DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 91-029
April 1991

Factorization of Bosonic String Scattering Amplitudes

J. Bolte, F. Steiner

II. Institut für Theoretische Physik, Universität Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

DESY 91-029
April 1991

ISSN 0418-9833

Factorization of Bosonic String Scattering Amplitudes

J. Bolte[1] and F. Steiner

II. Institut für Theoretische Physik
Universität Hamburg
Luruper Chaussee 149, 2000 Hamburg 50
Fed. Rep. Germany

Abstract

In an off-shell formulation, where scattering amplitudes for closed bosonic strings are defined as Polyakov path integrals over bordered world sheets, factorization of the amplitudes at the poles of exchanged particle states is shown in any order of perturbation theory. The same factorization is obtained for amplitudes defined via vertex operators, again for any number of loops.

[1]Supported by Doktorandenstipendium der Universität Hamburg

1 Introduction

Quantum mechanical scattering amplitudes that are derived from a unitary S-matrix must satisfy a factorization property: Consider the scattering of n external particles with momenta $p_1, \ldots, p_n$, described by the amplitude $\Gamma(p_1, \ldots, p_n)$. When m of these momenta, say $p_1, \ldots, p_m$, approach the mass shell of some particle of mass M that is being exchanged, the amplitude, as a function of $-k_m^2 = -(p_1 + \ldots + p_m)^2$, acquires a pole at M^2. The residue is the product of the two amplitudes describing the scattering of particles with momenta $p_1, \ldots, p_m, -k_m$ and $k_m, p_{m+1}, \ldots, p_n$.

In string theory this property has only been checked for some special (though important) cases within the usual approach to define the scattering amplitudes, i.e. when they are computed as correlation functions of vertex operators that describe an attaching of particle states to the (compact) string world sheet [1,2,3]. This formulation only works on the mass shell of the attached particles, as off-shell the vertex operators produce a conformal anomaly.

To be able to look at off-shell amplitudes, one can proceed slightly differently. For the closed, oriented bosonic string one defines the amplitude for the scattering of n external strings as a Polyakov path integral over bordered world sheets whose n bordering curves describe the external strings. If one shrinks closed curves of the world sheet to points, the respective amplitude describes the propagation of infinitely many particle states through the tube that has been pinched – either an "external leg" or an "internal line" of the world sheet viewed as a string analogue of a Feynman diagram. This programme was initiated in [4] and carried out further on in [5,6].

In [6] we showed in any order of string perturbation theory that the scattering amplitudes acquire poles in the external momenta or in the appropriate kinematical variables, when an "external" or "internal" closed curve on the world sheet is being pinched. In this article we further develop the formalism used in [6] and show that the residues of the poles found there factorize. Such a factorization has already been shown for some semi-off-shell amplitudes [4,7], where the off-shell formalism for one and two external strings has been combined with the insertion of vertex operators for the tree-level case. Here we give the result for any order in perturbation theory and any number of external strings with the exception of the two cases investigated in [4,7], for which the formalism presented in [6] does not apply.

Our paper is organized as follows: We first review the formalism of [4,5,6] for the treatment of string scattering amplitudes. We then show how one derives the desired factorization in this formalism. In the next section we give the relation of this formulation to the vertex operator approach and then show how the factorization works in that latter formalism. It turns out that both approaches produce formally identical formulae (compare (14) and (30)), although the meaning of the amplitudes is different, due to their different definitions. In the last section we summarize our results.

2 Factorization of Scattering Amplitudes

In the formalism we want to use the amplitudes for the scattering of n closed, oriented bosonic strings described by the loops $c_1, \ldots, c_n$ are given as a Polyakov path integral over all world sheets that possess the c_i's as their bordering curves [4,5,6],

$$A(c_1,\ldots,c_n)=\sum_{p=0}^{\infty}A_p(c_1\ldots,c_n)\ ,\quad A_p(c_1,\ldots,c_n)=\int\mathcal{D}\pi\int\frac{\mathcal{D}X\mathcal{D}g}{Vol(H)}\,e^{-S_P[X,g]}\ . \tag{1}$$

The world sheets Σ of genus p are parametrized by the functions $X^\mu(\sigma)$ that describe the embedding of Σ into flat, Eulcidean space-time, and carry the Riemannian metric $g_{\alpha\beta}(\sigma)$. The Polyakov action is

$$S_P[X,g]=\frac{1}{2}\int_\Sigma d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\mu\ . \tag{2}$$

We work in the critical dimension $d=26$ and thus do not explicitly state the counter terms to the action required to gauge-fix the conformal degree of freedom. $H=Weyl(\Sigma)\times Diff(\Sigma)$ is the symmetry group of the action and $\mathcal{D}\pi$ represents an integration over the parametrization of the bordering curves of Σ. The details of the definition of this path integral and a thorough discussion of the boundary conditions for the fields involved may be found in [4,8,9]. In this article we restrict ourselves to the case where $\hat{p}:=2p+n-1\geq 2$, thus excluding the cases $(p,n)=(0,1)$ and $(0,2)$ for $n>0$. This is required by the method used in [6] and also guarantees the absence of conformal Killing vectors in (1).

