DEUTSCHES ELEKTRONEN – SYNCHROTRON
INSTITUT FÜR HOCHENERGIEPHYSIK

DESY 92-034
March 1992

Propagators in Magnetic String Backgrounds Calculated by Shifting the Angular Operator

H. J. Kaiser

Deutsches Elektronen-Synchrotron DESY
Institut für Hochenergiephysik IfH, Zeuthen

ISSN 0418-9833

PLATANENALLEE 6 · O-1615 ZEUTHEN

DESY 92-034
March 1992

ISSN 0418-9833

PROPAGATORS IN MAGNETIC STRING BACKGROUNDS CALCULATED BY SHIFTING THE ANGULAR OPERATOR†

Hans Jürgen KAISER
DEUTSCHES ELEKTRONEN-SYNCHROTRON **DESY**
Institut für Hochenergiephysik - IfH Zeuthen
Platanenallee 6, D-O-1615 Zeuthen, Germany

A shortcut is presented to derive the ghost propagator of a non-Abelian gauge theory in the background of a single magnetic string. The technique takes advantage of the fact that in cylindrical coordinates the presence of a magnetic string of strength β shifts the differential operator $\partial/\partial\varphi$ to $\partial/\partial\varphi - i\beta$ in the differential equation fulfilled by the propagator. By using an appropriately modified Fourier transformation we can immediately derive from the free propagator an integral representation of the propagator in the string background. The same procedure can be applied to the background of a pair of antiparallel strings of equal strength.

Introduction

One of the few nontrivial models of a non-Abelian field theory where an exact (ghost) propagator can be calculated is the quantum field theory in the background of a classical magnetic string *

$$F^a_{\mu\nu}(x) = \delta^{a3}\beta\epsilon^{\perp}_{\mu\nu}\delta(r_\perp)/r_\perp \tag{1}$$

corresponding to the vector potential

$$B^a_\mu(x) = -\delta^{a3}\beta\epsilon^{\perp}_{\mu\rho}x^{\perp}_\rho/r^2_\perp . \tag{2}$$

We use the notation

$$x_\perp = (x_1, x_2), \quad x_\parallel = (x_3, x_4), \quad r_\perp = |x_\perp|, \qquad \epsilon^{\perp}_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3}$$

† Talk given at the XXV International Symposium Ahrenshoop, Gosen (Sept. 1991)
* Another treatable background is the homogeneous field $B^a_\mu(x) = -\delta^{a3}(\beta_\perp\epsilon^{\perp}_{\mu\rho}x^{\perp}_\rho + \beta_\parallel\epsilon^{\parallel}_{\mu\rho}x^{\parallel}_\rho)$ [2–3].

$B(x)$ is part of the covariant derivative

$$D^{ab}_\mu = \delta^{ab}\partial_\mu + \epsilon^{abc}B^c_\mu(x). \tag{4}$$

involved in the ghost kernel

$$K^{ab} = -(D^2)^{ab}. \tag{5}$$

We take the Euclidean 4-dimensional case and the gauge group $SU(2)$ in adjoint representation.

The ghost propagator G fulfils

$$K^{ab}G^{bc}(x,x') = \delta^{ac}\delta(x - x'). \tag{6}$$

The kernel with background (2) reads in cylindrical coordinates

$$\begin{aligned} K^{ab} &= -\delta^{ab}\Delta + 2\epsilon^{ab3}\frac{\beta}{r^2_\perp}\frac{\partial}{\partial\varphi_\perp} + (\delta^{a1}\delta^{b1} + \delta^{a2}\delta^{b2})\frac{\beta^2}{r^2_\perp}, \\ \Delta &= \Delta_\parallel + \frac{\partial^2}{\partial r^2_\perp} + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} + \frac{1}{r^2_\perp}\frac{\partial^2}{\partial\varphi^2_\perp}, \\ \Delta_\parallel &= \frac{\partial^2}{\partial x^2_3} + \frac{\partial^2}{\partial x^2_4}. \end{aligned} \tag{7}$$

Diagonalization in the group indices leads to the expression

$$G^{ab}_\beta = (\delta^{a1}\delta^{b1} + \delta^{a2}\delta^{b2})\Re G_\beta + \epsilon^{ab3}\Im G_\beta + \delta^{a3}\delta^{b3}G_0 \tag{8}$$

with

$$\Delta_\beta G_\beta(x,x') = -\delta(x - x') \tag{9}$$

where

$$\Delta_\beta = \Delta_\parallel + \frac{\partial^2}{\partial r^2_\perp} + \frac{1}{r_\perp}\frac{\partial}{\partial r_\perp} + \frac{1}{r^2_\perp}\left(\frac{\partial}{\partial\varphi_\perp} - i\beta\right)^2. \tag{10}$$

The standard approach uses the inversion formula

$$G(x,x') = \sum_n^{(\lambda_n \neq 0)} \frac{\psi_n(x)\psi_n(x')^*}{\lambda_n}. \tag{11}$$

involving the eigenfunctions and eigenvalues of K. To succeed in the construction of the propagator in closed form it is essential that the differential equation is separable in all 4 coordinates, that the resulting ordinary differential equations are solvable in terms of known function, and that the sum (11) can be performed exactly. All these conditions are met for the background (2) and the closed form of the propagator has been found in [1].

