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Study of the Charged Three-Pion System in the Mass Region of 1400-1900 MeV, and the J^P Quantum Numbers of the $A_3(1640)$ [$]

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From an experiment on the reaction $\pi^+ p \to \pi^+\pi^+\pi^- p$ at 11.7 GeV/c incident momentum we present a spin and parity analysis of the charged 3π system in the mass range of 1400-1900 MeV. We find a broad (Γ = 200-400 MeV) enhancement in the $J^P = 2^-$, $f\pi$(S wave) state, centered at about 1650 MeV, and no structure in the other $f\pi$ partial waves. We thus determine the J^P quantum numbers of $A_3 \to f\pi$ to be $J^P = 2^-$.

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1. Introduction and Summary

The A_3, also called $\pi(1640)$, is an enhancement with mass $M \approx 1650$ MeV seen in the three-pion spectra of several experiments [1] on the reaction $\pi^{\pm}p \rightarrow \pi^{\pm}\pi^{+}\pi^{-}p$. The dominant decay mode has been found [2] to be $f\pi$, from which isospin $I = 1$ and G-parity -1 are established. The spin and parity of this resonance are not yet known. They are usually assumed to be $J^P = 2^-$, since the center of the A_3 peak is only 240 MeV above $f\pi$ threshold and thus S-wave decay is favored. Bartsch et al. [3] and Caso et al. [4] have analyzed the A_3 decay with special assumptions made on the nature of the nonresonant background. They obtained good fits with $J^P = 2^-$ but could not exclude other J^P values of the unnatural series. Evidence against $J^P = 1^+$ has recently been reported by Paler et al. [5].

In this paper we describe the result of a similar analysis as the one by Bartsch et al. [3]. We too obtain, from our data, the best fit for $J^P = 2^-$ of the A_3. An addition - and this is the main topic of this paper - we present a more general analysis of the J^P quantum numbers of the 3π states in the mass range around the A_3 mass [6]. In this analysis we do not treat resonance and background separately. Rather, we make partial wave fits in order to determine the contributions to the total cross section of the various spin and parity states, as a function of the mass of the three-pion system. The reaction studied is $\pi^{+}p \rightarrow \pi^{+}\pi^{+}\pi^{-}p$ at 11.7 GeV/c incident momentum. Here, the signal to background ratio (about 1:2) for the A_3 bump is better than in most other experiments.

As a result we find that, if we exclude the possibility of $J \geq 4$, then $J^P = 2^-$ is the only possible quantum number assignment for the $A_3 \rightarrow f\pi$ state. Indeed in the 2^- $f\pi$ partial wave we find a broad enhancement, 200-400 MeV wide, starting from $f\pi$ threshold. We find no significant evidence for either 2^- $\rho\pi$ or 2^- $\varepsilon\pi$ states in the A_3 mass region. Therefore we cannot exclude the possibility that the A_3 is a $f\pi$ Deck-type enhancement [7]. Besides the 2^- $f\pi$ state a leading partial wave is

1^+ $\rho\pi$ (S wave), throughout the mass region studied. No significant contributions from natural J^P states are found. The assignment of $J^P = 2^-$ to the $A_3(1640)$ fits well into the quark model scheme, where it is interpreted to be the isospin 1 member of the L = 2, spin singlet, C = +1 octet.

2. The Three-Pion Mass Spectrum

The data are from 200 000 pictures of the CERN 2m hydrogen bubble chamber exposed to a 11.7 GeV/c π^+ beam. Of the reaction

$$\pi^+ p \rightarrow \pi^+\pi^+\pi^- p, \qquad (1)$$

7520 events were obtained (4c fits), corresponding to a cross section of (1.45 ± 0.15) mb. Other aspects of reaction (1) have already been published [8]. Fig. 1 shows the distribution of the effective mass of the three pions from this reaction. One sees a broad enhancement between 1000 MeV and 1400 MeV, and then the A_3 as a distinct peak at 1650 MeV. In the π^+p effective-mass distribution [8] one observes strong $\Delta^{++}(1236)$ production. We remove the $\Delta^{++}(1236)$ by eliminating all events that have $M(p\pi^+_\ell) < 1450$ MeV and $\cos\Theta(p\pi^+_\ell) > 0.97$, where $\Theta(p\pi^+_\ell)$ is the CMS angle between the initial state proton and the final $p\pi^+_\ell$ system; by π^+_ℓ we denote the π^+ in the final state that has the larger four momentum transfer from the incident π^+. This leaves 4252 events, shown hatched in fig. 1. From fits to this histogram in the A_3 region we obtain values of $M = (1660 \pm 25)$ MeV and $\Gamma = (190 \pm 100)$ MeV for the A_3 enhancement; the production cross section is $\sigma_{A_3} = (50^{+20}_{-15})\,\mu b$. The quoted errors include uncertainties due to the background subtraction.

3. Formalism

In this section we give a brief outline of the analysis procedure. The formalism and methods employed are similar to those used in other three-particle partial wave analyses [9].

We factorize the total amplitude for the reaction

$$\pi_{in} + N_{in} \rightarrow (3\pi) + N_{out}$$

into a production amplitude $A_{KM}(\vec{x}_p)$ and a decay amplitude $B_{KM}(\vec{x}_d)$ of the three-pion system (fig. 2)*:

$$\text{amplitude} = \sum_{KM} A_{KM}(\vec{x}_p)\, B_{KM}(\vec{x}_d) \qquad (2)$$

Here $\vec{x}_p$ is the vector of the production observables ($s = (\pi_{in} + N_{in})^2$, $t = (N_{in} - N_{out})^2$, $m_{3\pi}^2 = (\pi_{in} + N_{in} - N_{out})^2$, and the spin and isospin projections of π_{in}, N_{in}, and N_{out}), while $\vec{x}_d$ is shorthand notation for the three-pion observables (for which we may take e.g., the total mass $m_{3\pi}$, two of the three two-pion effective masses, the Euler angles Φ , Θ , and ψ

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* This factorization is a purely formal property. It does not imply dynamical assumptions, such as the absence of quasi-two-body production of the type $\pi_{in}\, N_{in} \rightarrow (2\pi)(\pi N)$. It is believed that this follows from a generalization, to four particles, of the properties of three-particle states (G.C. Wick, Ann. Physics 18, 65(1962)). Dynamics enters if e.g. one terminates the partial-wave sum, say at small J; if one assumes relations between phases and magnitudes of some of the partial waves (as would be appropriate for resonant (3π) states); or if one associates a physical picture with the parts of the amplitudes we called "production" and "decay" parts.

that describe the spatial orientation of the three-pion configuration with respect to the "production configuration" of the particles π_{in}, N_{in} and N_{out}, and the charges of the pions). K is a set of quantum numbers of the three-pion state,

$$K = \{I, J, L_a, I_a, \ell_a\} \qquad (3)$$

with I = isospin, J = total angular momentum, L_a = orbital angular momentum of pion a with respect to the dipion (bc); I_a and ℓ_a are isospin and orbital angular momentum of the dipion (bc); finally a,b,c is an even permutation of 1,2,3. Further quantum numbers are the spin projection $M = J_z$ (with the quantization axis fixed with respect to $\vec{\pi}_{in}$, $\vec{N}_{in}$ and $\vec{N}_{out}$), and the isospin three-component $I_3 = i_{31} + i_{32} + i_{33}$, the latter being fixed by the isospin-three-components i_{3a} of the three pions. The sum in equation (2) is done such that the Bose symmetry requirements for the pions are satisfied.