Integrating out the X-variable and gauge-fixing the symmetry degrees of freedom leads to

$$A_p(c_1,\ldots,c_n)=\int\mathcal{D}\pi\int_{\mathcal{M}_{p,n}}d\mu_{WP}\ N\ [det'(P_1^\dagger P_1)]^{1/2}[det'(-\Delta)_D]^{-13}e^{-S_{cl}}\ , \tag{3}$$

where $d\mu_{WP}$ denotes the Weil-Petersson measure, used to integrate over the moduli space $\mathcal{M}_{p,n}$ of Riemann surfaces of genus p with n bordering curves. $P_1^\dagger P_1$ is the ghost operator and Δ_D is the Dirichlet-Laplacian on Σ. S_{cl} is the classical action, $S_P[X_{cl},g]$ evaluated with the classical field X^μ_{cl}, $\Delta X^\mu_{cl}=0$. It only depends on the value of X_{cl} on the boundary $\partial\Sigma$ [4,5,6]. We are interested in the pinching limit of the amplitudes, where the lengths $l_i:=l(c_i)$ of the bordering curves shrink to zero. This limit has been described in [6]: When the bordering curves $c_1,\ldots,c_n$ shrink to points $x_1,\ldots,x_n$ the leading contribution to the momentum-space amplitude is in that limit given by

$$\begin{aligned}A_p(p_1,\ldots,p_n)\ &:=\ \int d^{26}x_1\ldots d^{26}x_n\,A_p(x_1,\ldots,x_n)\,e^{i\sum_{k=1}^n p_kx_k}\\ &\simeq\ \delta^{26}\left(\sum_{k=1}^n p_k\right)\prod_{i=1}^n\sum_{j=0}^{\infty}\frac{a_j}{p_i^2+4\pi(j-1)}\int d\mu^{reg}_{WP}e^{-\tilde{S}}\ .\end{aligned} \tag{4}$$

The a_j's are expansion coefficients of some known function (see [6]), while the integral contains the regular parts in the pinching limit.

In this article we want to study these amplitudes in the limit where an internal closed curve c on the world sheet is pinched in such a way that in the end Σ separates into two surfaces Σ_1 and Σ_2 of genera g_1 and g_2 and with additional punctures q_1 and q_2 on each part respectively. This division should take place such that $c_1,\ldots,c_m$ are bordering curves on Σ_1 and $c_{m+1},\ldots,c_n$ on Σ_2. Therefore the momentum that flows from Σ_1 to Σ_2 in the pinching limit $l:=l(c)\longrightarrow 0$ is $k_m:=\sum_{i=1}^m p_i$. In [6] we already showed that the amplitude develops in that limit a singularity structure of the form

$$\sum_{r=0}^{\infty}\frac{d_r}{k_m^2+4\pi(r-1)}\ , \tag{5}$$

where the d_r's are known expansion coefficients (see (10)). In the following we shall show the factorization of the residues at the poles in (5).

A typical world sheet that we want to study is depicted in figure 1. The pinching indicated

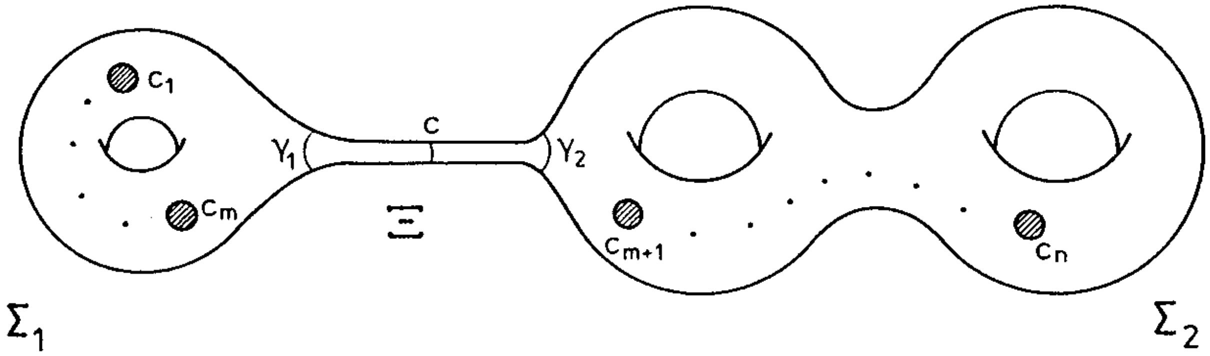

Figure 1: A world sheet Σ that is pinched into its parts Σ_1 and Σ_2. Ξ is the tube ("degenerating collar"), which develops in the pinching process.

in figure 1 is a well-investigated process in the mathematical literature (see [10]). As long as l stays finite, Σ will be dissected into the three parts Σ_1, Σ_2 and the "degenerating collar" Ξ, present in the mathematical description of the pinching process.

