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Abstract

Using the Callan-Symanzik equations as a tool to study the short distance behaviour of the gauge invariant axial vector current in quantum electrodynamics, we conclude that as a consequence of Ward-identities the axial vector current is required to scale non-canonically. The Wilson-Crewther calculation of the anomaly can't be carried out, although its use is legitimate in phenomenological applications treating electromagnetism to lowest order.

Broken scale invariance has been studied recently as a tool to understand the short distance behaviour of renormalizable field theory [1]. The properties of the gauge invariant axial vector current $J^5_\mu(x)$ in quantum electrodynamics (QED) in the Gell-Mann Low (GML) limiting theory, which is supposed to describe the short distance limit, have been of particular interest. Using the Adler-Bardeen (AB) low energy theorem [2),3)], stating the non-renormalization of the axial vector current anomaly, and the Federbush-Johnson [4)] theorem (FJ), Schroer [5)] and Adler et al. [6)] have argued that $J^5_\mu(x)$ behaves in a singular fashion even in the finite GML theory.

In this paper we analyse this phenomenon in some detail by constructing the preasymptotic theory via the Callan-Symanzik equations [7)8)]. The AB theorem is found not to be universally applicable to massless QED. A consistent state of affairs can be achieved, if $J^5_\mu(x)$ is allowed to scale in a non-trivial manner.

We also discuss the Wilson-Crewther [9)] calculation of the anomaly and find that it cannot be carried out in exact QED.

We start considering the complete unrenormalized gauge invariant two photon axial vector vertex function $\frac{-ie_0^2}{(2\pi)^4} R'_{\sigma\rho\mu}(k_1,k_2) = -e^2\int d^4x\, d^4y\, e^{-ik_1x-ik_2y}\langle 0|T^*[J_\sigma(x)J_\rho(y)J^5_\mu(0)]|0\rangle^{prop}$ which satisfies the Ward identity [2)]

$$-(k_1+k_2)^\mu R'_{\sigma\rho\mu}(k_1,k_2) = 2m_0 R'_{\sigma\rho}(k_1,k_2) + S'_{\sigma\rho}(k_1,k_2) \tag{1}$$

where $\frac{-ie_0^2}{(2\pi)^4} R'_{\sigma\rho}(k_1,k_2) = -e^2 \int d^4x\, d^4y\, e^{-ik_1x-ik_2y} \langle 0|T[J_\sigma(x) J_\rho(y) J^5(0)]|0\rangle^{prop}$
is the corresponding pseudoscalar vertex function and the anomaly is

$$\mathcal{S}'_{\sigma\rho}(k_1,k_2) = 8\pi^2 k_1^\xi k_2^\tau \varepsilon_{\xi\tau\sigma\rho} + O(\alpha_0^2) \tag{2}$$

We renormalize these Green functions in the following way [2),10)]

$$\tilde{R}'_{\sigma\rho\mu} = Z_5 R'_{\sigma\rho\mu} \tag{3}$$

and

$$\tilde{\mathcal{S}}'_{\sigma\rho} = \mathcal{S}'_{\sigma\rho} + (1 - Z_5)(k_1+k_2)^\mu R'_{\sigma\rho\mu} \tag{4}$$

The renormalization constant $Z_5 = 1 + \frac{3}{4}\left(\frac{\alpha}{\pi}\right)^2 \ln(\Lambda^2/m^2) +$
$+ \alpha^2 \times$ finite constant $+ O(\alpha^3)$
is calculated from graphs of the type

Fig. 1

where $\otimes$ denotes the lowest order contribution to $\mathcal{S}'_{\sigma\rho}(k_1,k_2)$.

From equs. (5) and (6) we obtain the Callan-Symanzik equations [8),11)] for $\tilde{R}'_{\sigma\rho\mu}$ and $\tilde{\mathcal{S}}'_{\sigma\rho}$ by varying the physical mass m and keeping α_0 and the cut off Λ fixed [12)].

