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Linear Stochastic Processes and Intensity Correlations of Photon States

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Abstract

We introduce linear stochastic processes on certain function spaces pertinent to the measurement of the intensity correlations that characterize quantized electromagnetic radiation. Next, we construct the characteristic functional of the basic process by considering second quantization as a functor. Finally, we calculate and discuss the characteristic functionals for specific photon states which are: (1) coherent states, (2) quasi-free states, and (3) n-particle states.

I. Introduction

It is characteristic of electromagnetic radiation that states with a definite photon number n are not likely to occur in nature. One frequently encounters a distribution of n and similar quantities which may well be described by an incoherent superposition of pure states with definite n, hence by a density matrix. Here, the most prominent example is provided by Gaussian light. On the other hand, due to the absence of a superselection rule with respect to the photon number, coherent superpositions of photon states with different n also play a significant rôle. The extensively studied class of coherent photon states illuminate this fact.

It is for these reasons that correlation experiments of ever increasing complexity are needed to gain sufficient knowledge of the observed radiation and its source. Since the classical correlation experiments for ordinary light intensities by Hanbury Brown and Twiss the quantum theory of optical coherence has been forcefully developed [1] and the present formulation owes much to the work of Glauber [2]. Finally, a quantum stochastic formulation of the theory of arrival times of light quanta has been attempted by Davies [3].

This line of reasoning suggests introducing local observables for the prediction of the correlation between the output of several photon detectors. Unfortunately, given any state of the radiation field, there is no appropriate definition for "the number of photons (or the energy) in $\mathcal{O}$" if $\mathcal{O}$ is a bounded open subset of Minkowski space. One may, however, pass to the asymtotic measurements and introduce global observables as is adequate for an S-matrix theory of quantum electrodynamics. In the present paper, we do this by ignoring largely the local nature of a realistic experiment an, in particular, by ignoring its finite extension in time. Instead, we emphasize the rôle played by the four-momenta and the polarisations of single particle events.

We are thus led to include the aperture, as viewed from the source, and the frequency range of the accepted photons into the specification of a photon detector [4]. We do this abstractly by associating a number operator N(S) to every Borel subset S of Σ with compact closure, where Σ is the momentum-helicity space of a single photon. The operators N(S) exist in infrared representations although a total number operator does not exist. Similarly, an energy-momentum operator can be associated with the Borel set S. Since all these observables commute, the predictions fall within the scope of classical stochastic analysis. In particular, given a state of the radiation field and any arrangement of several detectors, there is a joint probability measure which determines the distribution of measured values. As a result, coherent photon states exhibit no correlation at all. Crudely speaking, we find the opposite behavior of photon states in momentum space as compared to the behavior in position space.

The infinite set of joint probability distributions is seen to be generated by a linear stochastic process on a certain function space. For coherent states, this process is of the Poisson type as would be anticipated. However, it is not of the Gauss or Wiener type if the state is quasi-free.

Since the formulation of the theory within the framework of classical measurement theory and stochastic processes is the easiest way to present the underlying physical ideas without having to go through all the details of the operator theory and inequivalent representations, we shall present this formulation first (Section II and III). Then we shall pass to the relationship between this scheme and field theory. Here, the ultimate objective is the determination of the characteristic functional of the basic process for a given state of the radiation field (Section IV and V).

II. The Basic Stochastic Process

In most cases, it is a reasonable idealization to assume that a photon detector absorbes photons confined to a certain frequency range and solid angle (as seen from the source). In addition, the detector may discriminate against a specified helicity. We thus define the acceptance of an ideal detector to be a compact subset S of the space Σ of single particle events $(\mathbf{k}, \lambda)$:

$$\Sigma = \{(\mathbf{k},\lambda) \mid \mathbf{k} \in \mathbb{R}^3 - \{0\}, \lambda = \pm 1\} \tag{1}$$

Here, $\mathbf{k}$ denotes the three-momentum and λ the helicity of a single photon in the corpuscular picture. Obviously, $|\mathbf{k}|$ is the frequency in the wave picture. To be specific, let us assume our detector counting the number $X(S)$ of photons with the property S. As the points $(0, \pm 1)$ are excluded from the space Σ, no infrared problems arise. No doubt, $X(S)$ is a random variable, its possible values being the non-negative integers. The probability $P(n)$ of observing exactly n photons with the property S is then given by the characteristic function:

$$E\left(e^{itX(S)}\right) = \sum_{n=0}^{\infty} P(n)\, e^{itn} \tag{2}$$

where $\mathbb{E}(.)$ stands for the expectation value in the sense of probability theory [5; §27]. We must not expect to be able to define $X(\Sigma)$, as the total photon number may be infinite. A suitable choice for the class of sets for which $X(S)$ is assumed to be defined is then the class C of Borel sets S with compact closure in Σ.

Suppose now that several detectors are applied simultaneously and let $S_1, \ldots, S_r$ be the relevant sets from the class C. Then the joint probability distribution is given by:

$$E\left(\exp i\sum_{\alpha=1}^{r} t_\alpha X(S_\alpha)\right) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} P(n_1, \ldots, n_r)\, \exp i\sum_{\alpha=1}^{r} t_\alpha n_\alpha \tag{3}$$

This enumerable set of probability distributions is subject to severe restrictions. On physical grounds, these conditions are explicitly as follows:

(a) $X(S_1 \cup S_2) = X(S_1) + X(S_2)$ if $S_1 \cap S_2 = \emptyset$

(b) $X(S) \geq 0$

Therefore, the set function $S \mapsto X(S)$ whose domain of definition is the class C has most of the properties of an unbounded measure[1]. In particular, $X(\emptyset) = 0$ as follows from (a).

