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Abstract

An eikonal expansion method of the potential scattering transition amplitude in impact parameter representation is considered and evaluated, without approximation, through third order in the inverse momentum. A sequence of four approximations to the exact impact parameter representation is obtained which consists of the eikonal representation of Glauber and three systematic corrections to the Glauber approximation. The correction terms agree with the form conjectured by Wallace. Numerical results are given for the exponential and Yukawa potential. The sequence of eikonal amplitudes shows systematic improvement at all angles by comparison with partial wave calculations.

I. Introduction

Eikonal or straight line approximations as used in geometrical optics have been successfully applied to high energy quantum scattering problems. The basic idea is that the propagation of a high energy projectile is essentially unaltered in a very smooth potential, so that it can be described as a weakly modulated plane wave. Smoothness of the scattering potential means, in practice, that the scale R on which it changes its value appreciably is large compared to the wave length of the incident particle, i.e. $K \cdot R \gg 1$. Since the eikonal method is applicable to atomic, nuclear and high energy collisions a detailed study of the range of validity of the eikonal approximation is highly desirable.

Among the various versions of the eikonal representation that have been proposed for nonforward scattering the original eikonal amplitude of Glauber [1] has proved to be the simplest and most accurate representation in practical calculations. The eikonal path of Glauber is parallel to the average momentum $\vec{L} = \frac{1}{2}(\vec{K}_i + \vec{K}_f)$. The amplitude for a given momentum transfer $Q = 2K \cdot \sin\frac{\theta}{2}$ takes the form of a two-dimensional Fourier transform

$$T(Q) = \frac{K}{2\pi} \cdot \int d^2\vec{B} \; e^{i\vec{Q}\vec{B}} \cdot T(B) \qquad (1.1)$$

where $B = |\vec{B}|$ is the impact parameter and in the approximation of Glauber

$$T\,(B) \approx T_G\,(B) = i\left[e^{i\chi_0(B)} - 1\right] . \tag{1.2}$$

The phase $\chi_0(B)$ is given by the expression

$$\chi_0(B) = -\frac{M}{K} \cdot \int_{-\infty}^{+\infty} dz\; V(|\vec{X}|) \tag{1.3}$$

where $V(|\vec{X}|)$ is the potential and $\frac{K}{M}$ is the velocity of the incident particle. Due to the particular choice of the straight line path the Glauber amplitude is timereversal invariant and, for spherically symmetric potentials, assumes the form of a Fourier Bessel transform

$$T(Q) = K \int_0^\infty dB \cdot B\; J_0(BQ)\; T(B) \; . \tag{1.4}$$

Abarbanel and Itzykson (2) have suggested the form

$$T_{AI}(B) = i\cos\frac{Q}{2}\left[e^{i\frac{\chi_0(B)}{\cos\frac{Q}{2}}} - 1\right] \tag{1.5}$$

$T_{AI}(B)$ differs from $T_G(B)$ by the $\cos\frac{Q}{2}$ - factors which arise from using a momentum $L = K\cos\frac{Q}{2}$ along Glauber's eikonal path rather than Glauber's K. $T_{AI}(B)$ is not a Fourier-Bessel representation since it still depends on the momentum transfer via $\cos\frac{\Theta}{2} = \sqrt{1 - \frac{Q^2}{4K^2}}$. As $T_{AI}(B)$ varies strongly in the backward region due to the $\cos\frac{\Theta}{2}$ factors the AI-amplitude is inferior to Glauber's at large angles. This will be substantiated numerically.

Several authors have tried to derive corrections to the eikonal amplitude (2,3). Apart from being very complicated to evaluate, these amplitudes, however, turn out to be worse than Glauber's in practical calculations (4).

In a very elaborate paper Wallace (5) has calculated correction terms to the eikonal approximation using the approach developed by Abarbanel and Itzykson. He notes that certain terms which arise from the expansion of $\cos\frac{\Theta}{2}$ in powers of $\frac{Q^2}{K^2}$ cancel with other terms occurring in the eikonal expansion. The remaining correction terms are calculable in practice and are in surprisingly good agreement with the exact amplitude at all angles.

As the findings of Wallace are very involved it is desirable to cast some more light on this promising work from a different point of view. Moore (6) and Swift (7) have shown that if the high energy limit of each order of the Born series at fixed momentum transfer is calculated and the resulting series is summed the eikonal amplitude of Glauber is obtained. The high energy limit implies that $k = K\,R \gg 1$, where R is the range of the potential. In realistic problems nonleading terms in each order of the Born series may prove to be important. In this paper we use k^{-1} as the appropriate expansion parameter and systematically calculate nonleading correction terms up to order k^{-3} in each order of the Born series. The Born series of the correction terms can be summed in closed form so that the transition amplitudes are valid for weak and strong couplings as well. With minor modifications we obtain the corrections proposed by Wallace.

In Section 2 the eikonal expansion method is developed and the first three eikonal corrections to the Glauber amplitude are calculated. In Section 3 numerical tests of the sequence of improved eikonal amplitudes are given for the exponential and Yukawa potential which illustrate the angular range of validity of the theory.

2. Eikonal Expansion

Let us consider the nonrelativistic scattering of a spinless particle of mass M by a spherically symmetric potential $V(|\vec{X}|) = V_0 \cdot U(|\vec{X}|)$ of range R. The energy of the projectile is given by $E = \frac{K^2}{2M}$ where $K = |\vec{K}_i| = |\vec{K}_f|$ is its wave number. We use coordinates $\vec{X} = (\vec{B}, Z)$ where the Z-axis lies half way between the initial and the final direction. The transition amplitude is given by the expression

$$T(Q) = \frac{M}{2\pi} \cdot \int d^2\vec{B} \int d^2\vec{B}' \int dZ \int dZ' \, e^{i\vec{Q}\vec{B}} \, e^{-i\vec{K}_i\vec{X}} \, T(\vec{X},\vec{X}') \, e^{i\vec{K}_i\vec{X}'} , \qquad (2.1)$$

where $\vec{Q} = \vec{K}_i - \vec{K}_f$ is the momentum transfer and $T(\vec{X},\vec{X}')$ fulfills the Lippmann-Schwinger equation

$$T(\vec{X},\vec{X}') = \delta^3(\vec{X}-\vec{X}')\cdot V(|\vec{X}|) + \int d^3\vec{X}''\, V(|\vec{X}|)\, \mathcal{G}(\vec{X},\vec{X}'')\, T(\vec{X}'',\vec{X}') \qquad (2.2)$$

with the free Green's function

$$\mathcal{G}(\vec{X},\vec{X}') = (2\pi)^{-3}\cdot 2M\cdot \int d^3\vec{P}'\, \frac{e^{i\vec{P}'(\vec{X}-\vec{X}')}}{K^2-P'^2+i\varepsilon} \qquad (2.3)$$

It is appropriate to introduce dimensionless variables and parameters by measuring all lengths and wave numbers in units of the range and inverse range of the potential, respectively,

$$z = \frac{Z}{R}\;,\quad \vec{b} = \frac{\vec{B}}{R}\quad;\quad k = K\cdot R\;,\quad \vec{q} = \vec{Q}\cdot R \qquad (2.4)$$

and the dimensionless 'coupling constant'

$$\lambda = \frac{V_0\cdot R\cdot M}{K} = k\cdot\varepsilon\quad;\quad \varepsilon = \frac{V_0}{2E} \qquad (2.5)$$

We rewrite eqs. (2.1) and (2.2) in the form of an impact parameter representation

$$T(q) = \frac{K\cdot R^2}{2\pi}\cdot\int d^2\vec{b}\; e^{i\vec{q}\vec{b}}\cdot f(b) \qquad (2.6)$$

where

$$f(b) = \int dz \int dz' \int d^2\vec{b}' \, f(\vec{x}, \vec{x}') \quad . \tag{2.7}$$

$f(\vec{x}, \vec{x}')$ is the solution of the integral equation

$$f(\vec{x}, \vec{x}') = \lambda\, \mathcal{U}(|\vec{x}'|)\, \delta^3(\vec{x}-\vec{x}') + \lambda\, \mathcal{U}(|\vec{x}'|) \int d^3\vec{x}''\, G(\vec{x}, \vec{x}'')\, f(\vec{x}'', \vec{x}') \tag{2.8}$$

with the modified Green's function

$$G(\vec{x}, \vec{x}') = \frac{K}{M} R^2\, e^{-i \vec{K}_i \vec{X}}\, \mathcal{G}(\vec{X}, \vec{X}')\, e^{i \vec{K}_i \vec{X}'} \quad , \tag{2.9}$$

which by use of the average momentum

$$\vec{\ell} = \frac{1}{2}(\vec{k}_i + \vec{k}_f) \quad ; \quad \ell = \eta \cdot k \, , \quad \eta = \sqrt{1 - \frac{q^2}{4k^2}} \tag{2.10}$$

can be put in the form

$$G(\vec{x}, \vec{x}') = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{k}{\vec{k}_i \cdot \vec{p} - \frac{1}{2}p^2 + i\varepsilon}\, e^{-i\vec{p}(\vec{x}-\vec{x}')} \tag{2.11a}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{k}{\ell \cdot p_z - \frac{1}{2}(\vec{p}-\vec{q}) \cdot \vec{p} + i\varepsilon}\, e^{-i\vec{p}(\vec{x}-\vec{x}')} \quad , \tag{2.11b}$$

where in eq. (2.11b) the z-axis is fixed to the average momentum direction. The iteration of eq. (2.8) leads to the Born series

$$f(\vec{x},\vec{x}') = \sum_{n=0}^{\infty} f_{n+1}(\vec{x},\vec{x}') \tag{2.12}$$

with

$$f_{n+1}(\vec{x},\vec{x}') = \lambda^{n+1} \cdot \langle \vec{x} | \mathcal{U} \, (G\,\mathcal{U})^{n} | \vec{x}' \rangle \, . \tag{2.13}$$

In high energy problems it may be profitable to expand $f_{n+1}(\vec{x},\vec{x}')$ in powers of k^{-1}

$$f_{n+1}(\vec{x},\vec{x}') = \sum_{m=0}^{\infty} f_{n+1}^{(m)}(\vec{x},\vec{x}') \tag{2.14a}$$

$$f^{(m)}(\vec{x},\vec{x}') = \sum_{n=0}^{\infty} f_{n+1}^{(m)}(\vec{x},\vec{x}') \quad , \tag{2.14b}$$

where $f_{n+1}^{(m)}$ is of order $\lambda^{n+1} \cdot k^{-m}$. In the following the leading term and the first three correction terms of eq. (2.14a) are calculated. It will be shown that under the integrals of eqs. (2.6) and (2.7) the summation (2.14b) can be done in closed form so that the resulting amplitude is also reliable for a strong coupling constant λ .