We now write down the explicit expression for the decay amplitude

$$B_{KM}(\vec{x}_d) \equiv \langle \vec{p}_a i_{3a}\ \vec{p}_b i_{3b}\ \vec{p}_c i_{3c} | T | K\ I_3\ M \rangle \qquad (4)$$

using suitable projection operators. We rotate coordinates by Φ, Θ and Ψ, project on to an a+(bc) helicity state [10], and sum over all helicities λ_a of the dipion (bc). We then project the (bc) diparticle state with definite angular momentum ℓ_a and helicity λ_a on to a plane wave state. Finally we have to project out a definite orbital angular momentum L_a and an isospin I_a by suitable Clebsch-Gordan coefficients. The result is

$$B_{KM}(\vec{x}_d) = \frac{1}{4\pi} \sqrt{(2\ell_a+1)(2L_a+1)}\ \langle 1 i_{3b}; 1 i_{3c} | I_a I_{3a} \rangle\ \langle I_a I_{3a}; 1 i_{3a} | I I_3 \rangle$$
$$\sum_{\lambda_a} \left\{ \langle L_a 0; \ell_a, -\lambda_a \,|\, J, -\lambda_a \rangle\ D^{\ell_a *}_{-\lambda_a 0}(\varphi_a, \vartheta_a, -\varphi_a) \left[\sum_m D^{J*}_{m,-\lambda_a}(\phi_a, \theta_a, -\phi_a) \right. \right.$$
$$\left.\left. D^J_{mM}(\Phi, \Theta, \Psi) \right] \right\} T^K(m^2_{3\pi};\ m^2_{ab},\ m^2_{bc}), \qquad (5)$$

where Θ_a, Φ_a are polar and azimuthal angles of $\vec{p}_a$ in the rigid-body system; ϑ_a, φ_a polar and azimuthal angles of $\vec{p}_b$ in the rest frame of the diparticle (bc), taking the direction of $\vec{p}_a$ to be the z axis. All of these angles only depend on the Dalitz plot variables m_{ab}^2 and m_{bc}^2 and on the total three pion mass $m_{3\pi}$. The reduced matrix element (the partial wave amplitude) T^K contains the dynamical dependence on the effective masses of the pions. It is characterized by the quantum number set K but is independent of I_3 and M.

We assume the dynamical dependence of the amplitude to be determined by centrifugal barrier penetration factors [11] and a Watson final-state factor [12] for the interacting pion pair (bc). Thus we take the reduced T matrix element to be

$$T^K(m_{3\pi}^2; m_{12}^2, m_{23}^2) = p_a^{L_a + \frac{1}{2}} \; q_a^{-\ell_a - 1} \; \exp(i\delta_{bc}) \sin\delta_{bc}$$

where p_a is the momentum of pion a in the three-pion rest system, q_a the momentum of b in the (bc) rest frame, and δ_{bc} the elastic bc scattering phase shift. We took into account only the three resonance-dominated $\pi^+\pi^-$ interactions described by δ_0^0, δ_1^1 and δ_2^0. The phase shifts δ_ℓ^I were parametrized as ℓ-wave Breit-Wigner resonances [11] ε, ρ, and f, with masses and widths of $m_\varepsilon = 700$ MeV, $\Gamma_\varepsilon = 450$ MeV; $m_\rho = 765$ MeV, $\Gamma_\rho = 130$ MeV; and $m_f = 1265$ MeV, $\Gamma_f = 150$ MeV; a radius of interaction of 1 F was assumed.

The observed distribution of the three-pion variables $\vec{x}_d$ is obtained by integrating/summing the modulus squared of the amplitude (2) over a region in $\vec{x}_p$ space, at fixed $m_{3\pi}$:

$$\sigma(\vec{x}_d) \propto \int d\vec{x}_p \left| \sum_{KM} A_{KM}(\vec{x}_p) \, B_{KM}(\vec{x}_d) \right|^2$$

$$= \sum_{KM,K'M'} \rho_{KM,K'M'} \, B_{KM}(\vec{x}_d) \, B^*_{K'M'}(\vec{x}_d) \qquad (6)$$

where

$$\rho_{KM,K'M'} = \int d\vec{x}_p \, A_{KM}(\vec{x}_p) \, A^*_{K'M'}(\vec{x}_p) \qquad (7)$$

is the (unnormalized) density matrix of the three-pion state produced. Parity conservation halves the number of independent elements of this matrix.

A completely general spin parity analysis of the three-pion system would involve determining this density matrix $\rho_{KM,K'M'}$ from the data, as a function of $m_{3\pi}$. Actually this is a hopeless task. Using only the most important partial waves (the ones we actually took into account in this work, see below), one would have to determine > 1000 independent real parameters at each $m_{3\pi}$ value.

In order to reduce the number of parameters one integrates over some of the kinematic variables. Integrating e.g. over the Dalitz plot one keeps only the Euler angle (Φ,Θ, and Ψ) distribution [13]. We have studied this distribution in a general way but did not reach unambiguous conclusions from it (see end of section 4).

Another possibility is to integrate over Φ,Θ, and Ψ, thus considering only the Dalitz plot distribution. This turned out to be more successful in this experiment. Due to the orthogonality of the $D^J_{mM}(\Phi,\Theta,\Psi)$ only terms with $J = J'$, $M = M'$ in (6) survive, and the integral of the product $B_{KM}\, B^*_{K'M'}$ becomes independent of M. Therefore after integration over the Euler angles, (6) takes the form

$$\sigma(m_{3\pi};m^2_{12},m^2_{23}) \propto \sum_{\substack{K,K' \\ (\text{with } J=J')}} \mathrm{Tr}\rho_{KK'}(m_{3\pi})F_{KK'}(m_{3\pi};m^2_{12},m^2_{23}) \qquad (8)$$

where the trace is with respect to M, and where

$$F_{KK'}(m_{3\pi};m^2_{12},m^2_{23}) = \int d\Phi d\cos\Theta d\Psi \quad B_{KM}(\vec{x}_d)\, B^*_{K'M'}(\vec{x}_d). \qquad (9)$$

Using invariance under space reflection one finds further that only states with the same J^P interfere in (8). As a simplification we assumed that the density matrix factorizes in the form

$$\mathrm{Tr}\rho_{KK'}(m_{3\pi}) \propto a_K(m_{3\pi})\, a^*_{K'}(m_{3\pi}).$$

This is equivalent to assuming complete coherence between the three-pion partial waves with the same J^P but different L_a and ℓ_a, in the restricted subspace of production variables considered(i.e. a small range of t and $m_{3\pi}$).