A very useful concept in measure theory is that of a simple function. In the present context, a function $u: \Sigma \to \mathbb{R}$ is called simple if it takes on only a finite number of values different from zero, each on a set from the class C. The simplest example of a simple function is provided by the characteristic function 1_S of a set $S \in C$. In fact, any simple function may, though not uniquely, be written as a linear combination of characteristic functions of this kind:

$$u = \sum_{\alpha=1}^{r} t_\alpha 1_{S_\alpha} \tag{4}$$

Given u, the random variable X_u defined by

$$X_u = \sum_{\alpha=1}^{r} t_\alpha X(S_\alpha) \tag{5}$$

is independent of the representation (4) of u and therefore unambiguously defined. We shall find it convenient to employ the following very suggestive notation:

$$X_u = \int u \, dX \tag{6}$$

Let us finally introduce the functional

$$F(u) = \mathbb{E}(e^{iX_u}) \tag{7}$$

A glance at (3) reveals that F already determines all probability

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1) Measures taking values in a space of random variables were termed "random measures" by Gelfand and Vilenkin [6; Ch.III.3.4]

distributions under consideration.

It will prove to be a reasonable assumption that X_u is defined on a much larger class of functions: We shall assume that X_u is defined even if $u: \Sigma \to \mathbb{R}$ is an arbitrary bounded Borel function of compact support. The assumptions are explicitly as follows:

$$(a) \quad X_{u_1+u_2} = X_{u_1} + X_{u_2}$$

$$(b) \quad X_{tu} = t X_u \quad , \quad t \in \mathbb{R}$$

$$(c) \quad u_1 \geq u_2 \quad \Rightarrow \quad X_{u_1} \geq X_{u_2}$$

The map $u \mapsto X_u$ is thus said to be a <u>linear stochastic process on $\mathcal{U}$</u>, the real linear space of bounded Borel functions with compact support. As the functional F completely determines this process, it is called the <u>characteristic functional</u> of X_u. Evidently, F is normalized and positive definite:

$$(a) \quad F(0) = 1$$

$$(b) \quad \sum_{n,m=1}^{N} \bar{c}_n c_m F(u_m - u_n) \geq 0$$

for all $u_1, \ldots, u_N \in \mathcal{U}$ and all complex numbers $c_1, \ldots, c_N$.

III. The Derived Process

In the preceding section we confined ourselves to discussing the photon number and found it convenient to introduce the linear process X_u. Yet the general scope of the theory and the diversity of its results are not impaired by this restriction as we shall now demonstrate.

If the momentum of a single photon is $k = (k^\mu) = (|\mathbf{k}|, \mathbf{k})$, then kn is the momentum of n photons of this kind. Similarly, the set function $S \mapsto Y^\mu(S)$ given by the indefinite integral

$$Y^\mu(S) = \int_S k^\mu dX \tag{1}$$

yields the momentum of photons within each compact subset S of Σ. Clearly, the integral (1) is defined, for

$$(\mathbf{k}, \lambda) \mapsto \begin{cases} k^\mu & (\mathbf{k}, \lambda) \in S \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

is a bounded Borel function of compact support. For fixed S, $Y^\mu(S)$ is a random vector, i.e. a vector whose components are random variables. By construction, Y is an additive set function and satisfies the spectrum condition. Explicitly:

(a) $Y^\mu(S_1 \cup S_2) = Y^\mu(S_1) + Y^\mu(S_2)$ if $S_1 \cap S_2 = \emptyset$

(b) $x_\mu Y^\mu(S) \geq 0$ if[1] $x = (x^\mu) \in V_+$

(c) $Y^\mu(\emptyset) = 0$

Let $S_1, \ldots, S_r$ be sets from the class C associated with r detectors. If in a measurement, these detectors indicate the energy (or any other component of the momentum) rather than the photon number, we should be able to predict the joint probability $P(B_1, \ldots, B_r)$ that the output of the nth detector is in the Borel subset B_n of the momentum space. In probability theoretic language, P is a measure on the cartesian product $\mathbb{R}^4 \times \ldots \times \mathbb{R}^4$ (r factors) determined

1) V_+ denotes the closed forward light cone in Minkowski space.

by

$$\mathbf{E}\left(\exp i\sum_{n=1}^{r} x_\mu^n Y^\mu(S_n)\right) = \int P(dp_1,\dots,dp_r)\, \exp i\sum_{n=1}^{r} x_\mu^n p_n^\mu \tag{3}$$

This formula suggests constructing another linear process as follows. Let $\mathcal{V}$ be the linear space of bounded Borel functions $v:\Sigma\to\mathbb{R}^4$ with compact support. For each $v\in\mathcal{V}$ let $Y_v = X_{v^*}$ where $v^*\in\mathcal{U}$ is defined by $v^*(k,\lambda) = k^\mu v_\mu(k,\lambda)$ Then $v\mapsto Y_v$ is a linear process on $\mathcal{V}$ with the characteristic functional

$$\mathbf{E}\left(e^{iY_v}\right) = F(v^*) \tag{4}$$

In particular, if v is the simple function

$$v = \sum_{n=1}^{r} x^n\, 1_{S_n} \tag{5}$$

then $F(v^*)$ coincides with the left of equation (3). Thus Y_v , derived from the basic process X_u, is closest to the measurement of four-momenta.