The X-integration will now be treated as in [11], namely split into an integration over Σ_1, Σ_2 and Ξ respectively. The boundary $\partial\Xi = \gamma_1 \cup \gamma_2$ will be kept fixed and subsequently be integrated over,

$$\int_\Sigma \mathcal{D}X\, e^{-S[X]} = \int \mathcal{D}\gamma_1 \mathcal{D}\gamma_2 \int_{\Sigma_1} \mathcal{D}X_1\, e^{-S[X_1]} \int_\Xi \mathcal{D}X_c\, e^{-S[X_c]} \int_{\Sigma_2} \mathcal{D}X_2\, e^{-S[X_2]}\,. \tag{6}$$

In (6) the integral over the collar Ξ is an integral over cylinder-shaped world sheets. Inserting the g-integration, the part corresponding to Ξ is the propagator $\mathcal{A}(\gamma_1,\gamma_2)$ discussed in [4]. In the pinching limit, where γ_1 and γ_2 shrink to the punctures q_1 and q_2, this amplitude acquires the form [4]

$$\mathcal{A}(\bar{x}_1,\bar{x}_2) \simeq \text{const.} \int_0^\infty dl\, l^{11} e^{\frac{4\pi^2}{l}} e^{-\frac{l}{4\pi}(\bar{x}_1-\bar{x}_2)^2} \prod_{k=1}^\infty \left(1 - e^{-\frac{4\pi^2}{l}k}\right)^{-24}\,, \tag{7}$$

where $\bar{x}_{1,2}$ are the coordinates of $q_{1,2}$ in space-time.

At this stage we also pinch the external curves $c_1,\ldots,c_n$. Therefore Σ_1 and Σ_2 become surfaces of genera g_1 and g_2 with punctures $x_1,\ldots,x_m,\bar{x}_1$ and $\bar{x}_2,x_{m+1},\ldots,x_n$ respectively. As in (4) we perform a Fourier transformation to momentum-space and thus introduce momenta $p_1,\ldots,p_m$ on Σ_1 and $p_{m+1},\ldots,p_n$ on Σ_2. Calling the restriction to $\Sigma_{1,2}$ of the classical action S_{cl} (appearing in (3)) $S_{cl}^{(1,2)}$, we arrive at (see [6])

$$\int d^{26}x_1 \ldots d^{26}x_m \exp\left[-S_{cl}^{(1)} + i\sum_{k=1}^m p_k x_k\right] =$$

$$= \int d^{26}x_1 \dots d^{26}x_m \exp\left\{-\frac{1}{2}\sum_{k,l=1}^{m}(x_k - \bar{x}_1) Im\, \sigma^{(1)}_{kl}(x_l - \bar{x}_1) + i\sum_{k=1}^{m} p_k + O(V)\right\} \quad (8)$$

$$\simeq (2\pi)^{13m}\left(det\, Im\, \sigma^{-1}_{(1)}\right)^{13} \exp\left\{-\frac{1}{2}\sum_{k,l=1}^{m} p_k p_l (Im\, \sigma_{(1)})^{-1}_{kl} - i\bar{x}_1 k_m\right\} .$$

where $(Im\,\sigma)^{-1}$ denotes the central block of the period matrix for the surfaces $\hat{\Sigma}$ that are the closed doubles of the bordered world sheets Σ (see [6] for details of the construction). σ_1 is the part of σ that belongs to Σ_1. An analogous formula holds for Σ_2.

Altogether one finds in the pinching limit

$$\begin{aligned}
&\int d^{26}x_1 \dots d^{26}x_n \int_\Sigma \mathcal{D}X\, e^{-S[X]} \simeq \\
&\text{const.}\left[det'(-\Delta_{(1)})\right]^{-13} (det\, Im\, \sigma^{-1}_{(1)})^{13} \exp\left\{-\frac{1}{2}\sum_{k,l=1}^{m} p_k p_l (Im\, \sigma_{(1)})^{-1}_{kl}\right\} \\
&\cdot\left[det'(-\Delta_{(2)})\right]^{-13} (det\, Im\, \sigma^{-1}_{(2)})^{13} \exp\left\{-\frac{1}{2}\sum_{k,l=m+1}^{n} p_k p_l (Im\, \sigma_{(2)})^{-1}_{kl}\right\} \\
&\cdot\int_0^\infty dl\, l^{11}\int d^{26}\bar{x}_1 d^{26}\bar{x}_2 e^{-i\bar{x}_1 k_m - i\bar{x}_2 k'_m} e^{-\frac{l}{4\pi}(\bar{x}_1 - \bar{x}_2)^2} \\
&\cdot\sum_{r=0}^{\infty} d_r e^{-\frac{4\pi^2}{l}(r-1)} .
\end{aligned} \quad (9)$$