We now briefly summarize our formalism. We consider the Dalitz plot distribution of the three-pion system in reaction (1). As quantum numbers we use the relative orbital angular momentum L between π_1^+ and the pair $(\pi_2^+\pi^-)$; the angular momentum ℓ of the pair $(\pi^+\pi^-)$; and the spin J of the three-pion state, with $\vec{J} = \vec{L} + \vec{\ell}$. The parity is then given by $P = (-1)^{L+\ell+1}$. We take the isospin of the positively charged three-pion system to be 1. These quantum numbers are sufficient to define the kinematical behaviour of the matrix element for decay of the three-pion system. As for dynamics, we assume a final state $\pi^+\pi^-$ interaction dominated by the pion-pion partial waves that resonate in the region of available energies, i.e., by δ_0^0, δ_1^1 and δ_2^0 (dominated by the ε, ρ^0 and f resonances, respectively). We further assume that the matrix element is proportional to the threshold barrier penetration factor appropriate to angular momentum L.

4. Results

In our partial wave analysis we have included all amplitudes that have $L + \ell \leq 3$, assuming that the higher orbital angular momenta are suppressed by centrifugal barrier factors. (We have however not restricted L to the smallest possible value for given ℓ and J^P). Since the A_3 has dominant $f\pi$ decay, the amplitude with $J^P = 0^-$, $L = \ell = 2$, and $I_{\pi^+\pi^-} = 0$ was also included. This gives 16 different partial waves. The expression (8) was fitted to the data in each 1oo-MeV bin of the three-pion mass separately, using the maximum-likelihood method. We have left the phases between the different amplitudes free in the fit. This leads to a total of 26 (real) parameters to be determined in each $m_{3\pi}$ bin. In most $m_{3\pi}$ bins we found that many of the parameters were consistent with zero. In particular this was the case for the $J^P = 1^+$, $\rho\pi(L = 2)$ and for all 3^+ partial waves, as well as for the whole natural spin-parity series.* We therefore repeated the fits without these contributions, and whenever we got results as good as before, we used the results from the fit with the reduced set of partial waves (15 parameters). To check the uniqueness of the fits, various different sets of initial values of the parameters were used. We always obtained the same solutions.

Since we define the best fit by the maximum of the likelihood function, we do not obtain a goodness-of-fit parameter. In figs. 3-5 we therefore present various 1-dimensional distributions of the data and compare them with the result calculated from the fitted amplitudes. It is seen that there is satisfactory agreement between the data and the fit. Table I substantiates this by showing the χ^2 values for the histograms of figs. 3-5. Except for the mass bin of 1800-1900 MeV (where higher partial waves may indeed enter) the fits are excellent.

In fig. 6 we present the fitted contributions of the more important partial waves as a function of $m_{3\pi}$ in the mass range from 1400 MeV to 1900 MeV. The ordinate scale is number of events per bin for the partial wave; note that these do not exactly add up to the total number of events because of the interference terms. Those partial amplitudes not shown in fig. 6 were found to be negligible.

We see from fig. 6 that the $\rho\pi$ states predominantly have $J^P = 1^+$ (L = 0) in the whole mass range. The $f\pi$ states which are known [1,2] to contain the bulk of the A_3 decay, are seen

* Note that of the natural series, only $J^P = 2^+, 4^+, \ldots$ are non-exotic states because of $I \geq 1$ and $G = -1$.

to be predominantly $J^P = 2^-$, S wave. This partial wave shows a broad peaking, starting from $f\pi$ threshold at 14oo MeV and centered roughly at 165o MeV with a width of the order of 3oo MeV. We do not see peaking in any other $f\pi$ partial wave. On the other hand, some structure is seen in the $J^P = 0^-$ $\epsilon\pi$ contribution. We do not yet know how to interprete this latter effect, which appears to be quite significant in our data. However, we note that the $J^P = 0^-$ $\epsilon\pi$ matrix element is peculiar in the respect that it does not lead to any characteristic structure (nodes, zeros or peaks) of the Dalitz-plot density distribution. It therefore resembles a phase-space distribution and could perhaps be "faked" by the addition of many incoherent small effects with all characteristic Dalitz-plot structure washed out.

Turning back to the $J^P = 2^-$ partial wave, we observe no evidence for significant $\rho\pi$ or $\epsilon\pi$ contributions to the A_3 (see fig. 6).

We conclude that if (as is generally accepted) we define the A_3 as the state responsible for the $f\pi$ enhancement near threshold, then it has quantum numbers $J^P = 2^-$ (unless $J \geq 4$) and a rather large width of 2oo-4oo MeV.

To have an independent check of our result that the A_3 bump can only be ascribed to a 2^- state, we made a second type of fit. Here we fitted the Dalitz-plot distribution of reaction (1) in a $m_{3\pi}$-<u>dependent</u> way from 13oo to 21oo MeV, to an incoherent sum of the following contributions:

(a) A Breit-Wigner tail of the A_2.

(b) 3π, $\rho^o\pi^+$ and $f\pi^+$ backgrounds, with the $m_{3\pi}$ dependence described by phase space multiplied by polynomials whose coefficients were fitted.

(c) A Breit-Wigner resonance for A_3, with possible decay modes $f\pi$,$\rho\pi$ and $\epsilon\pi$ (including interference terms in the overlap regions), using the three-pion decay-matrix elements for I = 1 with a definite J^P.

This is not as general as the previous fit, since we use the additional assumption that the "background" under the A_3 has some smooth (polynomial) dependence on $m_{3\pi}$ (apart from the $f\pi$ threshold factor), and that only one J^P contributes to the 165o MeV A_3 bump. We indeed find the best fit for $J^P = 2^-$. Some $m_{3\pi}$ distributions (with various cuts) and the $m_{\pi^+\pi^-}$ mass distributions in various $m_{3\pi}$ ranges, as obtained from the best fit, are compared with the data in fig.7. The next-best solution obtained had $J^P = 1^+$, but its likelihood was much smaller (factor $\sim 1o^{-4}$). All other J^P were found unacceptable in this type of fit.