If $Y^0(\Sigma)$ (the least upper bound of all $Y^0(S)$ with S compact) exists, the total energy is said to be finite. We emphasize that the total energy may be finite even if the total photon number is infinite.

IV. Characteristic Functionals and Field Theory

The main purpose of this section will be the calculation of characteristic functionals of the process X_u for a certain class of states of the radiation field. The Fock space of photons is traditionally described in terms of n-particle subspaces. Yet the structure of this space which such a calculation depends upon is that of a Hilbert exponential. It is for that reason that we shall formulate the theory in terms of Hilbert exponentials from the outset. For convenience of the reader, we expound the necessary material in the Appendix.

The classical solutions of Maxwell's equation are in 1-1 correspondence with functions $f:\Sigma\to\mathbb{C}$. The relativity principle provides us with an invariant measure μ on Σ:

$$(f,g) = \int d\mu\, \bar{f} g = (2\pi)^{-3}\int\frac{dk}{2|k|}\sum_{\lambda}\overline{f(k,\lambda)}\,g(k,\lambda) \qquad (1)$$

and hence with a Hilbert space L_o of square integrable functions. The Fock space of photons will then be identified with $\mathbb{E}L_o$, the Hilbert exponential of L_o. In order to attribute a physical meaning to vectors in $\mathbb{E}L_o$, it suffieces to specify the representation of the radiation field [4 and 7], preferrably in terms of the bounded Weyl operator. We write

$$W_o(f)\,[g] = e^{-\frac{1}{2}\|f\|^2-(f,g)}\,[f+g] \quad , \qquad (2)$$

extend W_o to all of $\mathbb{E}L_o$ by linearity, and thus obtain the familiar Fock representation, which we shall now consider in detail.

Unitary operators V and L_o give rise to unitary operators $\mathbb{E}V$ on $\mathbb{E}L_o$. We focus our attention to the following case: $t\mapsto\mathbb{E}e^{it}$ is a one-parameter group of unitary operators on $\mathbb{E}L_o$. Since $t\mapsto e^{it}\mapsto\mathbb{E}e^{it}$ as a composition of continuous maps, is continuous, there exists an unbounded selfadjoint operator N such that

$$e^{itN} = \mathbb{E}e^{it} \qquad (3)$$

This reproduces the standard construction of the number operator N from gauge transformations of the first kind.

For any $u \in \mathcal{U}$, let e^{iu} denote the unitary operator on L_o given by pointwise multiplication and, for any $S \in C$, let $N(S)$ be the selfadjoint operator defined by

$$e^{itN(S)} = \mathbb{E}\, e^{it1_S} \tag{4}$$

If S' is the complement of S in Σ and if $K = 1_S L_o$ and $K' = 1_{S'} L_o$, then $L_o = K \oplus K'$ and the Fock space $\mathbb{E}L_o$ is canonically isomorphic to the Hilbert tensor product $\mathbb{E}K \otimes \mathbb{E}K'$, the isomorphism carrying the operator $\mathbb{E}e^{it1_S}$ into $\mathbb{E}e^{it} \otimes I$. This way it is seen that N(S) is the number operator associated with the subsystem of photons whose degrees of freedom are confined to S. It is worth noticing that number operators associated with different subsets S commute.

Next, we take u to be simple and recall that $\mathbb{E}$ is a functor:

$$\mathbb{E}e^{iu} = \mathbb{E}\prod_{\alpha=1}^{r} e^{it_\alpha 1_{S_\alpha}} = \prod_{\alpha=1}^{r} \mathbb{E}e^{it_\alpha 1_{S_\alpha}} = \prod_{\alpha=1}^{r} e^{it_\alpha N(S_\alpha)}$$

If ω is a normal state on the von Neumann algebra $\mathcal{L}(\mathbb{E}L_o)$ of bounded operators on $\mathbb{E}L_o$ and if $P(n_1, \ldots, n_r)$ is the probability of the joint event $(n_1, \ldots, n_r)$ in the state

$$\mathcal{E}(f) = \omega(W_o(f)) \tag{5}$$

of the radiation field triggering r counters associated with the Borel sets $S_1, \ldots, S_r$, then

$$\omega\Big(\prod_{\alpha=1}^{r} e^{it_\alpha N(S_\alpha)}\Big) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_r=0}^{\infty} P(n_1,\ldots,n_r)\, \exp i\sum_{\alpha=1}^{r} t_\alpha n_\alpha \tag{6}$$

according to the rules of quantum mechanics. This shows that

$$F(u) = \omega(\mathbb{E}e^{iu}) \tag{7}$$

is the characteristic functional of the process X_u which corresponds to the state (5).