Here $k'_m := \sum_{l=m+1}^{n} p_l$, and the d_r's are the expansion coefficients in

$$\prod_{n=1}^{\infty}(1-q^n)^{-24} = \sum_{n=0}^{\infty} d_n q^n . \quad (10)$$

In the pinching limit $\int \mathcal{D}\gamma_1 \mathcal{D}\gamma_2$ degenerates to $\int d^{26}\bar{x}_1 d^{26}\bar{x}_2$, thus

$$\int d^{26}\bar{x}_1 d^{26}\bar{x}_2\, e^{-i\bar{x}_1 k_m - i\bar{x}_2 k'_m}\, e^{-\frac{l}{4\pi}(\bar{x}_1 - \bar{x}_2)^2} = (2\pi)^{26}\left(\frac{4\pi^2}{l}\right)^{13} \delta^{26}(k_m + k'_m)\, e^{-\frac{\pi}{l}k_m^2} . \quad (11)$$

Therefore the l-integration in (9) yields

$$\begin{aligned}
&\int_0^\infty dl\, l^{11}\int d^{26}\bar{x}_1 d^{26}\bar{x}_2\, e^{-i\bar{x}_1 k_m - i\bar{x}_2 k'_m - \frac{l}{4\pi}(\bar{x}_1-\bar{x}_2)^2} \sum_{r=0}^{\infty} d_r e^{-\frac{4\pi^2}{l}(r-1)} \\
&= \text{const.}\, \delta^{26}(k_m + k'_m)\int_0^\infty \frac{dl}{l^2}\sum_{r=0}^{\infty} d_r e^{-\frac{4\pi^2}{l}(r-1) - \frac{\pi}{l}k_m^2} \\
&= \text{const.}\, \delta^{26}(k_m + k'_m)\sum_{r=0}^{\infty}\frac{d_r}{k_m^2 + 4\pi(r-1)} .
\end{aligned} \quad (12)$$

We are now in a position to collect everything: The Weil-Petersson measure (which factorizes also in the pinching limit), the ghost determinants and the pole factors from (4) for Σ_1 and Σ_2, and (12), arriving at

$$A_p(p_1, \dots, p_n) \;\simeq\; C_p\, \delta^{26}\left(\sum_{k=1}^{n} p_k\right)\prod_{i=1}^{m}\sum_{j=0}^{\infty}\frac{a_j^{(1)}}{p_i^2 + 4\pi(j-1)}\int d\mu^{reg}_{WP_1}\, e^{-\tilde{S}_{(1)}}$$

$$\cdot \prod_{i=m+1}^{n} \sum_{j=0}^{\infty} \frac{a_j^{(2)}}{p_i^2 + 4\pi(j-1)} \int d\mu_{WP_2}^{reg} e^{-\hat{S}_{(2)}} \tag{13}$$
$$\cdot \sum_{r=0}^{\infty} \frac{d_r}{k_m^2 + 4\pi(r-1)} \; .$$

We "amputate" the external propagator poles and leave out the δ-function indicating momentum conservation. The remaining, amputated amplitudes we call $\Gamma_p(p_1, \ldots, p_n)$. For these the *factorization rule*

$$\Gamma_p(p_1, \ldots, p_n) \simeq \Gamma_{g_1}(p_1, \ldots, p_m, -k_m) \sum_{r=0}^{\infty} \frac{C_p d_r}{k_m^2 + 4\pi(r-1)} \Gamma_{g_2}(k_m, p_{m+1}, \ldots, p_n) \tag{14}$$

holds in the above specified pinching limit, since e.g. $\int d\mu_{WP_1} e^{-\hat{S}_{(1)}}$ is the amputated amplitude on the world sheet Σ_1 of genus g_1 with punctures $x_1, \ldots, x_m, \bar{x}_1$ to which the momenta $p_1, \ldots, p_m, -k_m$ were attached (see (9)). (14) is exactly the desired factorization property that is expected for a scattering amplitude in a unitary theory. In addition one can see that the wave function renormalization constant for the r-th mass-level in p-th order of perturbation theory is $Z_r^{(p)} = C_p d_r$ and thus factorizes in an r- and a p-dependent part.

3 Comparison with the Vertex Operator Approach

The formalism used in section 2 describes the scattering of strings – the loops c_i. In the pinching limit these degenerate to pointlike strings. On the world sheet the picture of letting the lengths of the external strings shrink to zero is conformally equivalent to stretching the cylinder-like parts of the world sheet that connect the loops c_i with the rest of the surface to infinity, therefore moving the incoming and outgoing strings that take part in the scattering process to infinity.