We obtain [12)]

$$\left(m\frac{\partial}{\partial m} + \alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right)\tilde{R}'_{\sigma\rho\mu} = \eta(\alpha)\tilde{R}'_{\sigma\rho\mu} + \Delta\tilde{R}'_{\sigma\rho\mu} \tag{7}$$

and

$$\left(m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}\right)\tilde{S}'_{\sigma\rho}=-\eta(\alpha)(k_1+k_2)^{\mu}\tilde{R}'_{\sigma\rho\mu}+\Delta\tilde{S}'_{\sigma\rho} \tag{8}$$

where

$$\beta(\alpha)=\frac{m}{Z_3}\frac{dZ_3}{dm}=\frac{2\alpha}{3\pi}+O(\alpha^2),\ \eta(\alpha)=\frac{m}{Z_5}\frac{dZ_5}{dm}=-\frac{3}{2}\left(\frac{\alpha}{\pi}\right)^2+O(\alpha^3)$$

$$\Delta\tilde{S}'_{\sigma\rho}=m\frac{dm_0}{dm}\frac{\partial}{\partial m_0}S'_{\sigma\rho}+(k_1+k_2)^{\mu}(1-Z_5)m\frac{dm_0}{dm}\frac{\partial}{\partial m_0}R'_{\sigma\rho\mu}$$

$$\Delta\tilde{R}'_{\sigma\rho\mu}=Z_5 m\frac{dm_0}{dm}\frac{\partial}{\partial m_0}R'_{\sigma\rho\mu} \tag{9}$$

From standard arguments [8),11),13)] it follows that β and η are cut-off independent, so that $\Delta\tilde{S}'_{\sigma\rho}$ and $\Delta\tilde{R}'_{\sigma\rho\mu}$ share the same property. This can also be veryfied using Zimmermann's cut-off independent normal-product formalism [14)].

Integrating equs. (7) and (8), we now construct Symanzik's preasymptotic theory along the lines of Ref. (7), reproduced for our particular case for the readers convenience. Weinberg's theorem [15)] tells us that for large external momenta k_1 and k_2, $\tilde{R}'_{\sigma\rho\mu}(k_1,k_2)$ grows linearly apart from logarithms, whereas $\Delta\tilde{R}'_{\sigma\rho\mu}(k_1,k_2)$ has at most logarithmic growth. We are thus motivated to introduce a preasymptotic vertex function $\tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2)$, defined as a solution of the homogeneous equation obtained from equ. (7) by dropping the term $\Delta\tilde{R}'_{\sigma\rho\mu}$:

$$\left(m\frac{\partial}{\partial m}+\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}-\eta(\alpha)\right)\tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m,\alpha)=0 \tag{10}$$

$\tilde{R}'^{as}_{\sigma\rho\mu}$ is normalized such that it asymptotically approaches $\tilde{R}'_{\sigma\rho\mu}$ for large λ:

$$\tilde{R}'_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha) - \tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha) = O(\ln^{\alpha}\lambda/\lambda) \qquad (11)$$

where we used dimensional analysis to write:

$$\tilde{R}'_{\sigma\rho\mu}(\lambda k_1,\lambda k_2;m,\alpha) = \lambda\,\tilde{R}'_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha) \qquad (12)$$

Integrating equ. (7) we obtain the vertex function of a massless non-scale-invariant theory [7]:

$$\tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha) = z(\alpha)\,\Phi_{\sigma\rho\mu}[k_1,k_2;-\ln m+\ln\lambda+\rho(\alpha)] \qquad (13)$$

where $$z(\alpha) = \exp\left[\int_0^{\alpha} dx\,\frac{\eta(x)}{x\beta(x)}\right] = 1-\frac{9\alpha}{4\pi}+O(\alpha^2) \qquad (14)$$

and $$\rho(\alpha) = \int^{\alpha} dx\,\frac{1}{x\beta(x)} \qquad (15)$$

$z(\alpha)$ exists, because $\eta(x)$ is of second order and $\beta(x)$ of first order in x for $x \sim 0$.