From this correspondence between states and characteristic functionals it becomes evident that the incoherent mixture of states corresponds to the convex combination of their characteristic functionals.
We emphasize that, in all functionals so far considered, u may be taken a constant function and thus $X(\Sigma)<\infty$ which signalizes that the state belongs to the Fock sector. We may pass to infrared representations [1)] by considering sequences of functions f_n in $L_o \subset L_1$ that converge to some $\eta \in L_1$. Then

$$\lim W_o(f_n)^* W_o(g) W_o(f_n) = W_o(g)\, e^{i 2 \operatorname{Im}(\eta, g)} =: W_\eta(g) \qquad g \in L_{-1}$$

and

$$\lim W_o(f_n)^* \, \mathbb{E} e^{iu} W_o(f_n) = e^{i\zeta_u} \, \mathbb{E} e^{iu} W_o(\xi_u) =: U_\eta(u) \qquad u \in \mathcal{U}$$

in the strong operator topology, where we used the abbreviations:

$$\xi_u = (1 - e^{-iu})\eta \in L_o \qquad , \quad \zeta_u = \operatorname{Im}(\eta, (e^{iu} - 1)\eta)$$

We therefore conclude that F is the characteristic functional for the state $\mathcal{E}$, if

$$F(u) = \omega(U_\eta(u))$$

$$\mathcal{E}(f) = \omega(W_\eta(f))$$

and ω as above.
No such answer is known in the general case, where $f \mapsto W(f)$ is an arbitrary Weyl system. However, from the rôle played by the unitary representation $u \mapsto U(u)$ of the group $\mathcal{U}$, we must demand:

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1) For these representations and the significance of the spaces $L_{\pm 1}$ the reader is referred to [7].

$$\mathcal{U}(u)\, W(f) = W(e^{iu} f)\, \mathcal{U}(u) \tag{8}$$

This equation thus provides the essential link between the stochastic process X_u and the radiation field.

V. Examples

1. Coherent radiation

In the study of external current models, one is naturally led to introduce coherent states $\mathcal{E}_\eta$, where $\mathcal{E}_\eta = ([0], W_\eta(f)[0])$ with η from L_1 and where η is linearly dependent on the current. In this case

$$\mathcal{E}_\eta(f) = \exp\{-\tfrac{1}{2}\|f\|^2 + 2i\,\mathrm{Im}(\eta,f)\} \tag{1}$$

$$F_\eta(u) = \exp(\eta,(e^{iu}-1)\eta) \tag{2}$$

as follows from the general result in the preceding section. We immediately realize that X_u is a Poissonian process. Consequently for each $S \in \mathcal{C}$, the random variable $X(S)$ has a Poisson distribution with mean photon number $\|1_S\eta\|^2$. This particularly shows that $X(\Sigma)$ is undefined unless $\eta \in L_0$. Further, any two variables $X(S_1)$ and $X(S_2)$ are statistically independent for disjoint S_1 and S_2. Generally, a process X_u is said to be <u>decomposable</u>, if the latter property holds.

2. Incoherent mixture of coherent radiation

In the preceding example, the vector η may be randomly distributed. Motivated by the theory of black body radiation, we shall study the following special case. Introducing the Gaussian measure on $\mathbb{C}^N$ (see Appendix), we may consider its image m under a linear map $A : \mathbb{C}^N \to L_1$. By construction, m is a probability measure on L_1 and a routine calculation yields:

$$\mathcal{E}_A(f) := \int m(d\eta)\,\mathcal{E}_\eta(f) = \exp\{-(f,(\tfrac{1}{2}+AA^*)f)\} \quad f \in L_{-1} \tag{3}$$

$$F_A(u) := \int m(d\eta)\,F_\eta(u) = \det(1 - A^*(e^{iu}-1)A)^{-1} \quad u \in \mathcal{U} \tag{4}$$

where the adjoint A^* is regarded as a map from L_{-1} (the dual of L_1) into $\mathbb{C}^N$. Note that $e^{iu}-1$ takes L_1 into L_{-1}. As we see, the process X_u is no longer decomposable and correlations become significant. The distribution of a single variable is obtained from the identity

V. Summary and Discussion

We have constructed a completely relativistic model which incorporates the coupling of the ρ-meson to 4 channels, namely $\pi\pi$, $K\overline{K}$, $\pi\omega$ and $N\overline{N}$ and which produces, with reasonable assumptions on $g^2_{\omega N\overline{N}}/4\pi$, a width $\Gamma_\rho \simeq 200$ MeV. Since we have included self-energy corrections into the meson and nucleon propagators, we expect that we now can perform rather realistic calculations for electromagnetic form factors. Especially the vertex renormalization constant Z_1 can be taken equal zero, so that form factors decreasing as some power for $k^2 \to -\infty$ are likely to emerge. This is not possible within the bare propagator ladder approximation, if the "potential" is smooth, since then the integral in (16) is well convergent.

The relation of this approach to previous work on $\pi\pi$-scattering is not simple. In the bootstrap method the large ρ-width obtained usually (2) seems to be a consequence of two facts: First of all with cut-off masses in the GeV-range there are in the $\pi\pi$ channel no really short range forces present, and secondly the ρ coupling in the crossed channel has to be large in order to provide enough attraction. These facts are no special features of N/D-calculations, but persist also in the relativistic Schrödinger equation (9). The inclusion of the $\pi\omega$-channel does not change the situation drastically (9,10) as long as the $\pi\rho\omega$-coupling is kept at its physical value, since the admixture of the $\pi\omega$-state is not very large and one still needs the strong attraction of ρ-exchange with a large coupling constant $g_{\rho\pi\pi}$. In our model the effective attraction is strongly enhanced by the propagator corrections, as indicated in eq. (26), and we can allow for a corresponding weakening of the on-shell $\pi\pi$-potential by lowering the cut-off mass Λ. A small ρ width can be obtained in the relativistic Schrödinger equation (11), if the necessary cut-off (ρ-exchange provides a singular potential) is chosen around 10 M. Although we are far from understanding our parameter Λ, such high cut-off masses seem difficult to interprete as vertex corrections alone within the BSE.

$$\mathcal{E}_A(f) = tr\, \varrho W_0(f) \tag{11}$$

It is customary to call any state of the form (3) a quasifree state of the radiation field. Yet the class of quasifree states obtained from integrals over $\mathbb{C}^N$ is a little too narrow in that, so far, we did not allow N to be infinite. As a consequence, AA^* is an operator of finite rank.

