DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 73/58
December 1973

Massless ϕ^4 Theory in 4-ε Dimensions

by

K. Symanzik

2 HAMBURG 52 · NOTKESTIEG 1

Massless ϕ^4 Theory in $4-\varepsilon$ Dimensions

K. Symanzik

Abstract

The Green's functions of massless ϕ^4 theory in $4-\varepsilon$ dimensions are in perturbation theory infrared singular at certain positive rational ε. By analyzing the small-mass behaviour of the massive-theory functions and performing the limit to zero mass with care, we obtain terms nonanalytic in the coupling constant in addition to the usual perturbation theoretical ones. In this new expansion, the infrared singularities cancel out. This phenomenon is related to the nonvanishing of the bare mass in the zero-physical-mass theory.

Introduction

The role of the massless $(\phi^4)_{4-\varepsilon}$ model for the theory of critical phenomena has been discussed in detail at this Summer School.* The construction of that model via ε expansion is

*See the lectures of E. Brézin, G. Parisi, B. Schroer, and K. Wilson.

limited to infinitesimal ε. For finite ε, the use of conformal invariance* or of constructive renormalization group methods [1]

*See e.g. the lectures of I. Todorov at this Summer School.

is still in its infancy. It is then worthwhile to examine the perturbation expansion in the bare or a renormalised coupling constant. The scale invariant theory [1] hereby corresponds to infinite bare coupling constant, or to the renormalised coupling constant taking the critical fixed point value. The long-distance properties of the massless theory, however, should be the same [2] also for finite bare coupling constant, or for renormalised coupling constant in the range from arbitrarily small till possibly beyond the fixed point.

However, perturbation theory for massless $(\phi^4)_{4-\varepsilon}$ leads to infrared (UR) divergences: for any positive rational ε, there are (infinitely many) perturbation theoretical terms that do not exist at that ε, e.g. for $\varepsilon = 1$ or $\varepsilon = 2$, already the lowest-order self-energy part (see (1.4) below) does not, and these UR singularities do not cancel out. Also the construction of the massless theory from the massive one by change of the bare mass, which succeeds [3] for $\varepsilon = 0$, has* in perturbation theory the UR-divergence problem.

*This was pointed out to the author by J. Zinn-Justin.

On the other hand, there are arguments for massless $(\phi^4)_{4-\varepsilon}$ not being UR divergent. Mack pointed out to the author that the conformal invariant $(\phi^4)_{4-\varepsilon}$ theory, although massless, does apparently not [4] suffer from UR divergence. Parisi presented an argument closely related to the one of Sect. 3 and 4 below. Wilson stressed that in the construction of massless $(\phi^4)_{4-\varepsilon}$ by his renormalisation group method [1] no UR divergences arise at any stage of the calculation.

We show here* that the UR divergences disappear in an improved quasi-perturbation expansion, which has besides the usual terms additional ones nonanalytic in the coupling constant and involving as factors singular functions of ε not fully

*A summary of these results was given in [5].

computable in perturbation theory. These new terms are obtained by performing the transition, by change of the bare mass, from massive $(\phi^4)_{4-\varepsilon}$ to the massless one with the necessary care, and their UR singularities cancel the ones of the usual terms. All the considerations in this lecture are immediately extendable to the $(\phi_i\phi_i)^2$ theory with N components and with little effort to $(\phi^6)_{3-\varepsilon}$ theory.

In Sect. 1 we briefly discuss massless $(\phi^4)_{4-\varepsilon}$ in usual perturbation theory, which leads to the mentioned UR divergences. Sect. 2 deals with the bare masses in massive $(\phi^4)_{4-\varepsilon}$. In Sect. 3 we describe the small-mass behaviour of the unrenormalized perturbation expansion and analyse the phenomena that arise if the mass is made to vanish. Sect. 4 presents some familiar estimates to argue that the mass-switch-off process of Sect. 3 does lead to UR finite results. In Sect. 5 we interpret, and discuss some aspects of, the new quasi-perturbation expansion.

1. Massless $(\phi^4)_{4-\varepsilon}$ in Perturbation Theory

Integration in $4-\varepsilon$ dimensions has been described in the literature [6]. We only cite the formula*

*We use Minkowskian metric 1, $(-1)^{3-\varepsilon}$. Throughout this lecture we will consider functions at Euclidean momenta (energy components imaginary, space components real) except where renormalisation conditions, as in (1.3a) and (4.1a-c), are imposed at other momenta, but use Minkowskian notation for ease of comparison with the references.

$$\int dk[m^2 - k^2 - i0]^{-\alpha}$$
$$= i\pi^{2-\frac{1}{2}\varepsilon}\,\Gamma(\alpha)^{-1}\,\Gamma(\alpha-2+\tfrac{1}{2}\varepsilon)m^{4-2\alpha-\varepsilon} \qquad (1.1)$$

for later reference.

The amputated one-particle-irreducible parts of the Green's functions (ϕ_B is the unrenormalised field, m the physical particle mass, and g_B the bare coupling constant)

$$G_B(x_1 \ldots x_{2n}, y_1 \ldots y_\ell;\ m^2, g_B, \varepsilon)$$
$$= 2^{-\ell}\langle(\phi_B(x_1)\ldots\phi_B(x_{2n})\phi_B(y_1)^2 \ldots \phi_B(y_\ell)^2)_+\rangle_{conn}$$

are called vertex functions (VFs) and denoted by $\Gamma_B(x_1 \ldots x_{2n}, y_1 \ldots y_\ell;\ m^2, g_B, \varepsilon)$. Their Fourier transforms are introduced by

$$(2\pi)^{4-\varepsilon}\delta(\Sigma p+\Sigma q)\,\Gamma_B(p_1\dots p_{2n},q_1\dots q_\ell;\ m^2,g_B,\varepsilon)$$
$$=\int \Pi dx\, e^{ipx}\,\Pi dy\, e^{iqy}\,\Gamma_B(x_1\dots x_{2n},y_1\dots y_\ell;m^2,g_B,\varepsilon),$$

with

$$\dim \Gamma_B(p_1\dots p_{2n},q_1\dots q_\ell;m^2,g_B,\varepsilon)$$
$$= 4-2n-2\ell+\varepsilon(n-1)\ . \qquad (1.2)$$

$\Gamma_B(p(-p);\ m^2,g_B,\varepsilon)$ is the negative inverse propagator. The momenta set $p_1\dots p_{2n}$, $q_1\dots q_\ell$ will be abbreviated as $(2n),(\ell)$. For the massless theory, where $m^2=0$, we omit this argument.

The renormalisation conditions for the VFs are

$$\Gamma_B(p(-p);\ m^2,g_B,\varepsilon)\Big|_{p^2=m^2} = 0 \qquad (1.3a)$$

$$(p^2)^{-1}\,\Gamma_B(p(-p);\ m^2,g_B,\varepsilon)\Big|_{p^2\to-\infty} = i \qquad (1.3b)$$

$$\Gamma_B(p_1\dots p_4;\ m^2,g_B,\varepsilon)\Big|_{p_1\dots p_4\to\infty} = -i\,g_B \qquad (1.3c)$$

$$\Gamma_B(p_1p_2,q_1;\ m^2,g_B,\varepsilon)\Big|_{p_1p_2q_1\to\infty} = 1 \qquad (1.3d)$$

where in (1.3c) and (1.3d) the momenta should go to infinity in a nonexceptional way (to be explained later). The Feynman rules read off from (1.3) are, due to $\varepsilon>0$, the ones of a merely mass-renormalised, otherwise unrenormalised, theory. They yield VFs meromorphic [7] in ε in every order of expansion with respect to g_B, with poles only* at $\varepsilon = 0$ and at certain negative rational

---

*For $n = 1$, $\ell = 0$, $[\partial/\partial p^2]\, \Gamma_{\mathcal{B}}(p(-p)_{;}; m^2, g_B, \varepsilon)$ should be considered.

ε if $m^2 > 0$.