As a first step in this direction the Green's function $G(\vec{x},\vec{x}')$ is expanded in powers of k^{-1}. In order to obtain an impact parameter representation for the transition amplitudes, this expansion is derived from the expression (2.11b) for $G(\vec{x},\vec{x}')$. It reads

$$G(\vec{x},\vec{x}') = \frac{k}{i} \cdot \sum_{m=0}^{\infty} \left(\frac{i}{k}\right)^m \cdot g_m(\vec{x},\vec{x}') \qquad (2.15)$$

where

$$g_m(\vec{x},\vec{x}') = \left(-\frac{i}{2}\right)^m \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\left[(\vec{p}_\perp-\vec{q})\cdot\vec{p}_\perp + p_z^2\right]^m}{(p_z+i\varepsilon)^{m+1}} e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')} \qquad (2.16)$$

By use of the relation

$$\int \frac{dp_z}{(2\pi)} \frac{e^{-i\vec{p}_z\cdot(z-z')}}{(p_z+i\varepsilon)^{m+1}} = (-i)^{m+1} \cdot \frac{(z-z')^m}{m!} \theta(z-z') \qquad (2.17)$$

the lowest terms can be put into the form

$$g_0(\vec{x},\vec{x}') = -i\ \theta(z-z')\,\delta^2(\vec{b}-\vec{b}') \qquad (2.18a)$$

$$g_1(\vec{x},\vec{x}') = -\frac{i}{2}\left[(z-z')\,\theta(z-z')\cdot(\nabla_\perp^2+i\vec{q}\nabla_\perp) + \delta(z-z')\right]\delta^2(\vec{b}-\vec{b}') \qquad (2.18b)$$

$$\begin{aligned} g_2(\vec{x},\vec{x}') = -\frac{i}{4}\Big[&\frac{1}{2}(z-z')^2\theta(z-z')\left(\nabla_\perp^2+i\vec{q}\nabla_\perp\right)^2 \\ &+2\ \theta(z-z')(\nabla_\perp^2+i\vec{q}\nabla_\perp) + \frac{d}{dz}\delta(z-z')\Big]\delta^2(\vec{b}-\vec{b}') \end{aligned} \qquad (2.18c)$$

$$g_3(\vec{x},\vec{x}') = -\frac{i}{8}\Big[\frac{1}{6}(z-z')^3\Theta(z-z')(\nabla_\perp^2+i\vec{q}\nabla_\perp)^3$$
$$+3(z-z')\Theta(z-z')(\nabla_\perp^2+i\vec{q}\nabla_\perp)^2 \tag{2.18d}$$
$$+3\delta(z-z')(\nabla_\perp^2+i\vec{q}\nabla_\perp)$$
$$+\frac{d^2}{dz^2}\delta(z-z')\Big]\cdot\delta^2(\vec{b}-\vec{b}') \quad .$$

In order to get the complete expansion of $G(\vec{x},\vec{x}')$ in powers of k^{-1}, ℓ^{-1} has to be expanded. After collecting like powers of k^{-1} we get from eq. (2.15) the series

$$G(\vec{x},\vec{x}') = \sum_{m=0}^{\infty}\left(\frac{i}{k}\right)^m G_m(\vec{x},\vec{x}') \quad , \tag{2.19}$$

where the lowest terms explicitly read

$$G_0(\vec{x},\vec{x}') = g_0(\vec{x},\vec{x}') \tag{2.20a}$$

$$G_1(\vec{x},\vec{x}') = g_1(\vec{x},\vec{x}') \tag{2.20b}$$

$$G_2(\vec{x},\vec{x}') = g_2(\vec{x},\vec{x}') - \frac{1}{8}\vec{q}^{\,2}\cdot g_0(\vec{x},\vec{x}') \tag{2.20c}$$

$$G_3(\vec{x},\vec{x}') = g_3(\vec{x},\vec{x}') - \frac{1}{4}\vec{q}^{\,2}\cdot g_1(\vec{x},\vec{x}') \quad . \tag{2.20d}$$

At first glance the expressions (2.20c) for $G_2(\vec{x},\vec{x}')$ and (2.20d) for $G_3(\vec{x},\vec{x}')$, which combine $g_2(\vec{x},\vec{x}')$ with $\frac{1}{8}\vec{q}^{\,2}\cdot g_0(\vec{x},\vec{x}')$ and $g_3(\vec{x},\vec{x}')$ with $\frac{1}{4}\vec{q}^{\,2}g_1(\vec{x},\vec{x}')$, seem not to be justified for large $\vec{q}^{\,2}$. It is shown, however, in the following calculations that under the integral in eq. (2.6) those portions which result from $\frac{1}{8}\vec{q}^{\,2} g_0(\vec{x},\vec{x}')$ and $\frac{1}{4}\vec{q}^{\,2}\cdot g_1(\vec{x},\vec{x}')$ completely cancel with certain terms which result from $g_2(\vec{x},\vec{x}')$ and $g_3(\vec{x},\vec{x}')$, respectively. It should be noted that the Green's functions $G_n(\vec{x},\vec{x}')$ are singular functions which always act upon 'smooth' functions in order to be well defined. It is in this sense that $G_n(\vec{x},\vec{x}')$ is 'small' compared to $k\,G_{n-1}(\vec{x},\vec{x}')$ when $k \gg 1$.

Inserting eq. (2.19) in eq. (2.13) we obtain the series (2.14a). The leading term and the first three correction terms are

$$f^{(0)}_{n+1}(\vec{x},\vec{x}') = \lambda^{n+1}\cdot\langle\vec{x}|\,U\,(G_0 U)^n\,|\vec{x}'\rangle \tag{2.21a}$$

$$f^{(1)}_{n+1}(\vec{x},\vec{x}') = \frac{i}{k}\,\lambda^{n+1}\sum_P\langle\vec{x}|\,U G_1 U\,(G_0 U)^{n-1}|\vec{x}'\rangle \tag{2.21b}$$

$$\begin{aligned} f^{(2)}_{n+1}(\vec{x},\vec{x}') = \left(\frac{i}{k}\right)^2\lambda^{n+1}\sum_P\langle\vec{x}|\,[\,&U G_2 U\,(G_0 U)^{n-1} \\ &+ U G_1 U G_1 U\,(G_0 U)^{n-2}\,]\,|\vec{x}'\rangle \end{aligned} \tag{2.21c}$$

$$\begin{aligned} f^{(3)}_{n+1}(\vec{x},\vec{x}') = \left(\frac{i}{k}\right)^3\lambda^{n+1}\sum_P\langle\vec{x}|\,[\,&U G_3 U\,(G_0 U)^{n-1} \\ &+ U G_2 U G_1 U (G_0 U)^{n-2} + U(G_1 U)^3 (G_0 U)^{n-3}\,]\,|\vec{x}'\rangle, \end{aligned} \tag{2.21d}$$

where $\sum_P$ denotes the sum over all different permutations of the Green's functions G_n. The corresponding transition amplitudes are

$$T_{n+1}^{(m)}(q) = K R^2 \cdot \int \frac{d^2\vec{b}}{(2\pi)} e^{i\vec{q}\vec{b}} \cdot f_{n+1}^{(m)}(b)$$

$$= K R^2 \int \frac{d^2\vec{b}}{(2\pi)} e^{i\vec{q}\vec{b}} \cdot \int d^2\vec{b}' \int dz \int dz' \, f_{n+1}^{(m)}(\vec{x},\vec{x}') \, . \tag{2.22}$$

The leading order (2.21a) yields the expression

$$T_{n+1}^{(0)}(q) = K \cdot R^2 \cdot \int \frac{d^2\vec{b}}{(2\pi)} e^{i\vec{q}\vec{b}} \cdot i\,(-i\lambda)^{n+1}$$

$$\cdot \int_{-\infty}^{+\infty} dz_1 \cdots \int_{-\infty}^{+\infty} dz_{n+1}\, U(b,z_1)\,\theta(z_1-z_2)\,U(b,z_2) \cdots \theta(z_n - z_{n+1})\, U(b,z_{n+1}) \, , \tag{2.23}$$

from which one obtains

$$f_{n+1}^{(0)}(b) = i \, \frac{(i\chi_0(b))^{n+1}}{(n+1)!} \tag{2.24}$$

where

$$\chi_0(b) = -\lambda \int_{-\infty}^{+\infty} dz \; U(b,z) \qquad . \tag{2.25}$$

Clearly, the summation over n with the expression (2.24) yields the Glauber amplitude (1.2).

