamma_\nu$. With these choices, half of the terms disappear because $S_p(\tau_+\tau_-\tau) = 2$, $S_p(\tau_-\tau_+\tau) = 0$.
The remaining two terms are then

$$-2i\int \sum_{\pm} \pm \Big(S_p\big(\gamma_\mu S^{(\pm)}(x-y)\,\gamma_\nu\, S_R(y-z)\,\varphi(z)\,S^{(\mp)}(z-x)\big) + S_p\big(\gamma_\mu S^{(\pm)}(x-y)\,\gamma_\nu\, S^{(\mp)}(y-z)\,\varphi(z)\,S_A(z-x)\big)\Big)\, d^4z =$$

$$= \frac{-2i}{(2\pi)^6}\int \sum_{\pm} \pm \Big(S_p\Big(\gamma_\mu (i\gamma p - m)\gamma_\nu \frac{i\gamma(q+k)-m}{(q+k)^2+m^2}\,\tilde{\varphi}(k)\,(i\gamma q - m)\Big) e^{i(p-q)\xi + iky} +$$
$$+ S_p\Big(\gamma_\mu (i\gamma p - m)\gamma_\nu (i\gamma q - m)\,\tilde{\varphi}(k)\,\frac{i\gamma(q-k)-m}{(q-k)^2+m^2}\Big) e^{i(p-q)\xi + ikx}\Big)\, d\Omega^{(\pm)}(p)\, d\Omega^{(\mp)}(q)\, d^4k$$

In the second half, we make the transformation $p, q \to -p, -q$ and apply the charge conjugation matrix inside the trace with the result

$$S_p\left(\gamma_\nu (i\gamma p - m)\gamma_\mu \frac{i\gamma(q+k)-m}{(q+k)^2+m^2}(-\tilde{\varphi}(k))(i\gamma q - m)\right) e^{-i(p-q-\frac{1}{2}k)\xi + \frac{i}{2}kX} \, d\Omega^{(\pm)}(p)\, d\Omega^{(\mp)}(q)$$

The extra minus sign from φ^c disappears upon rearranging the sum over $\pm$. This last symmetry allows us to simplify the calculation in the two cases of interest.

If $\nu=\mu$, the two trace terms will be identical and smearing with a function $f'(\xi)g(X)$ gives

$$-\frac{i}{\pi^6}\int \sum_{\pm} \pm S_p(\gamma_\mu(i\gamma p - m)\gamma_\mu \frac{i\gamma(q+k)-m}{(q+k)^2+m^2}\, \tilde{\varphi}(k)\,(i\gamma q - m)\tilde{f}'_S(\vec{p}-q-\tfrac{1}{2}k)\, \tilde{g}(k)\, d\Omega^{(\pm)}(p)\, d\Omega^{(\mp)}(q)\, d^4k$$

where $f_S(\vec{\xi}) = \frac{1}{2}(f(\xi)+f(-\xi))$. We will want to choose again $f'(\xi) = f(\xi)\phi(\xi_0)$ with ϕ symmetric so that the smearing function f must be symmetrized. This has the effect of insuring that derivatives of δ-functions occur only to even orders. The other case of interest is $\mu = 0 \neq \nu$ (spacelike) where only the terms antisymmetric in $\vec{\xi}$ will be calculated (the terms predicted by Schwinger). The choice of an antisymmetric f requires us to subtract the trace terms giving

$$-\frac{2i}{(2\pi)^2}\int \sum_{\pm} \pm S_p([\gamma_0, i\gamma p - m, \gamma_\nu] \frac{i\gamma(q+k)-m}{(q+k)^2+m^2}\, \tilde{\varphi}(k)\,(i\gamma q - m))\tilde{f}'_A(p-q-\tfrac{1}{2}k)\, \tilde{g}(k)\, d\Omega^{(\pm)}(p)\, d\Omega^{(\mp)}(q)\, d^4k$$

In order to get rid of the summation over $\pm$ we make the transformation $p_0, q_0 \to \pm p_0, \pm q_0$. Ignoring the irrelevant factors and with the obvious schematization, these expressions acquire the form $d\Omega^{(+)}(p)\, d\Omega^{(-)}(q)\, X$

$$\sum_{\pm} \pm S_p(\cdots \pm \cdots)\tilde{\phi}(p_0 - q_0 \mp \tfrac{1}{2}k_0) = \sum_{\pm} \pm S_p(\cdots \pm \cdots)\tilde{\phi}_S(p_0-q_0; k_0) - \sum_{\pm} S_p(\cdots \pm \cdots)\tilde{\phi}_A(p_0 - q_0; k_0)$$

where $\phi_S(p,k) = \frac{1}{2}\sum_{\pm}\phi(p \pm k)$, $\phi_A(p,k) = \frac{1}{2}\sum \pm \phi(p \pm k)$, $\phi = \phi_A + \phi_S$. The calculation of $\sum_{\pm}(\pm)^\epsilon S_p(\cdots \pm \cdots)$ is carried out in appendix 2 and summarized in the following table where we make the transformation $q \to -q$ to have both integration variables on the forward mass shell. Here $\parallel$ and $\perp$ refer to the components parallel and perpendicular to the photon vector $\vec{k}$.

Commutator	$\varepsilon = 1$	$\varepsilon = 0$	
$\mu = \nu$	$\frac{2q_0(tq+m^2)\vec{q}_\perp\vec{\varphi}}{k_0(q_\perp^2+m^2)} + \frac{(t_0 q_\parallel - q_0 t_\parallel)\vec{q}_\perp\vec{\varphi}}{q_\perp^2+m^2}$	$\frac{2i(tq+m^2)\varphi_0}{k_0} - \left[\frac{2i(tq+m^2)q_\parallel}{k_0} - i(t_0 q_0 - t_\parallel q_\parallel)\right]\frac{\vec{q}_\perp\vec{\varphi}}{q_\perp^2+m^2} - i\vec{t}_\perp\vec{\varphi}$	$\times \frac{1}{\pi^2}$
$\mu = 0 \neq \nu$	$\frac{(m^2+q_\perp p)\vec{\varphi}_\perp\vec{e} + \vec{q}_\perp\vec{e}\,\vec{p}_\perp\vec{\varphi}}{q_\perp^2+m^2}\, q_0$	$\frac{(-m^2+\vec{q}_\perp\vec{p})\vec{\varphi}_\perp\vec{e} + \vec{q}_\perp\vec{e}\,p_\perp\vec{\varphi}}{q_\perp^2+m^2}\, q_0 + p_\parallel\vec{\varphi}_\perp\vec{e} - \vec{p}_\perp\vec{\varphi}\,\vec{k}_\perp\vec{e}$	$\times \frac{1}{2\pi^2}$

We now specify the δ-sequence to be inserted for ϕ in the computation; it will be formed from a function from $\mathcal{S}$ by dilation of the argument: $\phi_\alpha(t) = \alpha\phi(\alpha t)$; $\tilde{\phi}_\alpha(p) = \tilde{\phi}(\frac{p}{\alpha}) \longrightarrow \tilde{\phi}(0) = 1$, $(\alpha \to \infty)$. Because of the lack of symmetry in the argument that the function $\tilde{\phi}$ acquires, we make the following Taylor expansion:

$$\tilde{\phi}\left(\frac{p_0+q_0 \mp \frac{1}{2}k_0}{\alpha}\right) = \tilde{\phi}\left(\frac{2q}{\alpha}+\varepsilon_\mp\right) = \tilde{\phi}\left(\frac{2q}{\alpha}\right) + \varepsilon_\mp\tilde{\phi}'\left(\frac{2q}{\alpha}\right) + \cdots + \frac{\varepsilon_\mp^N}{N!}\tilde{\phi}^{(N)}\left(\frac{2q}{\alpha}+\vartheta\varepsilon_\pm\right)$$

$$0<\vartheta<1 \qquad \varepsilon_\pm = \delta \pm \frac{k_0}{2\alpha}$$

$$\delta = \frac{p_0+q_0}{\alpha} - \frac{2q}{\alpha} = \frac{p_0-q_0}{\alpha} + \frac{2q_0-q}{\alpha} = \frac{p_0^2-q_0^2}{\alpha(p_0+q_0)} + \frac{2m^2}{\alpha(q_0+q)}$$

The functions $\tilde{\phi}_S$ and $\tilde{\phi}_A$ will have the expansions ($\varepsilon_\pm = \delta \pm \frac{k_0}{2\alpha}$)

$$\tilde{\phi}_S(\cdots) = \tilde{\phi}\left(\frac{2q}{\alpha}\right) + \delta\,\tilde{\phi}'\left(\frac{2q}{\alpha}\right) + \frac{1}{2}\left(\delta^2 + \left(\frac{k_0}{2\alpha}\right)^2\right)\tilde{\phi}''\left(\frac{2q}{\alpha}\right) + \cdots$$

$$\tilde{\phi}_A(\cdots) = \frac{k_0}{2\alpha}\tilde{\phi}'\left(\frac{2q}{\alpha}\right) + \frac{k_0\,\delta}{4\alpha}\tilde{\phi}''\left(\frac{2q}{\alpha}\right) \qquad (\star)$$

To justify throwing away the remainder after a certain number of leading terms as well as to compute the limits of the rest, we will rely heavily on Lebesgue's bounded convergence criterion which states that if a sequence of functions $\{f_\alpha\}$ is bounded by an integrable function: $|f_\alpha| \leq g$, $\int g\,d\mu < \infty$ and if the functions f_α converge pointwise to 0: $f_\alpha(x) \to 0$ for almost all x, then the sequence of integrals $\int f_\alpha(x)\,d\mu(x)$ will also approach 0. To apply this, we will make first the transformation

$$\vec{p} \to \vec{r} = \vec{p} + \vec{q} - \tfrac{1}{2}\vec{k} \qquad d^3\vec{r} = d^3\vec{p}\ ; \qquad \vec{p} = \vec{r} + \tfrac{1}{2}\vec{k} - \vec{q}$$

and notice that the quantity $p_0^2 - q_0^2$ becomes $-2\vec{q}\,(\vec{r}+\tfrac{1}{2}\vec{k}) + |\vec{r}+\tfrac{1}{2}\vec{k}|^2$, a polynomial of first degree in $\vec{q}$. From this it follows that $\alpha\varepsilon_\pm = \alpha\delta \pm \frac{k_0}{2}$ is bounded by a polynomial in $\vec{r}$ and $\vec{k}$ since $(|\vec{q}| \leq q_0 \geq m)$

$$|\alpha\delta| \leq \frac{2|\vec{q}||\vec{r}+\frac{1}{2}\vec{k}| + |\vec{r}+\frac{1}{2}\vec{k}|^2}{p_0+q_0} + \frac{2m}{q_0+q} \leq 2|\vec{r}+\tfrac{1}{2}\vec{k}| + \tfrac{1}{m}|\vec{r}+\tfrac{1}{2}\vec{k}|^2 + 2 = P(\vec{r},\vec{k})$$

After the above transformation, the smearing functions without $\tilde{\phi}$ are $\tilde{f}(\vec{r})\,\tilde{g}(k)$ whose absolute values are integrable when multiplied by any polynomial. Thus the problem reduces to proving that the rest of the integrand satisfies Lebesgue's criterion in q. To do so, we make the substitution $\vec{q} \to \alpha\vec{q}$ so that the argument of each of the functions in (★) becomes 2q except for the remainder terms, having $2q + \varepsilon_\pm$. Since $\tilde{\phi}^{(N)}$ also falls off faster than any polynomial, the question of boundedness can be restricted to the rest of the integrand for the leading terms. As for the remainder, we first remark that $|\varepsilon_\pm'| \leq \frac{P(r,k)}{\alpha}$ so that for a certain value of N, the rest of the integrand will approach 0 pointwise and be bounded by a polynomial. Secondly, from the fact that $\tilde{\phi}^{(N)}$ is in $\mathcal{S}$, we may replace it by $(1 + |2q - |\varepsilon_\pm||)^{-m}$ $(1 + |2q - P(r,k)|)^{-m}$ when $\alpha > 1$. In addition, from the functions f and g, we may extract a factor of $(1 + P(r,k))^{-m}$ without disturbing their falloff properties, which, together with the first term gives a fall-off of

$$(1+|2q-P|)^{-m} \times (1+P)^{-m} \leq \left(1 + P + |2q - P|\right)^{-m} \leq (1+2q)^{-m}$$

which, for sufficiently large m, causes the integrand to satisfy Lebegue's criterion in q. The number of leading terms that must be considered will, of course depend on the polynomial strength of the remainder of the integrand in q since this determines how many powers of α will have to be cancelled by $(\varepsilon_\pm')^N$.