In addition to the analysis discussed, we have also done an analysis of the general moments

$$\sum_{\text{events}} D^j_{mn} \ (\Phi,\Theta,\Psi)$$

of the Euler angle distribution describing the orientation of the three-pion decay configuration. This distribution contains complementary information on the partial-wave amplitudes [13]. However, although we tried various different coordinate systems (s- and t-channel helicity frames, transverse, and non-transverse), it was found impossible to draw model-independent and unambiguous conclusions from the behavior of these moments in our experiment.

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Table I. χ^2 / number of bins, for the partial wave fit curves in the histograms of figs. 3-5.

(Note that χ^2 has not been minimized. We give it to display the amount of agreement, in various 1-dimensional projections, between the data and the partial wave fit to the whole Dalitz plot using the maximum-likelihood method. Therefore the numbers of n_{bin} cannot be taken as numbers of degrees of freedom).

quantity histogrammed \ $m(\pi^+\pi^+\pi^-)$ interval	(1.4-1.5) GeV χ^2 / n_{bin}	(1.5-1.6) GeV χ^2/n_{bin}	(1.6-1.7) GeV χ^2/n_{bin}	(1.7-1.8) GeV χ^2/n_{bin}	(1.8-1.9) GeV χ^2/n_{bin}
$m(\pi^+\pi^+)$	25.8 / 27	11.6 / 30	33.8 / 32	24.9 / 35	55.1 / 37
$m(\pi^+\pi^-)$	20.5 / 27	23.9 / 30	26.9 / 32	40.4 / 35	34.8 / 37
$\cos\vartheta_\rho$	15.8 / 20	14.2 / 20	15.6 / 20	25.8 / 20	32.5 / 20
$\cos\vartheta_f$	15.0 / 20	5.6 / 20	6.6 / 20	15.3 / 20	34.1 / 20
r	22.0 / 20	22.5 / 20	22.5 / 20	19.4 / 20	21.4 / 20
φ	26.3 / 18	11.2 / 18	9.0 / 18	16.8 / 18	17.7 / 18
Sum over 6 histograms	125.4 / 132	89.0 / 138	114.4 / 142	142.6 / 148	195.6 / 152

Figure Captions

Fig. 1 Distribution of the effective three-pion mass $m(\pi^+\pi^+\pi^-)$. The hatched histogram is the sub-sample used in the analysis (see text).

Fig. 2 Factorization of the amplitude for reaction (1) into "production" and "decay" parts.

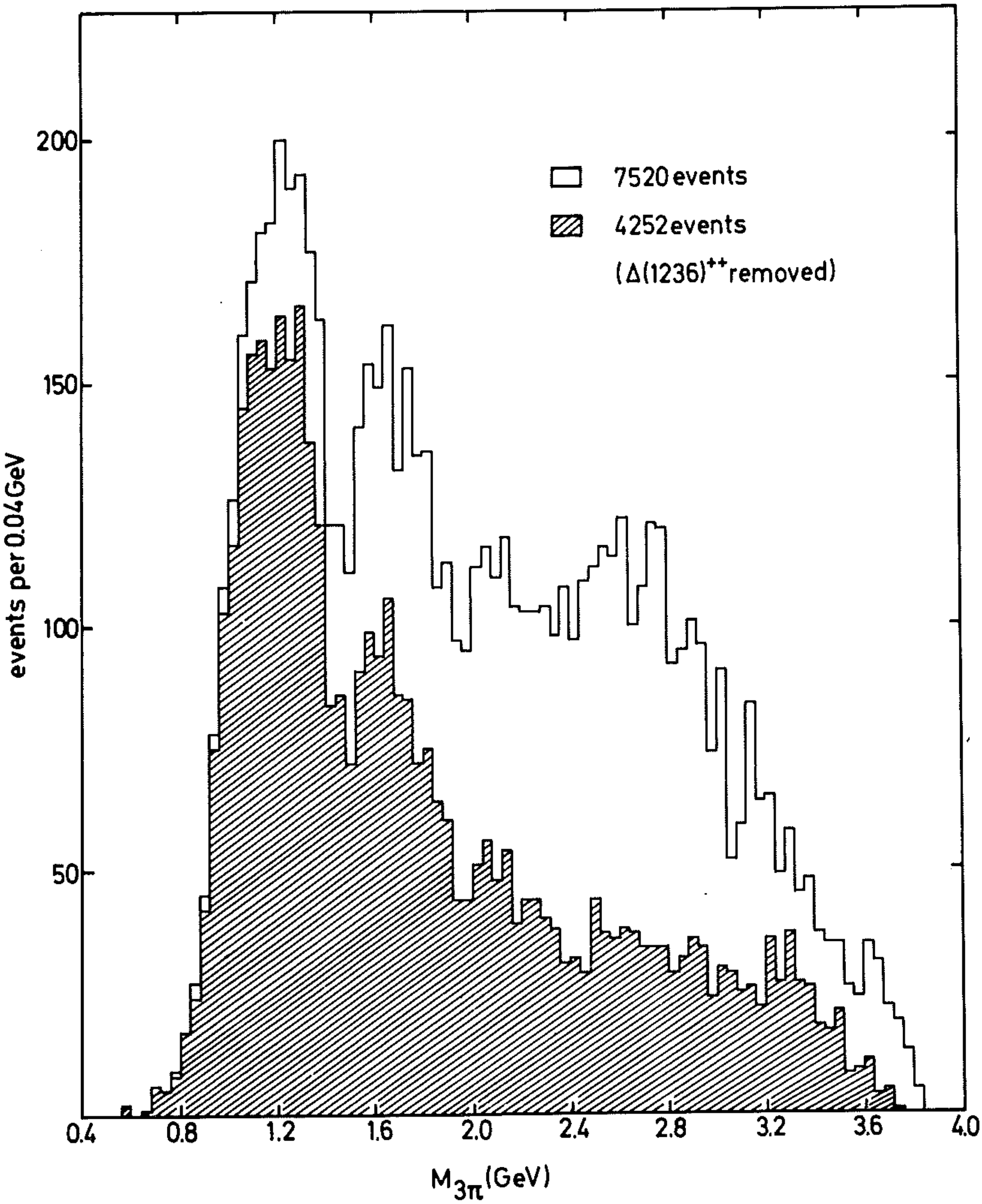
Fig. 3 Mass projections of the three-pion Dalitz plot (Δ^{++} excluded) in 1oo-MeV intervals of $m(\pi^+\pi^+\pi^-)$. The curves were calculated from the partial-wave fits to the Dalitz plot (see text), varying the 9 amplitudes of fig. 6 (i.e. 9 amounts plus 6 relative phases).