For $m = 0$, (1.3a) is not implementable in any perturbation theoretical order $N \geq 2\varepsilon^{-1}$. However, all perturbation theoretical terms, defined originally for ε sufficiently small positive, are meromorphic [7] in ε with poles, as for $m > 0$, at $\varepsilon = 0$ and at certain negative rational ε (these singularities will not be considered in the following and no longer be mentioned), and at certain positive rational ε. E.g., one finds

$$\Gamma_B(p(-p)_{;}; m^2, g_B, \varepsilon) = ip^2 + \frac{1}{6} i\, (4\pi)^{-4+\varepsilon}\, \Gamma(-1+\varepsilon)\Gamma(1 - \tfrac{1}{2}\varepsilon)^3\, \Gamma(3 - \tfrac{3}{2}\varepsilon)^{-1}\, g_B^2\, (-p^2 - i0)^{1-\varepsilon} + O(g_B^3) \qquad (1.4)$$

which obeys (1.3a) with $m = 0$ to second order only if $\varepsilon < 1$.

Defining the perturbation theoretical terms as meromorphic functions of ε by analytic continuation as explained is equivalent to the use of the bare propagator $i(p^2 + i0)^{-1}$ and of analytic subtractions* in the sense of 't Hooft [8]. The

---

*These amount to a) use of formulae such as (1.1), which imply for divergent (self-energy) (sub)graphs, subtraction at zero momentum, but no subtraction for convergent (even not if self energy$)_{\Lambda}^{\text{(sub)}}$ graphs, b) omission of graphs containing $\langle\phi_B^2\rangle$, by use of the additional prescription (3.15) below, as in usual perturbation theory.

---

perturbation series then takes the form

$$\Gamma_B^{\text{pert.th.}}((2n),(\ell);\, g_B,\varepsilon)$$
$$= ip^2\delta_{n1}\delta_{\ell 0} - ig_B\delta_{n2}\delta_{\ell 0} + \delta_{n1}\delta_{\ell 1}$$
$$+ \sum_{k=n+\delta_{n1}}^{\infty} g_B^k\, f_k((2n),(\ell);\varepsilon) \tag{1.5}$$

where according to (1.2) the f_k are homogeneous in the momenta of degree $4 - 2n - 2\ell + \varepsilon(n-k-1)$. Thus, the unrenormalised perturbation series is an ordering of contributions with respect to large-momenta behaviour. It follows that the singularities of the f_k cannot cancel, irrespective of whether the series (1.5) converges or not, provided the remainder to any finite sum of terms decreases more rapidly at large momenta than the terms of that sum itself.

We conclude with two remarks: First, the f_k in (1.5) have for generic ε singularities at exceptional momenta [2] [9], which are momenta sets such that a nontrivial even* partial sum of

---

*This means an even number of p and any number of q momenta.

momenta vanishes*, and are non-regular (in the analytical sense)

*In (1.3c) and (1.3d), all even partial sums of momenta should go to infinity at the same rate as the momenta in general, which is simplest achieved by scaling non-exceptional momenta up by a common factor.

at any vanishing partial sum of momenta [10]. The construction of the f_k directly at these momenta, using analytical subtractions, yields new meromorphic functions $\underline{f}_k$ as discussed in Sect. 3. Second, if one constructs the renormalised massless theory, specified by renormalisation conditions at finite nonzero Euclidean momenta (except for (1.3a)) in analogy* to the procedure of Gell-Mann and Low [11], the UR

*See, e.g., appendix B of [9], and [12].

divergence problem is not avoided: there are simple linear relations between the unrenormalised and the Gell-Mann-Low renormalised perturbation series terms, the latter having subtractions that remove the UR divergences at the normalisation points but not elsewhere.

2. Bare Masses in Massive $(\phi^4)_{4-\varepsilon}$

The model (1.2) with $m^2 > 0$ allows us to study the effect of change of the bare mass. The Schwinger action principle gives

$$[\partial/\partial m^2]\,\Gamma_B((2n),(\ell);\ m^2,g_B,\varepsilon)$$
$$= -i\varphi_B(g_B m^{-\varepsilon},\varepsilon)\,\Gamma_B((2n),(\ell)0;\ m^2,g_B,\varepsilon) \qquad (2.1)$$

where*

*All m^2 derivatives are meant with g_B fixed.

$$\varphi_B(g_B m^{-\varepsilon},\varepsilon) = \partial m_B^{*2}/\partial m^2 \ . \qquad (2.2)$$

Here m_B^{*2} is the bare mass squared defined by using in the Lagrangian density the interaction term $-\frac{1}{24}\, g_B\, \phi_B^{\ 4}$. The bare mass squared m_B^2 in the sense of Lehmann [13] is defined by using instead $-\frac{1}{24}\, g_B :\phi_B^4:$ where $:\phi_B^4: = \phi_B^{\ 4} - 6\phi_B^2 \langle\phi_B^2\rangle + 6\langle\phi_B^2\rangle^2 - \langle\phi_B^4\rangle$. Thus

$$m_B^{*2} = m_B^2 - g_B\,\Gamma_B(,0;m^2,g_B,\varepsilon) \qquad (2.3)$$

where (in the elementary sense, for $\varepsilon > 2$)

$$\Gamma_B(,0;m^2,g_B,\varepsilon) = \frac{1}{2}\langle\phi_B^{\ 2}\rangle$$
$$= -\frac{1}{2}\,(2\pi)^{-4+\varepsilon}\int dk\,\Gamma_B(k(-k),;m^2,g_B,\varepsilon)^{-1} \qquad (2.4)$$

is the derivative of the vacuum energy density $i\,\Gamma_B(,;m^2,g_B,\varepsilon)$ with respect to m_B^{*2}. $m_B^{\ 2}$ is also definable (in the elementary sense, for $\varepsilon > 1$) by

$$m_B^2 = [i\Gamma_B(p(-p),;m^2,g_B,\varepsilon) + p^2]\Big|_{p^2\to-\infty} \qquad (2.5)$$

and satisfies [13] $m_B^2 > m^2$ in the case of mass gap* below m.

---

*This gap, and one above m, have been demonstrated for $\varepsilon = 2$ and g_B sufficiently small by Glimm, Jaffe, and Spencer [14].

For ε positive integer, from the second Griffiths inequality [15] one may derive* that $\varphi_B \geq 0$. (2.3) implies that if $\varepsilon \to 2+0$,

---

*B. Simon (private communication).

$m_B^{*2} \to -\infty$ for fixed g_B and m^2 (or m_B^2).

From (2.5) and (2.1) follows

$$\partial m_B^2/\partial m^2 = \varphi_B(g_B m^{-\varepsilon},\varepsilon)\Gamma_B(p(-p),0;m^2,g_B,\varepsilon)\Big|_{p^2\to-\infty} . \qquad (2.6)$$

Thus, from (2.1-3) we obtain

$$\Gamma_B(p(-p),0;m^2,g_B,\varepsilon)\Big|_{p^2\to-\infty} = 1 - ig_B\Gamma_B(,00;m^2,g_B,\varepsilon) \qquad (2.7)$$

which has an obvious interpretation in terms of graphs and for which we will later indicate a different derivation. Note that the r.h.s. is unequal one in contrast to (1.3d), and that it does not exist for m = 0 in perturbation theory (see, however, (4.11) below). From (2.6-7) and the relation (in the elementary sense, for $\varepsilon > 1$)

$$[m_B^2 - m^2]\Big|_{m^2\to\infty} = 0 , \qquad (2.8)$$

due the form of the square bracket

$$c_1(\varepsilon)g_B^2 m^{2-2\varepsilon} + c_2(\varepsilon)g_B^3 m^{2-3\varepsilon} + \dots ,$$

we have

$$m_B^2 = m^2 + \int_{m^2}^{\infty} dx \left\{1 - \varphi_B(g_B x^{-\frac{1}{2}\varepsilon}, \varepsilon)[1 - ig_B \Gamma_B(,00;x,g_B,\varepsilon)]\right\} \tag{2.9}$$

while from (2.2) and the relation analogous to (2.8) for m_B^{*2} (in the elementary sense, for $\varepsilon > 2$) follows

$$m_B^{*2} = m^2 + \int_{m^2}^{\infty} dx[1 - \varphi_B(g_B x^{-\frac{1}{2}\varepsilon}, \varepsilon)] . \tag{2.10}$$

(2.9) and (2.10) are, for $\varepsilon > 1$ and $\varepsilon > 2$, respectively, convergent in perturbation theory. Note that from (2.3-4) and (2.1) follows

$$\partial(m_B^{*2} - m_B^2)/\partial m^2 = i\frac{1}{2} g_B \varphi_B(g_B m^{-\varepsilon}, \varepsilon) \cdot$$

$$\cdot (2\pi)^{-4+\varepsilon} \int dk\, \Gamma_B(k(-k),;m^2,g_B,\varepsilon)^{-2} \Gamma_B(k(-k),0;m^2,g_B,\varepsilon)$$

$$= ig_B \varphi_B(g_B m^{-\varepsilon}, \varepsilon) \Gamma_B(,00;m^2,g_B,\varepsilon)$$

in conformity with (2.9-10).