An interesting observation that may be made with the help of the expansions (★) is the effect that a cutoff would have on the computation. If the cutoff is to be of any use at all, it should restrict the integration to a bounded region and thereby allow the limit $\alpha \to \infty$ to be taken under the integral sign. However, any cut-off of this sort would allow all the terms in $\tilde{\phi}_A$ to be discarded and all but the leading one in $\tilde{\phi}_S$ since these terms all have at least one power of α in the denominator. Although the part that is left may depend on the more detailed nature of the cutoff, it is clear that in any case the contribution from $\tilde{\phi}_A$ would vanish.

A. Zeroth Components

Considering, now, the commutator of the 0th components of the $\tau_{\pm}$ currents, a glance at the preceeding table shows that the entire non gauge invariant part of this commutator, i.e. the contribution proportional to φ_0 comes precisely from the antisymmetric part of $\tilde{\phi}$. Since this particular contribution is not only easy to calculate, but also a source of several interesting results, we will complete the calculation of it explicitly. Putting together the various constituents and making the $\vec{p} \to \vec{r}$ transformation, we find the expression

$$-\frac{i}{\pi^2}\int \frac{2(tq+m^2)}{k_0}\,\frac{d^3\vec{r}\,d^3\vec{q}}{p_0\;\;q_0}\,\tilde{f}_s(\vec{r})\,\tilde{g}(k)\,\varphi_0(k)\,d^4k\left(\frac{k_0}{2\alpha}\tilde{\phi}'\left(\frac{2q}{\alpha}\right)+\frac{k_0}{4\alpha}\,\delta\,\tilde{\phi}''\left(\frac{2q}{\alpha}\right)+\cdots\right)$$

in which $t\gamma = \gamma_0 p \gamma \gamma_0$ i.e. $t = \mathcal{P}p$ so that $tq = p_0 q_0 + \vec{p}\vec{q} = q\sqrt{(\vec{q}-\vec{r}-\frac{1}{2}\vec{k})^2+m^2} -$

$$-\vec{q}^2+\vec{q}(\vec{r}+\tfrac{1}{2}\vec{k}) = q_0^2\sqrt{1+\frac{-2(\vec{r}+\frac{1}{2}\vec{k})\vec{q}+(\vec{r}+\frac{1}{2}\vec{k})^2}{q_0^2}} - q_0^2 + \vec{q}(\vec{r}+\tfrac{1}{2}\vec{k}) + m^2 =$$

$$= (\vec{r}+\tfrac{1}{2}\vec{k})^2 + m^2 - \frac{[(-2\vec{r}+\vec{k})\vec{q}+(\vec{r}+\frac{1}{2}\vec{k})^2]}{4q_0^2} + \cdots \qquad (q_0^2 \geq 3(\vec{r}+\tfrac{1}{2}\vec{k})^2)$$

To use this expansion and be able to majorize the remainder by the last written term, we break up the q-integration into the two regions: $0 \leq \vec{q}^2 < C^2$ and $C^2 \leq \vec{q}^2 < \infty$ with $C^2 = 4(\vec{r}+\frac{1}{2}\vec{k})^2$. The first of these will vanish as remarked above as a result of the cutoff. In the second integral, all terms past the first two can be grouped together as the remainder in the binomial expansion*) and easily seen to satisfy Lebesgue's criterion (replacing q_0 by m as often as necessary and

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*) The remainder after N terms in the binomial expansion is, up to a constant factor, $x^N(1+\vartheta x)^{n-N}$ which, if $0 \leq x \leq \beta < 1$, is bounded by $(1-\beta)^{n-N}|x|^N$.

making the transformation $\vec{q} \to \alpha\vec{q}$). The leading term

$$-\frac{i}{2\pi^2}\int\left((r+\tfrac{1}{2}k)^2+2m^2\right)\frac{q^2}{p_0 q_0}\,d\tilde{\phi}\left(\frac{2q}{\alpha}\right)\tilde{f}_s(\vec{r})\,g(k)\,\varphi_0(k)\,d^4k\,d^3\vec{r}\,d\omega_{q}$$

can be handled by setting $\frac{1}{p_0}=\frac{1}{q_0}+\frac{q_0^2-p_0^2}{p_0 q_0(p_0+q_0)}$ and bounding $\frac{q_0^2-p_0^2}{p_0 q_0(p_0+q_0)}$ by $\frac{q_0^2-p_0^2}{m q_0^2}$ where the numerator has already been seen to be of first degree in q. This second part will thus again fall away leaving

$$-\frac{2i}{\pi}\int\left[(\vec{r}+\tfrac{1}{2}\vec{k})^2+2m^2\right]\tilde{f}_s(\vec{r})\,\tilde{g}(k)\,\varphi_0(k)\,d^4k$$

multiplied by the q-integration $\int_{\frac{\varepsilon}{\alpha}}^{\infty}\frac{q^2}{q^2+(\frac{m}{\alpha})^2}\,d\tilde{\phi}(2q)$ which converges to $\int_0^{\infty}d\tilde{\phi}(2q)=-\tilde{\phi}(0)=-1$. Throwing away odd powers of $\vec{r}$ because of the symmetrization of the function f, and writing the above result in x-space, we find an extra term of

$$\left(2m^2+\vec{\nabla}_{\xi}^2\right)\delta(\vec{\xi})\cdot\varphi_0(X) \;+\; \frac{1}{4}\,\delta(\vec{\xi})\,\vec{\nabla}_X^2\varphi_0(X)$$

Using the same procedure as above, we find that the higher order terms in the expansion of $\tilde{\phi}_A(\star)$ gives no contribution, and the result above cannot be cancelled by any of the other terms since they are proportional to the transversal part of the electromagnetic field.

Before completing the calculation of the extra terms for the 0th components, we may discuss an interesting feature of the commutator of two spacelike components of these currents. More specifically, to retain simplicity, we consider the sum over μ, $-\frac{1}{2}\left[j^{(x)}_{\gamma_\mu a\tau_+},\,j^{(y)}_{\gamma_\mu a\tau_-}\right]$. Here $-\frac{1}{2}\gamma_\mu(i\gamma p-m)\gamma_\mu=-2ip-m$ i.e. $t=-2p$ so that $tq+m^2=-4p_0q_0-2(\vec{p}\vec{p})q+m^2$. As the last calculation shows, the terms (Pp)q and m lead to finite extra terms; on the other hand, the term $p_0 q_c$ will contribute in the amount of

$$\int_C^{\infty}\frac{p_0 q_0}{k_0}\,\frac{d^3\vec{r}}{p_c}\,\frac{d^3q}{q_0}\,\tilde{f}_s(\vec{r})\,\tilde{g}(k)\,\varphi_0(k)\,d^4k\,\left(\frac{k_0}{2\alpha}\,\phi'\left(\frac{2q}{\alpha}\right)+\cdots\right)$$

the first term of which, without the smearing functions, is

$$C\int_0^{\infty}\frac{d^3q}{\alpha}\,\phi'\left(\frac{2q}{\alpha}\right)=C'\alpha^2\int_0^{\infty}q^2\,d\tilde{\phi}(q)=C''\alpha^2\int_0^{\infty}q\,\tilde{\phi}(q)\,dq.$$

Thus each of the individual commutators j_k, j_k must have an α^2 divergence proportional to the integral $\int_0^\infty q\tilde{\phi}(q)\,dq$. Although it is an easy matter to make this integral vanish, there is no simple prescription in x space, such as keeping the times separate as they approach each other, that will automatically insure convergence of the commutator. The higher order terms in the expansion again give finite limits which will not be calculated here.

In completing the calculation of the j_0, j_0 commutator, the rest of the contributions from $\tilde{\phi}_A$ vanish. In the $\tilde{\phi}_S$ expansion, the first term must be treated differently from the others, and it will be necessary to make yet another variable transformation and to expand some more square roots. The reason is clear from the following homogeneity argument. In all other terms, there is at least one power of α in the denominator to be cancelled against the one appearing from dq, and the remainder is an algebraic function of q for which the criterion of pointwise convergence is also the criterion that the pointwise limit be a polynomial in q whose degree depends on the number of α's that one had to start with. However, if none are present, then the integrands converging to a nonzero function will necessarily behave as q^{-1} at the origin, and will consequently not satisfy Lebesgue's criterion. There is no hope that a nice choice of test functions will help, since $\tilde{\phi}(0)$ must be 1. However writing the expression

$$\int \frac{-2(tq+m^2)q_0 + t_0\vec{q}\vec{k} - q_0\vec{t}\vec{k}}{k_0 p_0 q_0} \frac{\vec{q}_\perp\vec{\varphi}}{m^2+q_\perp^2} \tilde{\phi}\left(\frac{2q}{\alpha}\right) d^3\vec{q}$$

we notice that because of symmetry in q, the $t_0\vec{q}\vec{k}$ term and the t_0q_0 part of $tq+m^2$ will disappear having allowed a cancellation of the unsymmetric p_0 in the denominator, and that the terms may be recollected as follows:

$$\frac{-2\left(m^2+\vec{p}\left(\vec{q}-\tfrac{1}{2}\vec{k}\right)\right)\vec{q}_\perp\vec{\varphi}}{k_0p_0\left(m^2+q_\perp^2\right)} \tilde{\phi}\left(\frac{2q}{\alpha}\right)$$

Making the transformation $\vec{q} \longrightarrow \vec{s} = \vec{q} + \frac{1}{2}\vec{k}$ $(\vec{q}_\perp = \vec{s}_\perp)$ we obtain the expression

$$\int -\frac{2\,(m^2+(\vec{r}-\vec{s})\vec{s})\,\vec{s}_\perp\vec{\varphi}}{k_0 p_0 (m^2+s_\perp^2)}\,\tilde{\phi}\left(\frac{2}{\alpha}\sqrt{(\vec{s}-\tfrac{1}{2}\vec{k})^2}\right) d^3\vec{s}$$

in which the integrand, with the exception of the smearing function $\tilde{\phi}$ is antisymmetric in r and s simultaneously. Thus making the expansion

$$\tilde{\phi}\left(\tfrac{2}{\alpha}\sqrt{(\vec{s}-\tfrac{1}{2}\vec{k})^2}\right) = \tilde{\phi}\left(\tfrac{2s}{\alpha}\right) + \frac{\delta}{\alpha}\tilde{\phi}'\left(\tfrac{2s}{\alpha}\right) + \cdots + \frac{\delta^N}{N!\,\alpha^N}\tilde{\phi}^{(N)}\left(\tfrac{2s}{\alpha}+\vartheta\tfrac{\delta}{\alpha}\right)$$

$$\delta = \sqrt{(\vec{s}-\tfrac{1}{2}\vec{k})^2} - s = \frac{-sk+\frac{1}{4}k^2}{s+\sqrt{(\vec{s}-\frac{1}{2}\vec{k})^2}}$$

the first term will vanish, and the others, each containing at least one inverse power of α can be handled by the methods used above. By successively employing the device

$$\frac{1}{p_0} = \frac{1}{s_0} + \frac{s_0^2-p_0^2}{s_0 p_0 (s_0+p_0)} = \frac{1}{s_0} + \frac{\vartheta}{s_0 p_0 (s_0+p_0)}\,; \quad \vartheta = s_0^2 - p_0^2 = 2\vec{r}\vec{s} - \vec{r}^2$$

we arrive at the expansion

$$\frac{1}{p_0} = \frac{1}{s_0} + \frac{\vartheta}{2s_0^3} + \frac{3\vartheta^2}{8s_0^5} + \vartheta^3\sum_{i+j+k=7}\frac{C_{ijk}}{s_0^i p_0^j (s_0+p_0)^k} + \vartheta^4\sum_{i+j+k=9}\frac{D_{ijk}}{s_0^i p_0^j (s_0+p_0)^k} \quad (C_{ijk}\,D_{ijk}\ \text{real})$$