In the calculation of the correction terms the $\vec{q}$-dependence of the Green's functions g_n can always be dropped by integration by part

$$\int d^2\vec{b}\, e^{i\vec{q}\vec{b}}\, v(b)\cdot(\nabla_\perp^2 + i\vec{q}\nabla_\perp)\, u(b) = -\int d^2\vec{b}\, e^{i\vec{q}\vec{b}} (\nabla_\perp v(b))(\nabla_\perp u(b)) \quad . \quad (2.26)$$

a) <u>First Eikonal Correction</u>

The Green's function $G_1(\vec{x}, \vec{x}')$ can be inserted in n possible ways into $f^{(1)}_{n+1}$. Employing the notation

$$\int_{-\infty}^{z} dz_1\, U(b,z_1) \int_{-\infty}^{z_1} dz_2\, U(b,z_2) \cdots \int_{-\infty}^{z_{\ell-1}} dz_\ell\, U(b,z_\ell)$$

$$= \frac{1}{\ell!} \left[\int_{-\infty}^{z} dz_1\, U(b,z_1) \right]^\ell = \frac{1}{\ell!} \left(\tau_+(b,z) \right)^\ell \qquad (2.27)$$

and

$$\int_{z}^{\infty} dz_1\, U(b,z_1) \int_{z_1}^{\infty} dz_2\, U(b,z_2) \cdots \int_{z_{\ell-1}}^{\infty} dz_\ell\, U(b,z_\ell)$$

$$= \frac{1}{\ell!} \left[\int_{z}^{\infty} dz_1\, U(b,z_1) \right]^\ell = \frac{1}{\ell!} \left(\tau_-(b,z) \right)^\ell \qquad (2.28)$$

where

$$\tau_+(b,z) + \tau_-(b,z) = \int_{-\infty}^{+\infty} dz_1\, U(b,z_1) = -\frac{1}{i} \chi_0(b) \quad , \qquad (2.29)$$

the first eikonal correction takes the form

$$T_{n+1}^{(1)}(q) = K\cdot R^2 (-i)^{n-1} \lambda^{n+1} \cdot \left(\frac{i}{k}\right) \cdot \int \frac{d^2\vec{b}}{(2\pi)} e^{i\vec{q}\vec{b}} \int d^2\vec{b}' \int dz \int dz' \cdot$$

$$\cdot \sum_{\ell=1}^{n} \frac{1}{(\ell-1)!}\,\frac{1}{(n-\ell)!} \cdot \left(\tau_+(b,z)\right)^{\ell-1} \mathcal{U}(b,z)\, g_1(\vec{x},\vec{x}')\,\mathcal{U}(b',z')\left(\tau_-(b',z')\right)^{n-\ell}, \qquad (2.30)$$

from which by use of the relation

$$\left(\tau_\pm(b,z)\right)^{\ell} \mathcal{U}(b,z) = \pm\frac{1}{(\ell+1)} \cdot \frac{d}{dz}\left(\tau_\pm(b,z)\right)^{\ell+1} \qquad (2.31)$$

and eqs. (2.18b) and (2.26) follows

$$T_{n+1}^{(1)}(q) = K R^2 i (-i\lambda)^{n+1} \cdot \left(\frac{i}{k}\right)\frac{1}{2} \cdot \int \frac{d^2\vec{b}}{(2\pi)} e^{i\vec{q}\vec{b}} \cdot \int dz \int dz'$$

$$\cdot \sum_{\ell=1}^{n} \frac{1}{(\ell-1)!}\,\frac{1}{(n-\ell)!} \Big\{ \left(\tau_+(b,z)\right)^{\ell-1} \cdot \left(\tau_-(b,z')\right)^{n-\ell} \mathcal{U}(b,z)\,\mathcal{U}(b,z')\,\delta(z-z') \qquad (2.32)$$

$$+ \frac{1}{\ell\cdot(n+1-\ell)} \left[\frac{d}{db}\frac{d}{dz}\left(\tau_+(b,z)\right)^{\ell}\right](z-z')\,\Theta(z-z')\left[\frac{d}{db}\frac{d}{dz'}\left(\tau_-(b,z')\right)^{n+1-\ell}\right]\Big\}.$$

After performing an integration by part in the z-variables the sum in eq. (2.32) is of binomial expansion type and can be carried out with the result

$$f_{n+1}^{(1)}(b) = i\,\frac{\left(i\,\chi_0(b)\right)^{n-1}}{(n-1)!} \cdot i\,\chi_1(b) \qquad , \qquad (2.33)$$

where

$$\chi_{,1}(b) = -\frac{1}{2}\,\frac{\lambda^2}{k}\cdot\int dz\left[\,\mathcal{U}^2(b,z) + z\frac{d}{dz}\left(\frac{d\tau_+(b,z)}{db}\cdot\frac{d\tau_-(b,z)}{db}\right)\right] \tag{2.34}$$

By use of eq. (2.31) with $\ell = 0$ and the relation

$$z\,\frac{d}{db}\,\mathcal{U}(r) = b\frac{d}{dz}\,\mathcal{U}(r)\;, \tag{2.35}$$

which holds for spherically symmetric potentials, and integration by part we finally obtain for the first phase correction

$$\chi_{,1}(b) = -\frac{1}{k}\lambda^2\,\frac{1}{2}\left(1 + b\,\frac{d}{db}\right)\int_{-\infty}^{+\infty} dz\,\mathcal{U}^2(b,z)\;. \tag{2.36}$$

b) Second Eikonal Correction

Second order corrections in $f_{n+1}(\vec{x},\vec{x}')$ arise from terms with one G_2 -function and two G_1-functions. A new type of correction terms already appears in the second Born approximation

$$f_2^{(2)}(\vec{x},\vec{x}') = \left(\frac{1}{k}\right)^2\lambda^2\,\mathcal{U}(|\vec{x}|)\;G_2(\vec{x},\vec{x}')\;\mathcal{U}(|\vec{x}'|) \tag{2.37}$$

Inserting the explicit form of $G_2(\vec{x},\vec{x}')$ and using twice the eq. (2.26) and the relation

$$\left(\partial_i \partial_j u(b)\right)\left(\partial_i \partial_j v(b)\right) = \frac{d^2 u(b)}{db^2} \cdot \frac{d^2 v(b)}{db^2} + \frac{1}{b^2} \frac{du(b)}{db} \frac{dv(b)}{db} \tag{2.38}$$

one obtains for the transition amplitude

$$\begin{aligned} T_2^{(2)}(q) = K \cdot R^2 \cdot i \frac{\lambda^2}{4k^2} \cdot \int \frac{d^2\vec{b}}{2\pi} e^{i\vec{q}\vec{b}} \int dz \int dz' \\ \cdot \Big[\frac{1}{2}(z-z')^2 \Big(\frac{d^2 U(b,z)}{db^2} \cdot \frac{d^2 U(b,z')}{db^2} + \frac{1}{b^2} \frac{dU(b,z)}{db} \frac{dU(b,z')}{db} \Big) \Theta(z-z') \\ - 2 \frac{dU(b,z)}{db} \frac{dU(b,z')}{db} \Theta(z-z') + U(b,z) \frac{dU(b,z)}{dz} \delta(z-z') \\ - \frac{1}{2} \vec{q}^{\,2} U(b,z) \Theta(z-z') U(b,z') \Big] \end{aligned} \tag{2.39}$$

The z and z' factors can be removed from eq. (2.39) by use of the following relation (and a corresponding relation in the z' variable)

$$\begin{aligned} \int_{-\infty}^{+\infty} dz \, z^n \frac{d^m u(b,z)}{db^m} \Theta(z-z') = -\int_{-\infty}^{+\infty} dz \Big[(n-1) z^{n-2} \frac{d^{m-1}(b\,u(b,z))}{db^{m-1}} \Theta(z-z') \\ + z^{n-1} \frac{d^{m-1}(b\,u(b,z))}{db^{m-1}} \delta(z-z') \Big] , \end{aligned} \tag{2.40}$$

which holds for a spherically symmetric function $u\left(\sqrt{b^2+z^2}\right)$ and follows from eq. (2.35) and an integration by part. The result is

$$f_2^{(2)}(b) = i \Big[-w_2(b) + \frac{1}{8k^2} (\vec{q}^{\,2} + \nabla_\perp^2) \frac{(i\chi_0(b))^2}{2!} \Big] , \tag{2.41}$$

where

$$\omega_2(b) = \frac{1}{8k^2}\; b\; \frac{d\chi_0(b)}{db} \cdot \nabla_\perp^2 \chi_0(b) \quad . \tag{2.42}$$

The second term in eq. (2.41) does not contribute to the scattering amplitude as it involves the null operator of the Fourier transform, that is

$$\int d^2\vec{b}\; e^{i\vec{q}\vec{b}}\,(\nabla_\perp^2 + \vec{q}^{\,2})\, h(b) = 0 \quad , \tag{2.43}$$

which holds if $\nabla_\perp^2 h(b)$ satisfies the usual conditions for Fourier transformation.

The second term in eq. (2.41) displays the cancellation of that portion which results from $\frac{1}{8}\vec{q}^{\,2} g_c(\vec{x},\vec{x}')$ with certain terms which result from $g_2(\vec{x},\vec{x}')$. From this we conclude that, even at large $\vec{q}^{\,2}$, $g_2(\vec{x},\vec{x}')$ can <u>not</u> be regarded as small compared with $\frac{1}{8}\vec{q}^{\,2}\cdot g_c(\vec{x},\vec{x}')$.