Meromorphy, and regularity for $\varepsilon > 0$, of φ_B at least in perturbation theory follow, e.g. from the consequence of (2.1) and (1.3a)

$$\varphi_B(g_B m^{-\varepsilon}, \varepsilon)$$

$$= -i\left\{\Gamma_B(p(-p),0;m^2,g_B,\varepsilon)^{-1}[\partial/\partial p^2]\Gamma_B(p(-p),;m^2,g_B,\varepsilon)\right\}\Big/_{p^2=m^2}.$$

These analytic properties hold, from (2.9) and (2.10), also for m_B^2 and m_B^{*2}, apart from simple poles of these latter for certain $\varepsilon > 0$ as discussed in Sect. 4.3 below.

3. Small-mass Behaviour in $(\phi^4)_{4-\varepsilon}$

3.1 A small-mass expansion

The perturbation series of $(\phi^4)_{4-\varepsilon}$ VFs allows the m^2 dependence of its terms to be expanded as

$$\Gamma_B^{\text{pert.th.}}((2n),(\ell);m^2,g_B,\varepsilon)$$

$$= \sum_{r=-a}^{\infty} \sum_{s=0}^{\infty} m^{2r-\varepsilon s} f_{rs}((2n),(\ell);g_B,\varepsilon) \ . \tag{3.1}$$

Here a = N-2 if $p_1 \dots p_{2n}, q_1 \dots q$ allow a partition into at most N nontrivial even momenta sets of sum zero each (a = 0 if N = 0). Each f_{rs} is a power series in g_B with terms meromorphic in ε. If the set $(2n),(\ell)$ is nonexceptional, then a = 0 and in addition $f_{os} \equiv 0$ for $s > 0$, with f_{oo} being the series (1.5).

While the perturbation series terms on the l.h.s. of (3.1) are meromorphic in ε with poles only at zero and at negative rationals, the terms on the r.h.s. have poles also at positive rationals. This is due to the fact that in the decomposition on the r.h.s., at any ε such that $2r - \varepsilon s = 2r' - \varepsilon s'$, a common single pole in f_{rs} and $f_{r's'}$ at that ε may cancel out, a double pole may do so if three terms obtain the same m-dependence etc., and this in each g_B order (within the f_{rs}) separately.*

---

*In actual computations, it is convenient to perform a Mellin transform on m, with transform variable t. The transformed function is meromorphic in t, with only simple poles for generic ε. The residua of these poles are meromorphic in ε. Letting ε go to the pole in the residuum of a t-pole makes the t-pole coalesce with one or more other t-poles to form a higher order pole, with finite residuum, giving rise to logarithms of m. A simple model for this behaviour is discussed in Sect. 5.

This is an explanation of the UR singularities in the f_k in (1.5). Note that for s too large also the $r > 0$ terms in (3.1) do not vanish in the $m \to$ limit.

We will now analyse for nonexceptional momenta the $r = 1$ terms in (3.1) and extract from them an altogether nonvanishing contribution in the $m \to 0$ limit. Hereby we shall encounter cases of exceptional momenta with $a = 0$ and find that the f_{os}, all s, factorise in a simple way, leading to such factorisations of the f_{1s}, all s, for nonexceptional momenta. All this could be pursued to higher r but we do not do that here.

3.2 Integral representation of massless $(\phi^4)_{4-\varepsilon}$ VFs

The properties (1.3b-d) are preserved if we change the bare mass in the Lagrangian density, since this does not affect the asymptotic behaviour (which is here the same as saying that it leaves the operators canonical). Thus (2.1) yields

$$\Gamma_B((2n),(\ell);g_B,\varepsilon) = \Gamma_B((2n),(\ell);m^2,g_B,\varepsilon) + i\int_0^{m^2} dx\, \varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon)\, \Gamma_B((2n),(\ell)0;x,g_B,\varepsilon) \qquad (3.2)$$

to be valid for nonexceptional momenta, and in Sect. 4 we shall argue for the convergence of the integral then. Since the integrand is expected to be holomorphic in the ε plane cut along the negative real axis and the integral is over a finite range, the massless theory VF will be holomorphic at least whenever the integral converges uniformly at $x=0$: the momenta of the VF in the integrand are exceptional and also φ_B is not expected to exist in general at $x=0$.

If we insert (3.1) in the integrand in (3.2) and expand also φ_B, we encounter integrals

$$\int_0^{m^2} dx\; x^{r-\frac{1}{2}\varepsilon s} = (1+r-\tfrac{1}{2}\varepsilon s)^{-1}\, m^{2+2r-\varepsilon s} \qquad (3.3)$$

convergent for $2r-\varepsilon s > -2$. Thus, (3.3) will converge also for large s if only ε is sufficiently small. In view of the discussion in Sect. 1, it is obvious that (1.5) is obtained by modifying (3.2) to

$$\Gamma_B^{\text{pert.th.}}((2n),(\ell);g_B,\varepsilon) = \Gamma_B((2n),(\ell);\, m^2,g_B,\varepsilon)$$
$$+\, i \;\text{"}\int_0^{m^2} dx\text{"}\; \varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon)\Gamma_B((2n),(\ell)0;x,g_B,\varepsilon) \quad (3.4)$$

where the "flip-flop" integral is understood as follows: expand the integrand in powers of x, which yields terms $x^{r-\frac{1}{2}\varepsilon s}$; integrate from 0 to m^2 if $r - \frac{1}{2}\varepsilon s > -1$ and from ∞ to m^2 if $r - \frac{1}{2}\varepsilon s < -1$, the singularity at $\varepsilon = 2(r+1)/s$ being obtained by analytic continuation. The prescription (3.4) isolates the $r = 0$ terms (with $f_{os} \equiv 0$ for $s > 0$ due to nonexceptionality) in (3.1).

From (3.2) and (3.4) we have

$$\Gamma_B((2n),(\ell);g_B,\varepsilon) - \Gamma_B^{\text{pert.th.}}((2n),(\ell);g_B,\varepsilon)$$
$$= i \;\text{"}\int_0^{\infty} dx\text{"}\; \varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon)\, \Gamma_B((2n),(\ell)0;x,g_B,\varepsilon) \quad (3.5)$$

where the integral may be defined as follows: expand the integrand in powers $x^{r-\frac{1}{2}\varepsilon s}$ as before; integrate from 0 to ∞ if $r - \frac{1}{2}\varepsilon s < -1$ and discard that term if $r - \frac{1}{2}\varepsilon s > -1$. This allows to redefine the integral as follows: from the unexpanded integrand, subtract all terms that, in its expansion in powers of x, are not integrable at ∞; this difference is integrated from 0 to ∞ without proviso. Since the subtraction terms are integrable at 0, the new integrand is integrable there if the one in (3.2) was.

3.3 Exceptional momenta and Wilson expansions*

*Most of this section follows closely the analogous $\varepsilon = 0$ analysis of [9].

We have been led to analyse the x dependence of the integrand in (3.2). The momenta are exceptional, but still $a = 0$ in (3.1). We will only construct the now nontrivial f_{os} terms. They are obtained as follows: In

$$[\partial/\partial m^2]\,\Gamma_B((2n),(\ell)0;m^2,g_B,\varepsilon)$$
$$= -i\varphi_B(g_B m^{-\varepsilon},\varepsilon)\,\Gamma_B((2n),(\ell)00;m^2,g_B,\varepsilon) \qquad (3.6)$$

the VF on the r.h.s. is subjected to a Wilson short-distance expansion, which takes the form (we omit the m^2, g_B, ε arguments)

$$\Gamma_B((2n),(\ell)00)$$
$$= \Gamma'_B((2n)00,(\ell))\,\Gamma_B(,000)\,\Gamma_B(00,0)^{-1}$$
$$+ \Gamma_{Brem}((2n),(\ell)00) \qquad (3.7)$$

where

$$\Gamma'_B((2n)00,(\ell)) = \Gamma_B((2n)00,(\ell))$$
$$+ \sum_{\text{partitions}} \Gamma_B(\ldots 0, \ldots)G_B(\Sigma'(-\Sigma'),)\,\Gamma_B(\ldots,\ldots)$$
$$\ldots\; G_B(\Sigma''(-\Sigma''),)\,\Gamma_B(\ldots 0, \ldots). \qquad (3.8)$$

Here the sum is over all different partitions of the momenta into nonempty even sets such that the two zero-momenta are in the first and the last set; the sets are the arguments of VFs ordered in a chain, with connecting propagators transferring momenta as obvious, in a graphical picture, from momentum conservation.