Noting that the argument of $\tilde{\phi}$ is in any case symmetric in $\vec{s}_\perp$, we may throw away the s_0^{-1} term and retain only those parts of the remaining terms which are symmetric in $\vec{r}$ and $\vec{s}_\perp$. The numerators of the second two are

$$(m^2-s^2+\vec{r}\vec{s})\,\vec{s}_\perp\vec{\varphi}\,\vartheta = (m^2-s^2+\vec{r}\vec{s})(2\vec{r}\vec{s}-r^2)\,\vec{s}_\perp\vec{\varphi} \underset{r}{\cong} -(m^2-s^2)r^2\vec{s}_\perp\vec{\varphi} + 2(\vec{r}\vec{s})^2\,\vec{s}_\perp\vec{\varphi} \underset{s_\perp}{\cong}$$
$$\underset{s_\perp}{\cong} 2r_\parallel s_\parallel (\vec{r}_\perp\vec{s})(\vec{s}_\perp\vec{\varphi})$$

$$(m^2-s^2+\vec{r}\vec{s})\,\vec{s}_\perp\vec{\varphi}\,\vartheta^2 = (m^2-s^2+\vec{r}\vec{s})(2\vec{r}\vec{s}-\vec{r}^2)^2\,\vec{s}_\perp\vec{\varphi} \underset{r}{\cong} (m^2-s^2)(r^4+4(\vec{r}\vec{s})^2)\,\vec{s}_\perp\vec{\varphi} -$$
$$- 4(\vec{r}\vec{s})^2 r^2\,\vec{s}_\perp\vec{\varphi} \underset{s_\perp}{\cong} 4(\vec{r}\vec{s})^2(m^2-s^2-r^2)\,\vec{s}_\perp\vec{\varphi} \underset{s_\perp}{\cong}$$
$$\underset{s_\perp}{\cong} -4r_\parallel s_\parallel (\vec{r}_\perp\vec{s})(\vec{s}_\perp\vec{\varphi})$$

The higher order terms in this latest expansion will all give 0 because $|\vartheta| \le r^2 + 2|r||s|$. The whole coefficient of $\frac{\delta}{\alpha}\tilde{\phi}'\left(\frac{2s}{\alpha}\right)$ becomes

$$-\frac{2\vec{s}_\perp\vec{\varphi}}{k_0(m^2+s_\perp^2)}\,\frac{r_\parallel s_\parallel(\vec{r}_\perp\vec{s})}{s_0^3}\left(1-\frac{3}{2}\,\frac{s^2-r^2-m^2}{s_0^2}\right) = \frac{r_\parallel s_\parallel(\vec{r}_\perp\vec{s})(\vec{s}_\perp\vec{\varphi})}{k_0 s_0^3(m^2+s_\perp^2)}\left(1+\frac{3(2m^2-r^2)}{s_0^2}\right) \;\Big|\; \times d^3\vec{s}$$

and the limit is (after $\vec{s}\to\alpha\vec{s}\,;\ \vec{n}=\frac{\vec{s}}{|\vec{s}|}$)

$$\int\frac{r_\parallel s_\parallel(\vec{r}_\perp\vec{s})(\vec{s}_\perp\vec{\varphi})}{k_0\, s^3 s_\perp^2}\left(\frac{-sk}{2s}\right)\tilde{\phi}'(2s)\,s^2 ds\, d\omega(\vec{s}) = -\int\frac{(n_\parallel)^2}{2}\,r_\parallel\,\vec{r}_\perp\vec{\varphi}\int_0^\infty\tilde{\phi}'(2s)\,ds\; d\omega(\vec{n}) = \frac{\pi r_\parallel\vec{r}_\perp\vec{\varphi}}{3}$$

Since all the symmetry considerations used above hold also for the terms $\frac{\delta^N}{\alpha^N N!}\tilde{\phi}^{(N)}(\frac{2q}{\alpha})$, and since this integrand converged, the rest will all vanish because of the greater number of α's in their denominators.

Having taken care of the leading term in the (★) expansion, we must now consider the next two. The last of these will also vanish in absolute value and thus because of the remark made earlier in the footnote, the entire remainder from this point on will vanish in the equal time limit. The two terms are

$$\left(\frac{-2(tq+m^2)\,q_0 + t_0\vec{q}\vec{k} + q_0\vec{p}\vec{k}}{p_0 q_0 k_0}\right)\frac{\vec{q}_\perp\vec{\varphi}}{q_\perp^2+m^2}\left(\frac{\delta}{\alpha}\,\tilde{\phi}'\!\left(\frac{2q}{\alpha}\right) + \frac{\delta^2+k_0^2}{2\alpha^2}\,\tilde{\phi}''\!\left(\frac{2q}{\alpha}\right)+\cdots\right)d^3\vec{q}$$

and may be rewritten as

$$\left(-\frac{2(tq+m^2)}{p_0 k_0} + \left(\frac{\vec{q}}{q_0}+\frac{\vec{p}}{p_0}\right)\frac{\vec{k}}{k_0}\right)\frac{\vec{q}_\perp\vec{\varphi}}{m^2+q_\perp^2}\,q^2\left[\frac{1}{2}\left(\frac{p_0^2-q_0^2}{p_0+q_0}+\frac{2m^2}{q_0+q}\right)d\tilde{\phi}\!\left(\frac{2q}{\alpha}\right)+\cdots\right]d\omega(\vec{q})$$

To see that the $\tilde{\phi}''$ contribution goes out, we remember that $tq + m^2$ does not grow in q, and that

$$\frac{\vec{p}}{p_0}+\frac{\vec{q}}{q_0} = \vec{p}\left(\frac{1}{p_0}-\frac{1}{q_0}\right)+\frac{\vec{q}+\vec{p}}{q_0} = \frac{\vec{p}}{p_0}\left(\frac{1-p_0^2/q_0^2}{1+p_0/q_0}\right)+\frac{\vec{r}+\frac{1}{2}\vec{h}}{q_0}$$

which also falls as q_0^{-1}. Thus the first term in parentheses remains bounded and converges to 0 whereas the rest remains bounded by $\tilde{\phi}''(2q)$ after the transformation $\vec{s}\to\alpha\vec{s}$. By Lebesgue's criterion, it will give no contribution to the limit. To get the contribution from $\tilde{\emptyset}'$, we may take the limit of each factor, i.e. no complicated symmetry arguments are necessary. Designating $\vec{q}/q$ by $\vec{n}$, we find a limit of

$$\left(-4m^2 - r^2 + \tfrac{1}{4}k^2 + (\vec{r}\vec{n})^2 - \tfrac{1}{4}(\vec{k}\vec{n})\right)\frac{\vec{n}_\perp\vec{\varphi}}{k_0^2\, n_\perp^2}\,\frac{\vec{k}\vec{n}}{8}\,d\omega(\vec{n}) \cong$$

$$\cong (\vec{r}\vec{n})^2\,\frac{\vec{n}_\perp\vec{\varphi}}{k_0 n_\perp^2}\,\frac{\vec{k}\vec{n}}{8}\,d\omega(\vec{n}) \cong \frac{1}{4}\, r_\parallel (n_\parallel)\,\frac{\vec{r}_\perp\vec{n}\,\vec{n}_\perp\vec{\varphi}}{n_\perp^2}\,d\omega(\vec{n}) = \frac{(n_\parallel)^2}{8}\, r_\parallel\, \vec{r}_\perp\vec{\varphi}\, d\omega(\vec{n})$$

The integral of $(\vec{n}_\parallel)^2 d\omega(\vec{n}) = \int 2\pi\cos^2\vartheta\, d\cos\vartheta = \frac{4\pi}{3}$ so the sum of all contributions from $\tilde{\emptyset}_S$ to the commutator is $\frac{4\pi}{8}\left(\frac{3}{8} r_\parallel \vec{r}_\perp\vec{\varphi}\right) = \frac{\pi}{2} r_\parallel \vec{r}_\perp\vec{\varphi}$
To within an unessential factor, this becomes

$$\vec{A}_\perp(x)\,\overleftarrow{\nabla}\,\partial_k\,\delta(\xi)$$

in x-space.

B. Schwinger Terms

We take first the contribution from $\tilde{\emptyset}_A$ which we find in the table under $\varepsilon = 0$

$$\left[-(m^2 + \vec{q}_\perp\vec{p})\,\vec{\varphi}_\perp\cdot\vec{e} + \vec{q}\vec{e}\;\vec{p}_\perp\vec{\varphi}\right]\frac{q_\parallel}{q_\perp^2 + m^2} + p_\parallel\,\vec{\varphi}_\perp\vec{e} - \vec{p}_\perp\vec{\varphi}\;\vec{k}\cdot\vec{e}$$

As usual, this is to be multiplied by

$$\frac{q^2}{q_0 p_0}\,\frac{k_0}{2}\,d\tilde{\emptyset}\left(\frac{2q}{\alpha}\right)d\omega(\vec{n}) = \frac{k_0 q^2}{2 p_0 q_0}\,d\tilde{\emptyset}\left(\frac{2q}{\alpha}\right)d\omega(\vec{n})\,\left(1 + \Delta + \Delta^2 + \cdots\right)$$

to get the leading term. Retaining only the terms antisymmetric in $\vec{r}$ and which do not vanish on integration over $\vec{q}$, we find for term containing $\Delta^0 = 1$ the integrand

$$\left(r_{\parallel}\,\vec{\varphi}_{\perp}\cdot\vec{e} - \vec{r}_{\perp}\vec{\varphi}\,e_{\parallel}\right)\frac{k_0 q^2}{2q_0^2}\,d\tilde{\phi}\left(\frac{2q}{\alpha}\right)d\omega(\vec{q})$$

whose limit is

$$-2\pi\left(\vec{r}\cdot\vec{k}\;\vec{\varphi}_{\perp}\cdot\vec{e} - \vec{r}_{\perp}\vec{\varphi}\;\vec{k}\cdot\vec{e}\right)$$

The rest of the expansion in Δ can be collected to

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_0}\left(1 - \frac{p_0}{q_0}\right) = \frac{1 - \frac{p_0^2}{q_0^2}}{p_0\left(1 + \frac{p_0}{q_0}\right)} = \frac{(2\vec{r}+\vec{k})\vec{q} - (\vec{r}+\frac{1}{2}\vec{k})^2}{p_0\, q_0^2\left(1 + \frac{p_0}{q_0}\right)}$$

which makes the whole term have the same finite limit as (the term with p_o replaced by q_o since, by standard arguments, the difference will have a limit of 0)

$$\Big[-[(m^2 - q_\perp^2 + q_\perp r)\vec{\varphi}_\perp\vec{e} - \vec{q}_\perp\vec{e}\,(\vec{r}_\perp - \vec{q}_\perp)\vec{\varphi}\,]\frac{q_\parallel}{q_\perp^2+m^2} + (r + \tfrac{1}{2}k - q)_\parallel\,\vec{\varphi}_\perp\cdot\vec{e} - (\vec{r}-\vec{q})_\perp\vec{\varphi}\,e_\parallel\Big]\times$$

$$\times\;\frac{(\vec{r}+\frac{1}{2}\vec{k})\vec{q}}{q_0}\times\frac{k_0 q^2}{2q_0^2}\,d\tilde{\phi}\left(\frac{2q}{\alpha}\right)d\omega(\vec{n})$$