3. n-Photon-States

We now decompose the Fock space $\mathbb{E}L_0$ according to the photon number:

$$\mathbb{E}L_0 \cong \mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$$

Each $\phi \in \mathbb{E}L_0$ is now represented by a sequence $\{\phi_0, \phi_1, \phi_2, \cdots\}$ of wave functions $\phi_n(k_1\lambda_1, .., k_n\lambda_n)$ such that $[f]$ corresponds to $\{1, f, \frac{1}{\sqrt{2}} f^{\otimes 2}, \cdots\}$ where

$$f^{\otimes n}(k_1\lambda_1, \ldots, k_n\lambda_n) := \prod_{\alpha=1}^{n} f(k_\alpha \lambda_\alpha)$$

From $\mathbb{E}e^{iu}[f] = [e^{iu}f]$ and the fact that vectors of the form $f^{\otimes n}$ are total in $\mathcal{H}_n$, we obtain

$$[\mathbb{E}e^{iu}\phi]_n(k_1\lambda_1, \cdots, k_n\lambda_n) = \phi_n(k_1\lambda_1, \cdots, k_n\lambda_n) \exp i\sum_{\alpha=1}^{n} u(k_\alpha\lambda_\alpha)$$

for any $\phi \in \mathbb{E}L_0$ and thus

$$F(u) =$$

$$(2\pi)^{-3n}\int\frac{dk_1}{2|k_1|} \cdots \int\frac{dk_n}{2|k_n|} \sum_{\lambda_1} \cdots \sum_{\lambda_n} |\phi_n(k_1\lambda_1, \cdots, k_n\lambda_n)|^2 \exp i\sum_{\alpha=1}^{n} u(k_\alpha\lambda_\alpha) \tag{12}$$

for the characteristic functional of the normalized n-photon state ϕ_n. Choosing simple functions, we get

$$E(e^{itX(S)}) = \sum_{\nu=0}^{n} P_\nu(S)\, e^{i\nu t} \qquad 0 \le P_\nu(S) \le 1 \tag{13}$$

and for disjoint S_k

$$\mathbf{E}\left(X(S_1)\cdots X(S_m)\right) = 0 \qquad m > n \tag{14}$$

Both relations express in statistical language that there can be at most n particles in the state Φ_n. The strong correlation present in (14) becomes even more striking if expressed as

$$\mathbf{E}\left(Y^{\mu_1}(S_1)\dots Y^{\mu_m}(S_m)\right) = 0 \qquad m > n \tag{15}$$

since (15) demonstrates that the particle number can be obtained from a correlation experiment, even though the detectors do not count particles at all, but measure, say, the energy.

Appendix

For any complex linear space L, let UL denote its underlying set of vectors. Clearly, U sends the direct sum of two vector spaces L and M to the cartesian product of their underlying sets: $U(L \oplus M) = UL \times UM$.

Conversely, for any set A, let VA denote the free vector space over A, i.e. the linear space of all formal finite sums $\sum \alpha_n [a_n]$ with $\alpha_n \in \mathbb{C}$ and $a_n \in A$. By linearity, $[a,b] \mapsto [a] \otimes [b]$ can be extended to an isomorphism between $V(A \times B)$ and the algebraic tensor product $VA \otimes VB$.

As is well known [9], U and V form an adjoint functor pair between the category of sets and the category of linear spaces, V being left adjoint to U. The functor E = VU sends every linear space L to some linear space EL with a selected base point $[0] \in EL$. As $E(L \oplus M) \cong EL \otimes EM$ for any two linear spaces L and M, EL will be called the exponential of L and E the exponential functor. Being a covariant functor, E associates a linear mapping $Ef : EL \to EM, \sum \alpha_n [a_n] \mapsto \sum \alpha_n [fa_n]$ with every linear mapping [1] $f : L \to M$ such that identities and compositions are respected: $E(1_L) = 1_{EL}$, $E(f \circ g) = Ef \circ Eg$ if $g : L \to M$ and $f : M \to N$. Our construction of E also provides two universal mappings

$$L \underset{\varepsilon}{\overset{\eta}{\rightleftarrows}} EL$$

given by $\eta : x \mapsto [x]$ and $\varepsilon : \sum \alpha_n [a_n] \mapsto \sum \alpha_n a_n$. We stress the fact that ε is linear, whereas there is no L for which η becomes a linear mapping.