Fig. 4 Distribution of the helicity anlge $\cos\vartheta$ of the $\pi^+\pi^-$ decay for the ρ^o and f bands in the Dalitz plot (Δ^{++} excluded). (The bands are defined by 665 MeV < $m(\pi^+\pi^-)$ < 865 MeV and 114o MeV < $m(\pi^+\pi^-)$ < 139o MeV, respectively.) The curves were calculated from the partial-wave fits as in fig. 3.

Fig. 5 Distributions of the polar coordinates r,φ of the Dalitz plot (Δ^{++} excluded). The radial coordinate r has been normalized such that $r = 1$ at the boundary. The angle φ is folded because of Bose symmetry. $\varphi = 0$ at maximum energy of the π^-. The curves were calculated from the partial-wave fits as in fig. 3.

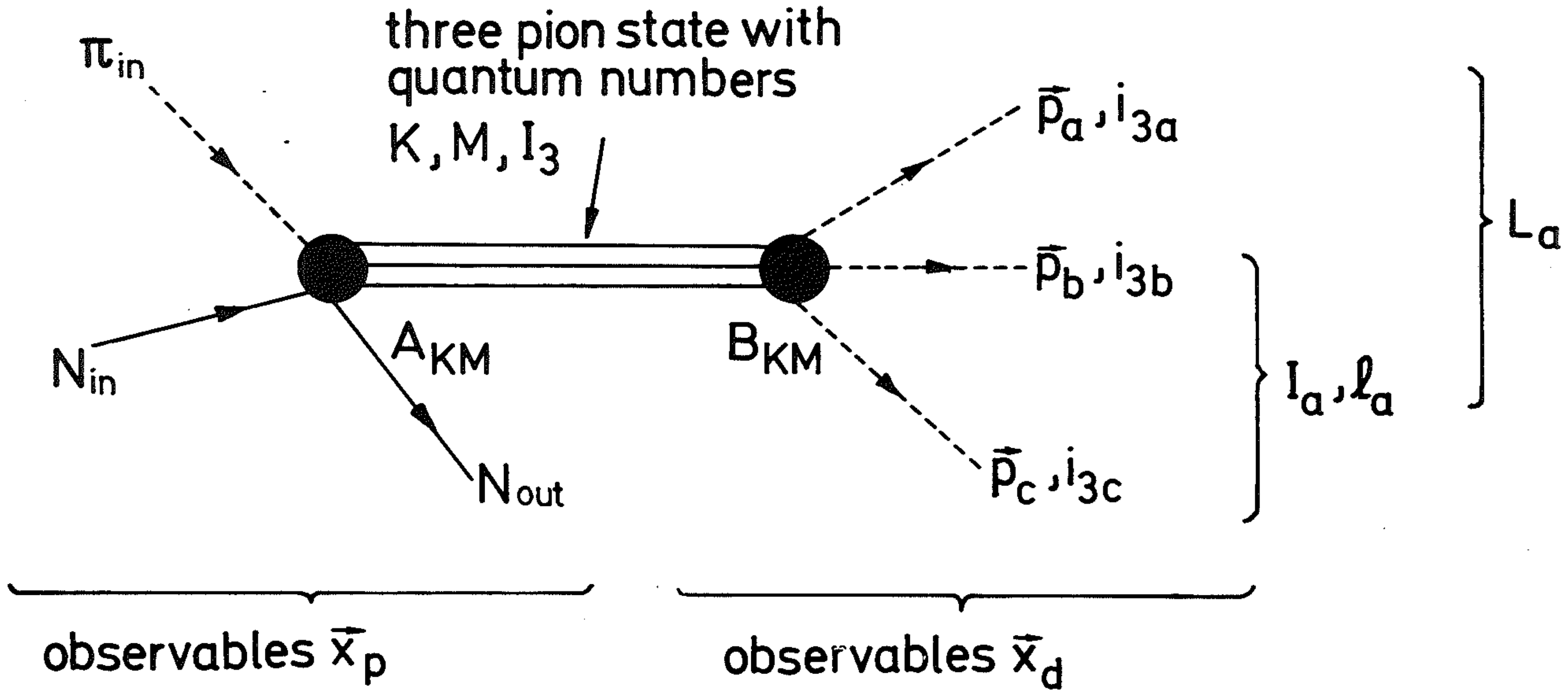
Fig. 6 Contributions of the partial waves as a function of $m(\pi^+\pi^+\pi^-)$.

Fig. 7 Distribution of the three-pion mass $m(\pi^+\pi^+\pi^-)$ with various cuts on the two-pion masses, and of $m(\pi^+\pi^-)$ in different regions of the three-pion mass (Δ^{++} excluded). (The ρ^o and f bands used are defined by 64o MeV < $m(\pi^+\pi^-)$ < 88o MeV and 112o MeV < $m(\pi^+\pi^-)$ <14oo MeV, respectively.) The curves were calculated from the $m(\pi^+\pi^+\pi^-)$-dependent fit.



7520 events
4252 events
(Δ(1236)++ removed)
events per 0.04GeV
200
150
100
50
0.4
0.8
1.2
1.6
2.0
2.4
2.8
3.2
3.6
4.0
M3π (GeV)

Fig.1



three pion state with quantum numbers
K, M, I3
πin
Nin
AKM
Nout
BKM
p⃗a, i3a
p⃗b, i3b
p⃗c, i3c
La
Ia, ℓa
observables x⃗p
observables x⃗d

