DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 68/5
January 1968

Strong Focussing in Linear Accelerators

by

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Summary

To focus large emittance positron beams in linear accelerators, both a solenoid field and a FODO channel consisting of a set of quadrupoles with alternating gradients may be used. A thin lens theory for a FODO channel in accelerated particle systems was given by R. H. Helm.

In this paper, the transformation matrices for such a FODO channel consisting of quadrupoles of finite magnetic length are derived. It was found that the thin lens theory gives too optimistic results with respect to the transmitted beam emittance. The deviations from the thin lens theory are shown in diagrams for various parameters.

A. Object and Scope

A cylindrically symmetric particle beam is characterized by the area of the ellipse occupied in the transverse phase plane (y, y') or (y, p_y). The term emittance ε will be used as the area of the phase ellipse in the plane (y, y') whereas the emittance E is the corresponding area in the plane (y, p_y).

According to Liouville's theorem, the emittance E remains constant, and for a right ellipse we have:

$$E = p_o \cdot \varepsilon = p_o r r' \pi = \text{const.} \qquad (1)$$

This emittance E is very small in linear accelerators for electrons because of their small initial energy in the gun (50 - 150 keV).

$$E_- = (0{,}005 - 0{,}01)\ \pi\ \left(\frac{\text{MeV}}{c}\right)(\text{cm}) \qquad (2)$$

For positrons, the emittance E is one to two orders of magnitude higher:

Frascati	(Am 63)	$E_+ = 0{,}3\ \pi$ (MeV/c) (cm)
		$p = 10$ (MeV/c)
NINA	(CM 65)	$E_+ = 0{,}25\ \pi$ (MeV/c) (cm)
		$p = 10$ (MeV/c)
SLAC	(He 62)	$E_+ = 0{,}2\ \pi$ (MeV/c) (cm)
		$p = 16$ (MeV/c)
DESY	(Wi 66)	$E_+ = 0{,}6\ \pi$ (MeV/c) (cm)
		$p = 6$ (MeV/c)

The emittances of the above positron sources are in the range:

$$E_+ = (0{,}1 - 0{,}6)\ \pi\ \left(\frac{\text{MeV}}{c}\right)\ (\text{cm}) \qquad (3)$$

With solenoid focussing, the magnetic field needed to hold a beam of emittance E within a cylinder of radius 2ρ is given by:

$$B = \frac{5}{3} \frac{E}{\pi \rho^2} \tag{4}$$

To get high positron intensities, the whole free aperture in the accelerating structure is used so that $2\rho = R$.

Equ. (4) shows that only a weak magnetic field is necessary to focus electrons. For positrons, however, the solenoid fields with $2\rho \approx 1$ cm are in the range of some thousand Gauss.

The FODO channel proposed by Helm (He 62) represents a more effective focussing system. Helm's analysis of such a FODO channel in "thin lens approximation" results in a constant field gradient for all quadrupoles and the following equation for the length Ln of the n-th drift spacc:

$$\frac{Ln}{d} = \frac{e^{b_o} - 1}{b_o} \; e^{b_o n} \qquad\qquad b_o = \frac{\alpha \cdot d}{p_o} \tag{5}$$

In this paper, the finite length of the quadrupoles will be taken into account, leading to a modification of this result.

B. Formulation of the Transformation Matrices.

We start from the well known equation for the motion of a charged particle in an electric and magnetic field.

$$\dot{\vec{p}} = e\vec{E} + \frac{e}{c}\left[\dot{\vec{r}} \times \vec{B}\right] \tag{6}$$

Inserting $\vec{r} = (z, x, s)$, $\vec{E} = (0, 0, \frac{\alpha}{e})$; $\vec{B} = (gx, gz, 0)$ and $p = p_o + \frac{\alpha}{c} s$, Equ. (6) gives:

$$\ddot{y}(p_o + \frac{\alpha}{c} s) + \alpha\dot{y} + ecgy = 0 \tag{7}$$

y being x or z. From (7) we obtain for highly relativistic particles with s = ct:

$$y'' + \frac{\eta_o}{1 + \eta_o s} y' + \frac{k_o}{1 + \eta_o s} y = 0 \qquad \eta_o = \frac{\alpha}{p_o c} \tag{8}$$

With $\tau = \dfrac{1 + \eta_o s}{\eta_o}$, the solution of this differential equation is given by:

$$y = C_1 \cdot I_o (2\sqrt{\kappa \cdot \tau}) + C_2 Y_o (2\sqrt{\kappa \cdot \tau}) \tag{9}$$

$\kappa = k_o/\eta_o$; C_1, C_2 are constants

I_o and Y_o are the first and second order Bessel functions, respectively, with the index zero and the argument $2\sqrt{\kappa \cdot \tau}$.