If L is a prehilbert space, so is EL with respect to the inner product

$$\left(\sum_n \alpha_n [a_n], \sum_m \alpha'_m [a'_m] \right)_{EL} = \sum_{n,m} \bar{\alpha}_n \alpha'_m \exp (a_n, a'_m)_L \tag{1}$$

1) Notice that $Vf : EL \to EM$ is linear even if $f : UL \to UM$ is a set function. This suggests admitting nonlinear transformations of L in the context of second quantizsation where one deals with a classical space L and its quantum mechanical analogue EL.

Indeed, $A_{nm} = \exp\,(a_n, a_m)_L$ are components of a positive semidefinite matrix A and $\alpha_n = 0$ if $\Sigma\,\alpha_n \exp\,(x, a_n)_L = 0$ for distinct $a_n \in L$ and all $x \in L$ [10; Lemma 8.2]. As an immediate consequence of (1), the mappings η and ε are continuous. However, η fails to be uniformly continuous in general.

For the following disucssion we needda preparatory result.

<u>Lemma 1.</u> Let L, M, N be prehilbert spaces and let $f: L \to M$ and $g: L \to N$ be linear maps such that $\| f\,x \|_M \leq \| g x \|_N$ for all $x \in L$. Then $\| Ef\,\phi \|_{EM} \leq \| Eg\,\phi \|_{EN}$ for all $\phi \in EL$
Proof. By assumption, $0 \leq |\Sigma \alpha_n|^2 + k^{-1} \| f x \|^2 \leq |\Sigma \alpha_n|^2 + k^{-1} \| g x \|^2$ for $x = \varepsilon\phi$, any integer k and $\phi = \Sigma \alpha_n [a_n]$. This inequality may be written as follows: $0 \leq s \leq t$, where s and t are finite matrices with components $s_{nm} = 1 + k^{-1} (f a_n, f a_m)$ and $t_{nm} = 1 + k^{-1} (g a_n, g a_m)$ respectively. It follows that $0 \leq s^{\otimes k} \leq t^{\otimes k}$. In particular, $\Sigma_{n,m} \overline{\alpha}_n \alpha_m (s_{nm})^k \leq \Sigma_{n,m} \overline{\alpha}_n \alpha_m (t_{nm})^k$ and $\Sigma_{n,m} \overline{\alpha}_n \alpha_m \exp(f a_n, f a_m) \leq \Sigma_{n,m} \overline{\alpha}_n \alpha_m \exp(g a_n, g a_m)$ in the limit $k \to \infty$, which establishes $\| Ef\phi \| \leq \| Eg\phi \|$.

As a simple application we prove:

<u>Proposition 1.</u> In order that $Ef : EL \to EM$ be continuous with $f: L \to M$, it is necessary and sufficient that $\| f x \|_M \leq \| x \|_L$ for all $x \in L$, i.e. that f is contractive.

Proof. Suppose Ef is continuous. Then $\| Ef\, [tx] \| \leq C \,\| [tx] \|$ for all $t \in \mathbb{R}$, $x \in L$, and some $C > 0$. Thus $\exp t^2 (\| fx \|^2 - \| x \|^2) \leq C^2$ for all t implying $\| f\,x \|^2 - \| x \|^2 \leq 0$. Conversely, suppose $\| f\,x \| \leq \| x \|$. Then $\| Ef\,\phi \| \leq \| \phi \|$ by Lemma 1.

As we have ssen, the functor E maps contractions into contractions and noncontractive functions into unbounded operators. Now, for any prehilbert space L, let CL denote its Hausdorff completion. As C is a covariant functor, so is $\mathbb{E} = CE$. To be specific, $\mathbb{E}$ is a covariant functor from the category of prehilbert spaces (with morphisms all linear contractions) to the category of Hilbert spaces with a selected base point.

If L and M are Hilbert spaces, let $L \oplus M$ denote their Hilbert sum. By construction,

$$\begin{aligned}([a,b],[a',b'])_{\mathbb{E}(L\oplus M)} &= \exp\left((a,a')_L + (b,b')_M\right)\\ &= ([a],[a'])_{\mathbb{E}L}\,([b],[b'])_{\mathbb{E}M}\\ &= ([a]\otimes[b],[a']\otimes[b'])_{\mathbb{E}L\otimes\mathbb{E}M}\end{aligned}$$

for all $a,a' \in L$ and $b,b' \in M$, so that the correspondence $[a,b] \mapsto [a]\otimes[b]$ uniquely extends to an isomorphism between $\mathbb{E}(L\oplus M)$ and the Hilbert tensor product $\mathbb{E}L \otimes \mathbb{E}M$. This property of $\mathbb{E}$ suggests calling $\mathbb{E}L$ the Hilbert exponential of L. The functor $\mathbb{E}$ has been constructed in a different way by I.E. Segal [11], by H. Araki and E.J. Woods [10; Chapter 5], and has subsequently been considered by J.R. Klauder [12], A. Guichardet [13] and others. Here we wish to emphasize its categorical significance.

Since $\mathbb{E}$ respects conjugation, $\mathbb{E}(f^*) = (\mathbb{E}\, f)^*$, it also preserves selfadjointness, positivity, unitarity, and polar decompositions of operators. If $\mathbb{E}f : \mathbb{E}L \to \mathbb{E}M$ has a bounded inverse, then $f: L\to M$, hence $\mathbb{E}f$, is an isomorphism. $\mathbb{E}$ maps projections into projections and, in particular, if $0 : L \to L$ is the zero operator, $P_o = \mathbb{E}0$ is the projection onto the one-dimensional subspace $\mathcal{H}_o \subset \mathbb{E}L$ generated by the base vector $[0]$. Then $P_oA = AP_o = P_o$ if $A = \mathbb{E}f$ and $f: L\to L$ is contractive. It should be also clear that $\mathbb{E}$ maps involutions into involutions, square roots into square roots and so on, but fails to be linear.