Clearly only the terms containing the maximum number of q's will survive leaving (with $\vec{q} = q\vec{n}$)

$$\left[(q_\perp^2\,\vec{\varphi}\cdot\vec{e} - \vec{q}_\perp\vec{e}\,\vec{q}_\perp\vec{\varphi})\frac{\vec{q}\vec{k}}{q_\perp^2} - \vec{n}\vec{k}\,\vec{\varphi}_\perp\vec{e} + \vec{n}_\perp\vec{\varphi}\;\vec{e}\cdot\vec{k}\right](\vec{r}\vec{n})\,(-d\omega(\vec{n})) =$$

$$\left(\frac{\vec{e}_\perp\vec{n}\;\vec{n}_\perp\vec{\varphi}}{n_\perp^2}\,n_\parallel^2\,\vec{r}\cdot\vec{k} - (\vec{r}_\perp\vec{n})(\vec{n}_\perp\vec{\varphi})\,\vec{e}\vec{k}\right)d\omega(\vec{n})$$

If $d\omega(\vec{n}) = d\cos\vartheta\, d\phi$, then, since $\int(\vec{a}_\perp\vec{n})(\vec{n}_\perp\vec{b})\,d\phi = \pi\,\vec{a}_\perp\vec{b}\sin^2\vartheta$ the angular integration results in:

$$\pi\int d\cos\vartheta\left(\vec{e}_\perp\vec{\varphi}\cos^2\vartheta\,\vec{r}\cdot\vec{k} - \vec{r}_\perp\vec{\varphi}\sin^2\vartheta\,\vec{e}\cdot\vec{k}\right) = \frac{2\pi}{3}\left(\vec{e}_\perp\vec{\varphi}\;\vec{r}\cdot\vec{k} - \vec{r}_\perp\vec{\varphi}\;\vec{e}\cdot\vec{k}\right)$$

The last of the contributions comes from the part

$$\frac{k_0\delta}{4\alpha}\,\phi''\left(\frac{2q}{\alpha}\right)\frac{d^3\vec{q}}{p_0 q_0} \cong \frac{q^2}{q_0 p_0}\,\frac{p_0^2 - q_0^2}{\alpha(p_0+q_0)}\,d\phi'\left(\frac{2q}{\alpha}\right) \cong \frac{q^2}{p_0 q_0}\,\frac{-(2r+k)q}{\alpha(p_0+q_0)}\,d\phi'\left(\frac{2q}{\alpha}\right)$$

of ϕ_A, all remaining ones not contributing to the limit. We may take the limits again straightforwardly without using symmetry considerations to eliminate the divergences obtaining

$$\left[\vec{n}_\perp\vec{\varphi}\;\vec{k}\vec{e} - \frac{\vec{e}_\perp\vec{n}\;\vec{n}_\perp\vec{\varphi}}{n_\perp^2}\,n_\parallel^2\right][-rq]\left[\tfrac{1}{2}\,d\phi'\left(\tfrac{2q}{\alpha}\right)\right] \longrightarrow \frac{\pi}{24}\left(\vec{e}_\perp\vec{\varphi}\;\vec{r}.\vec{k} - \vec{r}_\perp\vec{\varphi}\;\vec{e}.\vec{k}\right)$$

Writing these three terms, which differ only in their coefficients, in the less manifestly gauge invariant form

$$\vec{e}_\perp\vec{\varphi}\;\vec{r}\vec{k} - \vec{r}_\perp\vec{\varphi}\;\vec{e}.\vec{k} = \vec{e}\vec{\varphi}\;\vec{r}.\vec{k} - \vec{r}\vec{\varphi}\;\vec{e}.\vec{k} = \vec{r}\vec{k}\;\varphi_\nu - \vec{r}.\vec{\varphi}\;k_\nu$$

we may write the extra term coming from $\tilde{\phi}_A$ in x-space simply and without reference to the photon vector, to within a constant, as

$$\vec{\partial}\,\delta(\vec{\xi})\,\overleftarrow{\partial}A_\nu(X) - \vec{\partial}\delta(\vec{\xi})\,\partial_\nu\overleftarrow{A}(X) = \vec{\partial}\,\delta(\vec{\xi})\cdot\left(\overleftarrow{\partial}A_\nu(X) - \partial_\nu\overleftarrow{A}(X)\right)$$

The contribution from ϕ_S, likewise obtainable from the table,

$$\frac{(m^2+\vec{q}_\perp(\vec{r}-\vec{q}))\,\vec{\varphi}_\perp\vec{e} - \vec{q}_\perp\vec{e}\,(\vec{r}-\vec{q})_\perp\vec{\varphi}}{p_0}\;\frac{d^3\vec{q}}{q_\perp^2+m^2}\;\tilde{\phi}\left(\tfrac{2q}{\alpha}\right)$$

must be subjected to the same treatment as those for the j_0, j_0 commutator; the leading term must undergo another variable transformation $\vec{q} \rightarrow \vec{s} = \vec{q} - \frac{1}{2}\vec{k}$, and the function $\tilde{\phi}(\cdots)$ has to be expanded about a more symmetric argument. For this leading term, we then have the expression

$$\frac{\vec{s}_\perp(\vec{r}-\vec{s})\,\vec{\varphi}_\perp\vec{n} - \vec{s}_\perp\vec{n}\,(\vec{r}-\vec{s})_\perp\vec{\varphi}}{p_0 = \sqrt{(r-s)^2+m^2}}\;\frac{s^2\,ds\,d\omega(\vec{s})}{s_\perp^2+m^2}\left(\tilde{\phi}\left(\tfrac{2s}{\alpha}\right) + \frac{2}{\alpha}\left(\sqrt{(s+\tfrac{1}{2}k)^2}-s\right)\tilde{\phi}'\left(\tfrac{2s}{\alpha}\right)+\cdots\right)$$

where again the leading term vanishes for a reason similar to the one in the other commutator: the smearing function is symmetric in s as well as the measure; the rest of the integrand is symmetric in the two variables r and s simultaneously; thus the integral is symmetric in r whereas the smearing function in r was antisymmetrized; as a consequence, this contribution vanishes.

The next two terms will be

$$\frac{(\vec{r}-\vec{s})_\perp\,(\vec{\xi}\;\vec{\varphi}_\perp\overleftarrow{e} - \overleftarrow{\varphi}\;\vec{s}_\perp\overleftarrow{e})}{p_0}\;\frac{s^2\,d\omega(\vec{s})}{s_\perp^2+m^2}\left(\frac{ks+\frac{1}{4}k^2}{s+\sqrt{(s+\frac{1}{2}k)^2}}\;\frac{d}{2}\tilde{\phi}\left(\tfrac{2s}{\alpha}\right) + (\cdots)^2\frac{1}{2\alpha}\,d\phi'\left(\tfrac{2s}{\alpha}\right)+\cdots\right)$$

In the part containing $d\check{\phi}'$, the limit can be taken immediately giving a result independent of $\vec{r}$ and thus no contribution. The fact that this term converges absolutely means that the next will vanish absolutely, and that the remainder after this term will not contribute. Returning to the above expression, we may also throw away the k^2 term for the same reason and expand

$$\frac{1}{p_0} = \frac{1}{s_0} + \frac{s_0^2 - p_0^2}{s_0 p_0 (s_0 + p_0)} = \frac{1}{s_0} + \frac{2\vec{r}\vec{s} - r^2}{p_0 s_0 (p_0 + s_0)}$$

The leading term will drop out by antisymmetry in $s_{\parallel}$, and the other will again give an absolutely convergent limit of

$$\frac{-(s_\perp)^2\, \vec{\varphi}_\perp \overleftarrow{e} + \vec{\varphi}_\perp \overleftarrow{s}\; \vec{s}_\perp \overleftarrow{e}}{\vec{s}_\perp^{\,2}}\, d\omega(\vec{s})\; \frac{1}{2}\, \frac{\vec{k}\overleftarrow{s}\;\vec{r}\overleftarrow{s}}{s^2}\, \frac{d}{2}\check{\phi}(2s)$$

which, after the s integration, becomes

$$\frac{\pi}{6}\, \vec{r}\cdot\overleftarrow{k} \times \vec{\varphi}_\perp \overleftarrow{e}$$

Without reproducing here the details of the computation of the higher order $\check{\phi}_s$ contributions, which present absolutely no new problems, we state that they converge and amount to

$$-\frac{\pi}{6}\,\vec{r}\cdot\vec{k}\;\vec{\varphi}_\perp \vec{e} \qquad : \qquad \vec{\partial}\,\delta(\vec{\xi})\,\overleftarrow{\partial} A_{\perp\nu}(X)$$

We summarize briefly in the following table, the types of extra terms obtained in the two commutators which we separate into gauge invariant and non.

	j_0, j_0	Schw.
g.i.	$\vec{A}_\perp(x)\,\overleftarrow{\nabla}\partial_\parallel\,\delta(\vec{\xi})$	$\vec{\partial}\delta(\vec{\xi})\overleftarrow{\partial} A_{\perp\nu}$ $\quad \vec{\partial}\delta(\vec{\xi})\,\partial_\nu \overleftarrow{A}_\perp$
n.i.	$m^2 A_0\delta(\xi)$, $\delta(\vec{\xi})\vec{\nabla}^2 A_0$, $\vec{\nabla}^2\delta(\vec{\xi})A_0$	0

V. Summary and Discussion of Results

The calculation just performed shows first that the equal time commutators in Quantum Electrodynamics are not in general well defined objects even for vector currents, since we have obtained a divergence depending on the time smearing function in the j_i, j_i commutator for two non-conserved currents. At the same time, however, we notice increasingly nicer results when one first restricts the calculation to zeroth components, and then requires the current to be conserved. In fact, the equal time limit exists for zeroth components although extra terms appear which do not vanish unless the currents are conserved. Of these extra terms, we may ask what happens when one or both of the x space variables are integrated over three dimensional space. From locality of the currents, we must expect the extra terms to consist of δ functions and their spatial derivatives in $\vec{\xi}$ multiplied by some functions of X, which in our special case, will be made from the field $A_\mu(X)$. The necessary and sufficient condition that the current vanish after only one integration is then that there be no term proportional to an underived δ function. That this is not in general the case is shown by the extra term

$$(2m^2+\vec{\nabla}_\xi^2)\delta(\vec{\xi})A_0(X) + \frac{1}{4}\delta(\vec{\xi})\nabla^2 A_0(X) + \vec{A}_\perp(X)\overleftarrow{\nabla}\partial_\parallel\delta(\vec{\xi})$$

occuring in the zeroth components of the $\tau_\pm$ currents. After one integration, the result will be

$$\left(2m^2+\frac{1}{4}\nabla^2\right)A_0(y)$$

which is not zero. The second integration results in the single term

$$2m^2\int A_0(y)\,d^3\vec{y}$$

which we may show to diverge weakly.