From Equ. (9) the transformation through a quadrupole field in an accelerated particle system is found to be:

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \frac{\pi\sqrt{k_o}}{\eta_o} \begin{pmatrix} -I_o & Y_o \\ \dfrac{\sqrt{k_o}\, I_1}{\sqrt{1 + \eta_o \ell}} & \dfrac{\sqrt{k_o}\, Y_1}{\sqrt{1 + \eta_o \ell}} \end{pmatrix} \begin{pmatrix} Y_{10} & Y_{oo}/\sqrt{k_o} \\ I_{10} & I_{oo}/\sqrt{k_o} \end{pmatrix} \begin{pmatrix} y_o \\ y'_o \end{pmatrix} \tag{10}$$

with $Z_i = Z_i \left(\frac{2}{\eta_o} \sqrt{k_o(1 + \eta_o \ell)}\right)$ and $Z_{io} = Z_{io}\left(\frac{2}{\eta_o}\sqrt{k_o}\right)$

For a drift space with k = 0 we get:

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\eta_o} \log (1 + \eta_o L) \\ 0 & \frac{1}{1 + \eta_o L} \end{pmatrix} \begin{pmatrix} y_o \\ y'_o \end{pmatrix} \tag{11}$$

The Bessel functions in the transformation (10) may be expressed for large arguments in terms of circular and hyperbolic functions (JE 60).

In linear accelerators, the arguments are usually large enough ($\frac{2}{\eta_o}\sqrt{k_o} > 40 >> 1$) for this approximation which gives the following transformation matrices.

Focussing quadrupole:

$$M_+ = \begin{pmatrix} \sigma \cos (z - z_o) & \frac{\sigma}{\sqrt{k_o}} \sin (z - z_o) \\ -\sigma^3\sqrt{k_o} \sin (z - z_o) & \sigma^3 \cos (z - z_o) \end{pmatrix} \quad (12)$$

$$+ \begin{pmatrix} \frac{\sigma}{8}\left(\frac{3}{z_o} + \frac{1}{z}\right) \sin (z - z_o) ; & \frac{\sigma}{8\sqrt{k_o}}\left(\frac{1}{z_o} - \frac{1}{z}\right) \cos (z - z_o) \\ \frac{3\sigma^3}{8}\left(\frac{1}{z_o} - \frac{1}{z}\right)\sqrt{k_o} \cos (z - z_o) ; & -\frac{\sigma^3}{8}\left(\frac{1}{z_o} + \frac{3}{z}\right) \sin (z - z_o) \end{pmatrix}$$

Defocussing quadrupole:

$$M_- = \begin{pmatrix} \sigma\, ch(z - z_o) & \frac{\sigma}{\sqrt{k_o}}\, sh(z - z_o) \\ \sigma^3\sqrt{k_o}\, sh (z - z_o) & \sigma^3\, ch(z - z_o) \end{pmatrix} \quad (13)$$

$$+ \begin{pmatrix} \frac{\sigma}{8}\left(\frac{3}{z_o} + \frac{1}{z}\right) sh(z - z_o) ; & -\frac{\sigma}{8\sqrt{k_o}}\left(\frac{1}{z_o} - \frac{1}{z}\right) ch(z - z_o) \\ \frac{3\sigma^3}{8}\left(\frac{1}{z_o} - \frac{1}{z}\right)\sqrt{k_o}\, ch(z - z_o) ; & -\frac{\sigma^3}{8}\left(\frac{1}{z_o} + \frac{3}{z}\right) sh(z - z_o) \end{pmatrix}$$

Drift space:

$$M_o = \begin{pmatrix} 1 & \frac{1}{\eta_o} \log (1 + \eta_o L) \\ 0 & \frac{1}{1 + \eta_o L} \end{pmatrix} \tag{14}$$

with $z = \frac{2}{\eta_o} \sqrt{k_o \cdot (1 + \eta_o \ell)}$, $z_o = \frac{2}{\eta_o} \sqrt{k_o}\, \ell$ and $\sigma^4 = \frac{1}{1 + \eta_o \ell}$

Neglecting terms of relative magnitude $< 10^{-4}$, we get with

$z - z_o \simeq \sqrt{k_o} \cdot L = \phi$ and $\delta = \frac{1}{8}\left(\frac{3}{z_o} + \frac{1}{z}\right) \simeq \frac{1}{8}\left(\frac{1}{z_o} + \frac{3}{z}\right)$

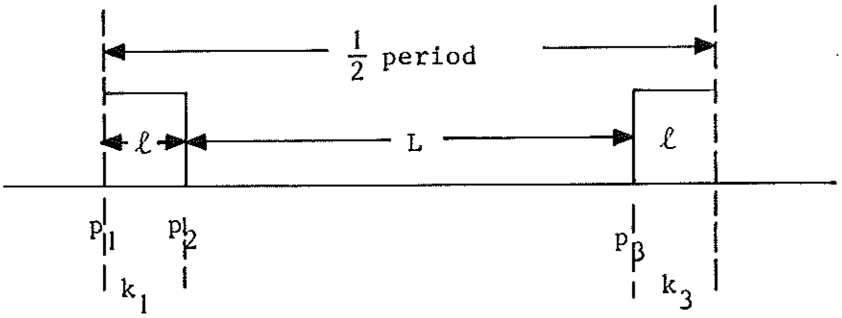
$$M_+ = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^3 \end{pmatrix} \left[\begin{pmatrix} \cos\phi & \frac{1}{\sqrt{k_o}} \sin\phi \\ -\sqrt{k_o} \sin\phi & \cos\phi \end{pmatrix} + \begin{pmatrix} \delta \sin\phi & 0 \\ 0 & -\delta \sin\phi \end{pmatrix} \right] \tag{12a}$$