In the physical context, the vectors $[x]$ were termed "coherent states" although the coherent states of optics constitute a much wider class. The function η, mapping L onto the coherent states in $\mathbb{E}L$, has the remarkable property of being entire analytic, that is, for arbitrary $a_1, \ldots, a_n \in L$, the function $F:\mathbb{C}^n \to \mathbb{E}L$ defined by $F(\alpha_1,\ldots,\alpha_n) = [\Sigma\alpha_k a_k]$ is entire analytic as follows from (1). Moreover, η as a mapping between Banach spaces is infinitely often differentiable. The first derivative $\eta'(o)$: $L\to\mathbb{E}L$

establishes an isomorphism between L and a subspace $\mathcal{H}_1$ of $\mathbb{E}L$. In cases, where one interpretes $\mathbb{E}L$ as the Fock space of a Bose particle, $\mathcal{H}_1$ is called the one-particle space. One is often led to identify L and $\mathcal{H}_1$. The various subspaces $\mathcal{H}_n$ associated with the nth derivation of η are most clearly visualized if one uses the isomorphism between $\mathbb{E}L$ and the symmetric tensor algebra over L, uniquely determined by the coorespondence

$$[a] \mapsto \{1, a, \dots, \frac{1}{\sqrt{n!}} a^{\otimes n}, \dots\}$$

This shows: If L is separable, so is $\mathbb{E}L$. In the sequel we shall restrict the discussion to separable Hilbert spaces L, M, ... The continuous linear mappings f: L→M form a Banach space $\mathcal{L}(L,M)$ with respect to their norm $\|f\| = \sup\{\|fx\| : x \in L, \|x\| = 1\}$. Then the set of contractions f : L→M is precisely the unit ball $\mathcal{L}_1(L, M)$ in that space.

Proposition 2. $\mathbb{E} : \mathcal{L}_1(L,M) \to \mathcal{L}_1(\mathbb{E}L, \mathbb{E}M)$ is continuous with respect to the strong operator topology.

Proof. As $\mathcal{L}_1(\mathbb{E}L, \mathbb{E}M)$ is an equi - continuous set of operators, its strong operator topology coincides with the topology of simple convergence on the total set $\{[a] : a \in L\}$ in $\mathbb{E}L$. The continuity of $\mathbb{E}$ is thus the continuity of $f \mapsto fa \mapsto [fa]$ which, as a composition of continuous maps, is indeed continuous. We now assert that the functor $\mathbb{E}$ preserves the partial ordering between positive operators.

Lemma 2. Let S, T $\in \mathcal{L}_1(L, L)$, then $0 \leq S \leq T$ implies $0 \leq \mathbb{E}S \leq \mathbb{E}T$.

Proof. Setting $f = S^{1/2}$ and $g = T^{1/2}$, we obtain $\|\mathbb{E}f\phi\| \leq \|\mathbb{E}g\phi\|$ for $\phi \in \mathbb{E}L$ from Lemma 1 and the continuity of $\mathbb{E}f$ and $\mathbb{E}g$. Now $(\mathbb{E}f\phi, \mathbb{E}f\phi) = (\phi, \mathbb{E}S\phi)$ and $(\mathbb{E}g\phi, \mathbb{E}g\phi) = (\phi, \mathbb{E}T\phi)$ and the assertion follows.

Before we continue the study of positive operators, it is useful to consider the simplest example, the space $\mathbb{E}\mathbb{C}$. There is a unique isomorphism between the spaces $\mathbb{E}\mathbb{C}$ and ℓ^2 such that [c] corresponds to $\{1, c, \ldots, (n!)^{-1/2} c^n, \ldots\}$. Any positive linear operator on $\mathbb{C}$ is of the form $c \mapsto \alpha c$ with $\alpha \geq 0$. If $\alpha \leq 1$, the operator $\mathbb{E}\alpha$ exists and $\mathbb{E}\alpha \geq 0$. Clearly,

$$\operatorname{tr} \mathbb{E}\alpha = \sum_{n=0}^{\infty} \alpha^n = (1-\alpha)^{-1} \in [1,\infty]$$

Proposition 3. Let $T \in \mathcal{L}_1(L, L)$ be completely continuous and positive and let $(\alpha_n)_{n\in N}$ be the eigenvalues of T. Then $0 \leq \alpha_n \leq 1$ and

$$\operatorname{tr} \mathbb{E}T = \Big(\prod_n (1-\alpha_n)\Big)^{-1} \tag{2}$$

(allowing $+\infty$ as a possible value); $\operatorname{tr} \mathbb{E}T < \infty$ if and only if $\operatorname{tr} T < \infty$ and $\| T \| < 1$.