In (3.7), the remainder term has in its m expansion powers $m^{2r-\varepsilon s}$, $r \geq 0$, while the first term on the r.h.s. encompasses all $m^{-2-\varepsilon s}$ terms (besides others), in view of $\Gamma_B(,000) = d_1(\varepsilon)m^{-2-\varepsilon s} + d_2(\varepsilon)g_B m^{-2-2\varepsilon s} + \ldots$. The stated property of the remainder can be proven by techniques*

*See Zimmermann [16]. A demonstration covering the case at hand can be given along the lines of [9] App. A.

familiar in renormalisation theory.

(3.7) leads us to consider

$$[\partial/\partial m^2]\Gamma'_B((2n)00,(\ell))$$
$$= -i\,\varphi_B[\Gamma_B((2n)00,(\ell)0) + \ldots] \qquad (3.9)$$

where we shall not need the terms indicated by dots, which stem from the partitioned terms in (3.8). Again we use a Wilson expansion

$$\Gamma_B((2n)00,(\ell)0)$$
$$= \Gamma'_B((2n)00,(\ell))\,\Gamma_B(00,00)\,\Gamma_B(00,0)^{-1}$$
$$+ \Gamma_{Brem}((2n)00,(\ell)0) \qquad (3.10)$$

for which what was said about (3.7) again holds, whereby

$$\Gamma_B(00,00) = e_1(\varepsilon)g_B m^{-2-\varepsilon} + e_2(\varepsilon)g_B^2 m^{-2-2\varepsilon} + \dots .$$

(3.9-10) with (2.1) yield

$$[\partial/\partial m^2]\left\{\Gamma'_B((2n)00,(\ell))\,\Gamma_B(00,0)^{-1}\right\}$$
$$= -i\varphi_B[\Gamma_{Brem}((2n)00,(\ell)0) + \dots]\,\Gamma_B(00,0)^{-1} \qquad (3.11)$$

where the r.h.s. has in its mass expansion only terms $m^{-\varepsilon s}$, $m^{2-\varepsilon s}$, etc. Thus the curly bracket on the l.h.s. has, like a non-exceptional-momenta VF, only terms m^{o}, $m^{2-\varepsilon s}$, $m^{4-\varepsilon s}$, etc. This suggests that outside of perturbation theory, the curly bracket should have an $m \to 0$ limit obtained e.g. in analogy to (3.2), and we write

$$\left\{\Gamma'_B((2n)00,(\ell);m^2,g_B,\varepsilon)\Gamma_B(00,0;m^2,g_B,\varepsilon)\right\}\Big|_{m\to 0}$$
$$\equiv \underline{\Gamma}'_B((2n)00,(\ell);g_B,\varepsilon), \qquad (3.12)$$

to be somewhat qualified later. In perturbation theory, $\underline{\Gamma}'_B$ is defined as the sum of the m-independent terms of the curly bracket.

Use of (3.11) and (3.7) with (2.1) in (3.6) yields

$$[\partial/\partial m^2]\Big\{\Gamma_B((2n),(\ell)0) - \Gamma_B(,00)\Gamma'_B((2n)00,(\ell))\Gamma_B(00,0)^{-1}\Big\}$$
$$= -i\varphi_B\Big\{\Gamma_{Brem}((2n),(\ell)00) - \Gamma_B(,00)[\Gamma_{Brem}((2n)00,(\ell)0)+\dots]\Gamma_B(00,0)^{-1}\Big\}. \qquad (3.13)$$

What was said about (3.11) applies also to (3.13) and leads us to write

$$\Big\{\Gamma_B((2n),(\ell)0;m^2,g_B,\varepsilon) - \Gamma_B(,00;m^2,g_B,\varepsilon)\Gamma'_B((2n)00,(\ell);m^2,g_B,\varepsilon)\Gamma_B(00,0;m^2,g_B,\varepsilon)^{-1}\Big\}\Big/_{m\to 0}$$
$$\equiv \underline{\Gamma}_B((2n),(\ell)0;g_B,\varepsilon) \qquad (3.14)$$

understood as (3.12) was and to be qualified below. Clearly, inside the curly bracket, (3.12) can already be used at least in perturbation theory.

(3.12) and (3.14) show that at exceptional momenta, and the two cases considered have still a = 0, in (3.1) the $r=0$ sum has terms also with $s > 0$ in contrast to nonexceptional momenta, but that these terms have simple factorisation properties as claimed before. As the estimates in Sect. 4 will show, the $r=0$ sum does in general not have a $m\to 0$ limit (since $\Gamma_B(00,0)$ and $\Gamma_B(,00)$ do not), and there is no reason why the $r>0$ sums should remedy this.

Definitions (3.12) and (3.14) are analogous to definitions (III.12a) and (III.14a) in [12], or equivalently to (III.8) and

(III.21) in [9]. However, while these latter definitions are proven to lead to finite functions, in (3.12) and (3.14) the $r > 0$ sums of (3.1) will require additional terms, which are expected to yield on the r.h.s. terms having explicit factors $g_B^{2/\varepsilon}$, $g_B^{4/\varepsilon}$, etc. by similar computations as in Sect. 3.4 below.

In perturbation theory, the $\underline{\Gamma}_B$ functions are obtained* from the

*This was pointed out to the author by G. 't Hooft.

Feynman integrals for the Γ_B functions by using the rule

$$\int dk[-k^2 - i0]^{-\alpha} = 0, \text{ any } \alpha \tag{3.15}$$

which contrasts with the $m \to 0$ limit of (1.1) if $4 - 2\alpha - \varepsilon \leq 0$. It is the evaluation (1.1) that leads to the $m^{-\varepsilon s}$ terms at exceptional momenta; use of (3.15) instead (and of similar ones for several-loop integrals [8]) amounts to omitting the terms with $s > 0$ and thus yields the $\underline{\Gamma}_B$ directly.*

*A definition in perturbation theory which involves a limit process, however, was given by H. Trute (private communication). In our notation, it is: choose in (3.1) $\varepsilon < 0$ but sufficiently small (depending on the order) and let $m \to 0$ provided $a = 0$. The result is f_{oo}, whose analytic continuation in ε is $\Gamma_B^{\text{pert.th.}}$, or $\underline{\Gamma}_B^{\text{pert.th.}}$, of the massless theory.

The relation to the "elementary recipe" of [9], proven to be applicable also to the present exceptional-momenta cases, is obvious.– Note, incidentally, that forming $\underline{\Gamma}_B$ and $\underline{\Gamma}'_B$ does not amount to omitting the graphs that are two-particle reducible between the (2n),(ℓ) and the 0,0 and 00, momenta, respectively. It is only parts extracted subtractively from them and strictly proportional to (3.15) that are discarded.

3.4 Evaluation of an integral

Use of (3.14) and (3.12) in (3.5) yields

$$\Gamma_B((2n),(\ell);g_B,\varepsilon) - \Gamma_B^{\text{pert.th.}}((2n),(\ell);g_B,\varepsilon)$$

$$= i["\int_0^\infty dx" \,\varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon)]\underline{\Gamma}_B((2n),(\ell)0;g_B,\varepsilon)$$

$$+ i["\int_0^\infty dx" \,\varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon)\, \Gamma_B(,00;x,g_B,\varepsilon)]$$

$$\cdot \Gamma'_B((2n)00,(\ell);g_B,\varepsilon) + \ldots \qquad (3.16)$$

where the dots stand for "$\int_0^\infty dx$" integrals of the r.h.s. of (3.13) and (3.11) integrated over m^2 from 0 to x and not repeated here to save space. The terms written out on the r.h.s. of (3.16) are the $m \to 0$ limit of the $r = 1$ part of (3.1), and the terms represented by the dots are the $m \to 0$ limit of the $r = 2,3$, etc. parts and could be analysed by using Wilson expansions to higher accuracy than in (3.7) and (3.10), for which Zimmermann [16] has given the requisite formulae. Actually, the discussed qualifications of (3.12) and (3.14) imply that the $r=1$, $r=2$ sums, etc. need not necessarily have $m\to 0$ limits separately, but only all of them together. This, however, does not affect the qualitative conclusions in Sect. 5.