$$M_- = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^3 \end{pmatrix} \left[\begin{pmatrix} \mathrm{ch}\,\phi & \frac{1}{\sqrt{k_o}} \mathrm{sh}\,\phi \\ \sqrt{k_o}\, \mathrm{sh}\,\phi & \mathrm{ch}\,\phi \end{pmatrix} + \begin{pmatrix} \delta\, \mathrm{sh}\,\phi & 0 \\ 0 & -\delta\, \mathrm{sh}\,\phi \end{pmatrix} \right] \tag{13a}$$

$$M_o = \begin{pmatrix} 1 & -\frac{4}{\eta_o} \log \sigma \\ 0 & \sigma^4 \end{pmatrix} \tag{14a}$$

It is easy to see from these matrices that vanishing acceleration leads to the well known transformation matrices.

C. Transformation through a FODO half period



The transformations derived in the last section are not only related to the magnetic length of the elements and to their strenghts, but also to the particle momentum. For a FODO half period, the transformation may therefore be written:

$$M_{+\frac{1}{2}} = M_{-}(\ell, k_3, p_3) \cdot M_o(L, p_2) \cdot M_{+}(\ell, k_1, p_1)$$

$$M_{-\frac{1}{2}} = M_{+}(\ell, k_3, p_3) \cdot M_o(L, p_2) \cdot M_{-}(\ell, k_1, p_1) \qquad (15)$$

The method of calculating beam envelopes with these matrices is well known*.

$$\begin{pmatrix} E_1^2 \\ E_1 E_1' \\ A_1^2 \end{pmatrix}_+ = \begin{pmatrix} A^2 & 2AB & B^2 \\ AC & AD + BC & BD \\ C^2 & 2CD & D^2 \end{pmatrix} \begin{pmatrix} E_o^2 \\ E_o E_o' \\ A_o^2 \end{pmatrix} \qquad (16a)$$

$$\begin{pmatrix} E_1^2 \\ E_1 E_1' \\ A_1^2 \end{pmatrix}_- = \begin{pmatrix} F^2 & 2FG & G^2 \\ FH & FI + GH & GI \\ H^2 & 2HI & I^2 \end{pmatrix} \begin{pmatrix} E_o^2 \\ E_o E_o' \\ A_o^2 \end{pmatrix} \qquad (16b)$$

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* see e.g. K. G. Steffen, "High Energy Beam Optics".

Here, $M_{+\frac{1}{2}} \neq M_{-\frac{1}{2}}$, i.e. some symmetric properties are lost due to the acceleration. With given quadrupole length 2ℓ, out of the three parameters k_1, k_3, L only k_3 and L are free variables, because k_o is already determined by the previous FODO half period. The parameters k_3 and L may be used to match maximum beam envelope $\hat{E}$ at the end of the half period to the available aperture with vanishing beam divergence ($\hat{E}' = 0$).

Then the minimum beam envelope $\check{E}$ as well as the beat factor $m = \hat{E}/\check{E}$ are no more free variables, but are given by the formalism and are varying from half period to half period.

D. Numerical Investigation of FODO Channels.

In the optimization procedure as described in the last section, the quadrupole strength k_o and the beat factor m at the beginning of the FODO channel are not determined. These parameters may be used to maximize the admittance of the FODO channel for given cost.

To do this, it is easiest to start from a FODO channel without acceleration, for which the following equations for the acceptance ε and the beat factor are given by Steffen (ST 65):

$$\varepsilon = \frac{m \quad R^2}{1 + m^2} \frac{\phi}{\ell} \left[\frac{\tan\phi\ (\mathrm{ctnh}\,\phi + \frac{L}{\ell}\phi) - 1}{1 + \mathrm{ctnh}\,\phi\ (\tan\phi + \frac{L}{\ell}\phi)} \right]^{1/2}$$

$$m^2 = \frac{1 + \tanh\phi\ (\tan\phi + \frac{L}{\ell}\phi)}{1 - \tan\phi\ (\tanh\phi + \frac{L}{\ell}\phi)} \quad ; \qquad \phi = \ell\sqrt{k} \tag{17}$$

For thin lenses, the value ε/R^2 reaches a maximum if the focal length of half a quadrupole is given by:

$$f_o = \sqrt{2} \cdot L \tag{18}$$

Then, the maximum acceptance of this FODO channel is:

$$\varepsilon_o = \frac{R^2}{4 \cdot L} \tag{19}$$

With the aid of Equ. (17) those FODO channels have been determined which, for a given quadrupole filling factor $2\ell/(L + 2\ell)$, have the maximum acceptances. The resulting focal length $f = \sqrt{k_o} \cdot \sin(\sqrt{k_o}\,\ell)$ in the focussing plane of the quadrupoles and the maximum acceptances ε of the optimized FODO channel are given in normalized units in the Figures 1 and 2 as functions of the reciprocal filling factor. It appears that for a given filling factor, these optimum parameters do not depend on the quadrupole length 2ℓ. This is true at least for quadruple lengths between 0.1 and 2 m and for drift spaces between 2ℓ and 100 m.