Recalling that

$$A_\mu(x) = \frac{1}{(2\pi)^3}\int\frac{d^3\vec{k}}{\sqrt{k_0}}\sum_\lambda e_\mu^{(\lambda)}(\vec{k})\,a^{(\lambda)}(\vec{k})\,e^{ikx} + \text{h.c.}$$

$$\langle 0|\{a^{\lambda *}(\vec{k}),\, a^{\lambda'}(\vec{k}')\}|0\rangle = \delta_{\lambda\lambda'}\,\delta(\vec{k}-\vec{k}')$$

we may compute the one particle matrix element $\langle 0 | A_0(x) | A_\nu(g_\nu)\Omega\rangle$ where g_ν is an arbitrary set of 4 test functions, and smear with a sequence $f_\alpha(x)$ which approaches 1, i.e. whose Fourier transforms form a δ sequence. Smearing over the time variable with a function f_T, we obtain for this matrix element

$$\int \frac{d^3\vec{k}\, d^3\vec{k}'}{\sqrt{k_0 k_0'}} \sum_{\lambda\lambda'} \tilde{f}_T(k_0)\, \tilde{f}_n(\vec{k})\, g_\nu(k')\, \overline{e_0^{(\lambda)}(k)}\; e_\nu^{(\lambda')}(k') \left(\langle 0|a^{(\lambda)}(k)^* a^{(\lambda')}(k')|0\rangle + \langle 0|a^{(\lambda')}(k')^* a^{(\lambda)}(k)|0\rangle \right)$$

$$= 2 \int \frac{d^3\vec{k}}{k_0}\, \tilde{f}_\alpha(\vec{k})\, \tilde{f}_T(k_0)\, g_\nu(k)\, \overline{e_0^{(4)}(k)}\; e_\nu^{(4)}(k')$$

where, choosing the normalization of [2] p 183, we have

$$g_\nu(k')\, \overline{e_0^{(4)}(k)}\, e_\nu^{(4)}(k') = g_0(k')$$

and conclude that because of the k_0^{-1} singularity in the measure, the matrix element will diverge for a δ-sequence. If we take explicitly the dilation $\tilde{f}_\alpha(\vec{k}) = \alpha^3 \tilde{f}(\alpha\vec{k})$, then after a simple change of variable, the integral becomes

$$2\alpha \int \frac{d^3\vec{k}}{k_0}\, \tilde{f}(\vec{k})\, \tilde{f}_T\left(\frac{k_0}{\alpha}\right) \tilde{g}_0\left(\frac{k}{\alpha}\right)$$

giving a divergence of α^1 for the matrix element if we take, for instance $\tilde{g}_0(0) = \int g(x)\,dx \neq 0$. A similar calculation for the norm $\| \int f_T(x_0) f_\alpha(\vec{x}) A_0(x)\Omega \|$ gives a divergence of α^2.

It is interesting to note that these divergences are peculiar to quantum electrodynamics with its non definite Hilbert space and absence of a mass gap. Because many results obtained in current algebras use this spatial integral, we would like to point out that when norm convergence is concerned, the answer will depend critically on the manner in which this ill defined concept of the integral over three dimensional space of the current operator is given a precise meaning. This is nicely illustrated by the somewhat less pathological case of an axiomatic field with a positive norm and a mass gap in the energy momentum spectrum in which case general, rigorous and explicit results may be obtained from the Källén-Lehmann representation of the two point function

$$\langle 0|j_\mu(x) j_\nu(y)|0\rangle = \int [d\mu(m)(\partial_\mu\partial_\nu + m^2 g_{\mu\nu}) + d\nu(m)]\, \Delta^{(+)}(x-y;m)$$

In the case of interest where $\mu = 0 = \nu$, the integral in p-space becomes

$$\int (\vec{p}^{\,2} d\mu(m) + d\nu(m))\, e^{ip(x-y)}\, d\Omega^{(+)}(p,m)$$

and, after smearing with $f_S \otimes f_T$ in each variable,

$$\int (\vec{p}^{\,2} d\mu(m) + d\nu(m)) \left| \tilde{f}_S(\vec{p})\, \check{f}_T(\sqrt{\vec{p}^{\,2}+m^2}) \right|^2 \frac{d^3\vec{p}}{\sqrt{\vec{p}^{\,2}+m^2}}$$

If we first let $f_S \to 1$ by dilation $f_S \leftarrow f_\alpha$; $f_\alpha(x) = f_S(\frac{x}{\alpha})$ then we substitute $\tilde{f}^{\alpha}_S(\vec{p}) = \alpha^3 \tilde{f}_S(\vec{p}\alpha)$ into the above integral and make immediately the transformation $\vec{p} \to \vec{q} = \alpha\vec{p}$ obtaining

$$\int \left(\frac{\vec{q}^{\,2}}{\alpha^2} d\mu(m) + d\nu(m) \right) \left| \tilde{f}_S(\vec{q})\, \check{f}_T(\sqrt{(\tfrac{\vec{q}}{\alpha})^2 + m^2}) \right|^2 \frac{d^3\vec{q}}{\sqrt{\frac{q^2}{\alpha^2}+m^2}}$$

In the case that j_μ is divergence free, i.e. $\partial_\mu j_\mu(x) = 0$, then $d\nu = 0$ and this quantity will disappear as α^{-2} with leading term

$$\frac{1}{\alpha^2} \int \vec{q}^{\,2} |f_S(\vec{q})|^2 d^3\vec{q} \int \frac{d\mu(m)\, |\check{f}_T(m)|^2}{m} .$$

Otherwise the norm will approach a constant

$$\int |f_S(\vec{q})|^2\, d^3\vec{q} \int \frac{d\nu(m)\, |\check{f}_T(m)|^2}{m} .$$

On the other hand, one might think of taking a function of the radius vector which is 1 for $r \leqslant R$, drops smoothly to zero, and remains there for $r \geqslant R+\varepsilon$. In this case, even a conserved current gives rise to a divergence going as R^2, the coefficient of which is zero only if the current is itself zero. The calculation is reproduced in [5] and will not be repeated here.

It will now be shown that there are physical states between which the non gauge invariant part of our answer does not vanish, i.e. the restrictions imposed by the Lorentz condition

$$(\partial_\mu A_\mu(x))^{(+)}\psi = 0 \qquad ; \qquad (a^{(3)}(k) + i\,a^{(4)}(k))\psi = 0$$

on a physical state ψ does not automatically make this part of the result unphysical. The state

$$\psi = \int (a^{(3)*}(k) + i\,a^{(4)*}(k))\,|0\rangle\, f(k)\, d^3\vec{k} \quad .$$

does not satisfy this condition; the cross terms vanish because the destruction operators can be commuted over to the vacuum, and the other terms give

$$(a^{(3)}(k)\,a^{(3)}(k')^* - a^{(4)}(k)\,a^{(4)}(k')^*)|0\rangle =$$

$$= [a^{(3)}(k), a^{(3)*}(k')]\,|0\rangle - [a^{(4)}(k), a^{(4)}(k')^*]\,|0\rangle =$$

$$= \delta(k-k')\,|0\rangle - \delta(k-k')\,|0\rangle = 0$$

The vacuum being another such physical state, we compute

$$\langle 0|A_0(x)|\psi\rangle = \langle 0| \int \frac{d^3\vec{k}}{\sqrt{k_0}} e^{ikx} a^{(4)}(k)\,(a^{(3)}(k')^* + i\,a^{(4)}(k')^*)\,|0\rangle\, f(k')\, d^3\vec{k}' =$$

$$= i \int \frac{d^3\vec{k}}{\sqrt{k_0}} e^{ikx} f(k')\, d^3k'\, \langle 0|\,[a^{(4)}(k), a^{(4)}(k')]\,|0\rangle =$$

$$= i \int \frac{d^3\vec{k}}{\sqrt{k_0}} e^{ikx} f(k) \neq 0$$

We also remark that the Lorentz condition holds rigorously for this matrix element since the creation part of the divergence $\partial_\mu A_\mu$ applied to ψ creates only two particle states (of the free field) which are orthogonal to the vacuum.

VI. Comparison with Existing Works

A. Comparison of Method

In the introduction, we have discussed how our approach differs in its physical aspects from previous ones. Before comparing the results, we would like to clarify the relation between our concept of equal times and other existing prescriptions for obtaining field theoretical quantities at equal times. Schroer and Stichel [5] have attempted to derive the Adler Weissberger relations from the assumption of certain charge commutation relations at equal times between states of rapid spatial falloff: the assumption that

$$\lim_{T\to 0}\ \lim_{R\to 0} \langle \cdots \mid [\dot{j}_\Lambda(f_R\otimes f_T),\ \dot{j}_\Sigma(f_R\otimes f_T)] \mid \cdots \rangle$$

have the canonically expected values for $\Lambda = \tau_+ \otimes \gamma_0\gamma_5$, $\Sigma = \tau_- \otimes \gamma_0\gamma_5$. If we were to apply this to current commutators as operators, we would have to leave off the states as well as the $R\to 0$ limit which corresponds to integrating first over both spatial variables and would result in charge commutators.

As was seen in the last section, these authors have used a definition of spatial integration which could not possibly be considered in the sense of operator convergence, even in the case of conserved currents, although there is another equally reasonable definition of this integral for which the divergence discussed by them never arises, and one obtains even strong convergence for the conserved currents. Since the time variables above are both smeared with identical functions which are both allowed to approach δ functions, the process of Schroer and Stichel corresponds to restricting both times to sharp values which are the same rather than restricting only the time difference to zero and allowing for a distribution character in the remaining variable $X_0 = \frac{1}{2}(x_0 + y_0)$. This done, however, one obtains on the right hand side of the commutation relations, among other things, the δ-function in the relative space coordinates times the current restricted to a sharp value of time. This has the disadvantage that this quantity, even in zeroth order, is no longer an operator in the original Hilbert space that we started with; the best that can be expected is that matrix elements between states of restricted momentum be finite, as illustrated by the free field current for which the matrix element between plane wave states is

$$\langle p,\alpha | \int \psi(\vec{x})\, j_\Lambda(\vec{x},0)\, d^3\vec{x}\, | q,\beta \rangle = \int \tilde{\psi}(p-q)\, \bar{u}_\alpha(p) \Lambda u_\beta(q)$$

which will not be square integrable in both variables simultaneously and therefore not represent the matrix element of an operator in the original Hilbert space. A similar consideration shows also that $\dot{j}_\Lambda(\vec{x},0)\Omega$ is no longer a vector in the original space although matrix elements to plane wave stats do exist. On the other hand, with the methods used here, the operators depart from the Hilbert space only in the case of a genuine divergence which appears as well in the matrix elements and is not a result of ignoring the distribution character of the currents. It seems not unreasonable to believe that the assumption of commutation relations of this form put strong requirements on the currents which arise principally from these mathematical considerations and not from physical ones. It should also be noted that the vanishing of the expection value of the commutator in TCP-invariant stated relies only on the fact that $\dot{j}_\Lambda^C = \dot{j}_\Lambda^*$ and the symmetry f(x,y) = f(-x,-y) of the smearing function so that a smearing of $f(x)f'(y)\varnothing(\xi_0)g(X_0)$ with all functions symmetric would, in the case considered in [5] also cause the vacuum expection values to vanish.

Johnson and Low [1] suggest writing

$$\langle \cdots | [j(x), j(y)] | \cdots \rangle = \int e^{i(k\xi + qX)} M(k,q)\, d^4k,q$$

and passing to equal times by integrating M(k,q) over k_0. As pointed out by these authors, there appear polynomials in the integrand which cannot be integrated in the naive sense, so it is therefore necessary to examine more carefully what is meant by the integral. With the requirement that the points be kept separate as they approach one another, this integral clearly can not be considered as the limit obtained by integrating over part of the axis

$$\int_{-A}^{B} M(k,q)\,dk_0 = \int_{-\infty}^{\infty} \chi_{AB}(k_0)\,M(k,q)\,dk_0 \qquad \chi_{AB}(k_0) \underset{df}{=} \begin{cases} 0 & k_0 < -A \\ 1 & -A \leq k_0 \leq B \\ 0 & B < k_0 \end{cases}$$

and letting the range of integration expand, since this corresponds to smearing the current in ξ_0 with the Fourier transform $\tilde{\chi}_{AB}(\xi_0)$ of the characteristic function of the range of integration, and in order that this function vanish in the neighborhood of the origin, it is necessary that

$$0 = \left(\frac{d^n}{d\xi^n}\tilde{\chi}_{AB}(\xi)\right)_{\xi=0} = \int_{-A}^{B} k^n\,dk$$

which certainly does not happen for even n. Thus it appears that in order to give this integral a definition for which the concept of keeping the points separate makes sense, one must introduce a damping function whose fourier transform vanishes in a neighborhood of the origin, and let this function approach 1. Adding differentiability and falloff, one is lead to the definition adopted here.

B. Comparison of Results

The results of Brandt agree with ours since he deals only with conserved currents and claims to obtain zero for the commutator of zeroth components, and the Schwinger terms are apparently not expected until fourth order. We recall that in first order, our method gives no extra terms for conserved currents.