The second correction term of the third order of the Born series reads

$$\begin{aligned} f_3^{(2)}(\vec{x},\vec{x}') = \left(\frac{i}{\hbar}\right)^2 \lambda^3 \int d^3x'' \, U(|\vec{x}|)\Big[& G_0(\vec{x},\vec{x}'')\, U(|\vec{x}''|)\, G_2(\vec{x}'',\vec{x}') \\ & + G_2(\vec{x},\vec{x}'')\, U(|\vec{x}''|)\, G_0(\vec{x}'',\vec{x}') + G_1(\vec{x},\vec{x}'')\, U(|\vec{x}''|)\, G_1(\vec{x}'',\vec{x}')\Big] U(|\vec{x}'|) \end{aligned} \tag{2.44}$$

which under the integrals in eq. (2.22) and by use of eqs. (2.26) and (2.38) can be written in a form which contains besides Θ- and

δ -functions factors of the form $z^n \cdot \frac{d^m U(b,z)}{d b^m}$. It is a straightforward but rather lengthy calculation to remove all z-factors from $f_3^{(2)}(b)$ by use of eq. (2.40). The result is

$$f_3^{(2)}(b) = i\left[i\chi_2(b) - i\chi_0(b)\,\omega_2(b) + \frac{(\nabla_\perp^2 + \vec{q}^{\,2})}{4k^2} \frac{(i\chi_0(b))^3}{3!} \right], \tag{2.45}$$

where $\omega_2(b)$ is defined in eq. (2.42) and

$$\chi_2(b) = -\frac{1}{k^2}\lambda^3 \frac{1}{2}\left(1 + \frac{5}{3} b \frac{d}{db} + \frac{1}{3} b^2 \frac{d^2}{db^2}\right)\int_{-\infty}^{+\infty} dz\, U^3(b,z) - \frac{1}{24k^2} b \left(\frac{d\chi_0(b)}{db}\right)^3. \tag{2.46}$$

Finally a strict analysis of the second correction of the (n+1)-th order of the Born series, similar to eq. (2.32), leads to the expression

$$\begin{aligned} f_{n+1}^{(2)}(b) = i\Big[& i\chi_2(b) \frac{(i\chi_0(b))^{n-2}}{(n-2)!} - \omega_2(b) \frac{(i\chi_0(b))^{n-1}}{(n-1)!} \\ & + \frac{(i\chi_1(b))^2}{2!} \frac{(i\chi_0(b))^{n-3}}{(n-3)!} + \frac{n}{8k^2}(\nabla_\perp^2 + \vec{q}^{\,2}) \frac{(i\chi_0(b))^{n+1}}{(n+1)!} \Big], \end{aligned} \tag{2.47}$$

where the last term displays that typical cancellations occur in each order of the Born series.

c) Third Eikonal Correction

Due to the increased complexity of the third correction the calculation of $f_{n+1}^{(3)}(b)$ is rather extensive but straightforward. New types of phase corrections only appear in the second, third and fourth Born approximation. By use of eqs. (2.26), (2.38) and the equation

$$\left(\partial_i \partial_j \partial_k u(b)\right)\left(\partial_i \partial_j \partial_k v(b)\right) = \frac{d^3 u(b)}{db^3}\,\frac{d^3 v(b)}{db^3} + \frac{3}{b^2}\,\frac{d^2 u(b)}{db^2}\,\frac{d^2 v(b)}{db^2} + \frac{3}{b^3}\left(\frac{1}{b} - \frac{d}{db}\right)\left(\frac{du(b)}{db}\,\frac{dv(b)}{db}\right) \tag{2.48}$$

we obtain

$$\begin{aligned} f_2^{(3)}(b) &= i\left[\, i\,\varphi_3(b) + \frac{1}{4k^2}\left(\vec{q}^{\,2} + \nabla_\perp^2\right)\left(i\chi_1(b)\right)\right] \\ f_3^{(3)}(b) &= i\left[\left(i\chi_0(b)\right)\left(i\varphi_3(b)\right) - \omega_3(b) + \frac{1}{4k^2}\left(\vec{q}^{\,2} + \nabla_\perp^2\right)\left(i\chi_0(b)\right)\left(i\chi_1(b)\right)\right. \\ f_4^{(3)}(b) &= i\left[\frac{\left(i\chi_0(b)\right)^2}{2!}\left(i\varphi_3(b)\right) + \left(i\chi_0(b)\right)\left(-\omega_3(b)\right) + i\chi_3(b)\right. \\ &\qquad \left. + \frac{1}{4k^2}\left(\vec{q}^{\,2} + \nabla_\perp^2\right)\frac{\left(i\chi_0(b)\right)^2}{2!}\left(i\chi_1(b)\right)\right] \end{aligned} \tag{2.49}$$

with the phase corrections of third order

$$\varphi_3(b) = -\frac{1}{k^3}\,\lambda^2\,\frac{1}{8}\left(1 + \frac{5}{3}\,b\frac{d}{db} + \frac{1}{3}\,b^2\frac{d^2}{db^2}\right)\int_{-\infty}^{+\infty} dz \left(\frac{d\,U(r)}{dr}\right)^2 \tag{2.50}$$

$$w_3(b) = \frac{1}{k^2}\,\frac{1}{8}\,b\left[\frac{d\chi_0(b)}{db}\nabla_\perp^2\chi_1(b) + \frac{d\chi_1(b)}{db}\nabla_\perp^2\chi_0(b)\right] \tag{2.51}$$

$$\chi_3(b) = -\frac{1}{k^3}\,\lambda^4\,\frac{5}{8}\left(1 + \frac{11}{5}b\frac{d}{db} + \frac{4}{5}b^2\frac{d^2}{db^2} + \frac{1}{15}b^3\frac{d^3}{db^3}\right)\int_{-\infty}^{+\infty} dz\, U^4(b,z)$$
$$- \frac{1}{8k^2}\,b\,\frac{d\chi_1(b)}{db}\cdot\left(\frac{d\chi_0(b)}{db}\right)^2 \quad . \tag{2.52}$$

With these phase corrections the third correction to the (n+1)-th order of the Born series is expressed by

$$f_{n+1}^{(3)}(b) = i\left[\frac{(i\chi_0(b))^{n-1}}{(n-1)!}(i\,\vartheta_3(b)) + \frac{(i\chi_0(b))^{n-2}}{(n-2)!}(-w_3(b))\right.$$
$$+\frac{(i\chi_0(b))^{n-3}}{(n-3)!}(i\chi_3(b)) + \frac{(i\chi_0(b))^{n-4}}{(n-4)!}(i\chi_1(b))(i\chi_2(b)) \tag{2.53}$$
$$\left.+\frac{(i\chi_0(b))^{n-5}}{(n-5)!}\,\frac{(i\chi_1(b))^3}{3!} + \frac{1}{4k^2}(\vec{q}^{\,2}+\nabla_\perp^2)\,\frac{(i\chi_0(b))^{n-1}}{(n-1)!}(i\chi_1(b))\right] .$$

Again typical cancellations under the integral in eq. (2.22) occur.

d) <u>Summing the Born series</u>

An important result of the foregoing analysis is that the impact parameter amplitude $f_n^{(m)}(b)$ differs from the Fourier Bessel transform $T_n^{(m)}(b)$ of $T_n^{(m)}(q)$ only by terms which involve the null operator $\nabla_\perp^2+\vec{q}^{\,2}$ of the Fourier transform (2.43). This behaviour was confirmed in a few leading terms. There is, however, a conjecture of Wallace [5] that it is maintained in general, i.e.,

$$f^{(m)}(b) = T^{(m)}(b) + \sum_r (\nabla_\perp^2+\vec{q}^{\,2})^r h_r^{(m)}(b) \tag{2.54}$$

for arbitrary m.

The Born series (2.14b) of the correction terms (2.33), (2.47) and (2.53) can be summed in closed form

$$\begin{aligned}
T^{(1)}(b) &= i\left[e^{i\chi_0(b)}(i\chi_1(b)) - 1\right]\\
T^{(2)}(b) &= i\left[e^{i\chi_0(b)}\left(i\chi_2(b) - \omega_2(b) + \frac{(i\chi_1(b))^2}{2!}\right) - 1\right]\\
T^{(3)}(b) &= i\left[e^{i\chi_0(b)}\left(i\chi_3(b) - \omega_3(b) + i f_3(b)\right.\right.\\
&\quad \left.\left. + (i\chi_1(b))(i\chi_2(b) - \omega_2(b)) + \frac{(i\chi_1(b))^3}{3!}\right) - 1\right]
\end{aligned} \tag{2.55}$$

.

As all necessary cross terms are present, exponentiation of the various phase corrections is suggested. However in the calculation of special terms of the fourth order eikonal correction $f_{n+1}^{(4)}(b)$ one finds the term $-i\,\frac{1}{2}(\omega_2(b))^2\,\frac{(i\chi_0(b))^{n-3}}{(n-3)!}$, with a minus sign, which would suggest the factor $\sqrt{1-2\omega_2(b)}$ instead of $e^{-\omega_2(b)}$. Indeed, this behaviour is confirmed by considering the classical limit $(\hbar\to 0,\ K\to\infty)$ of the scattering amplitude [8]. Thus we are led to the three phase corrected eikonal amplitudes

$$T_i(q) = k R^2 \int_0^\infty db \cdot b \, J_0(bq) \, T_i(b) \quad , \tag{2.56}$$

where

$$T_I(b) = i \left[e^{i(\chi_0(b) + \chi_1(b))} - 1 \right] \tag{2.57}$$

$$T_{II}(b) = i \left[e^{i(\chi_0(b)+\chi_1(b)+\chi_2(b))} \cdot \sqrt{1 - 2\omega_2(b)} - 1 \right] \tag{2.58}$$

$$T_{III}(b) = i \left[e^{i(\chi_0(b)+\chi_1(b)+\chi_2(b)+\chi_3(b)+\beta_3(b))} \cdot \sqrt{1 - 2(\omega_2(b)+\omega_3(b))} - 1 \right] , \tag{2.59}$$

which systematically add k^{-1}-, k^{-2}- and k^{-3}-corrections to the eikonal amplitude.