For a given acceptance the quadrupole strength k_o and the beat factor m of the corresponding optimized FODO channel without acceleration have been used as initial values for the optimization procedure of the FODO channel with acceleration, as described in the previous section. For practical purposes, this will yield the optimum solution since, at the beginning the two channels are almost identical.

Figure 3 shows for a given quadrupole length and different acceleration, the quadrupole spacings of the optimized FODO channel with finite quadrupole length as compared to those obtained in thin lens approximation. In Figure 4 and 5 the relative diviations from the thin lens approximation are shown for various filling factors.

From the results it appears that Equ. (5) is still valid, but with a somewhat modified value of the exponent:

$$\frac{Ln + 2\ell}{d} = \frac{e^{b} - 1}{b}\, e^{bn} \qquad b = \frac{\alpha d}{p_o} \tag{20}$$

The deviations $\frac{b_o - b}{b_o}$ shown in Figure 4 increases with increasing filling factor.

As shown in Figure 5, the optimum quadrupole strengths vary somewhat along the channel. Since, however, this variation does not exceed the order of 10%, the quadrupoles can still be powered in series, being individually adjustable by a regulated power transistor parallel to the coils.

E. Example.

The new 400 MeV linear accelerator for DESY incorporates a positron converter system which, in its final form, will have an emittance E of 0.6π (MeV/c)(cm) (Wi 65, Wi 66), corresponding to an emittance ε of 100π (mrad · cm) at 6 MeV initial positron energy. Figure 2 shows that it is practically impossible to transmit this high emittance through a FODO channel along a linear accelerator. Therefore, we use solenoid focussing up to a positron energy of about 100 MeV.

A detailed description is given in another paper (Wi 67); therefore, only the characteristic figures of the focussing system are given here:

Emittance of positronhorn: $\varepsilon = 100\ \pi$ mrad cm
Initial positron momentum: $p = 6 \pm 2$(MeV/c)

A 4kG solenoid field over the first 10 m of linac structure serves as an initial focussing system, until the emittance ε is reduced to $\varepsilon = 5.26\ \pi$ mrad · cm at $p_+ = 114$ (MeV/c).

By means of a quadrupole doublet, the solenoid emittance area is matched to the FODO channel, whose characteristic data are given below:

$$E = 0.6\ \pi\ \text{(MeV/c)(cm)}$$
$$d = 0.623\ \text{m}$$
$$2\ell = 0.303\ \text{m}$$
$$g_o = 0.374\ \text{kG/cm}$$
$$\bar{\alpha} = 9.17\ \text{MeV/m}$$

g_o is the field gradient of the first quadrupole and $\bar{\alpha}$ is the averaged acceleration along the FODO channel.

This ideal FODO channel, however, cannot precisely be realized in practice due to technical restrictions near the ends of the accelerator sections. By slight variations of quadrupole spacings and strengths it was possible to match the FODO channel to the practical requirements without any loss in acceptance.

It may be interesting to note that the 10 m solenoid is powered by 250 kW whereas the 25 FODO quadrupoles along 27 m of linac structure only needs about 60 kW.

Acknowledgement.

I wish to thank Dr. K. G. Steffen for useful discussions.

References

Am 63: F. Amman, R. Andreani LNF - 63/46

CM 65: M. C. Crowley-Milling EL/TM/27

He 62: R. H. Helm SLAC-2, Aug. 1962

JE 60: Jahnke-Emde-Lösch, Tables of Higher Functions.

ST 65: K. G. Steffen, "High Energy Beam Optics", I. Wiley 1965, New York

Wi 65: H. Wiedemann, DESY Internal Note H 9 - 1965

Wi 66: H. Wiedemann, DESY Internal Note H14 - 1966

Wi 67: H. Wiedemann, DESY 68/3, Jan. 1968.

Used Notations

A	:	angular envelope
α (MeV/cm)	:	acceleration
B (kG)	:	solenoid field
d	:	distance of quadrupole centers of the corresponding FODO channel with $\alpha = 0$ as defined by the required emittance ε and the available aperture.
E_i, E'_i	:	beam envelope and its derivative
E, E_-, E_+ (MeV/c) · (cm)	:	emittance or acceptance area of the phase ellipse in the phase plane (y, p_y)
ε (mrad · cm)	:	emittance or acceptance area of the phase ellipse in the phase plane (y, y')
g (kG/cm)	:	field gradient in the quadrupoles
k (m^{-2})	:	quadrupole strength
2ℓ	:	magnetic quadrupole length
L_i	:	drift length between quadrupoles
n	:	index of n-th quadrupole
p_o (MeV/c)	:	particle momentum at the entrance of the FODO-channel
R (cm)	:	minimum radius of linac aperture
ρ (cm)	:	maximum radius of trajectory spirals in the solenoid field.