In this paper I propose a much shorter derivation of the propagator using a technique which is applicable even to a (special case of) two-string background.

The Transformation $G_0 \to G_\beta$

We look again at the differential operator Δ_β in cylindrical coordinates (9). It is obtained from Δ_0 by the substitution

$$\frac{\partial}{\partial\varphi_\perp} \to \frac{\partial}{\partial\varphi_\perp} - i\beta. \tag{12}$$

The task is now to transform the well-known free (i.e. $\beta = 0$) propagator

$$G_0(x,x') = \frac{1}{4\pi^2(x-x')^2} \tag{13}$$

into $G_\beta(x,x')$. The claim is that we obtain G_β by means of

$$G_\beta(\dots,\Delta\varphi) = \sum_{p=-\infty}^{\infty} e^{ip\Delta\varphi} \int_{C_p} \frac{d\chi}{2\pi} e^{-i(p-\beta)\chi} G_0(\dots,\chi). \tag{14}$$

Indeed, by partial integration,

$$\left(\frac{\partial}{\partial\Delta\varphi} - i\beta\right)^2 G_\beta(\dots,\Delta\varphi) = \sum_{p=-\infty}^{\infty} e^{ip\Delta\varphi} \int_{C_p} \frac{d\chi}{2\pi} e^{-i(p-\beta)\chi} \frac{\partial^2}{\partial\chi^2} G_0(\dots,\chi) \tag{15}$$

provided we choose the path C_p in a manner to avoid boundary terms *.

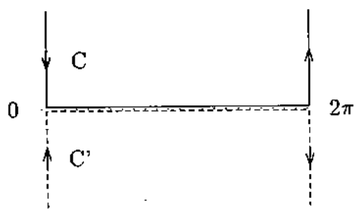

Fig. 1

The appropriate choice of the path is (see Fig. 1)

$$C_p = \begin{cases} C & \text{for } p < \beta \\ C' & \text{for } p > \beta. \end{cases} \tag{16}$$

In application to a single-string background we express the free propagator in cylindrical coordinates

$$G_0(x,x') = G_0(x_\parallel, x'_\parallel, r_\perp, r'_\perp, \Delta\varphi) = \frac{q}{\cosh\Phi - \cos\Delta\varphi} \tag{17}$$

where

$$q = \frac{1}{8\pi^2 r_\perp r'_\perp}, \quad \cosh\Phi = \frac{(x_\parallel - x'_\parallel)^2 + r_\perp^2 + {r'_\perp}^2}{2 r_\perp r'_\perp}, \quad \Delta\varphi = \varphi_\perp - \varphi'_\perp. \tag{18}$$

Furthermore we restrict β to the interval $0 \le \beta < 1$ **.

* $\int_0^{2\pi} d\chi \dots$ would contribute $(1-\exp(2\pi i\beta))[G_0\delta'(\Delta\varphi) + (G'_0 - i\beta G_0)\delta(\Delta\varphi)]$.
** The continuation to other values of β is given by $G_{\beta+m} = e^{im\Delta\varphi}G_\beta$.

The transformation (14), (16) applied to (17) leads to

$$G_\beta = \sum_{p=-\infty}^{0} e^{ip\Delta\varphi} \int_C \frac{d\chi}{2\pi} e^{-i(p-\beta)\chi} \frac{q}{\cosh\Phi - \cos\chi} + \sum_{p=1}^{\infty} e^{ip\Delta\varphi} \int_{C'} \frac{d\chi}{2\pi} e^{-i(p-\beta)\chi} \frac{q}{\cosh\Phi - \cos\chi}. \tag{19}$$