Fig.2

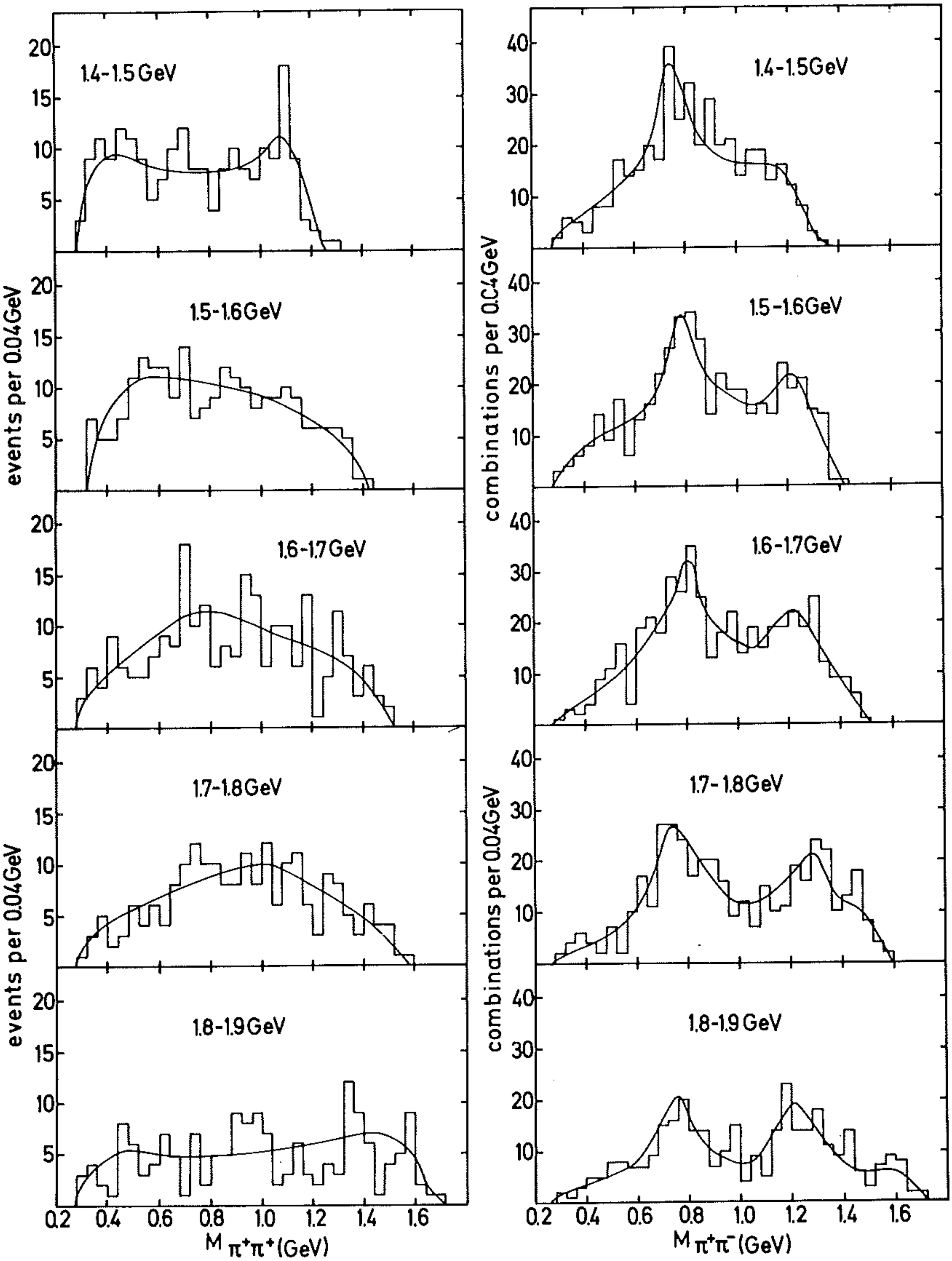

1.4-1.5 GeV
1.5-1.6 GeV
1.6-1.7 GeV
1.7-1.8 GeV
1.8-1.9 GeV
events per 0.04GeV
combinations per 0.04GeV
0.2 0.4 0.6 0.8 1.0 1.2 1.4 1.6
$M_{\pi^+\pi^+}$ (GeV)
$M_{\pi^+\pi^-}$ (GeV)

Fig.3

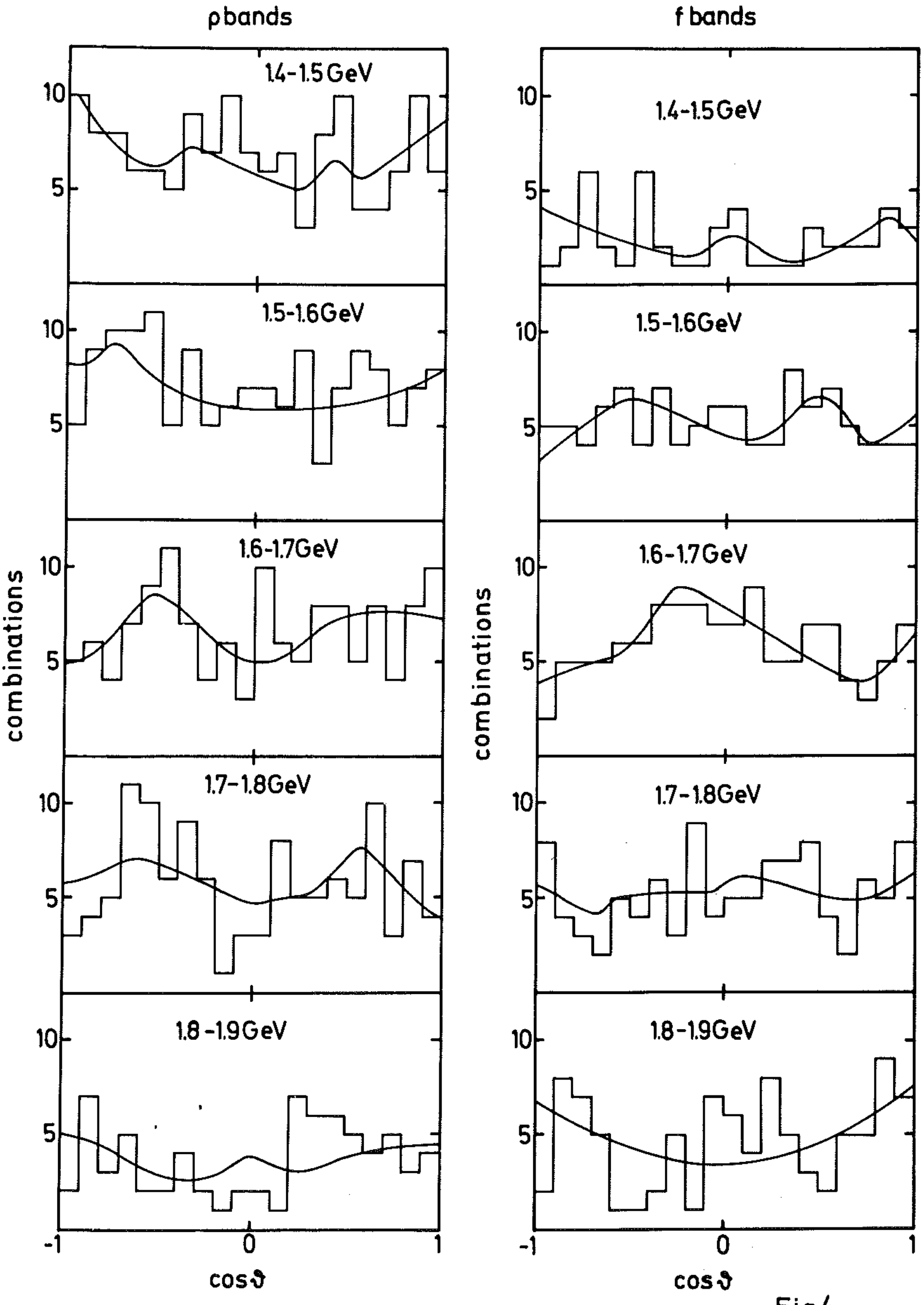

ρ bands
f bands
1.4-1.5 GeV
1.5-1.6GeV
1.6-1.7GeV
1.7-1.8GeV
1.8-1.9GeV
combinations
10
5
-1
0
1
cos ϑ