In (3.15),

$$"\int_0^\infty dx" \varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon) = \int_0^\infty dx[\varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon) - 1] = -m_{BO}^{*\,2} \tag{3.17a}$$

if $\varepsilon > 2$, and

$$"\int_0^\infty dx" \varphi_B(g_B x^{-\frac{1}{2}\varepsilon},\varepsilon)\Gamma_B(,00;x,g_B,\varepsilon) = \text{the ordinary integral}$$

$$= -ig_B^{-1}(m_{BO}^2 - m_{BO}^{*\,2}) = -i\,\Pi(,0;g_B,\varepsilon) \tag{3.17b}$$

if $\varepsilon > 2$, in view of

$\Gamma_B(,00;m^2,g_B,\varepsilon) = e_1(\varepsilon)m^{-\varepsilon} + e_2(\varepsilon)g_B m^{-2\varepsilon} + \ldots$, where m_{BO}^2 and $m_{BO}^{*\,2}$ are the bare masses of the massless theory, obtained by letting $m\to 0$ in (2.9) and (2.10). At least as far as the $r=1$ part is concerned, the convergence of the integral in (3.2) at 0 will have been shown (and the consequences, about removal of ε singularities, mentioned earlier verified) if the integrals in (3.17), or equivalently the integrals

$$m_{BO}^{*\,2} = g_B^{2/\varepsilon}\int_0^\infty dx[1 - \varphi_B(x^{-\frac{1}{2}\varepsilon},\varepsilon)] \equiv g_B^{2/\varepsilon}\mu^*(\varepsilon) \tag{3.18a}$$

and

$$m_{BO}^2 = g_B^{2/\varepsilon}\int_0^\infty dx\left\{1 - \varphi_B(x^{-\frac{1}{2}\varepsilon},\varepsilon)[1 - i\bar{g}_B\Gamma_B(,00;x\bar{g}_B^{2/\varepsilon},\bar{g}_B)]\right\}$$

$$\equiv g_B^{2/\varepsilon}\mu(\varepsilon) \tag{3.18b}$$

converge at 0 for $\varepsilon > 2$ and $\varepsilon > 1$, respectively, the integrand in (3.18b), since dimensionless, being independent of $\bar{g}_B$. These integrals, which obviously do not converge in perturbation theory, are discussed in the next section.

4. Conjectured true small-mass behaviour

4.1 Renormalised $(\phi^4)_{4-\varepsilon}$

The estimation of the small-x behaviour of the integrands in (3.18), on the basis of Wilson's ideas, is by now familiar* and

*See the lectures of E. Brézin, G. Parisi, and B. Schroer at this Summer School, where further references are given.

will be sketched only briefly. The (mass shell) renormalised VFs $\Gamma((2n),(\ell);m^2,g,\varepsilon)$ are defined by imposing the renormalisation conditions

$$\Gamma(p(-p),;m^2,g,\varepsilon)\Big|_{p^2=m^2} = 0 \tag{4.1a}$$

$$[\partial/\partial p^2]\Gamma(p(-p),;m^2,g,\varepsilon)\Big|_{p^2=m^2} = i \tag{4.1b}$$

$$\Gamma(p_1 \dots p_4,;m^2,g,\varepsilon)\Big|_{p_ip_j=\frac{1}{3}(4\delta_{ij}-1)m^2} = -im^\varepsilon g \tag{4.1c}$$

$$\Gamma(00,0;m^2,g,\varepsilon) = 1 \tag{4.1d}$$

$$\Gamma(,00;m^2,g,\varepsilon) = 0\ . \tag{4.1e}$$

They satisfy*

*The easiest proof of (4.2) is by observing that renormalisation theory implies a relation of the form (4.3). Inserting (4.3) in (2.1) yields (4.2), the coefficients being β, γ, η, κ, and φ from (4.4-5). That for the Γ the limit $\varepsilon \to 0$ can be performed in (renormalised) perturbation theory by analytic continuation implies that it can for (4.2) also. This derivation of the $\varepsilon = 0$ form of (4.2) is more elegant, and for gauge field theories much simpler, than the one by Pauli-Villars regularisation [17], but directly applicable only in perturbation theory since $g_W(0) = 0$.

$$Op_{2n,\ell}\Gamma((2n),(\ell);m^2,g,\varepsilon) + i\delta_{no}\delta_{\ell 2}\kappa(g,\varepsilon)m^{-\varepsilon}$$
$$= -im^2\varphi(g,\varepsilon)\Gamma((2n),(\ell)0;m^2,g,\varepsilon) \tag{4.2a}$$

where

$$Op_{2n,\ell} = m^2[\partial/\partial m^2] + \beta(g,\varepsilon)[\partial/\partial g]$$
$$- 2n\gamma(g,\varepsilon) + \ell(2\gamma(g,\varepsilon) + \eta(g,\varepsilon)) \ . \tag{4.2b}$$

The expressions for $\beta(g,\varepsilon)$, $\gamma(g,\varepsilon)$, $\eta(g,\varepsilon)$, $\varphi(g,\varepsilon)$, and $\kappa(g,\varepsilon)$ obtained from consistency of (4.2) with (4.1a-e) suggest these functions to be holomorphic in the ε plane cut along the negative real axis.

It is easy to prove that

$$\Gamma_B((2n),(\ell);m^2,g_B,\varepsilon)$$
$$= a(g,\varepsilon)^{-n+\ell}h(g,\varepsilon)^{\ell}\Gamma((2n),(\ell);m^2,g,\varepsilon)$$
$$+ i\delta_{no}\delta_{\ell 2}k(g,\varepsilon)m^{-\varepsilon}g^{-1}\exp[\tfrac{1}{2}\varepsilon\bar{\rho}(g,\varepsilon)] \tag{4.3}$$

and

$$\varphi_B(g_B m^{-\varepsilon},\varepsilon) = a(g,\varepsilon)^{-1}h(g,\varepsilon)^{-1}\varphi(g,\varepsilon) \ , \tag{4.4}$$

with

$$\bar{\rho}(g,\varepsilon) = \int_0^g dg'[\beta(g',\varepsilon)^{-1} + 2\varepsilon^{-1}g'^{-1}] \tag{4.5a}$$

$$a(g,\varepsilon) = \exp[2\int_0^g dg'\beta(g',\varepsilon)^{-1}\gamma(g',\varepsilon)] \qquad (4.5b)$$

$$h(g,\varepsilon) = \exp[\int_0^g dg'\beta(g',\varepsilon)^{-1}\eta(g',\varepsilon)] \qquad (4.5c)$$

$$k(g,\varepsilon) = \int_0^g dg'\beta(g',\varepsilon)^{-1}a(g',\varepsilon)^2 h(g',\varepsilon)^2 \cdot$$

$$\cdot\, \kappa(g',\varepsilon)g' \exp[-\tfrac{1}{2}\varepsilon\bar{\rho}(g',\varepsilon)] \qquad (4.5d)$$

and

$$g_B = m^\varepsilon g \exp[-\tfrac{\varepsilon}{2}\bar{\rho}(g,\varepsilon)] \ . \qquad (4.6)$$

From (4.6) and (4.5a) and in view of

$$\beta(g,\varepsilon) = -\tfrac{1}{2}\varepsilon g + b_o(\varepsilon)g^2 + \ldots, \quad b_o(0) = 3(32\pi^2)^{-1}$$

it follows that

$$\partial \ln g_B/\partial \ln g = -\tfrac{1}{2}\varepsilon g\beta(g,\varepsilon)^{-1} = 1 - 2\varepsilon^{-1}b_o(\varepsilon)g + \ldots > 0 \qquad (4.7)$$

for $0 \leq g < g_W(\varepsilon)$, with $g_W(\varepsilon)$ the first positive zero of $\beta(g,\varepsilon)$, such that g_B and g are there related* monotonically. Outside of this range, which corresponds to $0 \leq g_B < \infty$, neither side of (4.3) makes sense directly.