Focal length of optimal FODO-channels

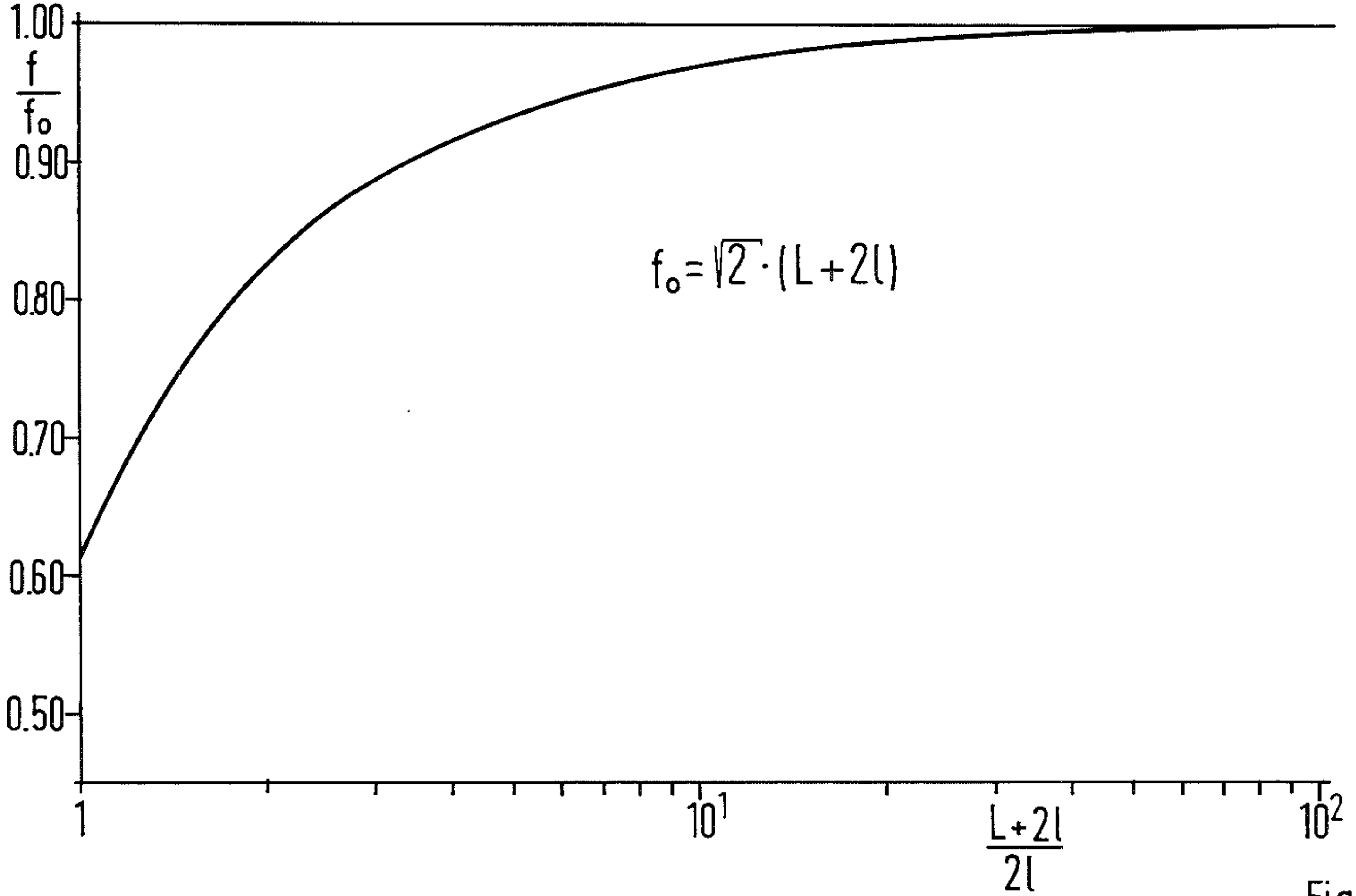


Fig.1

Emittances of optimal FODO-channels

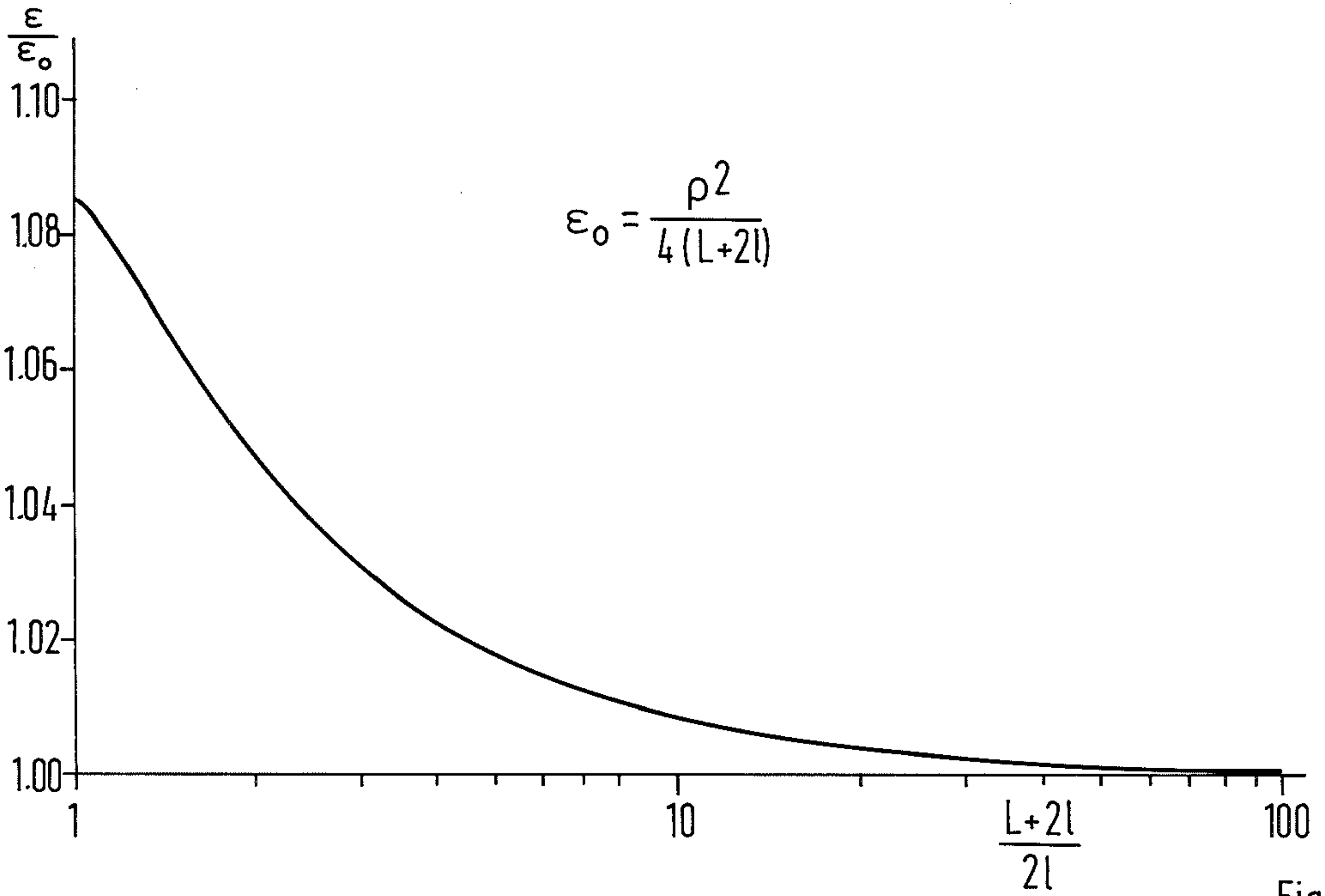


Fig.2

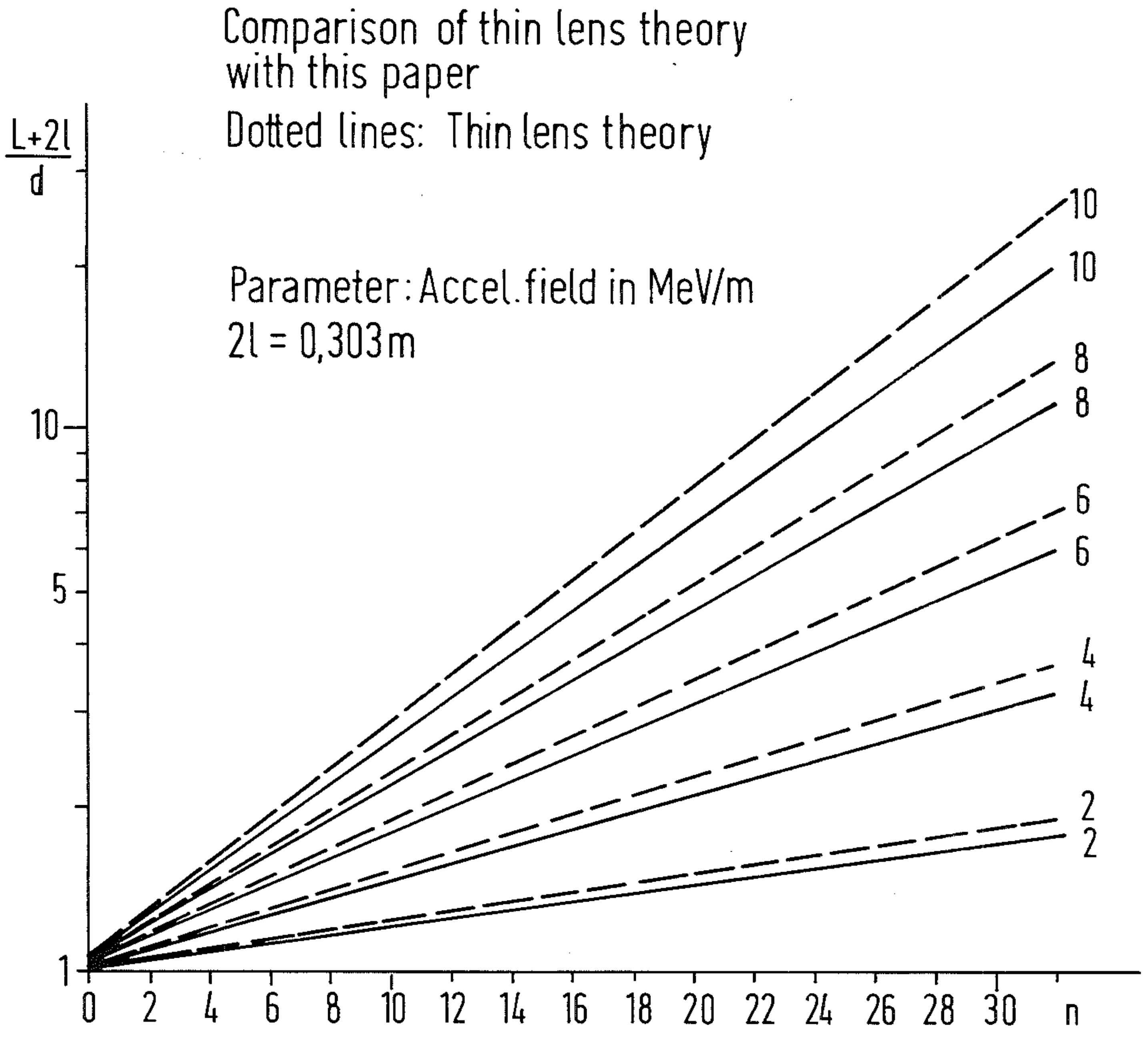