Fig.4

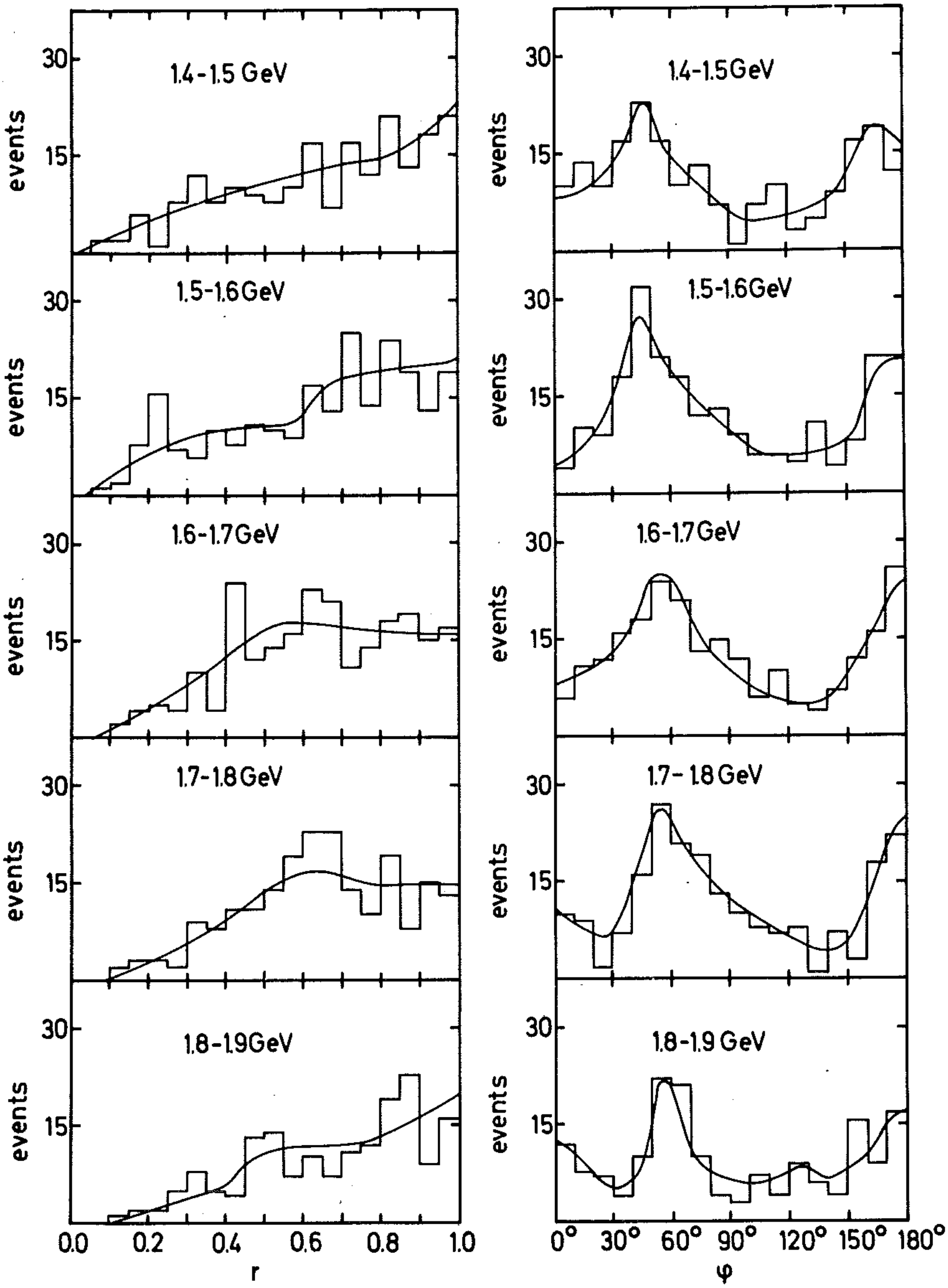

1.4-1.5 GeV
1.5-1.6 GeV
1.6-1.7 GeV
1.7-1.8 GeV
1.8-1.9 GeV
events
30
15
0.0
0.2
0.4
0.6
0.8
1.0
r
0°
30°
60°
90°
120°
150°
180°
φ