*Transcribing (3.1) using (4.3) and (4.6) into a small-mass expansion for the renormalised VFs yields the 4-ε analog of the expansion (0.2) (with $r_i = 0$) of [9].

4.2 Assumptions and their consequences

Following Wilson [1]* one assumes that $\beta(g,\varepsilon)$, $\gamma(g,\varepsilon)$, $\eta(g,\varepsilon)$,

*Wilson's assumptions are analogous to the present ones, but for functions $\hat{\beta}$, $\hat{\gamma}$, $\hat{\eta}$, and $\hat{\kappa}$ of the zero-mass theory described at the end of Sect. 1. There are no analogs of (3.18) in zero--mass-theory quantities, which is the reason for the difficulty of removing the UR singularities in perturbation-theoretically considered zero-mass theory.

$\kappa(g,\varepsilon)$, and $\varphi(g,\varepsilon)$ are left-continuous, and at least $\beta(g,\varepsilon)$ even left-differentiable, at $g = g_W(\varepsilon)$. The rationale hereto is that these properties very likely hold for ε small and that there is apparently no known reason why they should cease to hold if ε increases, and that deductions from this hypothesis appear to be in accord [1] with experiment as well as with computer calculations. Then for $m \to 0$ at fixed g_B, or equivalently $g \to g_W(\varepsilon) - 0$ at fixed m, (4.6) yields easily

$$g_B = m^{\varepsilon} f_1(g,\varepsilon) \cdot [g_W(\varepsilon) - g]^{-\frac{1}{2}\varepsilon\{[\partial/\partial g]\beta(g,\varepsilon)^{-1}\}|_{g_W(\varepsilon)-0}} \qquad (4.8a)$$

with $f_1(g_W(\varepsilon)-0,\varepsilon)$ finite, and then (4.5 b-d) give

$$a(g,\varepsilon) = f_2(g,\varepsilon)[g_B m^{-\varepsilon}]^{-4\varepsilon^{-1}\gamma(g_W(\varepsilon),\varepsilon)} \qquad (4.8b)$$

$$h(g,\varepsilon) = f_3(g,\varepsilon)[g_B m^{-\varepsilon}]^{-2\varepsilon^{-1}\eta(g_W(\varepsilon),\varepsilon)} \qquad (4.8c)$$

and

$$k(g,\varepsilon) = \begin{cases} f_4(g,\varepsilon) & \text{if } \theta > \frac{1}{2}\varepsilon \\ f_5(g,\varepsilon)[g_B m^{-\varepsilon}]^{1-2\varepsilon^{-1}\theta} & \text{if } \theta < \frac{1}{2}\varepsilon \\ f_6(g,\varepsilon)\,\ln[g_B m^{-\varepsilon}] & \text{if } \theta = \frac{1}{2}\varepsilon \end{cases} \qquad (4.8d)$$

where

$$\theta \equiv 4\gamma(g_W(\varepsilon),\varepsilon) + 2\eta(g_W(\varepsilon),\varepsilon).$$

Then from (4.4)

$$\varphi_B(g_B m^{-\varepsilon},\varepsilon) = f_7(g,\varepsilon)[g_B m^{-\varepsilon}]^{\varepsilon^{-1}\theta}, \tag{4.9}$$

from (4.1d) and (4.3)

$$\Gamma_B(00,0;m^2,g_B,\varepsilon) = h(g,\varepsilon) = (4.8c) \tag{4.10}$$

and from (4.1e) and (4.3)

$$\Gamma_B(,00;m^2,g_B,\varepsilon) = i\ g_B^{-1}k(g,\varepsilon) = i\ g_B^{-1}\ (4.8d). \tag{4.11}$$

In these formulae, the $f_i(g,\varepsilon)$, $i = 2 \ldots 7$ do not have power behaviour in $g_W(\varepsilon) - g$, and e.g. have limits for $g \to g_W(\varepsilon) - 0$ if the functions $\gamma(g,\varepsilon)$, $\eta(g,\varepsilon)$, $\varphi(g,\varepsilon)$, $\kappa(g,\varepsilon)$ are left--differentiable at $g_W(\varepsilon)$.

4.3 <u>On existence and properties of $\mu^*(\varepsilon)$ and $\mu(\varepsilon)$</u>

It is now seen that (3.18a) converges at $x = 0$ provided

$$\theta < 2 \tag{4.12a}$$

and that (3.18b) converges there provided in addition

$$\theta > -2 + \varepsilon \tag{4.12b}$$

holds. (4.12) appears likely to be satisfied*, from the ε-

---

*Insertion of (3.14) with (3.12) into (3.2) yields an estimate to the small-mass behaviour of the massive-theory VFs, replacing the formal expansion (3.1): the x-integrals converge at 0 if (4.9) and (4.11) hold and (4.12) is satisfied, and lead to power laws in m, cp. [19].

expansion of θ, for not too large ε.

In view of (4.19) which implies

$$\dim :\phi^2: = \begin{cases} 2 - \varepsilon + \theta & \text{if } \theta < \frac{1}{2}\varepsilon \\ 2 - \frac{1}{2}\varepsilon & \text{if } \theta \geq \frac{1}{2}\varepsilon \end{cases} \tag{4.13}$$

(4.12) is equivalent to the condition that the long-distance dimension of the massless-theory $:\phi^2:$ satisfies* (assuming $\varepsilon < 4$)

*The conditions (4.12) are analogous to the consistency conditions derived in [9] Sect. IV, in fact in both cases assumptions of a certain behaviour at a fixed point in Wilson's [18] sense are being tested.

$$0 < \dim :\phi^2: < 2 - \frac{1}{2}\varepsilon . \tag{4.14}$$

Also $\dim :\phi^2: > 1 - \frac{1}{2}\varepsilon$ must hold for positivity reasons. Furthermore, from the second Griffiths inequality [15] follows*

*B. Simon (private communication).

(for positive integer ε)

$$\Gamma_B(00,0;m^2,g_B,\varepsilon) > 0 \tag{4.15a}$$

and from the Lebowitz inequality* (for positive integer ε)

*See, e.g., [14] .

$$\Gamma_B(00,0;m^2,g_B,\varepsilon) \leq 1 . \tag{4.15b}$$

(4.15) with (4.10) gives (for positive integer ε)

$$\eta(g_W(\varepsilon),\varepsilon) \geq 0 \tag{4.16a}$$

while from (4.18) for positivity reasons follows

$$\gamma(g_W(\varepsilon),\varepsilon) \geq 0 \ . \qquad (4.16b)$$

The integrals (3.18) are written, as stated there, for $\varepsilon > 2$ and $\varepsilon > 1$, respectively. Otherwise, further subtractions in the integrands of terms proportional $x^{-\varepsilon/2}$, $x^{-\varepsilon}$, etc. have to be made. Obviously, an analytic continuation [20] of $\mu^*(\varepsilon)$ and $\mu(\varepsilon)$ beyond simple poles, with computable residua, at $\varepsilon = 2,1,\frac{2}{3}$ ... and $\varepsilon = 1, \frac{2}{3}, \frac{1}{2}$ ..., respectively, is thereby made. In view of the presumed analyticity in ε of the integrand and the uniform convergence (to the extent that (4.12) is satisfied and the $f_i(g,\varepsilon)$ in (4.8-9) obey a mild condition) of the integrals at 0, we are led to conjecture that $\mu^*(\varepsilon)$ and $\mu(\varepsilon)$ are meromorphic in the ε plane cut along the negative real axis, with only the mentioned poles at positive ε. $m^{-2}m_B^2$ and $m^{-2}m_B^{*2}$ in (2.9-10) should afortiori have these analytic properties, and the same poles and residua.

If one writes [5] the integrals (3.18) in terms of functions $\beta(g,\varepsilon)$, etc. using the formulae of this section, one may derive from the perturbation expansions of these functions, and suitable additional assumptions, expansions for $\mu^*(\varepsilon)$ and $\mu(\varepsilon)$. However, an illuminating one has not been found yet.