If one wants to describe the scattering of particle states, one has to attach wave functionals of given states to the boundary curves. The state space $\mathcal{H}^{\text{off}}$ of these wave functionals has been investigated in [9]. For $\psi_1, \ldots, \psi_n \in \mathcal{H}^{\text{off}}$ the scattering amplitude in p-loop order is given by

$$\begin{aligned} A_p(\psi_1, \ldots, \psi_n) &= \int \mathcal{D}c_1 \ldots \mathcal{D}c_n \; \psi_1[c_1] \ldots \psi_n[c_n] \, A_p(c_1, \ldots, c_n) \qquad (15) \\ &= \int_{\mathcal{M}_{p,n}} d\mu_{WP} \frac{(det' \, P_1^\dagger P_1)^{1/2}}{[det' \, (-\Delta)_D]^{-13}} \prod_{i=1}^{n} \int \mathcal{D}c_i \, \psi_i[c_i] \exp\left[-\frac{1}{2} \int_{c_i} d\lambda X_i^\mu(\lambda) \partial_n X_i^\mu(\lambda)\right] \; , \end{aligned}$$

where in the integrand the amplitude (1) occurs, and

$$\begin{aligned} S_{cl} = S[X_{cl}, g] &= \frac{1}{2} \int d^2\sigma \, \sqrt{g} g^{\alpha\beta} \partial_\alpha X_{cl}^\mu \partial_\beta X_{cl}^\mu \\ &= \sum_{i=1}^{n} \frac{1}{2} \int_{c_i} d\lambda X_i^\mu(\lambda) \partial_n X_i^\mu(\lambda) \; , \qquad (16) \end{aligned}$$

which only depends on the value $X_i(\lambda)$ of $X_{cl}(\sigma)$ on each c_i. The factor $e^{-S_{cl}}$ becomes in the pinching limit $e^{-S_{pt}}$, see [6], which has to be included in ψ_i. If c_i shrinks to the point P_i one gets

$$A_p(\psi_1,\ldots,\psi_n) \to \int_{\mathcal{M}_p^{(n)}} d\mu_{WP} \frac{(det'\, P_1^\dagger P_1)^{1/2}}{[det'(-\Delta)]^{13}} \prod_{i=1}^n \psi_i(P_i) \ . \tag{17}$$

Here the integration runs over the moduli space $\mathcal{M}_p^{(n)}$ of Riemann surfaces of genus p with n punctures. In [12] it was shown that this amplitude is identical to the following amplitude

$$A_p^V(p_1,\ldots,p_n) = \int_{\mathcal{M}_p} d\mu_{WP} \frac{(det'\, P_1^\dagger P_1)^{1/2}}{[det'(-\Delta)]^{13}} < \prod_{i=1}^n \int d^2\sigma_i \sqrt{g(\sigma_i)} W_i(\sigma_i,p_i) > , \tag{18}$$

defined by integrating the vertex operator W_i over a compact world sheet that has been gotten by removing the punctures, see [12] for details.

It is a well-known fact that this approach works only for external momenta p_i that are on the mass shell of the particle that is being described by W_i. In the sequel we will for simplicity only work with tachyon vertex operators $W_T(\sigma,p) = e^{ipX(\sigma)}$, but everything goes through for any other vertex operator too. We call $V_T(p) := \int d^2\sigma \sqrt{g} W_T(\sigma,p)$. Then it was shown in [13,3] that

$$< \prod_{i=1}^n V_T(p_i) >= \delta^{26}\left(\sum_{k=1}^n p_k\right) \int_{\Sigma^n} \prod_{k\neq l} |E(z_k,z_l)|^{\frac{p_k p_l}{2\pi}} \exp\left\{-\frac{1}{2}\sum_{k,l=1}^n p_k p_l g_\Sigma(z_k,z_l)\right\} . \tag{19}$$

In this formula an UV-cut-off has been performed by subtracting the singular part of the Green function for the Laplacian on the world sheet, leading to the omission of the $(k=l)$–term in the product in (19). In this formula $E(z,w)$ denotes the prime form, which is a $(-\frac{1}{2},-\frac{1}{2})$–form on $\Sigma\times\Sigma$, see [10]. Furthermore we put

$$g_\Sigma(z,w) := \sum_{k,l=1}^p Im \int_w^z \omega_k \,(Im\,\Omega)^{-1}_{kl}\, Im \int_w^z \omega_l \ , \tag{20}$$

where Ω is the period matrix of Σ and $(\omega_1,\ldots,\omega_p)$ are p linearly independent Abelian differentials of the first kind (i.e. holomorphic one-forms) on Σ [10]. It is necessary to introduce (20) to make the integrand in (19) single-valued on Σ.

We do not want to go into the details of the pinching process, but only sketch the necessary calculations. In [10] one can find the mathematical description of the pinching. One introduces the (complex) parameter t that is related to the length $l = l(c)$ being pinched through $\frac{2\pi^2}{l} = -\log|t|$. Therefore $l \to 0$ corresponds to $t \to 0$.