Fig.5

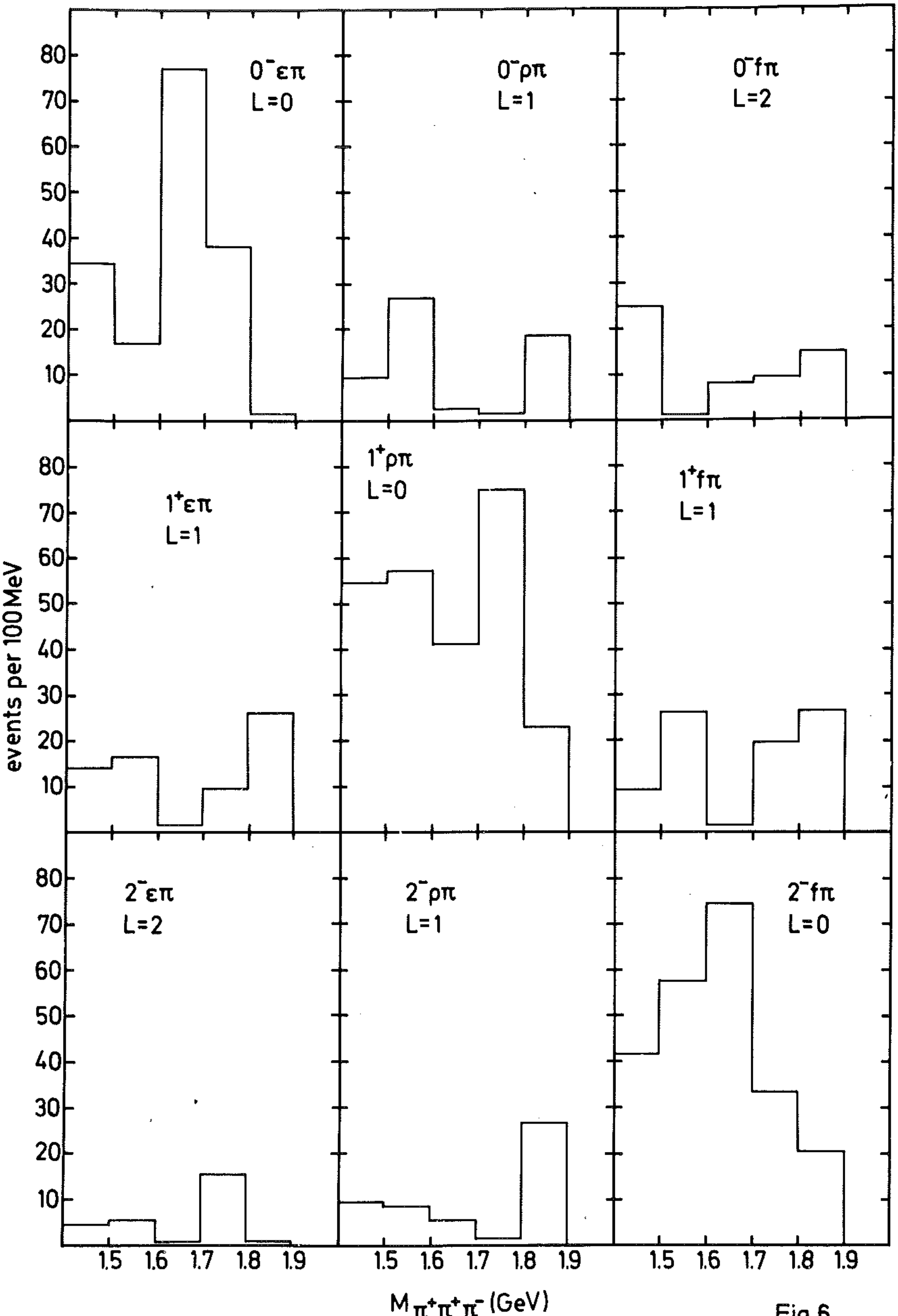
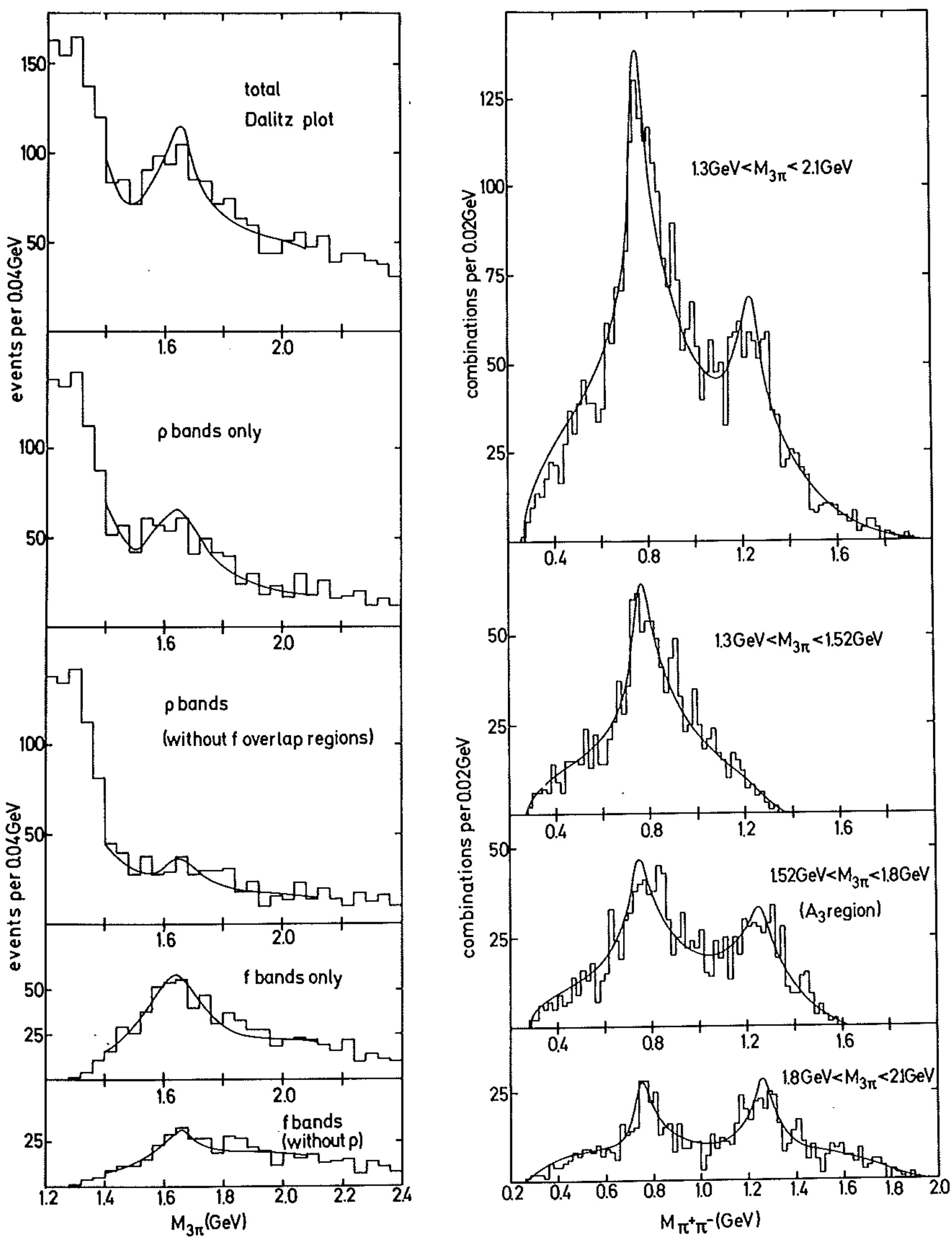

0⁻επ L=0
0⁻ρπ L=1
0⁻fπ L=2
1⁺επ L=1
1⁺ρπ L=0
1⁺fπ L=1
2⁻επ L=2
2⁻ρπ L=1
2⁻fπ L=0
events per 100 MeV
10
20
30
40
50
60
70
80
1.5
1.6
1.7
1.8
1.9
$M_{\pi^+\pi^+\pi^-}$ (GeV)

Fig. 6



total
Dalitz plot
ρ bands only
ρ bands
(without f overlap regions)
f bands only
f bands
(without ρ)
events per 0.04GeV
M3π(GeV)
1.3GeV<M3π<2.1GeV
1.3GeV<M3π<1.52GeV
1.52GeV<M3π<1.8GeV
(A3region)
1.8GeV<M3π<2.1GeV
combinations per 0.02GeV
Mπ+π-(GeV)

Fig.7