We can close the integration path at infinity (Fig. 2) and get

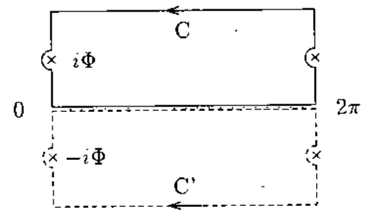

Fig. 2

$$G_\beta = \frac{q}{\sinh\Phi}\left[e^{-\beta\Phi}\sum_{p=0}^{\infty} e^{-(\Phi+i\Delta\varphi)p} + e^{\beta\Phi}\sum_{p=1}^{\infty} e^{-(\Phi-i\Delta\varphi)p}\right] = \frac{q}{\sinh\Phi}\left[\frac{e^{-\beta\Phi}}{1-e^{-\Phi-i\Delta\varphi}} + \frac{e^{-(1-\beta)\Phi+i\Delta\varphi}}{1-e^{-\Phi+i\Delta\varphi}}\right] \tag{20}$$

and finally

$$G_\beta(x,x') = \frac{1}{4\pi^2(x-x')^2} \frac{\sinh(1-\beta)\Phi + e^{i\Delta\varphi}\sinh\beta\Phi}{\sinh\Phi} \tag{21}$$

in accordance with the result presented in [1].

In more complicated cases (in particular in treating the two-string background) it may be useful to interchange in (14) summation and integration. Note that the sum

$$\sum_{p=0}^{\infty} e^{ip\varphi} = \pi \sum_{k=-\infty}^{\infty} \delta(\varphi + 2\pi k) + \frac{1}{2} + \frac{i}{2}\widetilde{\cot}(\frac{\varphi}{2}) \tag{22}$$

involves besides the δ distribution the distribution $\widetilde{\cot}(\varphi/2)$ oscillating violently around $\cot(\varphi/2)$ within the limits 0 and $2\cot(\varphi/2)$. Fig. 3 shows approximations to $\frac{1}{2}\widetilde{\cot}(x/2)$, i.e. $\sum_{n=0}^{N}\sin(nx)$ for $N = 50, 1000, 20000$.

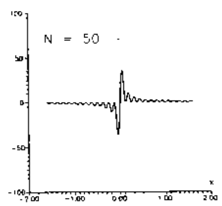

Fig. 3a

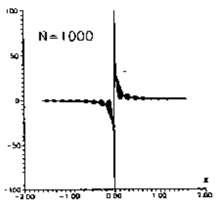

Fig. 3b

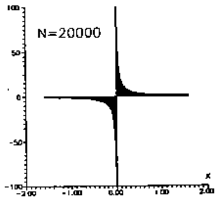

Fig. 3c

Under an integral with a smooth testfunction we can replace $\widetilde{\cot}$ by the usual cot function.

Then from (14) and (16)

$$G_\beta(\ldots,\Delta\varphi) = \int_C \frac{d\chi}{2\pi} e^{i\beta\chi} G_0(\ldots,\chi)\left(\pi\delta(\Delta\varphi-\chi)+\frac{1}{2}-\frac{i}{2}\widetilde{\cot}(\frac{\Delta\varphi-\chi}{2})\right) + \int_{C'} \frac{d\chi}{2\pi} e^{i\beta\chi} G_0(\ldots,\chi)\left(\pi\delta(\Delta\varphi-\chi)-\frac{1}{2}+\frac{i}{2}\widetilde{\cot}(\frac{\Delta\varphi-\chi}{2})\right). \quad (23)$$

Noting that

$$1 \mp i\cot\frac{\Delta\varphi-\chi}{2} = \frac{e^{\mp i\chi}-e^{\pm i\Delta\varphi}}{\cos\chi-\cos\Delta\varphi} \quad (24)$$

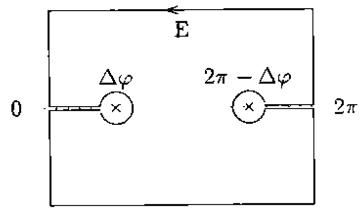

Fig. 4

we recognize that the contributions from the poles at $\chi = \Delta\varphi$ and $\chi = 2\pi - \Delta\varphi$ compensate those from $\delta(\Delta\varphi - \chi)$. There remain the contributions from the poles of $G_0(\ldots,\chi)$ inside the path E (see Fig. 4)

$$G_\beta(\ldots,\Delta\varphi) = \int_E \frac{d\chi}{4\pi} e^{i\beta\chi} \times G_0(\ldots,\chi)\frac{e^{-i\chi}-e^{i\Delta\varphi}}{\cos\chi-\cos\Delta\varphi}. \quad (25)$$

As a check we insert (17) into (25)

$$\begin{aligned} G_\beta &= \int_E \frac{d\chi}{4\pi} e^{i\beta\chi}\frac{q}{\cosh\Phi-\cos\chi}\frac{e^{-i\chi}-e^{i\Delta\varphi}}{\cos\chi-\cos\Delta\varphi} \\ &= \frac{iq}{2}\operatorname*{Res}_{\chi=\pm i\Phi}\frac{e^{i\beta\chi}}{\cosh\Phi-\cos\chi}\frac{e^{-i\chi}-e^{i\Delta\varphi}}{\cos\chi-\cos\Delta\varphi} \\ &= \frac{q}{2\sinh\Phi}\frac{1}{\cosh\Phi-\cos\Delta\varphi}\left[e^{-\beta\Phi}(e^{\Phi}-e^{i\Delta\varphi})-e^{\beta\Phi}(e^{-\Phi}-e^{i\Delta\varphi})\right] \end{aligned} \quad (26)$$

and regain (21).