As in section 2 one dissects the world sheet Σ into three parts: Σ_1, Σ_2 and the degenerating collar Ξ, see figure 2. For $p_0 \in \Sigma$ the Jacobi map I maps every point $z\in\Sigma$ onto the Jacobian torus $\mathbb{C}^p/L_\Omega$, $L_\Omega := \mathbb{Z}^p + \Omega\mathbb{Z}^p$; $I(z)_k := \int_{p_0}^z \omega_k$, $k = 1,\ldots,p$. The prime form $E(z,w)$, $z,w\in\Sigma$, can then be expressed by

$$E(z,w)^{-2} = \frac{\sum_{k=1}^p \omega_k(z)\partial_{z_k}\vartheta[\alpha](0;\Omega)\sum_{l=1}^p \omega_l(w)\partial_{z_l}\vartheta[\alpha](0;\Omega)}{\{\vartheta[\alpha](I(z-w);\Omega)\}^2} \ . \tag{21}$$

In this formula $\vartheta[\alpha](\vec{z};\Omega)$, $\vec{z}\in\mathbb{C}^p$, denotes a theta function with (odd) characteristic $\alpha = \begin{pmatrix} a \\ b \end{pmatrix}$, see [10],

$$\vartheta[\alpha](\vec{z};\Omega) = \sum_{n\in\mathbb{Z}^p} \exp\left\{i\pi(n+a)^t\Omega(n+a) + 2\pi i(n+a)^t(\vec{z}+b)\right\} . \tag{22}$$

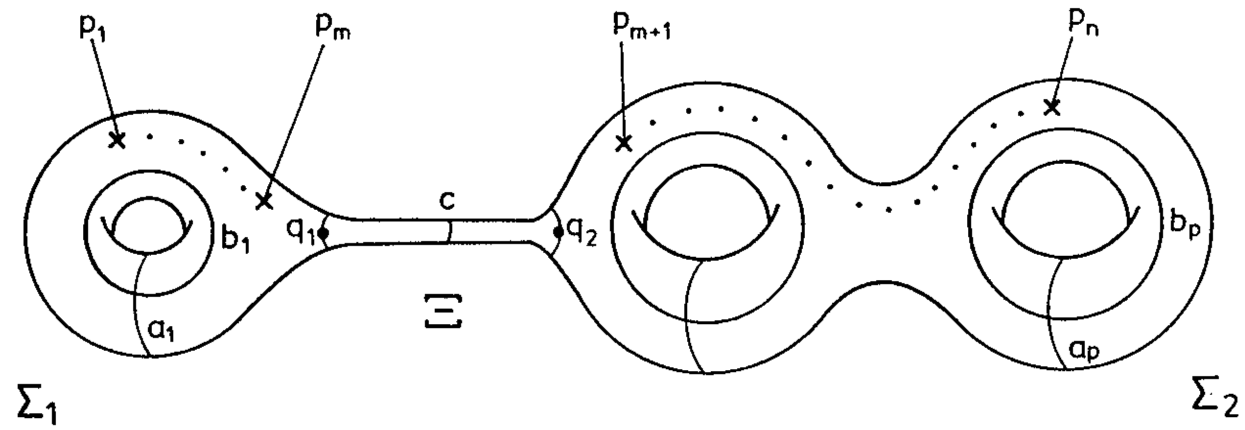

Figure 2: A compact world sheet Σ with vertex operator insertions, that will be pinched.

In [10] one finds the expansion of Ω and ω_k in powers of the pinching parameter t and can then expand $E(z,w)^{-2}$ in powers of t. For the period matrix the expansion is given by [10]

$$\Omega = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + t \begin{pmatrix} 0 & \Omega_{12} \\ \Omega_{12}^t & 0 \end{pmatrix} + O(t^2) \ . \tag{23}$$

One has to distinguish two cases:

1. z and w are on the same part of Σ; $z, w \in \Sigma_1$ say. Then one finds

$$|E(z,w)| = |E_1(z,w)| + O(t) \ ,$$

   where $E_1(z,w)$ denotes the prime form in which one uses all expressions in (21) defined only on Σ_1.

2. z and w are on different parts of Σ; $z \in \Sigma_1$, $w \in \Sigma_2$, say. Then one finds

$$|E(z,w)| = |t|^{-1/2} \, |E_1(z,q_1)E_2(q_2,w)| + O(t^{1/2}) \ ,$$

   where $q_{1,2}$ are defined as in section 2, see also figure 2.

Next, one has to insert the expression (23) into (20) to find the expansion for $g_\Sigma(z,w)$:

1. $z, w \in \Sigma_1$,

$$g_\Sigma(z,w) = g_{\Sigma_1}(z,w) + O(t) \ ,$$

2. $z \in \Sigma_1$ and $w \in \Sigma_2$,

$$g_\Sigma(z,w) = g_{\Sigma_1}(z,q_1) + g_{\Sigma_2}(q_2,w) + O(t) \ .$$

As described in figure 2, the first m external momenta $p_1, \ldots, p_m$ will be attached to Σ_1, and the remaining ones $p_{m+1}, \ldots, p_n$ to Σ_2; thus

$$G_\Sigma(p_1, \ldots, p_n) := \sum_{1 \le k \ne l \le n} p_k p_l \left\{ \log |E(z_k, z_l)|^{-2} + 2\pi g_\Sigma(z_k, z_l) \right\}$$

$$
\begin{aligned}
&= \sum_{1\le k\neq l\le m} p_k p_l \left\{\log|E_1(z_k,z_l)|^{-2} + 2\pi g_{\Sigma_1}(z_k,z_l)\right\} \\
&+ \sum_{m<k\neq l\le n} p_k p_l \left\{\log|E_2(z_k,z_l)|^{-2} + 2\pi g_{\Sigma_2}(z_k,z_l)\right\} \\
&+2\sum_{k=1}^{m}\sum_{l=m+1}^{n} p_k p_l \left\{\log|E_1(z_k,q_1)|^{-2} + 2\pi g_{\Sigma_1}(z_k,q_1)\right. \\
&\qquad\qquad \left. +\log|E_2(q_2,z_l)|^{-2} + 2\pi g_{\Sigma_2}(q_2,z_l) + \log|t|\right\} + O(|t|) \; .
\end{aligned} \tag{24}
$$