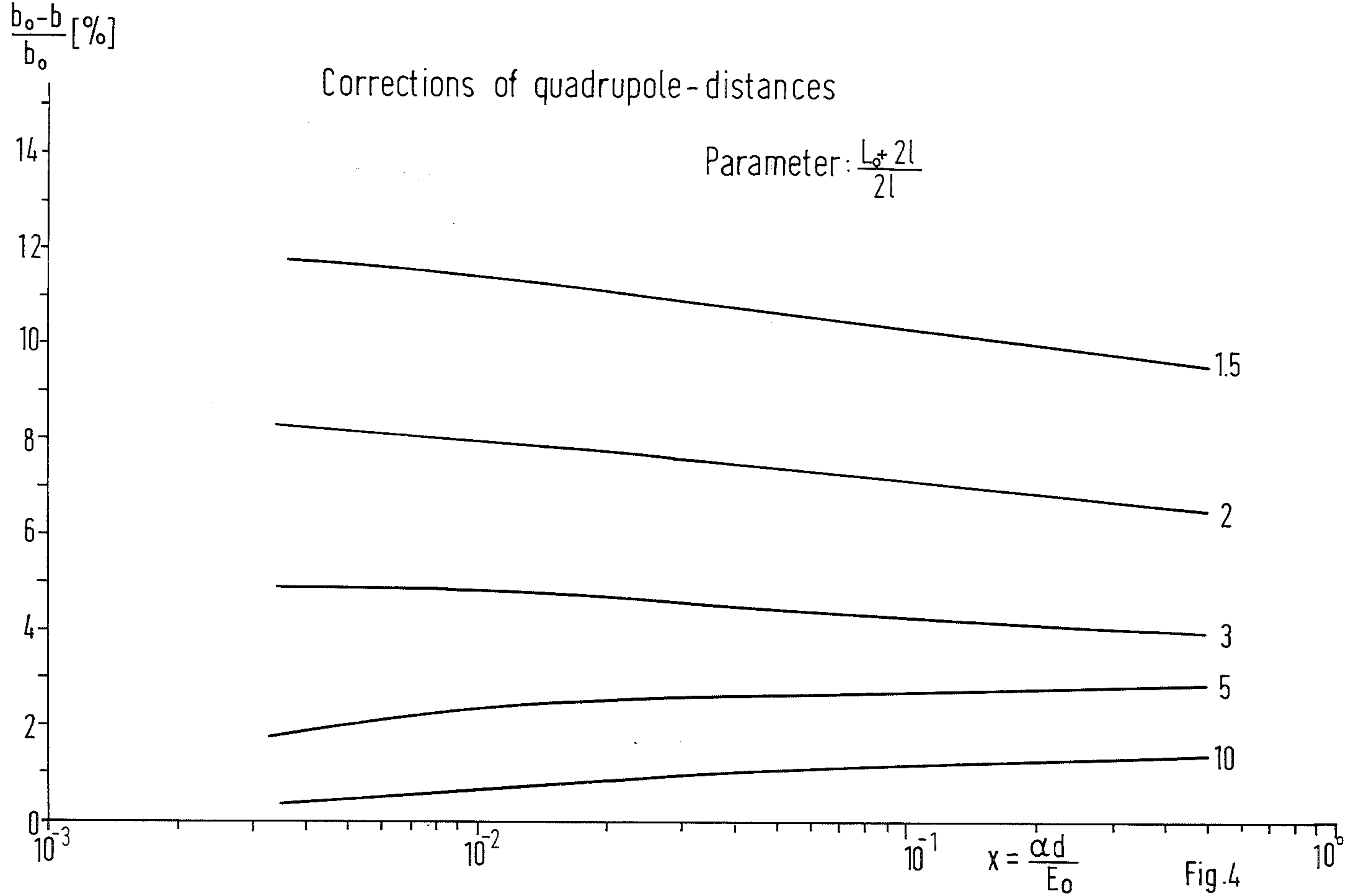
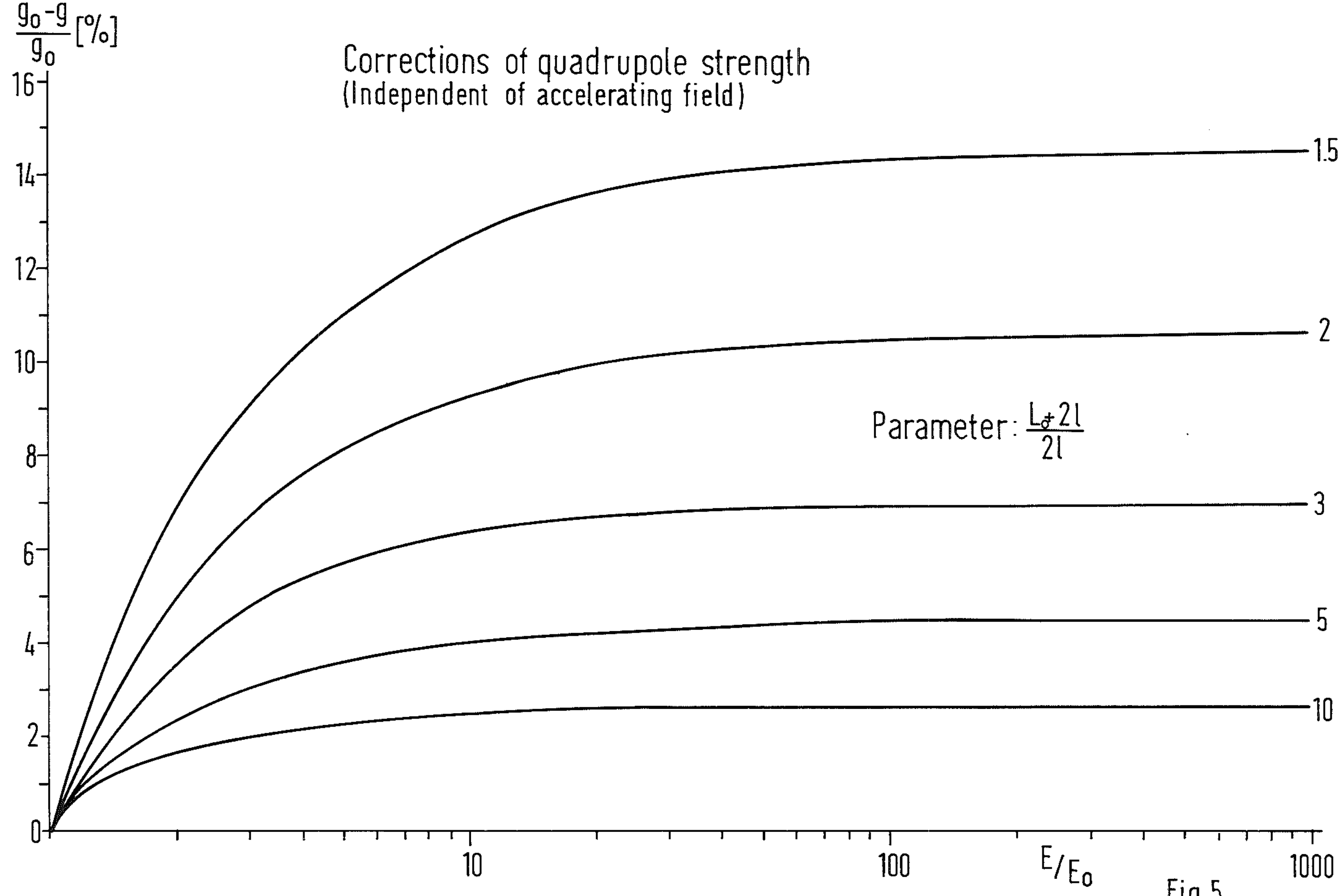

Comparison of thin lens theory
with this paper
Dotted lines: Thin lens theory
Parameter: Accel. field in MeV/m
2l = 0,303m
L+2l / d
10
5
1
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
n
10
10
8
8
6
6
4
4
2
2

Fig. 3



$\frac{b_o - b}{b_o}$ [%]
Corrections of quadrupole-distances
Parameter: $\frac{L_o + 2l}{2l}$
14
12
10
8
6
4
2
0
1.5
2
3
5
10
10^{-3}
10^{-2}
10^{-1}
10^{0}
$x = \frac{\alpha d}{E_o}$

Fig. 4



$\frac{g_0-g}{g_0}$ [%]
Corrections of quadrupole strength
(Independent of accelerating field)
Parameter: $\frac{L_0+2l}{2l}$
1.5
2
3
5
10
0
2
4
6
8
10
12
14
16
10
100
1000
E/E_0

Fig.5