An alternative form of the transformation (25) is

$$G_\beta(\ldots,\Delta\varphi) = e^{i\beta\Delta\varphi}G_0(\ldots,\Delta\varphi) + \frac{i}{4\pi}\left(e^{2\pi i\beta}-1\right)\int_{-\infty}^{\infty} d\chi e^{-\beta\chi}G_0(\ldots,i\chi)\frac{e^{\chi}-e^{i\Delta\varphi}}{\cosh\chi-\cos\Delta\varphi}. \quad (27)$$

where we take the principal value if poles of $G_0(\ldots,i\chi)$ are situated on the real χ axis.

Towards the Two-String Propagator

A two-string background can be defined by superimposing two strings of the type (1) located at $(x_1,x_2) = (\pm a, 0)$ respectively. The method to construct propagators proposed in this paper becomes applicable if we choose two strings of opposite strengths. Then the components of the vector potential for the background field are

$$\begin{aligned} B_1^3(x) &= \beta\left(\frac{x_2}{(x_1-a)^2+x_2^2}-\frac{x_2}{(x_1+a)^2+x_2^2}\right), \\ B_2^3(x) &= -\beta\left(\frac{x_1-a}{(x_1-a)^2+x_2^2}-\frac{x_1+a}{(x_1+a)^2+x_2^2}\right). \end{aligned} \quad (28)$$

We introduce bicylindrical coordinates in the (x_1,x_2)-plane

$$x_1 = \frac{a\sinh\tau}{\cosh\tau-\cos\sigma}, \qquad x_2 = \frac{a\sin\sigma}{\cosh\tau-\cos\sigma} \quad (29)$$

and find for the kernel

$$-(D^2)^{ab} = \delta^{ab}\Delta_\parallel + \left(\frac{\cosh\tau-\cos\sigma}{a}\right)^2\left[\delta^{ab}\left(\frac{\partial^2}{\partial\sigma^2}+\frac{\partial^2}{\partial\tau^2}-\beta^2\right)-2\beta\epsilon^{ab3}\frac{\partial}{\partial\sigma}\right]. \quad (30)$$

The propagator $G_\beta(x.x')$ fulfils (9) where now

$$\Delta_\beta = \Delta_\parallel + \left(\frac{\cosh\tau-\cos\sigma}{a}\right)^2\left[\frac{\partial^2}{\partial\tau^2}+\left(\frac{\partial}{\partial\sigma}-i\beta\right)^2\right]. \quad (31)$$

It is essential to remark that the simple structure of the kernel which involves β only as a shift of the angular operator $\partial/\partial\sigma$ as apparent in (31) is a consequence of the particular configuration of the two-string background: the strings must be chosen parallel and of oppositely equal strength. More general two-string configurations cannot be treated by the approach of this paper.

Since in the two-string case there is no translation invariance in the angular variable we must apply the transformation of type (14) twice, i.e. in σ and σ' separately *.

The free propagator is expressed in bicylindrical coordinates as follows

$$G_0(x_\parallel, x'_\parallel, \tau, \tau', \sigma, \sigma') = \frac{1}{4\pi^2}\frac{1}{(x_\parallel - x'_\parallel)^2 + a^2\Re\left(\coth\frac{\tau+i\sigma}{2}-\coth\frac{\tau'+i\sigma'}{2}\right)^2}. \quad (32)$$

The application of the transformation in the form (25) or (27) together with (8) provides an integral representation of the ghost propagator in the special two-string background (28). A detailed investigation is postponed to a forthcoming paper. It

* It is not very difficult to verify for the one-string case that the separate transformation in φ and φ' reproduces indeed the result (21).

seems improbable that the propagator can be expressed in terms of known functions. Nevertheless the knowledge of an integral representation for the propagator allows the study of its short- and long-distance behaviour and other limiting cases.

References

[1] H.J. KAISER: Proc. **XXIV** Symp. Ahrenshoop 1990, p.126

[2] H.J. KAISER, K. SCHARNHORST, E. WIECZOREK: J. Phys. **G16** (1990) 161

[3] C. EBERLEIN, H.J. KAISER, E. WIECZOREK: Annalen d. Physik (Leipzig) **48** (1991) 343