As in section 2 define $k_m := \sum_{l=1}^{m} p_l$, then momentum conservation, indicated by the δ-function in (19), yields $\sum_{l=m+1}^{n} p_l = -k_m$. Therefore

$$
\begin{aligned}
G_\Sigma(p_1,\ldots,p_n) &= G_{\Sigma_1}(p_1,\ldots,p_m) + G_{\Sigma_2}(p_{m+1},\ldots,p_n) \\
&\quad -2k_m\left\{\sum_{k=1}^{m} p_k\left[\log|E_1(z_k,q_1)|^{-2} + 2\pi g_{\Sigma_1}(z_k,q_1)\right]\right. \\
&\qquad \left. - \sum_{l=m+1}^{n} p_l\left[\log|E_2(q_2,z_l)|^{-2} + 2\pi g_{\Sigma_2}(q_2,z_l)\right]\right\} \\
&\quad -2k_m^2\log|t| + O(|t|) \\
&= G_{\Sigma_1}(p_1,\ldots,p_m,-k_m) + G_{\Sigma_2}(k_m,p_{m+1},\ldots,p_n) - 2k_m^2\log|t| + O(|t|) \; .
\end{aligned} \tag{25}
$$

In terms of these expressions the integrand in (19) reads $\exp\{-\frac{1}{4\pi}G_\Sigma(p_1,\ldots,p_n)\}$.

The integration over the $6p-6$ (real) moduli for Σ splits into an integration over $6g_1-6+2$ moduli for Σ_1, $6g_2-6+2$ moduli for Σ_2 and 2 moduli for Ξ. (The two moduli for $\Sigma_{1,2}$ that are added to the usual $6g_{1,2}-6$ ones for a compact surface correspond to the coordinates of the punctures $q_{1,2}$.) We choose Fenchel-Nielsen coordinates (l,θ) as the two moduli for the collar Ξ, see [4,6,11]. Therefore one finds in the pinching limit

$$
d\mu_{WP} \simeq d\mu_{WP}^{(1)} \wedge d\mu_{WP}^{(2)} \wedge l dl \wedge d\theta \wedge d^2q_1\sqrt{g(q_1)} \wedge d^2q_2\sqrt{g(q_2)} \tag{26}
$$

to hold. We thus obtain from (25), using the $q_{1,2}$-part from (26),

$$
\begin{aligned}
\int d^2q_1\sqrt{g(q_1)}\int d^2q_2\sqrt{g(q_2)} < \prod_{i=1}^{n} V_T(p_i) >_\Sigma &= < \prod_{i=1}^{m} V_T(p_i)V_T(-k_m) >_{\Sigma_1} \\
&\quad \cdot < V_T(k_m)\prod_{i=m+1}^{n} V_T(p_i) >_{\Sigma_2} |t|^{\frac{k_m^2}{2\pi}}\{1+O(|t|)\} \; .
\end{aligned} \tag{27}
$$

To treat the contribution of the determinants in (18), we proceed as in [11] and in section 2, equations (6)–(12). We also include the factor $|t|^{\frac{k_m^2}{2\pi}} = e^{-\frac{\pi}{l}k_m^2}$ of equation (27) and observe for that part of (18) which comes from the integration over the collar Ξ (including the integration over the moduli for Ξ)

$$
\begin{aligned}
\int d^{26}\bar{x}_1 d^{26}\bar{x}_2 \mathcal{A}(\bar{x}_1,\bar{x}_2)|t|^{\frac{k_m^2}{2\pi}} &= \text{const.} \int_0^\infty \frac{dl}{l^2}\; e^{-\frac{\pi}{l}k_m^2}\sum_{r=0}^{\infty} d_r e^{-\frac{4\pi^2}{l}(r-1)} \\
&= \text{const.} \sum_{r=0}^{\infty} d_r \int_0^\infty dx\; e^{-x[k_m^2+4\pi(r-1)]} \\
&= \text{const.} \sum_{r=0}^{\infty} \frac{d_r}{k_m^2+4\pi(r-1)} \; .
\end{aligned} \tag{28}
$$