3. Numerical Comparison of Eikonal and Exact Amplitudes

Analytic forms of the eikonal phase and various phase corrections can be obtained for exponential, Yukawa and Gauss potential. The phase corrections explicitly vanish for the Coulomb potential. It is a simple task to compute the impact parameter integral (2.56)

with the Glauber approximation $T_G(b)$ and the improved representations $T_I(b)$, $T_{II}(b)$ and $T_{III}(b)$.
In this section we compare the resulting amplitudes with partial wave calculations for the exponential and Yukawa potentials. We choose a system of units where $\hbar = c = 2M = 1$.

a) Exponential Potential

For the exponential potential

$$V(|\vec{x}|) = V_0 \cdot e^{-\frac{|\vec{x}|}{R}} = V_0 \, e^{-\sqrt{b^2+z^2}} \tag{3.1}$$

the various phases are expressed by standard integrals (9). The Glauber phase is

$$\chi_0(b) = -2\lambda \, b \, K_1(b) \quad , \tag{3.2}$$

where $K_n(b)$ is the K-Besselfunction of n-th order, and the phase corrections are

$$\chi_1(b) = -\frac{\lambda^2}{k}\left(b\,K_1(2b) - 2b^2 \cdot K_0(2b)\right)$$

$$\chi_2(b) = -\frac{\lambda^3}{k^2}\left((1+3b^2)\, b\, K_1(3b) - 6b^2 \cdot K_0(3b) + \frac{1}{3} b^4 \left(K_0(b)\right)^3\right)$$

$$\chi_3(b) = -\frac{\lambda^4}{k^3}\Big(\frac{5}{4}\, b\, K_1(4b) - 15 b^2 K_0(4b) + \frac{52}{3} b^3 K_1(4b) - \frac{16}{3} b^4 K_0(4b) + (3b\,K_0(2b) - 2b^2 K_1(2b))\, b^3 (K_0(b))^2\Big)$$

$$\omega_2(b) = \frac{1}{2}\frac{\lambda^2}{k^2}\Big(2b^2 (K_0(b))^2 - b^3 K_0(b)\, K_1(b)\Big) \qquad (3.3)$$

$$\omega_3(b) = \frac{1}{4}\frac{\lambda^3}{k^3}\, b^2\Big(24 K_0(b) K_0(2b) + 8b\, K_0(b) K_0(2b) - 28b\, K_0(b) K_1(2b) - 6b\, K_0(2b) K_1(b) + 4b^2 K_1(b) K_1(2b)\Big)$$

$$\varphi_3(b) = -\frac{\lambda^2}{k^3}\,\frac{1}{4}\, b\Big(K_1(2b) - 4b\, K_0(2b) + \frac{4}{3} b^2 \cdot K_1(2b)\Big).$$

In our first numerical example we choose the parameters $V_0 = -12$, $R^{-1} = 1.45$, $K = 5$ which are those of Berriman and Castillejo [4]. However our amplitude differs from their by a factor of $(2\pi)^2 \cdot M$. The expansion parameters are

$$\lambda = -\frac{24}{29} \quad ; \quad k^{-1} = 0.29 \quad , \qquad (3.4)$$

Berriman and Castillejo showed that for these parameters the simple Glauber amplitude is not worse than the correction amplitudes of Saxon and Schiff [3] and Blankenbecler and Sugar. In figs. 1 and 2 the Glauber amplitude T_G and the phase corrected amplitudes T_I ,

T_{II} and T_{III} are compared to the amplitude of Abarbanel and Itzykson (2) and the partial wave calculations T_{ex}. The figures clearly show the bad behaviour of T_{AI} at large angles. The phase corrected amplitudes are systematically improved at large angles and converge to T_{ex} for both real and imaginary part of the amplitude.

A similar comparison is made in figs. 3 and 4 for the parameters

$$\lambda = -\frac{24}{29} \quad ; \quad k^{-1} = 0.58 \quad . \tag{3.5}$$

Even in this case where k is not very large compared to 1 the convergence of the phase corrected amplitudes to the exact amplitude is surprisingly good.

b) Yukawa Potential

The eikonal phase and phase corrections of the Yukawa potential

$$V(|\vec{x}|) = V_0 \cdot \frac{e^{-\frac{|\vec{x}|}{R}}}{\frac{|\vec{x}|}{R}} := V_0 \frac{e^{-\sqrt{b^2+z^2}}}{\sqrt{b^2+z^2}} \tag{3.6}$$

are expressed by standard integrals. One obtains (5)

$$\chi_0(b) = -2\lambda \cdot K_0(b)$$

$$\chi_1(b) = 2\frac{\lambda^2}{k} K_0(2b)$$

$$\chi_2(b) = -3\,\frac{\lambda^3}{k^2}\left(K_0(3b) - \frac{1}{3b}K_1(3b) + \frac{b}{9}\left(K_1(b)\right)^3\right)$$

$$\chi_3(b) = \frac{\lambda^4}{k^3}\left(\frac{16}{3}K_0(4b) - \frac{4}{b}K_1(4b) + 2b\,K_1(2b)\left(K_1(b)\right)^2\right) \tag{3.7}$$

$$\omega_2(b) = -\frac{1}{2}\,\frac{\lambda^2}{k^2}\;b\,K_1(b)\,K_0(b)$$

$$\omega_3(b) = \frac{\lambda^3}{k^3}\cdot b\left(K_0(b)\,K_1(2b) + 2K_0(2b)\cdot K_1(b)\right)$$

$$\varphi_3(b) = -\frac{1}{6}\,\frac{\lambda^2}{k^3}\left(K_0(2b) + 2b\,K_1(2b)\right) .$$

Figs. 5-8 show the real and imaginary parts of the various eikonal scattering amplitudes corresponding to the expansion parameters

$$\lambda = -0.5 \quad , \quad k^{-1} = 0.2 \tag{3.8}$$

and

$$\lambda = -1 \quad , \quad k^{-1} = 0.2 \ . \tag{3.9}$$

The parameters are those of Byron, Joachain and Mund (10) in their figures 1-4. The examples are high energy problems for which the Born approximation does not dominate. For comparison the amplitudes T_{B2} given by the second Born approximation are displayed which yield a very poor result. The phase corrected amplitudes converge very well to the exact amplitudes for both real and imaginary part of the amplitude. The already quite good Glauber approximation is systematically improved at all angles.

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Figure Captions

Fig. 1. The real parts of the scattering amplitudes for an exponential potential of the form given in eq. (3.1) with $V_0 = -12$, $R^{-1} = 1.45$ and $K = 5$ $(\lambda = -\frac{24}{29}, k^{-1} = 0.29)$. T_{AI} = Abarbanel-Itzykson, T_{EX} = partial wave, T_G, T_I, T_{II} and T_{III} = Glauber and phase corrected eikonal amplitudes.

Fig. 2. The imaginary parts of the amplitudes of Fig. 1.

Fig. 3. The real parts of the scattering amplitudes for an exponential potential with $V_0 = -6$, $R^{-1} = 1.45$ and $K = 2.5$ $(\lambda = -\frac{24}{29}, k^{-1} = 0.58)$.

Fig. 4. The imaginary parts of the amplitudes of Fig. 3.

Fig. 5. The real parts of the scattering amplitudes for a Yukawa potential of the form given in eq. (3.6) with $V_0 = -5$, $R = 1$ and $K = 5$ $(\lambda = -0.5, k^{-1} = 0.2)$. T_{B2} = second Born approximation, T_{ex} = partial wave, T_G, T_I, T_{II} = Glauber and phase corrected eikonal amplitudes.

Fig. 6. The imaginary parts of the amplitudes of Fig. 5.

Fig. 7. The real parts of the scattering amplitudes for an exponential potential with $V_0 = -10$, $R = 1$ and $K = 5$ $(\lambda = -1, k^{-1} = 0.2)$.

Fig. 8. The imaginary parts of the amplitudes of Fig. 7.