The interpolating fine structure constant is introduced as

$$\alpha(\lambda) \equiv \rho^{-1}[\ln\lambda+\rho(\alpha)]$$

such that it has the properties:

$$\alpha(0) = 0\ ,\ \alpha(1) = \alpha \quad \text{and} \quad \alpha(\infty) = \alpha_{\infty}\ ,$$

where α_∞ is an assumed first zero of $\beta(x)$.
Eliminating Φ from equ. (13) we obtain the transformation formula

$$\tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m,\alpha) = z(\alpha)/z[\alpha(\lambda^{-1})]\,\tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1})) \tag{16}$$

Using equ. (11) and the fact that $\alpha(\lambda^{-1})$ and α differ only by an infinite series of logarithms [7], we finally obtain from equ. (16):

$$\tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m,\alpha) = \lim_{\lambda\to\infty}\left\{ z(\alpha)/z[\alpha(\lambda^{-1})]\,\tilde{R}'_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1}))\right\} \tag{17}$$

Similarly we drop the $\Delta\tilde{\mathfrak{F}}'_{\sigma\rho}$ term of equ. (8) to obtain a pre-asymptotic vertex function $\tilde{\mathfrak{F}}'^{as}_{\sigma\rho}$ by integrating now an inhomogeneous equation:

$$\tilde{\mathfrak{F}}'^{as}_{\sigma\rho}(k_1,k_2;\frac{m}{\lambda},\alpha) = -(k_1+k_2)^{\mu}\,\Phi_{\sigma\rho\mu}(k_1,k_2;-\ln m+\ln\lambda+\rho(\alpha))\,z(\alpha) + \Psi_{\sigma\rho}(k_1,k_2;-\ln m+\ln\lambda+\rho(\alpha)) \tag{18}$$

where $\Psi_{\sigma\rho}$ is a complete solution of the homogeneous equation.
The transformation formula for $\tilde{\mathfrak{F}}'_{\sigma\rho}$ is thus from equ. (18):

$$\tilde{\mathfrak{Z}}'^{as}_{\sigma\rho}(k_1,k_2;m,\alpha) = \tilde{\mathfrak{Z}}'^{as}_{\sigma\rho}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1})) - \tag{19}$$
$$- (k_1+k_2)^\mu \tilde{R}'^{as}_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1}))\,[\varkappa(\alpha)/\varkappa[\alpha(\lambda^{-1})] - 1]$$

Since by Weinberg's theorem [15] $R'_{\sigma\rho}$ drops out for large λ from equ. (1) we have:

$$\tilde{\mathfrak{Z}}'_{\sigma\rho}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1})) + (k_1+k_2)^\mu \tilde{R}'_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1})) = \tag{20}$$
$$= O(\lambda^{-2})$$

Inserting equ. (11) into the corresponding one for $\tilde{\mathfrak{Z}}'_{\sigma\rho}$ and using equ. (20) we get:

$$\tilde{\mathfrak{Z}}'^{as}_{\sigma\rho}(k_1,k_2;m,\alpha) =$$
$$= -\lim_{\lambda\to\infty}(k_1+k_2)^\mu \tilde{R}'_{\sigma\rho\mu}(k_1,k_2;m/\lambda,\alpha(\lambda^{-1}))\,\varkappa(\alpha)/\varkappa[\alpha(\lambda^{-1})] \tag{21}$$

and hence we obtain the validity of the Ward identity also for the preasymptotic theory:

$$-(k_1+k_2)^\mu \tilde{R}'^{as}_{\sigma\rho\mu} = \tilde{\mathfrak{Z}}'^{as}_{\sigma\rho} \tag{22}$$

so that the Ψ -term is actually zero in equ. (18).