Collecting the results from (26)–(28) and inserting these into (18) yields in the pinching limit

$$\begin{aligned} A_p^V(p_1,\ldots,p_n) \;\simeq\; & C_p^V\, \delta^{26}\left(\sum_{k=1}^{n} p_k\right) \int_{\mathcal{M}_{g_1}} d\mu_{WP}^{(1)} \frac{(det' P_1^\dagger P_1)^{1/2}}{[det'(-\Delta)]^{13}}|_{\Sigma_1} < \prod_{i=1}^{m} V_T(p_i) V_T(-k_m) >_{\Sigma_1} \\ & \cdot \int_{\mathcal{M}_{g_2}} d\mu_{WP}^{(1)} \frac{(det' P_1^\dagger P_1)^{1/2}}{[det'(-\Delta)]^{13}}|_{\Sigma_2} < V_T(k_m) \prod_{i=m+1}^{n} V_T(p_i) >_{\Sigma_2} \\ & \cdot \sum_{r=1}^{\infty} \frac{d_r}{k_m^2 + 4\pi(r-1)} \;. \end{aligned} \tag{29}$$

As in section 2 we introduce the amputated amplitudes $\Gamma_p^V(p_1,\ldots,p_n)$, where the momentum-conserving δ-function has been left out. For these amplitudes equation (29) yields the following *factorization rule*

$$\Gamma_p^V(p_1,\ldots,p_n) \simeq \Gamma_{g_1}^V(p_1,\ldots,p_m,-k_m) \sum_{r=0}^{\infty} \frac{C_p^V d_r}{k_m^2 + 4\pi(r-1)} \Gamma_{g_2}^V(k_m,p_{m+1},\ldots,p_n) \;. \tag{30}$$

This is exactly the factorization as in (14), but within the vertex operator formalism. Again the wave function renormalization constant $Z_r^{(p),V} = C_p^V d_r$ for the r-th mass level in p-loop order factorizes in an r- and a p-dependent part. Therefore both amplitudes, the off-shell formalism ones and the vertex operator formalism ones, share exactly the same factorization property, when the world sheets are being pinched into two parts.

4 Summary

We have investigated the factorization of scattering amplitudes in bosonic string theory using two different formulations. The first formalism [4,5,6] describes the scattering of n pointlike string states and uses methods developed previously [6], extending former results for some special situations [4,5,7], i.e. for the lowest orders in the topological perturbation expansion. The second formalism we analyzed is the usual vertex operator approach that describes scattering of on-shell particle states. There some results were obtained previously in [2,3], which have been extended to all possible cases in our paper.

We found that factorization holds to all orders in perturbation theory for both formalisms, and the final results in both cases look formally alike, see equations (14) and (30). It is not very surprising that this coincidence appears, since the techniques used to derive factorization are similar: one has to pinch the world sheets in order to separate the external momenta into two groups. This divides the world sheet into two parts that become the two (separated) world sheets of the factors on the r.h.s. of the factorization formulae (14) and (30). In addition, since both formalisms are connected in a specific manner, see (15)–(18), the similarity of the results could be expected.

We also take our result on the off-shell formulation as a hint for the consistency and the usefulness of this approach. It allows to study scattering amplitudes in a technically comparatively simple manner (as compared to string field theory) also off the mass shell for particle states.

References

[1] S. Weinberg, in: R.C. Hwa (ed.), Proc. of the Oregon Meeting, Eugene 1985 (World Scientific, Singapore 1986)

[2] G. Moore, Phys. Lett. **176B**(1986)369

[3] E. Verlinde and H. Verlinde, Nucl. Phys. **B288**(1987)357

[4] A. Cohen, G. Moore, P. Nelson and J. Polchinski, Nucl. Phys. **B267**(1986)143
A. Cohen, G. Moore, P. Nelson and J. Polchinski, Nucl. Phys. **B281**(1987)127

[5] S.K. Blau and M. Clements, Nucl. Phys. **B284**(1987)118
S.K. Blau, M. Clements and B. McClain, preprint UTTG-29-86, University of Texas, Austin 1986
S.K. Blau, M. Clements, S. Della Pietra, S. Carlip and V. Della Pietra, Nucl. Phys. **B301**(1988)205

[6] J. Bolte and F. Steiner, DESY-preprint 90-081 (July 1990), Nucl. Phys. **B** in print

[7] W. Weisberger, Nucl. Phys. **B294**(1987)113

[8] O. Alvarez, Nucl. Phys. **B216**(1983)125

[9] Z. Jaskólski, Commun. Math. Phys. **128**(1990)285
Z. Jaskólski, preprint ITP UWr 729/89, Univ. of Wrocław, Wrocław 1989

[10] J.D. Fay, Theta Functions on Riemann Surfaces, Lecture Notes in Mathematics, vol. 352 (Springer, Berlin, Heidelberg, New York 1973)
A. Yamada, Kodai Math. J. **3**(1980)114

[11] E. Gava, R. Iengo, T. Jayaraman and R. Ramachandran, Phys. Lett. **168B**(1986)207

[12] E. D'Hoker and S.B. Giddings, Nucl. Phys. **B291**(1987)90
S.B. Giddings, Phys. Rep. **170**(1988)167

[13] S. Hamadi and C. Vafa, Nucl. Phys. **B279**(1987)465