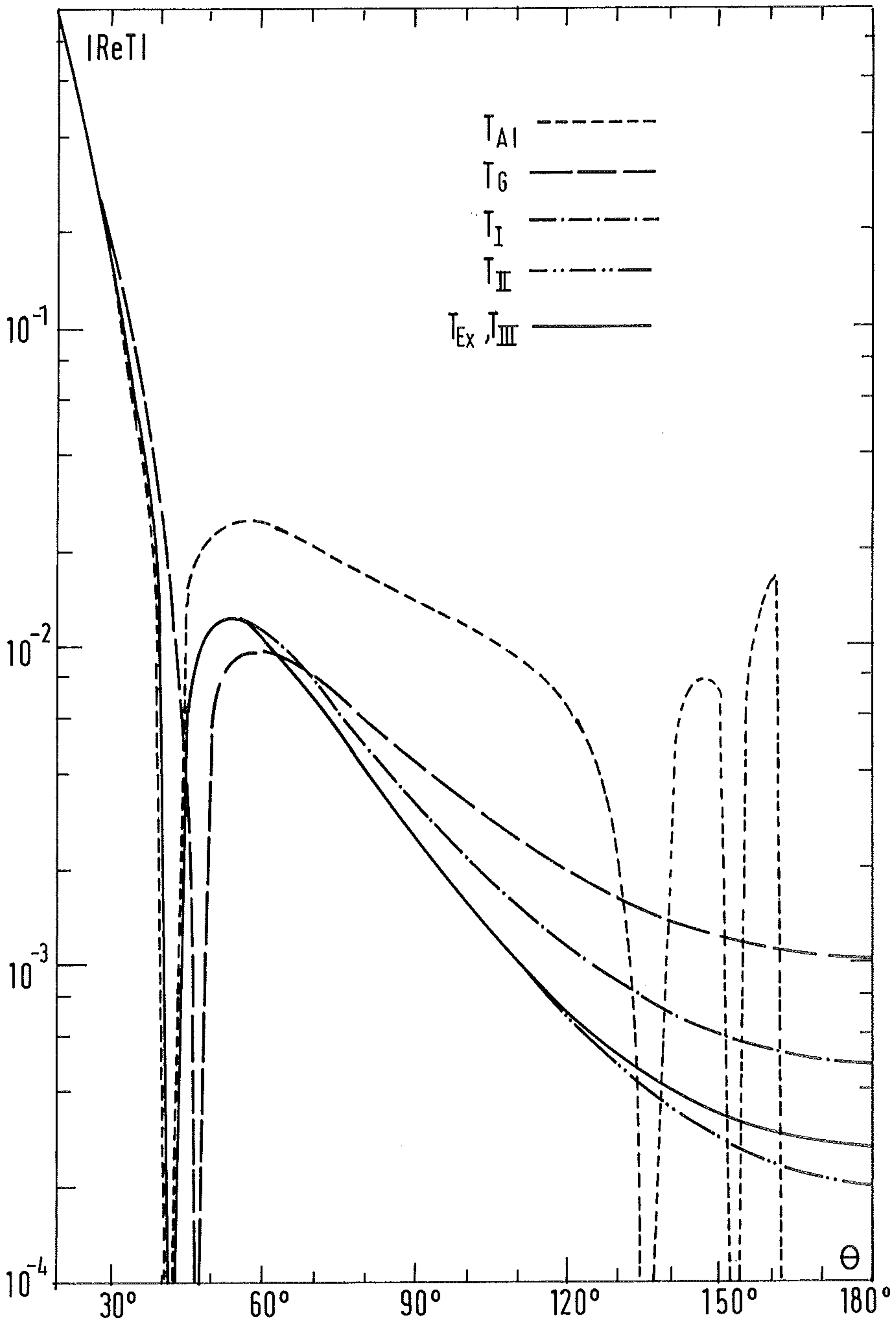

Fig. 1

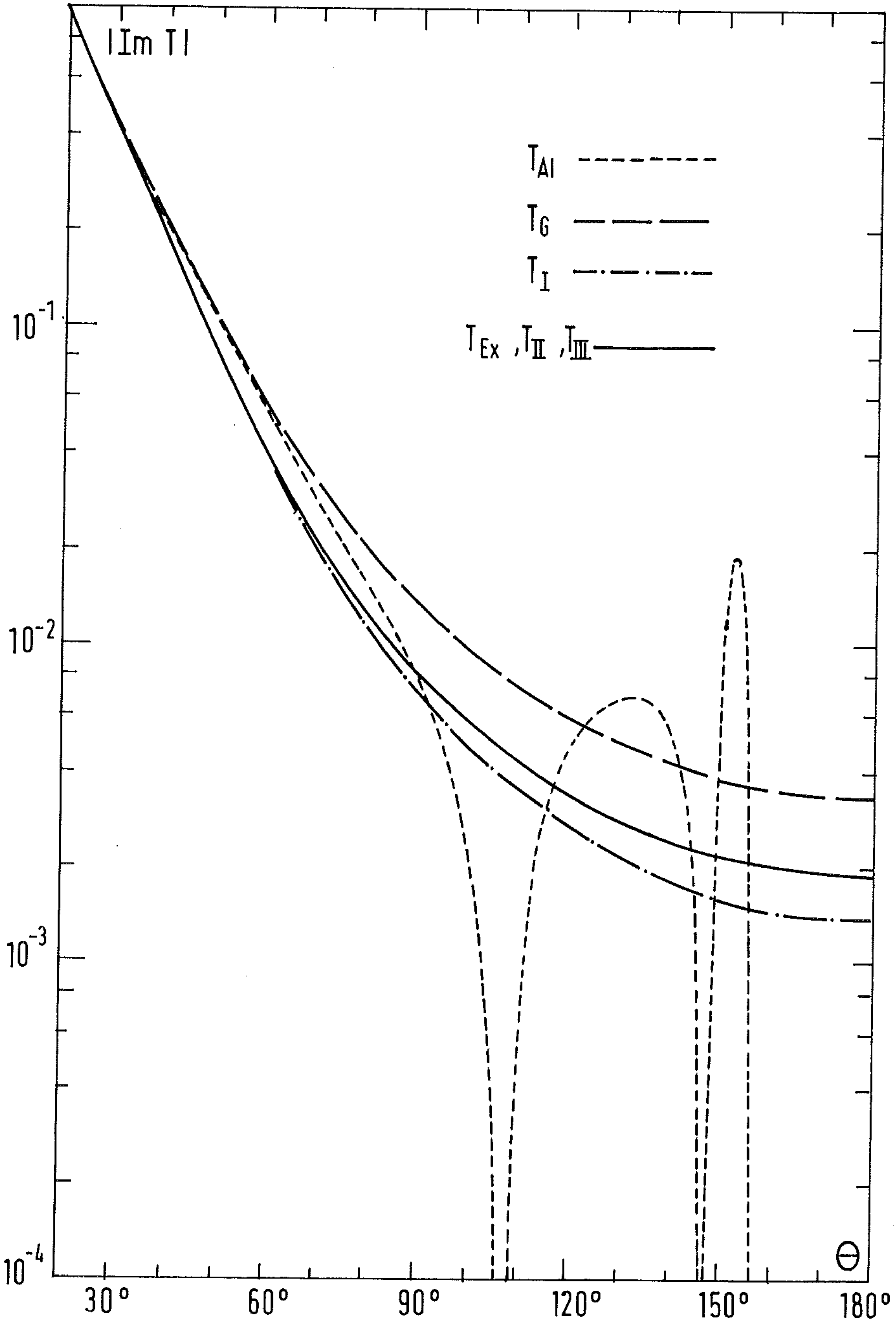

Fig. 2

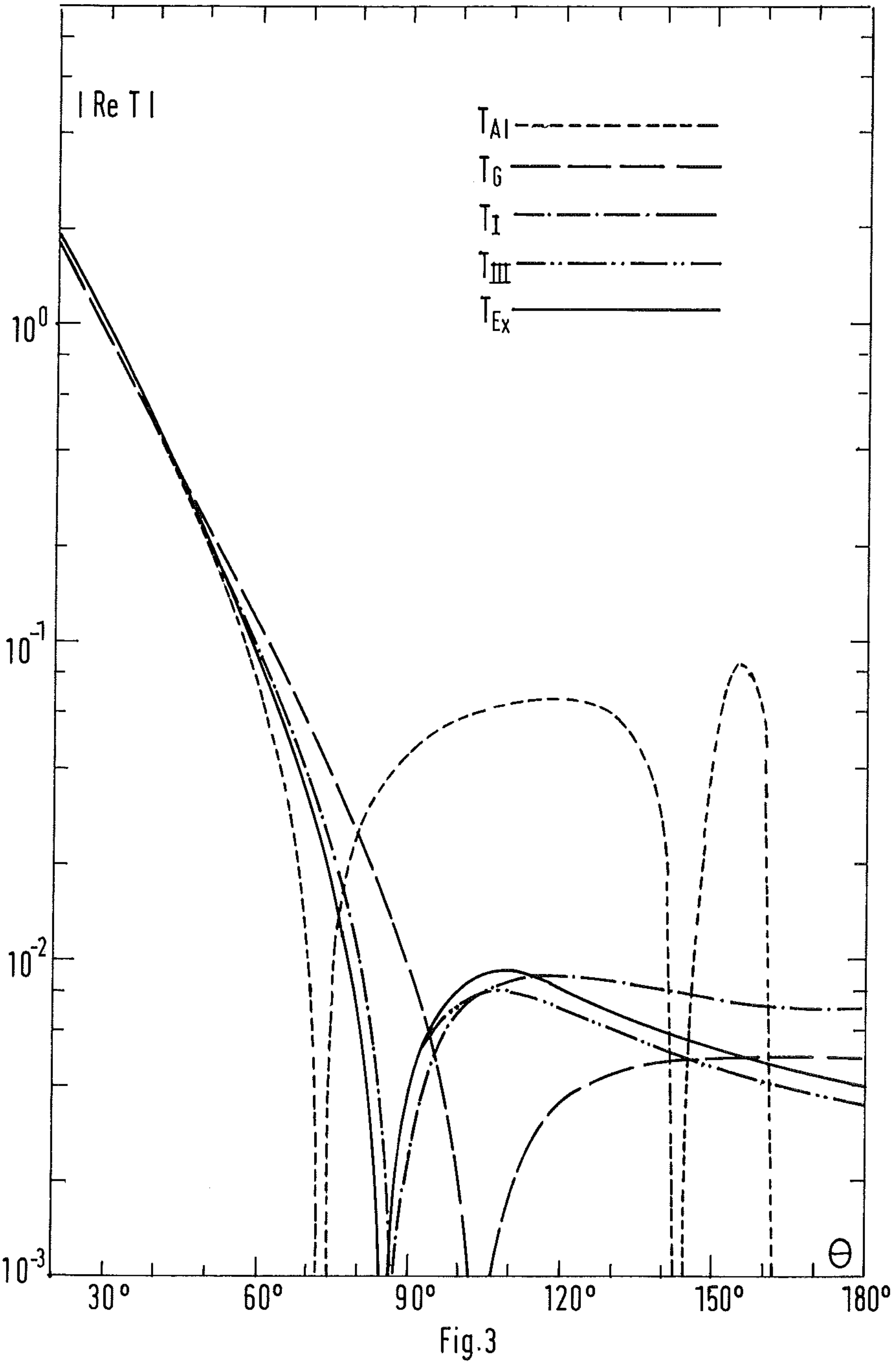