4.4 Small-momenta behaviour of massless-theory VFs

For the massless-theory VFs, (4.3) becomes

$$\Gamma_B((2n),(\ell);g_B,\varepsilon)$$
$$= a(g,\varepsilon)^{-n+\ell} h(g,\varepsilon)^{\ell}\, \Gamma_{as}((2n),(\ell);m^2,g,\varepsilon)$$
$$+ i\delta_{no}\delta_{\ell 2} k(g,\varepsilon) m^{-\varepsilon} g^{-1} \exp[\tfrac{1}{2}\,\varepsilon\bar{\rho}(g,\varepsilon)] \qquad (4.17)$$

where the Γ_{as} are the asymptotic forms of the Γ, defined in analogy to the $\varepsilon = 0$ case of [1][9][12]. Since the Γ_{as} satisfy (4.2b) with the r.h.s. replaced by zero, the r.h.s. of (4.17) is annihilated by $Op_{0,0}$, which means, in view of (4.7), that it is independent of m for fixed g_B, as the l.h.s. requires. The relations replacing (4.17) in the case of exceptional momenta, for the $\underline{\Gamma}'_B$ and $\underline{\Gamma}_B$ in (3.12) and (3.14), respectively, are formed in precise analogy to (III.25) and (III.27) of [12], which is the reason for the "renormalisation group invariance" of these constructions proven there.

The small-momenta behaviour of the massless-theory Γ_B (or $\underline{\Gamma}_B$) is obtained from the one of the Γ_{as} (or $\underline{\Gamma}_{as}$) via (4.17) (or the relations replacing it for exceptional momenta). The one of the latter functions one obtains, following Wilson [18], from their transformation formulae and the assumption that these latter functions exist for $g \to g_W(\varepsilon)$. Testing the consistency of these assumptions in parallel to [9] Sect. IV leads (at least in the $r = 0$ approximation) again to the conditions (4.12). One so obtains, e.g.,

$$-\Gamma_B(p(-p),;g_B,\varepsilon)^{-1} \approx \text{prop. } (-p^2)^{-1+2\gamma(g_W(\varepsilon),\varepsilon)} \qquad (4.18)$$

for $p^2 \to 0$, and

$$\Gamma_B(,q(-q);g_B,\varepsilon) \approx \begin{cases} \text{const.} & \text{if } \theta - \frac{1}{2}\varepsilon > 0 \\ \text{prop. } (-q^2)^{\theta - \frac{1}{2}\varepsilon} & \text{if } \theta - \frac{1}{2}\varepsilon < 0 \\ \text{prop. } \ell n(-q)^2 & \text{if } \theta - \frac{1}{2}\varepsilon = 0 \end{cases} \qquad (4.19)$$

for $q^2 \to 0$, under appropriate assumptions on $\gamma(g,\varepsilon)$, $\eta(g,\varepsilon)$, and $\kappa(g,\varepsilon)$ as before. The role of the behaviour (4.18), not obtainable in perturbation theory, in the present context is briefly commented upon in the next section.

From (4.18-19) and other formulae arrived at similarly, one extracts the critical exponents.* Since $\beta(g,\varepsilon)$, $\gamma(g,\varepsilon)$, etc. used hereby are expected to be holomorphic in the cut ε-plane as mentioned in Sect. 4.1, one infers holomorphy of the critical exponents at least inside a strip around the positive real ε-axis. This is the basis of the ε-expansion of these exponents, with no apparent source of nonanalytic terms such as $\varepsilon^{2r/\varepsilon}$. Concerning the VFs of the critical theory, however, we have not been able to arrive at a definite conclusion as to such terms.

*See the references given at the beginning of Sect. 4.1, and [19].

5. Discussion

That the terms on the r.h.s. of (3.16) have coefficients expressible, due to (3.17), directly in terms of the bare masses of the massless theory suggests the following interpretation: Consider constructing the massless-theory VFs, rather than using the bare propagators $i(p^2+i0)^{-1}$ and analytic subtractions as explained before (1.5), using, instead, the bare propagator

$$i(p^2 - m_{BO}{}^2 + i0)^{-1} = i(p^2+i0)^{-1} + im_{BO}{}^2(p^2+i0)^{-2} + \ldots \qquad (5.1)$$

in its expanded form, and analytic subtractions only to the extent as they are implied by (1.1), with graphs containing $\langle\phi_B^2\rangle$ parts set zero as required by the use of $m_{BO}{}^2$ rather than $m_{BO}^{*}{}^2$. With these rules, the first term on the r.h.s. of (5.1) gives $\Gamma_B^{\text{pert.th.}}((2n),(\ell);g_B,\varepsilon)$. In the Appendix we show that the second term gives rise to the written terms on the r.h.s. of (3.16) apart from terms proportional to $(m_{BO}{}^2)^r$, $r \geq 2$. These last terms, together with the ones from higher terms in (5.1), have factors $g_B{}^{2r/\varepsilon}$ and correspond to the terms in (3.16) represented by dots, stemming from the remaining sums in (3.1) (modulo qualifications, see Sects. 3.3-4).

Indeed, write the r-part of (3.1) as

$$g_B{}^{2r/\varepsilon} \sum_{s=0}^{\infty} [g_B m^{-\varepsilon}]^{s-2\varepsilon^{-1}r} g_B{}^{-s} f_{rs}((2n),(\ell);g_B,\varepsilon) \,. \qquad (5.2a)$$

If this expression has a limit as $m \to 0$, that limit will necessarily* be of the form

---

*Similar reasoning has been applied often; see, e.g. [21].

$$g_B^{2r/\varepsilon} f_r((2n),(\ell);g_B,\varepsilon) \tag{5.2b}$$

which, for $r=1$, is the form the written terms on the r.h.s. of (3.16) (apart from the nonanalytic terms in $\underline{\Gamma}_B$ and $\underline{\Gamma}'_B$, see Sect. 3.3) do have. The complete form of (3.16) will be

$$\Gamma_B((2n),(\ell);g_B,\varepsilon) - \Gamma_B^{\text{pert.th.}}((2n),(\ell);g_B,\varepsilon)$$
$$= \sum_{r=1}^{\infty} g_B^{2r/\varepsilon} \sum_{t=1}^{T(r)} \nu_{rt}(\varepsilon) \underline{f}_{rt}((2n),(\ell);g_B,\varepsilon) \; . \tag{5.3}$$

Here $\nu_{rt}(\varepsilon)$ $(r \geq 2)$ are meromorphic functions of ε similar to $\mu(\varepsilon)$ and $\mu^*(\varepsilon)$, with poles at $\varepsilon = 2/s$, $s \geq 1$, with computable residua. The $\underline{f}_{rt}$ are Laurent series in g_B with computable coefficients and only a finite number of negative-power terms, and for $r=1$ without a negative-power term if $n \geq 1$. $T(r)$ increases with r, with $T(1) = 2$. In view of the discussion of (3.1) at the beginning of Sect. 3.1, the persisting cancellation of ε singularities in the $m \rightarrow 0$ limit would mean that the ε singularities are not changed in the transition from (5.2a) to (5.2b) if the sums to different r had $m \rightarrow 0$ limits independently.

The explanation of cancellation of UR singularities on the basis of (5.1) is: The propagator (5.1) cannot lead to UR divergences unless all orders of perturbation theory are summed up because

$m_{B0}^{\;2}$ is so chosen that only the complete propagator becomes massless. That propagator, however, has small-p dependence (4.18) and leads, just as in the conformal invariant theory [4] whose consideration for this purpose is sufficient, not to UR divergences.