We are now in a position to analyse the behaviour of $\tilde{\mathfrak{Z}}'^{as}_{\sigma\rho}$ at $\lambda=0,\infty$. Namely

$$\lambda^{-2}\tilde{\mathcal{S}}'^{as}_{\sigma\rho}(\lambda k_1,\lambda k_2;m,\alpha) = \tilde{\mathcal{S}}'^{as}_{\sigma\rho}(k_1,k_2;m/\lambda,\alpha) =$$

$$= \tilde{\mathcal{S}}'^{as}_{\sigma\rho}(k_1,k_2;m,\alpha(\lambda))\; z(\alpha)/z[\alpha(\lambda)] \tag{23}$$

$$\underset{\lambda\to 0}{\longrightarrow}\; \tilde{\mathcal{S}}^{(0)}_{\sigma\rho}(k_1,k_2)\, z(\alpha)$$

$\tilde{\mathcal{S}}^{(0)}_{\sigma\rho}$ is the zeroth order term of equ. (2). It is the nontrivial factor $z(\alpha)$, peculiar to the preasymptotic theory, whose absence in the massive theory makes the derivation of the AB low energy theorem [2] go through. We now assume the existence of the GML limiting theory by performing $\alpha \to \alpha_\infty$ on the preasymptotic vertex functions. Since they satisfy homogeneous Callan-Symanzik equations without the term $\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}$ [16] they scale as

[17]

$$\mathcal{S}^{GML}_{\sigma\rho}(\lambda k_1,\lambda k_2) = \lambda^{2-\eta(\alpha_\infty)}\; \mathcal{S}^{GML}_{\sigma\rho}(k_1,k_2) \tag{24}$$

According to the foregoing discussion we have the following asymptotic relation for large λ:

$$\tilde{\mathcal{S}}'_{\sigma\rho}(\lambda k_1,\lambda k_2;m,\alpha) \simeq \lambda^2 z(\alpha)/z[\alpha(\lambda)]\;\tilde{\mathcal{S}}'^{as}_{\sigma\rho}(k_1,k_2;m,\alpha(\lambda)) \simeq$$

$$\simeq \lambda^{2-\eta(\alpha_\infty)}\exp\left[\int_{\alpha_\infty}^{\alpha} dx\,\frac{\eta(x)-\eta(\alpha_\infty)}{x\,\beta(x)}\right]\;\mathcal{S}^{GML}_{\sigma\rho}(k_1,k_2) \tag{25}$$

provided the exponential factor exists [18].

Let us now discuss which kind of possibilities for the short distance behaviour of the axial vector current may arise:

a) if $\eta(\alpha_\infty) \neq 0$, the anomalous scaling behaviour of $\mathfrak{S}^{GML}_{\sigma\rho}$ (equ. 24), prevents us from expanding about $k_1 = k_2 = 0$. An AB low energy theorem cannot be stated and consequently implies no constraint.

b) if the eigenvalue condition $\eta(\alpha_\infty) = 0$ holds and $\chi[\alpha(\infty)]$ is finite, Z_5 is finite as well [19]. Because we do not expect an interchange of the limits $\lambda \to 0$ and $\alpha(\lambda) \to \alpha_\infty$ to be allowed in equ. (23), we cannot draw conclusions on $\mathfrak{S}^{GML}_{\sigma\rho}(k_1, k_2)$. On the other hand, one can by accepted arguments[13] extend the FJ theorem to the one loop approximation, where electron-positron creation from internal photon lines is neglected. From gauge invariance it follows that $R^{GML}_{\sigma\rho\mu} = 0$ in the one loop approximation, whereas we know $\mathfrak{S}^{GML}_{\sigma\rho}$ to reduce to its zeroth order term in this case and thus cannot vanish in the GML limit. We thus reject this case since it contradicts the Ward identity equ. (22).

c) if $\eta(\alpha_\infty) = 0$ and $\chi[\alpha(\infty)]$ is either zero or infinity, we are unable to make a plausible statement. This possibility suggests itself due to Adler's paper Ref. 18.

In conclusion our results indicate that we are able to avoid the inconsistencies pointed out by Schroer [5] and Adler et al. [6] if either a) or c) is realized, i.e. the axial vector renormalization constant Z_5 cannot be finite [19]. Notice that in distinction to the authors of Ref. 6, we believe the FJ theorem to hold in QED with an external gauge invariant axial vector current, since it rests on a much firmer ground than the other statements.