Fig.3

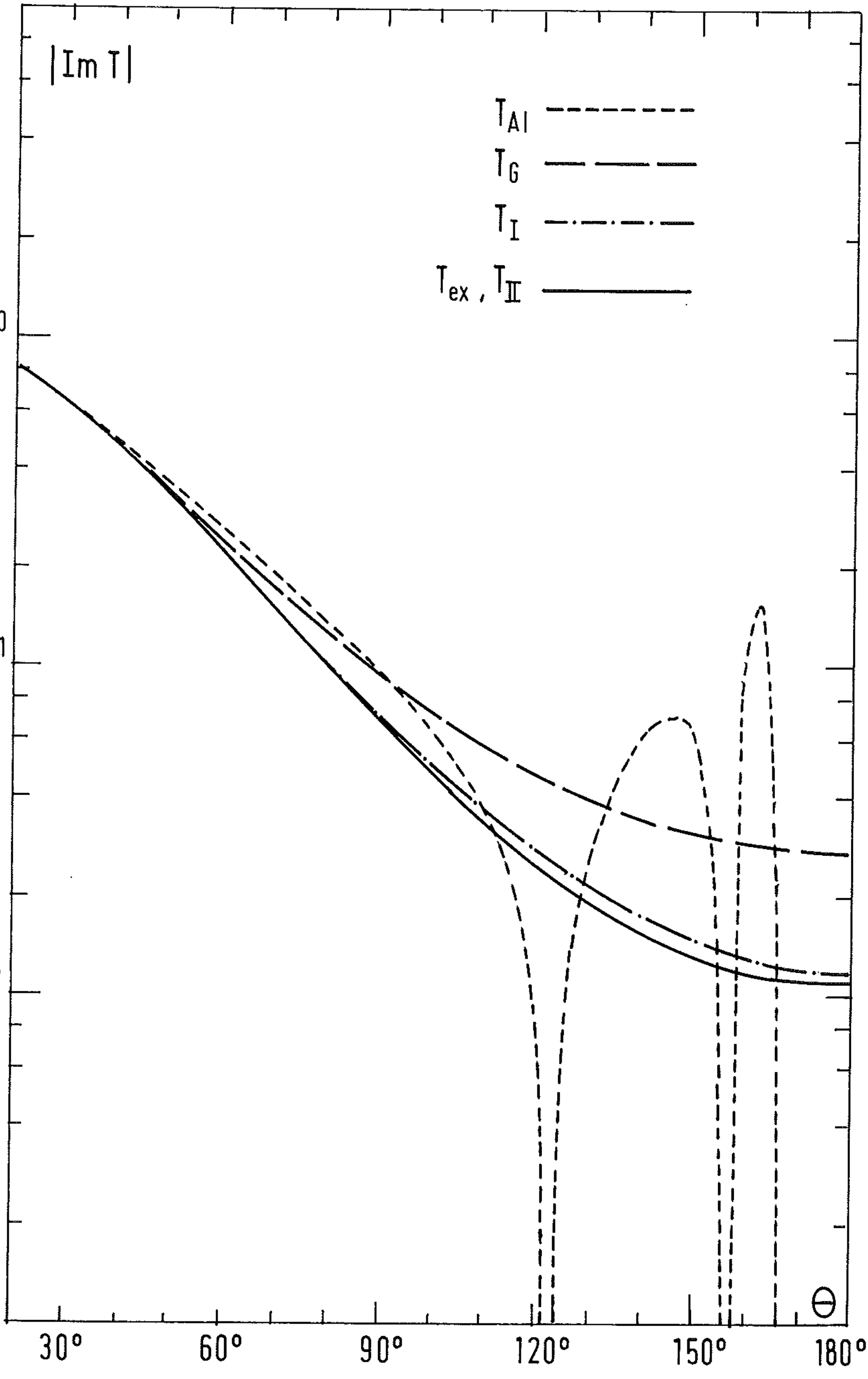

Fig.4

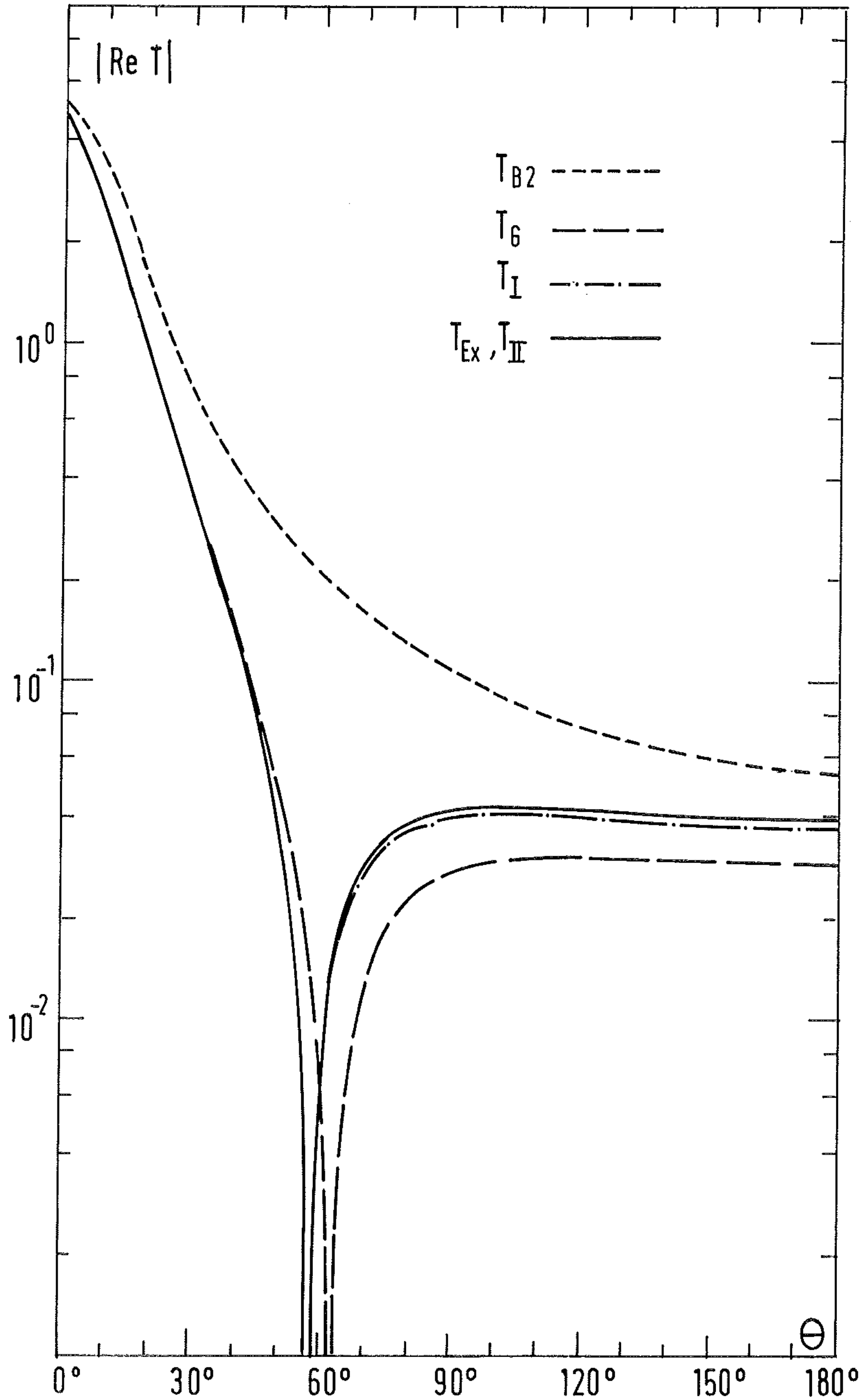

Fig. 5