It is amusing to consider a simple mathematical model of cancellation of UR divergences. In a field-theoretical model calculation, Parisi [22] obtained a series which we write

$$f^{\text{pert.th.}}(x,\varepsilon) = \sum_{n=0}^{\infty} (-1)^n \Gamma(1-\tfrac{1}{2}\varepsilon n)x^n \quad . \tag{5.4}$$

The n^{th} term has "UR" singularities at $\varepsilon = 2k/n$, $k \geq 1$. (We may think of x as $g_B(-p^2)^{-\varepsilon/2}$ as appears in $\Gamma_B^{\text{pert.th.}}(p(-p),;g_g,\varepsilon)$.) Now sum (5.4) formally to

$$\begin{aligned} f(x,\varepsilon) &= \int_0^{\infty} dy\; e^{-y}[1+xy^{-\varepsilon/2}]^{-1} \\ &= (2i)^{-1}\int_{c-i\infty}^{c+i\infty} ds[\sin \pi s]^{-1}\,\Gamma(1+\tfrac{1}{2}\varepsilon s)x^{-s} \end{aligned} \tag{5.5}$$

with $0 < c < 1$. $f(x,\varepsilon)$ is holomorphic in the ε plane cut along the negative real axis, with singularities at all negative rational ε, and $f(\infty,\varepsilon) = 0$. $f(x,\varepsilon)$ is not analytic in x at 0. Its correct expansion there is obtained by shifting in (5.5) the s-integration path to the left, with the result

$$\begin{aligned} &f(x,\varepsilon) - f^{\text{pert.th.}}(x,\varepsilon) \\ &= \sum_{k=1}^{\infty} (-1)^k 2\pi\varepsilon^{-1}\,\Gamma(k)^{-1}[\sin(2\pi k\varepsilon^{-1})]^{-1}x^{2k\varepsilon^{-1}} \quad . \end{aligned} \tag{5.6}$$

Each term on the r.h.s. again has poles at positive rational ε, at which, however, the pole in the term in $f^{\text{pert.th.}}(x,\varepsilon)$ obtaining at that pole the same x-dependence is just cancelled. (5.6) is a mimikry of (5.3). That in the present example ε-singularity compensating terms were obtained from $f^{\text{pert.th.}}(x,\varepsilon)$ alone suggests, in the light of our discussion of (5.2), that there is a close relation between the $r=0$ and the $r>0$ sums in (3.1), as there obviously is. It is here the step from (5.4) to (5.5) that is analogous to the step of reinterpreting the integral in (3.5), as described after that formula, and which produces a result transcending perturbation theory.

The cancellation in (5.3) of poles at positive rational ε leads at those ε themselves to logarithms of g_B. E.g., we find from (1.4) and (3.16-18)

$$\begin{aligned}\Gamma_B(p(-p),;g_B,\varepsilon{=}1) &= ip^2 - i(192\pi^2)^{-1}g_B{}^2\, \ln[(-p^2-i0)g_B{}^{-2}] \\ &\quad + i\,\text{const}\; g_B^2 - i\,[\partial/\partial\varepsilon][(\varepsilon-1)\mu(\varepsilon)]\Big|_{\varepsilon=1} g_B{}^2 + O(g_B^3)\end{aligned} \qquad (5.7)$$

where the constant is computable but the last g_B^2 term apparently not exactly. (5.7) is consistent with, but that term not obtainable from, dispersion theoretical calculation. We have not examined whether the fact of cancellation of ε singularities in (5.3) leads to information on the finite parts of $\mu(\varepsilon)$, $\mu^*(\varepsilon)$, and the $\nu_{rt}(\varepsilon)$, $r \geq 2$, of which we have not checked whether they are expressible by $\mu(\varepsilon)$ and $\mu^*(\varepsilon)$.

The expansion (5.7) is, however, usable only at large momenta. The small-momenta limit, which is to yield (4.18), is equivalent (upon extraction of a divergent factor from (5.7), as seen from (4.17) and (4.8b)) to $g_B \to \infty$.

Acknowledgement

The author is indebted to G. 't Hooft, G. Mack, G. Parisi, and K. Wilson for helpful discussions and suggestions, and to J. Zinn-Justin for stimulating remarks that started this investigation. He thanks Dr. E. Brézin for the invitation to lecture at the Cargèse Summer School, and Professor D. Holcomb for hospitality at the Physics Department of Cornell University where these lectures were written up.

Appendix

We here discuss the effect of the second term on the r.h.s. of (5.1). It yields for the r.h.s. of (3.16) the term
$-i\, m_{BO}^2\, \Gamma_B((2n),(\ell)0;g_B,\varepsilon)$ wherefrom its terms of $\langle\phi_B^2\rangle$ type must be subtracted, leaving
$-im_{BO}^2[1 - ig_B\Gamma_B(,00;g_B,\varepsilon)]^{-1}\,\Gamma_B((2n),(\ell)0;g_B,\varepsilon)$. Both VFs herein are (at least in perturbation theory) meaningless and we define them by a limit, e.g., by

$$-im_{BO}^2 \lim_{x\to 0}\left\{[1 - ig_B\Gamma_B(,00;x,g_B,\varepsilon)]^{-1}\Gamma_B((2n),(\ell)0;x,g_B,\varepsilon)\right\} .$$

For the last VF herein we use (3.14) and obtain

$$-i\bar{m}_B^{*2}\underline{\Gamma}_B((2n),(\ell)0;g_B,\varepsilon) + g_B^{-1}(m_{BO}^2 - \bar{m}_{BO}^{*2})\underline{\Gamma}'_B((2n)00,(\ell);g_B,\varepsilon) \qquad (A.1)$$

where

$$\bar{m}_{BO}^{*2} = [1 - ig_B\Gamma_B(,00;g_B,\varepsilon)]^{-1}m_{BO}^2 \ . \qquad (A.2)$$

(A.1) agrees with the written terms in (3.16), by virtue of (3.17), apart from $\bar{m}_{BO}^{*2} \neq m_{BO}^{*2}$ since

$$\bar{m}_{BO}^{*} - m_{BO}^2 = ig_B\bar{m}_{BO}^{*2}\Gamma_B(,00;g_B,\varepsilon) \neq -g_B\Gamma_B(,0;g_B,\varepsilon)$$

violating (2.3). However,

$$\Gamma_B(,0;g_B,\varepsilon) = -ig_B\Gamma_B(,00;g_B,\varepsilon)m_{BO}^{*2} + O(m_{BO}^{*4})$$
$$= -ig_B[1 - ig_B\Gamma_B(,00;g_B,\varepsilon)^{-1}]\,\Gamma_B(,00;g_B,\varepsilon)m_{BO}^2 + O(m_{BO}^4) \qquad (A.3)$$

from use of (5.1) and of its m_{BO}^{*2} analog, such that the error in (A.1) due to $\overline{m}_{BO}^{*2} \neq m_{BO}^{*2}$ is of higher order in m_{BO}^{2}, or of the $g_B^{2r/\varepsilon}$, $r \geq 2$ type.– The use of (A.3) is not entirely unambiguous, however, and the argument in Sect. 5 based on (5.1) ultimately is only of intuitive value; the calculations performed in Sect. 3 are, however, precise.

References

1. K. G. Wilson, J. B. Kogut, CLNS-245 (to appear in Physics Reports); K. G. Wilson, Lectures at this Summer School.
2. G. Mack, Lecture Notes in Physics, Vol. 17, Eds. W. Rühl, A. Vancura, Heidelberg, Springer 1973.
3. K. Symanzik, Springer Tracts in Modern Physics 57, 222 (1971).
4. G. Mack, I. T. Todorov, Phys. Rev. D8, 1764 (1973).
5. K. Symanzik, Lett. Nuovo Cimento (DESY 73/39).
6. K. G. Wilson, Phys. Rev. D7, 2911 (1973), App.; G. 't Hooft, M. Veltman, CERN 73-9, Lab. I, Theor. Stud. Div.
7. E. Speer, M. J. Westwater, Ann. de l'Inst. Poincaré A14, 1 (1971).
8. G. 't Hooft, CERN-Th. 1666 and Lectures at NATO Summer Institute, Capri, July 1973.
9. K. Symanzik, Comm. math. Phys. 23, 49 (1971).
10. D. Ruelle, Nuovo Cimento 19, 356 (1961).
11. M. Gell-Mann, F. E. Low, Phys. Rev. 95, 1300 (1954).
12. K. Symanzik, Comm. math. Phys. (DESY 73/6).
13. H. Lehmann, Nuovo Cimento 11, 342 (1954).
14. J. Glimm, A. Jaffe, T. Spencer, 1973 Erice Lectures, Springer Verlag.
15. R. B. Griffiths, J. Math. Phys. 10, 1559 (1969).
16. W. Zimmermann, Ann. Phys. (N.Y.) 77, 536, 570 (1973).
17. K. Symanzik, Comm. math. Phys. 18, 227 (1970).

Cf. also S. Coleman, R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).

18. K. G. Wilson, Phys. Rev. D3, 1818 (1971).
19. E. Brézin, D. J. Amit, J. Zinn-Justin (Saclay preprint).
20. I. M. Gelfand, G. E. Shilow, "Generalized Functions", Vol. 1, London, Academic Press, 1964.
21. T. D. Lee, K. Huang, C. N. Yang, Phys. Rev. 106, 1135 (1957).
22. G. Parisi, Lett. Nuovo Cimento 6, 450 (1973).