The work of Hamprecht, insofar as his results differ from those of Johnson and Low, is discussed in the next section where the relation to ours is commented upon.

It would be deceptive to compare the results of the text to those of Johnson and Low since it was impossible to retain sufficient generality with regard to the nature of the coupling to treat both cases simultaneously, and the results are qualitatively different. In order to have an idea of the relation between the two methods, however, we consider as an example the commutator of two zeroth components of vector currents in their model in the case $g_{ps} = 0$. All terms occurring in their calculation eventually take the form

$$q_0 S_p \left([\lambda_a, \lambda_b][m, g_s] \right) \times \text{divergence}$$

In order to make sure we do not get a term which these authors would claim to be able to cancel by the renormalization term on the right hand side, we consider the case in which the masses of the nucleons are not split (m is a scalar matrix) in which case the presence of the commutator causes all their terms to vanish. Assuming further that $S_p([\lambda_a, \lambda_b] g_s) \neq 0$, and proceeding in the same manner as in the calculation for photons, we obtain, in addition to the canonically expected renormalized current term, the contribution from the zeroth order term in the $\tilde{\phi}_s$ expansion

$$k_0 \int \frac{q_0 [2m^3 + m(\vec{t} + \vec{q})\vec{k}] - m(t_0 + q_0)(\vec{q}\vec{k} + \frac{\mu^2}{2})}{[k_0^2(q_\perp^2 + m^2) + \mu^2(q_\parallel^2 + \vec{q}\vec{k}) - \frac{\mu^4}{4}] p_0 q_0} \tilde{\phi}\left(\frac{2q}{\alpha}\right) d^3\vec{q} \qquad \times f_A(\vec{r}) d^3\vec{r}$$

Since in powers of α , it stands five to four in favor of the numerator, all other contributions, having at least one extra power of α in the denominator, are finite. The part of the numerator containing t_0 leads to a term independent of r which, multiplying the antisymmetric f_A gives no contribution to the integral. Collecting the remaining terms

$$\int \frac{[k_0(2m^3 - \frac{m\mu^2}{2}) + m\,\vec{t}\vec{k} k_0] d^3\vec{q}}{[\cdots] p_0} \tilde{\phi}\left(\frac{2q}{\alpha}\right) \times f_A(\vec{r}) d^3\vec{r}$$

and making the transformation $\vec{q} = \vec{s} - \vec{r} - \frac{1}{2}\vec{k}$ (remembering $\vec{p} = \vec{q} + \vec{r} + \frac{1}{2}\vec{k} \to \vec{s}$) we obtain

$$\int \frac{\left[k_0\left(2m^3 - \frac{m\mu^2}{2}\right) + m\vec{s}\vec{k}k_0\right] d^3\vec{q}}{\left[k_0^2\left((\vec{s}-\vec{r})_\perp^2 + m^2\right) + \mu^2(s_\parallel - r_\parallel)^2 + \frac{1}{4}\mu^2(\vec{k}^2-\mu^2)\right] s_0} \left(\tilde{\phi}\left(\frac{2s}{\alpha}\right) + \frac{\sqrt{(\vec{s}-\vec{r}-\frac{1}{2}\vec{k})^2} - s}{\alpha}\, \tilde{\phi}'\left(\frac{2s}{\alpha}\right) + \cdots\right)$$

in which the first half of the numerator gives no contribution. As before, the higher order terms in α^{-1} will yield only finite results (for $\mu \neq 0$), whereas the first term will diverge. We observe that the integral as it stands is symmetric in r and will therefore antisymmetrize in $r_\parallel$ using $\frac{\vec{q}\vec{k}}{a - 2\mu^2 \vec{q}_\parallel \vec{r}} + \frac{\vec{q}\vec{k}}{a + 2\mu^2 \vec{q}_\parallel \vec{r}} = \frac{2\mu^2 \vec{q}_\parallel \cdot \vec{r}}{[\cdots + \cdots][\cdots + \cdots]}$ with the result

$$2m\mu^2 k_0 \vec{k}\cdot\vec{r} \int \frac{d^3\vec{q}\, q_\parallel^2}{[\cdots+\cdots][\cdots+\cdots] q_0} \tilde{\phi}\left(\frac{2q}{\alpha}\right) \rightarrow 2m\mu^2 \vec{r}\cdot\vec{k}\, k_0 \int \frac{q_\parallel^2 q^2\, dq}{[k_0^2 q_\perp^2 + \mu^2 q_\parallel^2]^2 q}\, \tilde{\phi}(2q)$$

where the last arrow symbolizes the result of the following divergence criterion: If $0 \leq f_n \rightarrow f$ pointwise and $\int_X f\, d\mu = \infty$ then $\int_X f_n\, d\mu \rightarrow \infty$. To this we need only mention that the two terms in the denominator are positive for all values of $\vec{q}, \vec{r}$ and $\vec{k}$ in case $4m^2 \geq \mu^2$ and all $\vec{q}, \vec{r}$ in case $\vec{k}^2 \geq \mu^2$ showing that the divergence can never be avoided by a clever choice of the masses.

VII. Further Remarks Concerning Cutoffs

Since the only extra terms we have found which must disappear in the presence of a cutoff are gauge dependent and thus not physically observable, it may be of interest to display a cutoff procedure which makes the entire non-canonical contribution to the commutator vanish for two zeroth components of possibly nonconserved currents. A similar statement is made by Hamprecht [4] for the "extra terms" in the sense of Johnson and Low, i.e. those which are left after the removal of not only the canonically expected current, but also a divergence similar to the renormalization infinity that must be subtracted from the current itself. The difference is that we show the vanishing of not only the extra terms, but also this renormalization type infinity in the presence of this particular cutoff.

To begin with, we rewrite the one-photon terms

$$\sum_{\pm} \pm \left[S_p\left(\Lambda S^{(\pm)}(x-y)\Sigma S_R(y-z)\varphi(z)S^{(\mp)}(z-x)\right) + S_p\left(S^{(\pm)}(x-z)\varphi(z)S_A(z-y)\Sigma S(y-x)\Lambda\right)\right] - (x\Lambda \leftrightarrow y\Sigma)$$

with the help of the relation $2S^{(\pm)} = cS \pm S^{(1)}$ in the form

$$S_p\left(\Lambda S^{(1)}(x-y)\Sigma \left\{\begin{matrix} S_R(y-z)\varphi(z)S(z-x) \\ + \\ S(y-z)\varphi(z)S_A(z-x)\end{matrix}\right\}\right) - S_p\left(\left\{\begin{matrix} S_R(x-z)\varphi(z)S(z-y) \\ + \\ S(x-z)\varphi(z)S_A(z-y)\end{matrix}\right\}\Sigma S^{(1)}(y-x)\Lambda\right)$$

$$+ S_p\left(\Lambda S(x-y)\Sigma \left\{\begin{matrix} S^{(1)}(y-z)\varphi(z)S(z-x) \\ + \\ S_R(y-z)\varphi(z)S^{(1)}(z-x)\end{matrix}\right\}\right) - S_p\left(\left\{\begin{matrix} S^{(1)}(x-z)\varphi(z)S_A(z-y) \\ + \\ S_R(x-z)\varphi(z)S^{(1)}(z-y)\end{matrix}\right\}\Sigma S(y-x)\Lambda\right)$$

The argument of appendix 1 will be applicable to the terms containing $S^{(1)}(x-y)$ provided that the integration over the variable q appearing in the Fourier transform of the S-function is cut off symmetrically, so that these terms will have a limit of zero. The other two terms will have the finite limit

$$i\,\delta(\vec{\xi})\, S_p\left([\Lambda,\gamma_0,\Sigma]\left\{\hat{S}^{(1)}(x-z)\varphi(z)S_A(z-x) + S_R(x-z)\varphi(z)\hat{S}^{(1)}(z-x)\right\}\right)$$

where the $\hat{S}^{(1)}$-functions are the ordinary $S^{(1)}$-functions symmetrically cut off in momentum space. Considering one of the terms in momentum space

$$\int S_p\left([\Lambda,\gamma_0,\Sigma](i\gamma q - m)\,\tilde{\varphi}(k)\,\frac{i\gamma(q-k)-m}{(q-k)^2+m^2}\, d\hat{\Omega}(q)\, d^4k\right.$$

we may, again with the aid of an anticommutation, calculate the trace appearing therein and obtain the simple expression:

$$\tilde{A}_0(k)\,qk \;-\; k_0\, q_\mu A_\mu(k) - 2\, q_\mu \tilde{A}_\mu(k)\, q_0$$

Including $(q-k)^2+m^2$ in the denominator, we perform the summation over forward and backward mass shells as required by the measure $d\hat{\Omega}$ with the result

$$\tilde{A}_0(k) - \left[\tilde{A}_0(k) - \frac{q_{\parallel}\,\vec{q}_\perp\,\vec{\tilde{A}}(k)}{q_\perp^2+m^2}\right] - 2\,\frac{q_0^2\,\vec{q}_\perp\,\vec{\tilde{A}}(k)}{k^2+m^2}$$

which is sufficiently asymmetric to cause a symmetrically cut off integral to vanish.

It may be observed from a comparison of the cutoff limit and the renormalization infinity

$$\int S_p(\Lambda\{S^{(1)}(x-y)\,\varphi(y)\,S_A(y-x)+S_R(x-y)\,\varphi(y)\,S^{(1)}(y-x)\})\,d^4y$$

occurring in the first order current, that the calculation with the cutoff mentioned above leads formally to the statement

$$\left[\, j_{\Lambda}^{(r,1)}(x),\ j_{\Sigma}^{(r,1)}(y)\right]_{x_0=y_0} = i\,\delta(\vec{\xi})\, j^{(u,1)}_{[\Lambda,\gamma_0,\Sigma]}(X)$$

i.e. the commutator of two renormalized currents gives the canonically expected unrenormalized current without extra terms in the sense of Johnson and Low. This fact was pointed out by Hamprecht [4] who also discusses the cutoff dependence of the extra terms. It may also be of interest to note that the vacuum expectation value of the current commutator in zeroth order

$$\sum_{\pm} \pm\, S_p\,(\Lambda S^{(\pm)}(x-y)\,\Sigma\, S^{(\mp)}(y-x)) =$$

$$= -i\left[S_p(\Lambda S(\xi)\,\Sigma\, S^{(1)}(y-x)) - S_p(\Lambda S^{(1)}(x-y)\,\Sigma\, S(-\xi))\right]$$

will have an equal time limit when the $S^{(1)}$ functions are cut off that amounts to

$$\delta(\vec{\xi})\,S_p([\Lambda,\gamma_0,\Sigma]\,S^{(1)}(y-x)) = i\,\delta(\vec{\xi})\,\langle 0|\tfrac{i}{2}\left[\bar{\psi}(y),[\Lambda,\gamma_0,\Sigma]\,\psi(y)\right]|0\rangle$$

in which we recognize the vacuum expectation value of the unrenormalized current in zeroth order.