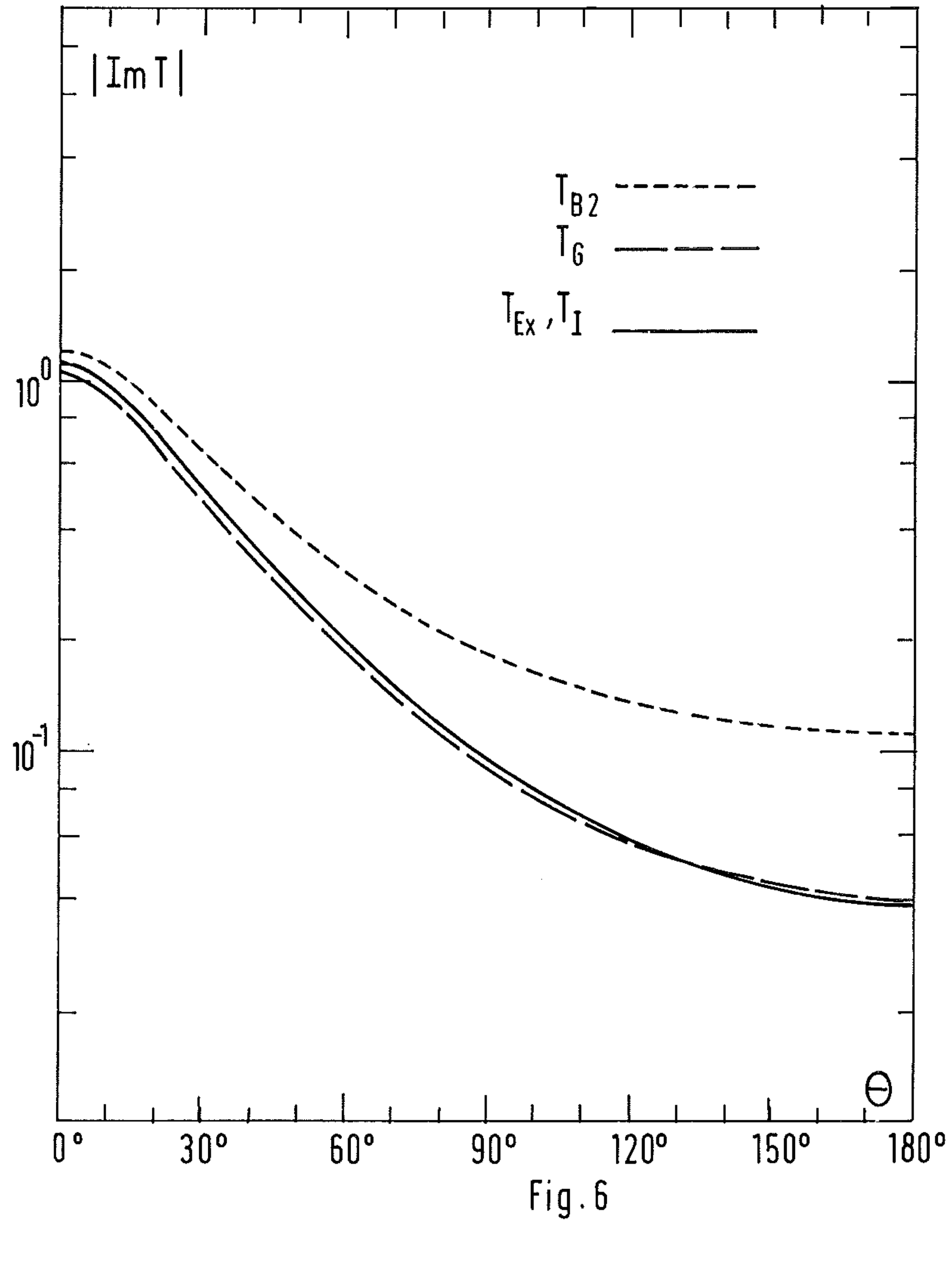

Fig. 6

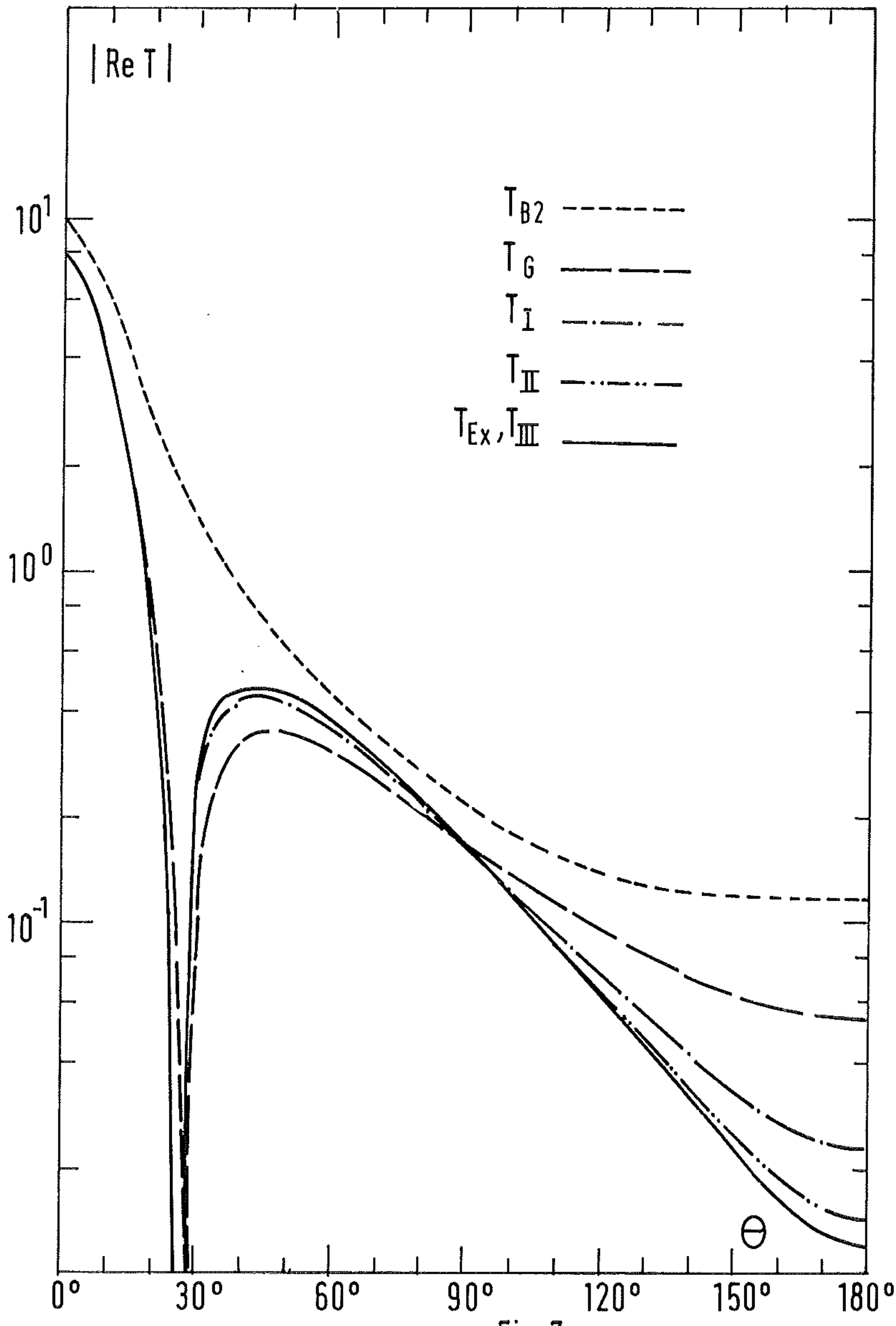

Fig.7

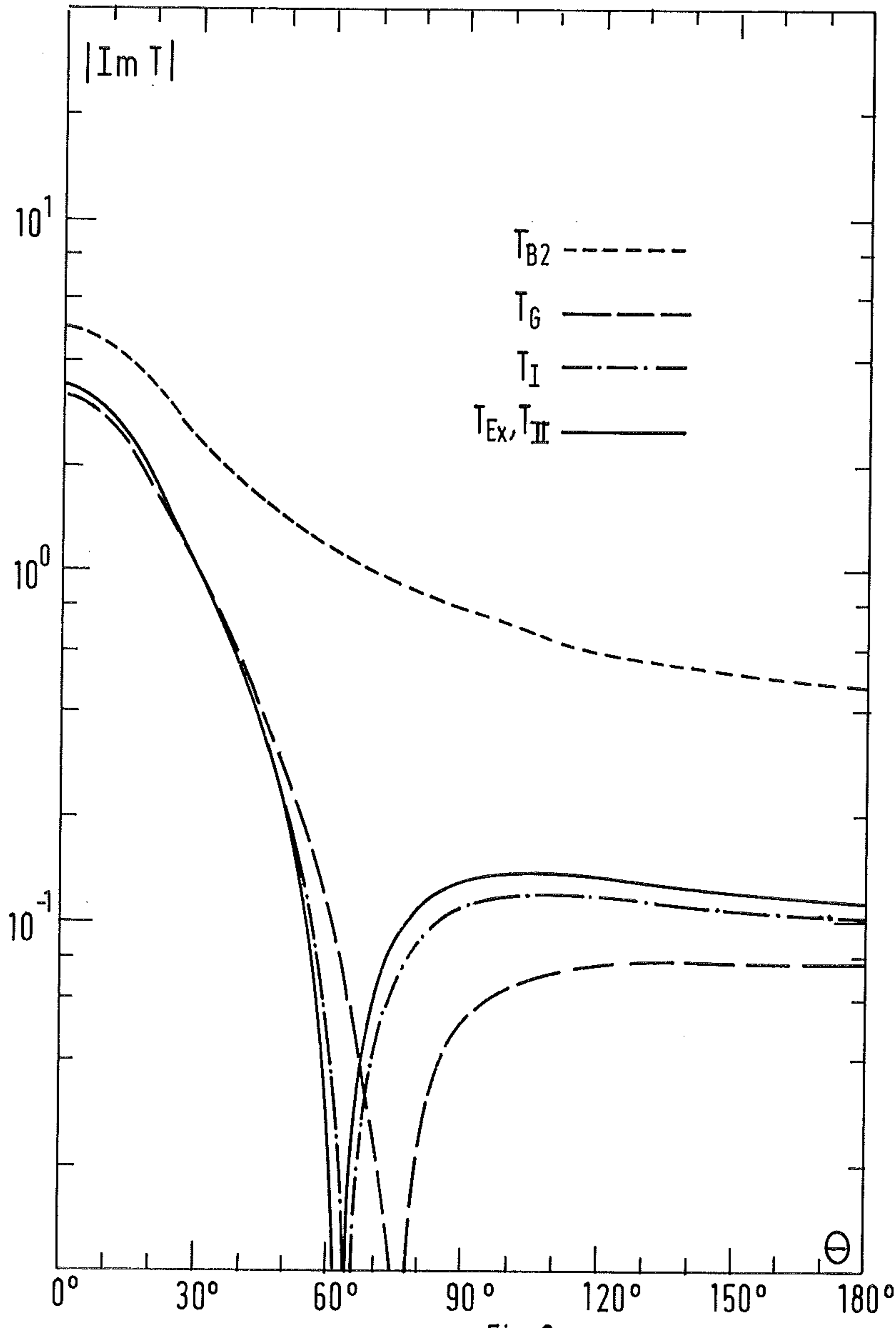

Fig. 8