Proof. There exists a basis $(e_n)_{n\in N}$ in L such that $Tx = \sum \alpha_n (e_n, x) e_n$ and $0 \leq \alpha_n \leq 1$ for all $x \in L$. Define $T_m x = \sum_1^m \alpha_n (e_n, x) e_n$. Then L may be represented as $\mathbb{C}^m \oplus L'$ such that T_m takes $\{c_1, \ldots, c_m, a\}$ into $\{\alpha_1 c_1, \ldots, \alpha_m c_m, 0\}$. Thus

$$\operatorname{tr} T_m = \operatorname{tr} \bigotimes_{n=1}^{m} \mathbb{E}\alpha_n = \Big(\prod_{n=1}^{m} (1-\alpha_n)\Big)^{-1} .$$

If $m \leq n$, then $0 \leq T_m \leq T_n$ and thus $0 \leq \mathbb{E}T_m \leq \mathbb{E}T_n$ by Lemma 2.

Since $\quad \| (T-T_m)x \| = \| \sum_{n=m+1}^{\infty} \alpha_n (e_n, x) e_n \| \leq \|x\| \sup_{n>m} \alpha_n$

for all $x\in L$, T_m tends to T strongly. By Proposition 2, $\mathbb{E}T_m \uparrow \mathbb{E}T$ strongly and $\operatorname{tr} \mathbb{E}T = \operatorname{tr} \sup \mathbb{E}T_m = \sup \operatorname{tr} \mathbb{E}T_m = (\prod(1-\alpha_n))^{-1}$ as the trace is a normal positive functional [14; Chapter I.§6, Théorème 5]. Then $\prod(1-\alpha_n) > 0$ if $\alpha_n < 1$ and $\sum \alpha_n < \infty$ |15; IV.7.5|. Thus $\operatorname{tr} \mathbb{E}T < \infty$ iff $\|T\| < 1$ and $\operatorname{tr} T < \infty$.

The physical significance of (2) is that, for a particular choice of L and T, $\operatorname{tr} \mathbb{E}T$ is the grand partition function of an ideal

Bose gas, whereas $\varrho = (\mathrm{tr}\ \mathbb{E}T)^{-1}\ \mathbb{E}T$ is the density matrix describing the grand canonical ensemble.

For any integer n, let $\mathcal{L}^n(L,M)$ denote the set of all elements $f \in \mathcal{L}(L,M)$ such that $\| f \|_n < \infty$ where

$$\| f \|_n^n = \mathrm{tr}\,(f^* f)^{n/2}$$

Since $\|\alpha f + \beta g\|_n \leq |\alpha|\|f\|_n + |\beta|\|g\|_n$, $\mathcal{L}^n(L,M)$ is a linear space; $\mathcal{L}^1(L,M)$ is known as the space of nuclear mappings.

Proposition 4. For $\mathbb{E}f: \mathbb{E}L \to \mathbb{E}M$ to be defined and to belong to $\mathcal{L}^n(\mathbb{E}L, \mathbb{E}M)$ for some n, it is necessary and sufficient that $f \in \mathcal{L}^n(L,M)$ and $\|f\| < 1$.

Proof. If $T = (f^* f)^{n/2}$, then $\|T\| = \|f\|^n$ and $(\mathbb{E}f^*\ \mathbb{E}f)^{n/2} = \mathbb{E}T$. Hence the assertion almost follows from Proposition 3. We only have to demonstrate that the complete continuity of T is implied by $\mathrm{tr}\ \mathbb{E}T < \infty$. This, however, is obvious as there is an isomorphism $U : L \to \mathcal{H}_1 \subset \mathbb{E}L$ that carries T into $\mathbb{E}T$ restricted to $\mathcal{H}_1$.

Finally, we wish to employ a functional representation of $\mathbb{E}\mathbb{C}^n$ which is due to Bargmann [16]. For any $\phi \in \mathbb{E}\mathbb{C}^n$ let $\hat{\phi}: \mathbb{C}^n \to \mathbb{C}$ be defined by $\hat{\phi}(a) = ([a], \phi)$. Similarly, for any operator Q acting on $\mathbb{E}\mathbb{C}^n$, we put $\hat{Q}(a,b) = ([a], Q[b])$. If μ is the normalized Gaussian measure on $\mathbb{C}^n$, i.e. $d\mu(a) = \pi^{-n} \exp\{-\|a\|^2\} da$, then $(\phi,\psi) = \int d\mu(a)\ \overline{\hat{\phi}(a)}\ \hat{\psi}(a)$ and $\widehat{Q\phi}(a) = \int d\mu(b)\ \hat{Q}(a,b)\ \hat{\phi}(b)$ so that $\mathbb{E}\mathbb{C}^n$ may be viewed as Hilbert space of functions with reproducing kernel

$$([a],\ I\ [b]) = e^{(a,b)}\ .$$

If Q is of trace class, then $\mathrm{tr}\ Q = \int d\mu(a)\ \hat{Q}(a,a)$. Now, Proposition 4 asserts that $\mathbb{E}A$ is of trace class if $\|A\| < 1$. We find $\widehat{\mathbb{E}A}(a,b) = \exp\,(a,Ab)$ and $\mathrm{tr}\ \mathbb{E}A = \pi^{-n}\int da\ \exp\,(a,(A-I)a) = \det\,(I-A)^{-1}$. We have thus proved:

<u>Proposition 5.</u>

For any complex (n, n) - Matrix A (considered as a linear map $\mathbb{C}^n \to \mathbb{C}^n$ where $\mathbb{C}^n$ is given the natural Hilbert space structure) such that $\|A\| < 1$, we have the identity: $\operatorname{tr} \mathbb{E}A = \det (I - A)^{-1}$,

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