This does of course not prevent the application of the Wilson-Crewther calculation [9] to theories treating electromagnetism to lowest order; to QED on the other hand it does not seem to be relevant as can be seen from the following argument.

Define the Adler-Bardeen constant as

$$S = -\frac{\pi^2}{12}\, \varepsilon^{\mu\nu\alpha\beta} \iint d^4x\, d^4y\, x_\mu y_\nu \langle 0| T\, [\, J_\alpha(x)\, J_\beta(0)\, 2m_0\, J^5(y)\,]\, |0\rangle \tag{26}$$

Since the pseudoscalar current $m_0 J^5(y)$ is a soft operator a small region $|x_0| < \varepsilon_1$, $|y_0| < \varepsilon_2$, $|x_0 - y_0| < \varepsilon_3$ may be cut out of the integrals in equ. (26) and S be defined as the limit $\varepsilon_i \to 0$, $i = 1, 2, 3$. Now we use the renormalized version of the Ward identity equ. (1) to replace $2m_0 J^5(y)$ by $\partial^\lambda \left[J^5_\lambda(y) - \frac{\alpha}{2\pi}\, \varepsilon_{\lambda\rho\sigma\tau}\, \hat{N}_3 [A^\rho(y)\, F^{\sigma\tau}(y)] \right] \equiv \partial^\lambda J^{5S}_\lambda(y)$ where the notation $\hat{N}_3$ is explained in Ref. 14. Since the dimension of the symmetry current $J^{5S}_\lambda(y)$ is always three, we can make the Schreier ansatz for $\langle 0| T^* [J_\alpha(x)\, J_\beta(0)\, J^{5S}_\lambda(y)\, |0\rangle$ fixing the proportionality factor by the Wilson-Crewther calculation. But now one is unable to make further progress since the $\hat{N}_3 [A^\rho(y)\, F^{\sigma\tau}(y)]$ term cannot be dropped under the integral in equ. (26), unless it is a seagull as is the case in the simplified models discussed by Adler et al. [6].

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12) Since no infrared problems arise in our problem, we set the photon mass equal to zero.

13) See also the following reference for a discussion of Callan-Symanzik equations:

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16) Due to equ. (22) this is also true for $\delta^{GML}_{\sigma\rho}$.

17) A rescaling of $J^5_\mu(x)$ by a factor $1 + O(\alpha^2)$, the $O(\alpha)$ factor being fixed by the Ward identity for the two fermion vertex function, can change $\eta(\alpha)$ by a finite amount of $O(\alpha^3)$. We of course require the rescaling factor to remain finite at the GML limit and hence not affecting $\eta(\alpha_\infty)$.

18) If $\beta(\alpha)$ vanishes at $\alpha = \alpha_\infty$ with an infinite order zero (see the loopwise summed solution in S.L. Adler, Phys. Rev. D5, 3021 (1972)) this may not be true. This however does not affect the validity of equ. (24). See the discussion following equ. (IV.8) of Ref. 7.

19) This follows from equ. (9), defining $\eta(x)$, according to which Z_5 satisfies a Callan-Symanzik equation identical in form to equ. (10). By a discussion analogous to the one leading to equ. (25) one obtains that $Z_5(\Lambda^2/m^2;\alpha) \simeq \chi[\alpha_\infty]^{-1}$ for the cut-off $\Lambda \to \infty$, assuming $Z_5(\alpha;\alpha_\infty)$ exists for some large, finite α.

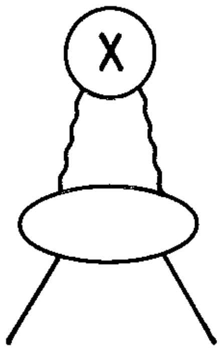

Fig. I. Diagrams contributing to the axial vector renormalization constant Z_5 .