Appendix 1

Smearing the term $\int :\bar{\psi}(y) \sum S_p(y-z)\, \varphi(z)\, S(z-x) \Lambda \psi(x)\, d^4 z$: with $f(\vec{\xi}, \vec{X})\, g(X_0)\, \delta(\xi_0)$ gives

$$\int \tilde{f}(\vec{p}+\vec{r}-\vec{q}+\vec{s},\, \vec{p}+\vec{q}+\vec{k})\, \tilde{g}(p_0+q_0+k_0) :\tilde{\bar{\psi}}(p) \sum \frac{i\gamma r - m}{r^2+m^2} \tilde{\varphi}(k)\, \delta(-r+k+s)(i\gamma s - m)\, \delta(s^2+m^2) \Lambda \tilde{\psi}(q): \times d^4 r, s, p, q, k.$$

where, because of the equal time limiting process, s_0 and r_0 do not appear in the smearing functions. With the identities

$$\frac{1}{r^2+m^2} = \int \frac{d\tau}{\tau}\, \delta((r+\tau e)^2+m^2) \quad \text{and} \quad 0 = \int d\tau\, \delta((r+\tau e)^2+m^2)$$

(e is the unit vector (1,0,0,0)), the part between $\sum$ and Λ may be written as

$$\int \frac{d\tau}{\tau} (i\gamma(r+\tau e) - m)\, \delta((r+\tau e)^2+m^2)\, \tilde{\varphi}(k)\, \delta(-r+k+s)\, (i\gamma s - m)\, \delta(s^2+m^2)$$

The transformations $r \to r-\tau e$, $s \to s-\tau e$, $\tau \to -\tau$ give it the form

$$-\int \frac{d\tau}{\tau} (i\gamma r - m)\, \delta(r^2+m^2)\, \tilde{\varphi}(k)\, \delta(-r+k+s)\, (i\gamma(s+\tau e) - m)\, \delta((s+\tau e)^2+m^2)$$

Again using the two identities, one completes the transformation of the first term into minus the second.

Appendix 2

A. Zeroth Components

We want to evaluate the sums

$$\sum_{\pm} (\pm)^{\varepsilon} S_p\left(\gamma_\mu (i\gamma p_{\pm} - m)\, \gamma_\nu \frac{i\gamma(q_{\pm}+k) - m}{(q_{\pm}+k)^2+m^2}\, \tilde{\varphi}\, (i\gamma q_{\pm} - m)\right) ; \sum_{\pm} (\pm)^{\varepsilon} S_p\left(\gamma_\mu (i\gamma p_{\pm} - m)\, \gamma_\nu (i\gamma q_{\pm} - m)\, \tilde{\varphi}\, \frac{i\gamma(q-k) - m}{(q_{\pm}-k)^2+m^2}\right)$$

with $\varepsilon = 0,1$ in the cases $\mu = \nu$ and $\mu = 0 \neq \nu$. The $\varepsilon = 1$ term will occur as the coefficient of $\tilde{\varnothing}$, and the $\varepsilon = 0$ term, the coefficient of $\tilde{\varnothing}_A$.

To clear the dependence on $\pm$ out of the denominator $2kq_{\pm} = 2k_0(\pm q_0 - q_{\parallel})$, we multiply both numerator and denominator by $\pm q_0 + q_{\parallel}$ obtaining $2k_0(q_{\perp}^2+m^2)$ in the denominator and simple summations in the numerator. To simplify the above terms, we make one commutation keeping in mind that $\{\gamma k, \tilde{\varphi}(k)\} = = 2\tau k^{\mu} A_{\mu} = 0$, $(i\gamma q - m)(i\gamma q + m) = q^2 + m^2 = 0$. Let $\Gamma = \gamma_{\mu}(i\gamma p_{\pm} - m)\gamma_{\nu}$, then

$$Sp(\Gamma(i\gamma(q+k)-m)\varphi(i\gamma q-m)) = 2iq\varphi\, Sp(\Gamma(i\gamma q-m)) + Sp(\Gamma\varphi(i\gamma(-q-k)-m)(i\gamma q-m)) =$$
$$= 2iq\varphi\, Sp(\Gamma(i\gamma q-m)) - i\,Sp(\Gamma\varphi\gamma k\,(i\gamma q-m))$$

$$Sp(\Gamma(i\gamma q-m)\varphi(i\gamma(q-k)-m)) = 2iq\varphi\, Sp(\Gamma(i\gamma q-m)) + i\,Sp(\Gamma(i\gamma q-m)\gamma k\,\varphi)$$

Making the transformation p,q $\rightarrow$ -p,-q in the second line, we notice that in the trace terms, the relative signs of $i\gamma q$ and m need not be changed since they affect only cross terms containing the trace of an odd number of gammas. Thus effectively $\Gamma \rightarrow \Gamma$ and $i\gamma(q-k)-m \rightarrow i\gamma(q+k)-m$ etc. under this transformation, and the second line reads

$$-2iq\varphi\, Sp(\Gamma(i\gamma q-m)) + i\,Sp(\Gamma(i\gamma q-m)\gamma k\,\varphi)$$

If we specialize to the case $\Gamma = i\gamma t - m$ as in the commutator with $\mu = \nu$, then this term is the negative of its partner above since γk anti-commutes with φ, and so the second half of this term is

$$i\,Sp(i\gamma t\; i\gamma q\; \gamma k\; \varphi) = -i\,Sp(i\gamma q\; i\gamma t\; \gamma k\; \varphi) - 2i\,tq\,Sp(\gamma k\,\varphi) =$$
$$= i\,Sp(i\gamma q\; i\gamma t\; \varphi k) = i\,Sp(\Gamma\varphi\gamma k\,(i\gamma q-m))$$

This justifies the statement made in the text that only the symmetric component of $f(\vec{\xi})$ contributes to the commutator. From here, the calculation is straightforward.

$$Sp((i\gamma t-m)(i\gamma q-m)) = -tq + m^2 \quad ; \quad Sp(i\gamma t\,\varphi\, i\gamma k\; i\gamma q) = -(t\varphi)(q\cdot k) + (tk)(\varphi q)$$

and thus each of the original terms becomes

$$-2i\,q\cdot\varphi\;(tq-m^2) \;+\; i\left(t\cdot\varphi\, qk - t\cdot k\; q\cdot\varphi\right)$$

Making the promised transformation, p,q $\rightarrow$ $\pm$ p,$\pm$ q and inserting $\pm q_0 \mp q_{\parallel}$ into the numerator, we carry out the summations

$$-\frac{2(tq+m^2)q\varphi}{2qk}: \quad \begin{array}{l} \varepsilon=1: \ \sum_{\pm} \pm \frac{(\pm q_0\varphi_0 - \vec{q}\overleftarrow{\varphi})(\pm q_0+q_\parallel)}{2(q_\perp^2+m^2)k_0} = \frac{q_0 q_\parallel \varphi_0 - \vec{q}\overleftarrow{\varphi} q_0}{k_0(q_\perp^2+m^2)} = -\frac{q_0\vec{q}_\perp\overleftarrow{\varphi}}{k_0(q_\perp^2+m^2)} \\ \varepsilon=0: \ \sum_{\pm} \frac{(\pm q_0\varphi_0 - \vec{q}\overleftarrow{\varphi})(\pm q_0+q_\parallel)}{2(q_\perp^2+m^2)k_0} = \frac{q_0^2\varphi_0 - q_\parallel\vec{q}\overleftarrow{\varphi}}{k_0(q_\perp^2+m^2)} = \frac{\varphi_0}{k_0} - \frac{q_\parallel\vec{q}_\perp\overleftarrow{\varphi}}{k_0(q_\perp^2+m^2)} \end{array}$$

$$\frac{t\varphi qk}{2qk}: \quad \begin{array}{l} \varepsilon=1: \ \sum_{\pm} \pm(\pm t_0\varphi_0 - \vec{t}\overleftarrow{\varphi}) = t_0\varphi_0 \\ \varepsilon=0: \ \sum_{\pm} (\pm t_0\varphi_0 - \vec{t}\overleftarrow{\varphi}) = \vec{t}\overleftarrow{\varphi} \end{array}$$

$$-\frac{tkq\varphi}{2qk}: \quad \begin{array}{l} \varepsilon=1: \ \sum_{\pm} \frac{\pm(\pm t_0 - t_\parallel)(\pm q_0\varphi_0 - \vec{q}\overleftarrow{\varphi})(\pm q_0+q_\parallel)}{2(q_\perp^2+m^2)} = t_\parallel \frac{q_0\vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2} + t_0\left(\varphi_0 - \frac{q_\parallel\vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2}\right) \\ \varepsilon=0: \ \sum_{\pm} \frac{(\pm t_0 - t_\parallel)(\pm q_0\varphi_0 - \vec{q}\overleftarrow{\varphi})(\pm q_0+q_\parallel)}{2(q_\perp^2+m^2)} = -t_0 \frac{q_0\vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2} - t_\parallel\left(\varphi_0 - \frac{q_\parallel\vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2}\right) \end{array}$$

The parts of the last two groups containing φ_0 will completely cancel leaving behind only $\vec{t}_\perp\overleftarrow{\varphi}$. Thus, as was mentioned in the text, the entire contribution comes from $\varepsilon = 0$ sum. Written together, the results will be:

$$\varepsilon = 1 \ : \quad \frac{2q_0\,(tq-m^2)\,\vec{q}_\perp\overleftarrow{\varphi}}{k_0\,(q_\perp^2+m^2)} + \frac{(t_0 q_\parallel - q_0 t_\parallel)\,\vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2}$$

$$\varepsilon = 0 \ : \ -\frac{2\,(tq-m^2)\,\varphi_0}{k_0} + \frac{2(q_\parallel\,\vec{q}_\perp\overleftarrow{\varphi})(tq-m^2)}{k_0\,(q_\perp^2+m^2)} + \frac{t_\parallel\, q_\parallel\, \vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2} - \frac{t_0\, q_0\, \vec{q}_\perp\overleftarrow{\varphi}}{q_\perp^2+m^2} - $$
$$- t_\perp\varphi.$$

B. Schwinger Terms

We make the commutation of $(i\gamma(q+k)-m)\varphi = 2iq\varphi + \varphi(-i\gamma(q+k)-m)$ in the two traces

$$Sp\,(\Gamma(i\gamma(q+k)-m)\,\varphi\,(i\gamma q-m)) \qquad Sp\,(\Gamma(i\gamma q-m)\,\varphi\,(i\gamma(q-k)-m)$$

which combined with $\frac{e^{i(p-q-\frac{1}{2}k)\xi}}{(q+k)^2+m^2}$ and $\frac{e^{i(p-q+\frac{1}{2}k)^2}}{(q-k)^2+m^2}$ resp., comprise the extra terms in the commutator $[j_0, j_\nu]^{(1)}_{;}\ \nu\neq 0$. We let $\Gamma = \gamma_0(i\gamma p - m)\gamma_\nu$ and, using standard manipulations with Dirac matrices, obtain for these quantities respectively

$$\begin{aligned}&Sp(\Gamma(i\gamma q-m))\,2iq\varphi + \\ &+ Sp(\Gamma\varphi[-i\gamma(q+k)-m](i\gamma q-m)) = \\ &= Sp(\Gamma(i\gamma q-m))\,2iq\varphi - \\ &- i\,Sp(\Gamma\varphi\gamma k(i\gamma q-m))\end{aligned}$$

$$\begin{aligned}&Sp(\Gamma(i\gamma(q-k)-m))\,2iq\varphi + \\ &+ Sp(\Gamma\varphi(i\gamma q-m)(i\gamma(q-k)-m)) = \\ &= Sp(\Gamma(i\gamma q-m))\,2iq\varphi - \\ &- i\,Sp(\Gamma\varphi\gamma k(i\gamma q-m)) + \\ &+ Sp(\Gamma\gamma k)(2q\varphi) - 2qk\,Sp(\Gamma\varphi)\end{aligned}$$

Making the transformation p,q $\rightarrow$ -p,-q in the second integral gives a -1 from rearranging the sum over $\pm$, taking into account that $d\Omega^{(\pm)}(-p) = d\Omega^{(\mp)}(p)$. The factor containing the exponential becomes $\frac{e^{-i(p-q-\frac{1}{2}k)\xi}}{(q+k)^2+m^2}$. We notice that the term $Sp(\Gamma(i\gamma q-m))$ is unchanged by this transformation, and that the -1 obtained is cancelled by the factor $q\cdot\varphi$ so that these terms give a contribution symmetric in ξ, i.e. do not give rise to Schwinger terms. On the other hand, $Sp(\Gamma\varphi\gamma k(i\gamma q-m))$ remains the same and has no coefficient so that the result is antisymmetric. We thus collect, along with this term, the antisymmetric parts of the last two obtaining

$$\left[-2i\,Sp(\Gamma\varphi\gamma k(i\gamma q-m)) + 2qk\,Sp(\Gamma\varphi) - 2q\varphi\,Sp(\Gamma\gamma k)\right]\frac{i\sin(p-q-\frac{1}{2}k)\xi}{(q+k)^2+m^2}$$

The term in brackets becomes

$$-2im^2\,Sp(\gamma_0\gamma_\nu\varphi\gamma k) + 2\,Sp(\Gamma\varphi\gamma k\gamma q) - 2qk\,Sp(\Gamma\varphi) + 2q\varphi\,Sp(\Gamma\gamma k)$$

From a well known rule, we have

$$\begin{aligned}Sp(\Gamma\varphi\gamma k\gamma q) = qk\,Sp(\Gamma\varphi) - q\varphi\,Sp(\Gamma\gamma k) + q_\nu Sp(\gamma_0 i\gamma q\varphi\gamma k) - \\ - iqp\,Sp(\gamma_0\gamma_\nu\varphi\gamma k) + q_0\,Sp(i\gamma p\gamma_\nu\varphi\gamma k)\end{aligned}$$

giving the result

$$2\left[(qp+m^2)\,Sp(\gamma_0\gamma_\nu\varphi\gamma k) - \left(q_\nu Sp(\gamma_0\gamma p\varphi\gamma k) + q_0 Sp(\gamma p\gamma_\nu\varphi\gamma k)\right)\right]\frac{\sin(p-q-\frac{1}{2}k)\xi}{(q+k)^2+m^2}$$

The divergence freedom of φ, $k\varphi = 0$, simplifies the traces considerably, (let $\vec{e}$ be a unit vector in the ν direction)

$$Sp(\gamma_0\gamma_\nu\varphi\gamma k) = k_0\vec{\varphi}\overleftarrow{e} - \varphi_0\vec{k}\cdot\overleftarrow{e} = k_0\left(\frac{\vec{\varphi}\overleftarrow{k}\;\vec{k}\cdot\overleftarrow{e}}{k_0} + \vec{\varphi}_\perp\overleftarrow{e}\right) - \varphi_0\vec{k}\cdot\overleftarrow{e} = k_0\vec{\varphi}_\perp\overleftarrow{e}$$

$$Sp(\gamma_0\gamma p\varphi\gamma k) = k_0 p\varphi - \varphi_0 pk = k_0(p_0\varphi_0 - \vec{p}\,\overleftarrow{\varphi}) - \varphi_0(p_0k_0 - \vec{p}\,\overleftarrow{k}) = -k_0\vec{p}_\perp\overleftarrow{\varphi}$$

$$Sp(\gamma p\gamma_\nu\varphi\gamma k) = pk\,\vec{\varphi}\overleftarrow{e} - p\varphi\,\vec{k}\overleftarrow{e} = p_0k_0\vec{\varphi}_\perp\overleftarrow{e} + \frac{p_0}{k_0}\vec{\varphi}\cdot\overleftarrow{k}\;\vec{k}\cdot\overleftarrow{e} - \vec{p}\overleftarrow{k}\,\vec{\varphi}\overleftarrow{e} +$$
$$+\vec{p}\overleftarrow{\varphi}\;\vec{k}\cdot\overleftarrow{e} - p_0\varphi_0\,\vec{k}\cdot\overleftarrow{e} = p_0k_0\,\vec{\varphi}_\perp\overleftarrow{e} - \vec{p}\overleftarrow{k}\;\vec{\varphi}_\perp\overleftarrow{e} + \vec{p}\overleftarrow{\varphi}\;\vec{k}\cdot\overleftarrow{e}$$

With the formulas

$$\sum_\pm \pm\frac{1}{(q_\pm+k)^2+m^2} = \sum_\pm \frac{\pm(\pm q_0+q_\parallel)}{2k_0(q_\perp^2+m^2)} = \frac{q_0}{k_0(q_\perp^2+m^2)} \quad ; \quad \sum_\pm \frac{1}{(q_\pm+k)^2+m^2} = \frac{q_\parallel}{k_0(q_\perp^2+m^2)}$$

we may evaluate the contribution of these three terms to the summations $\sum_\pm (\pm)^\varepsilon\, Sp(\cdots p_\pm\cdots q_\pm\cdots)$. In the first two and part of the third, the only quantity depending on $\pm$ is the denominator. These terms add up to

$$(qp+m^2)\,k_0\vec{\varphi}_\perp\overleftarrow{e} + \vec{q}\overleftarrow{e}\,k_0\vec{p}_\perp\overleftarrow{\varphi} - q_0p_0k_0\,\vec{\varphi}_\perp\overleftarrow{e} = (m^2-\vec{q}\vec{p})\,k_0\vec{\varphi}_\perp\overleftarrow{e} + \vec{q}\overleftarrow{e}\,k_0\vec{p}_\perp\overleftarrow{\varphi}$$

whereas the part of the numerator proportional to $\pm$ is $-q_0(\vec{p}_\perp\overleftarrow{\varphi}\,\vec{k}\overleftarrow{e} - \vec{p}\overleftarrow{k}\,\vec{\varphi}_\perp\overleftarrow{e})$. Thus the summation will result in

$$\left[(m^2-qp)\vec{\varphi}_\perp\overleftarrow{e} + \vec{q}\overleftarrow{e}\;\vec{p}_\perp\overleftarrow{\varphi}\right]\frac{q_0}{q_\perp^2+m^2} - \left[\vec{p}_\perp\overleftarrow{\varphi}\;\vec{k}\cdot\overleftarrow{e} - \vec{p}\overleftarrow{k}\;\vec{\varphi}_\perp\overleftarrow{e}\right]\frac{q_\parallel q_0}{q_\perp^2+m^2} \qquad (\varepsilon=1)$$

$$\left[(m^2-qp)\vec{\varphi}_\perp\overleftarrow{e} + \vec{q}\overleftarrow{e}\;\vec{p}_\perp\overleftarrow{\varphi}\right]\frac{q_\parallel}{q_\perp^2+m^2} - \left[\vec{p}_\perp\overleftarrow{\varphi}\;\vec{k}\overleftarrow{e} - \vec{p}\overleftarrow{k}\;\vec{\varphi}_\perp\overleftarrow{e}\right]\frac{q_0^2}{q_\perp^2+m^2} \qquad (\varepsilon=0)$$

After splitting $\vec{q}$ into $\vec{q}_\parallel + \vec{q}_\perp$, cancellations again occur, and these expressions become

$$(\varepsilon=1) \qquad \left[(m^2-\vec{q}_\perp \overleftarrow{p})\, \vec{\varphi}_\perp \overleftarrow{e} + \vec{q}_\perp \overleftarrow{e}\; \vec{p}_\perp \overleftarrow{\varphi}\right] \frac{q_0}{q_\perp^2+m^2}$$

$$(\varepsilon=0) \qquad \left[(m^2-\vec{q}_\perp \overleftarrow{p})\, \vec{\varphi}_\perp \overleftarrow{e} + \vec{q}_\perp \overleftarrow{e}\; \vec{p}_\perp \overleftarrow{\varphi}\right] \frac{q_\parallel}{q_\perp^2+m^2} + p_\parallel \vec{\varphi}_\perp \overleftarrow{e} - \vec{p}_\perp \overleftarrow{\varphi}\; \vec{k} \overleftarrow{e}$$

Gauge invariance results from the absence of the longitudinal part of the electromagnetic field, and locality from the absence of $q_\parallel$ and $p_\parallel$ in the $\varepsilon = 1$ sum.

Appendix 3 Gauge Invariance and Divergence Freedom

In the computation of the $[j_{0,\tau_+}, j_{0,\tau_-}]$ commutator, we have obtained a nongauge invariant term, that contained φ_0. In this section, we investigate some aspects of gauge invariance of the currents that we start with, and the connection with divergence freedom. We first note that for the conservation of the current $j_{\lambda\otimes\gamma_\mu}$ in view of the Dirac equation, $(i\gamma\partial - m)\psi = \tau\varphi\psi$ formally implies

$$0 = \partial^\mu \bar{\psi}(x)\gamma_\mu \otimes \lambda\psi(x) = \bar{\psi}(x)[\lambda, m]\psi(x) + \bar{\psi}(x)[\lambda,\tau]\varphi(x)\psi(x)$$

i.e. we should have $[\lambda, m] = 0 = [\lambda, \tau]$

The expression for the current in first order is

$$j_\Lambda^{(1)}(x) = \int :\bar{\psi}(x)\Lambda S_R(x-y)\tau\varphi(y)\psi(y): dy + \text{h.c.}$$

A necessary condition for gauge invariance is that if we make the transformation $(\Box\chi = 0)$

$$A_\mu \rightarrow A_\mu + \varepsilon\partial_\mu\chi \qquad \psi \rightarrow e^{i\varepsilon\tau\chi}\psi$$

the derivative with respect to ε should vanish at $\varepsilon = 0$. Taking this derivative and setting the term containing no photon field operators equal to zero, we find

$$0 = \int :\bar{\Psi}(x)\Lambda S_R(x-y)\tau\gamma\partial\chi(y)\,\psi(y): d^4y + \text{h.c.}$$

In p-space, the first integral becomes

$$\int :\bar{\psi}(p)\Lambda \frac{i\gamma(q+k)-m}{(q+k)^2+m^2}\,\tau\gamma k\,\psi(q):\tilde{\chi}(k)\,e^{i(p+q+k)x}\,d^4 p,q,k$$

or, assuming, as is the case in our problem, the commutativity of τ and m,

$$\int :\bar{\psi}(p)\Lambda\tau\,\frac{2iqk-\gamma k\,(i\gamma q+m)}{(q+k)^2+m^2}\,\psi(q): e^{i(p+q+k)x}\,\tilde{\chi}(k)\,d^4k,p,q =$$

$$= \int :\bar{\Psi}(p)\Lambda\tau\,\Psi(q):\tilde{\chi}(k)\,e^{i(p+q+k)x}\,d^4 k,p,q =$$

$$= \quad i\,\chi(x)\,j^{(0)}_{\Lambda\tau}(x)$$

Adding the hermitian conjugate, we find then the transformation

$$j^{(1)}_{\Lambda}(x) \longrightarrow j^{(1)}_{\Lambda}(x) + \varepsilon\chi(x)\,j^{(0)}_{i[\Lambda,\tau]}(x) + \varepsilon\,(\bar{\psi}\cdots\varphi\cdots\psi)$$

i.e., the current density itself will not be gauge invariant unless $[\Lambda,\tau] = 0$, the condition obtained above for divergence freedom.

Without carrying this calculation any further, we may also point out that conversely, if the current is conserved, then the extra terms disappear from the commutator and thus the entire result will be gauge invariant.

The fact that we obtain a non gauge invariant contribution to the commutator that is supposed to give the gauge invariant τ_3 current, and that this part disappears when a cutoff is made parallels the situation encountered in [2] pp 280-8 in the calculation of the current in the presence of an external electromagnetic field.

The divergent integral $K_{\mu\nu}(x-y)$ must be divergence free ($\partial_\mu K_{\mu\nu} = 0$) in order that the current be gauge invariant, and the first calculation indeed reduces the divergence freedom to the statement

$$\int p_\nu \delta(p^2+m^2)\, d^4p = 0$$

which is true if the integration is symmetrically cut off in $|\vec{p}|^2$. However, an explicit calculation of $K_{\mu\nu}$ involving several substitutions of variables leads to an explicitly non gauge invariant quantity which is subsequently rejected in favor of the first.

In our calculation, a similar integral appears which would have an equal time limit of this very function $K_{\mu\nu}$. If the calculation is to be done correctly, however, the cutoff must be renounced and those variable transformations made which enable us to eliminate the terms threatening divergence before the limit is taken, and it is therefore not surprising that our (sometimes convergent) result lies between the two extremes calculated, and that it is not gauge invariant.

Acknowledgments

I am grateful to Prof. H. Lehmann for having suggested this problem, and to him as well as Prof. R. Haag for critical discussions of the results.

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