DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 92–076
June 1992

Mass and Width of Unstable Gauge Bosons

H. Veltman

Deutsches Elektronen-Synchrotron DESY, Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

DESY 92-076
June 1992

ISSN 0418-9833

Mass and width of unstable gauge bosons

H. Veltman
Deutsches Elektronen Synchrotron - DESY
Notkestraße 85, D-2000 Hamburg 52

Abstract

It has been proposed that the mass of the Z^0 boson should be defined as the real part, and the width of the Z^0 as the imaginary part of the complex pole M_c, for the reason that M_c is gauge invariant. This definition is only useful if we can formulate both, the S-matrix in the non-perturbative energy region around the Z^0 resonance and M_c, in a gauge invariant way up to some order in perturbation theory. By considering an infinitesimal change of the gauge breaking term, we show this indeed to be the case, and we demonstrate that the gauge invariance of M_c is a consequence of a Ward identity.

1. Introduction

The measurements relating to the Z^0 vector boson at LEP are now approaching a precision [1] requiring consideration of two loop effects [2]. A problem presents itself with respect to the mass and width of the Z^0, because treatment of an unstable particle requires a partial summation of perturbation theory, and this poses questions of gauge invariance. The question is this: the experimentalist has available data pertaining to Z^0 production around the peak. What thus is the mass and width of the Z^0 as derived from this data?

One may define the mass as the center of the peak and the width as the half width of the peak; this may perhaps produce a result depending on the particular channel that is considered. Or one may use theoretical equations to deduce mass and width including various radiative corrections. That requires a clear theoretical framework so that different experiments produce the same numbers. Thus mass and width measurement through the process $e^+e^- \to \mu^+\mu^-$ would give the same as through $e^+e^- \to$ hadrons.

The general treatment of unstable particles in perturbation theory has been given a long time ago [3]. In defining the theoretical framework one would prefer a gauge independent definition of mass and width, although this is not a must. One could also use a gauge dependent definition and then specify the gauge; usually that would be the Feynman-'t Hooft gauge.

The most natural definition of mass and width theoretically is as the real and imaginary part of the location of the pole in the complex s-plane. This is the standard definition [4] and recently this usage has also been suggested [5-7] for a gauge theory, the Standard Model, arguing that this is a gauge invariant definition, because the S-matrix in the physical region ($s > 0$ and real), and thereby also its analytical continuation, is gauge invariant.

The situation is now as follows. The experiments measure the amplitude for values of s along the positive real axis. No one can make an analytic continuation of the data, and there is no direct connection between the data and the location of the pole. To deduce the location of the pole, one needs explicit functions which then can be fitted to the data. Is it possible to produce gauge invariant functions that can be used to determine the location of the pole order by order in perturbation theory? This then is the issue that we explore in this paper, and the answer is yes.

In this article we will spell out in detail the implication of this definition within the framework of perturbation theory. The important point is thereby to avoid gauge dependent terms of higher order, implicit or explicit. Section 6 contains a technical summary.

We consider the process $e^+e^- \to \mu^+\mu^-$ and by means of an infinitesimal change of the gauge breaking term, we investigate in section 2 the one loop properties of the theory, evaluated in the energy region far away from the resonance, where we know perturbation theory is valid. We assume the external fermions to be massless and for the purpose of simplicity the photon contribution is ignored, by setting the weak mixing angle θ_w equal to zero. In section 3 we consider the Z^0 resonance energy region. We derive the conditions for the complex pole to be gauge invariant in each order in g and demonstrate that they follow from perturbation theory, i.e. the results obtained in section 2 are rederived. In the same fashion, we show that the S-matrix in this non-perturbative energy region may be formulated in a gauge invariant manner up to a certain order in g. In section 4 we demonstrate that the gauge invariance of the complex pole is a consequence of a Ward identity. In section 5 we give the necessary details needed to incorporate the photon contribution, and in section 6 we briefly summarize what must be calculated to obtain the expressions for the location of the complex pole and the S-matrix element, up to the accuracy required by the LEP-data. Section 7 contains a discussion of the results.

Our metric is such that $p^2 = -m^2$ for a particle on mass-shell with mass m and momentum p.

2. A one loop calculation

2.1 Introduction

Before we investigate the gauge invariance of the S-matrix near the Z^0 resonance, the gauge invariance of the complex pole ... etc., it is very useful to understand first the properties of the theory as derived from the gauge invariance of the S-matrix far away from the Z^0 resonance, thus in the energy region where perturbation theory is valid. To this purpose we do an explicit one loop calculation, which can be described as follows. First we look at the S-matrix element at one loop for the process $e^+e^- \rightarrow \mu^+\mu^-$ in the perturbative energy region in the Feynman-'t Hooft gauge. There will be the vector boson self energy contribution, the vertex corrections, the external electron and muon line corrections and of course the box diagrams. We then make an infinitesimal change of the gauge breaking term, and we will obtain different expressions for each of these contributions. But we know that the S-matrix element is gauge invariant order by order in perturbation theory and thus also at the one loop order that we are considering. Therefore in the end the gauge dependence must drop out. We will investigate if the constraint of the gauge invariance of the S-matrix implies certain constraints on the gauge dependence of each of these contributions. For example consider the vector boson self energy at one loop, first in the Feynman-'t Hooft gauge

$$\Pi(k,g^2)_{\alpha\beta} = \delta_{\alpha\beta}\Pi(k^2,g^2) + k_\alpha k_\beta Q(k^2,g^2). \tag{2.1}$$

$\Pi_{\alpha\beta}$ is a function of the momentum k and the coupling constant g. Since we will only consider the case where the external fermions are massless, we do not have to worry about the $k_\alpha k_\beta$ term and in the following we will only consider the $\delta_{\alpha\beta}$ piece. If we now make an infinitesimal change of the gauge breaking term, then the self energy will change by an amount $\epsilon X(k^2,g^2)$

$$\Pi(k^2,g^2) \rightarrow \Pi(k^2,g^2) + \epsilon X(k^2,g^2), \tag{2.2}$$

where ϵ is infinitesimally small. Indeed, in order for gauge invariance of the matrix element at one loop to be maintained, we will find that $X(k^2,g^2)$ needs to satisfy the property that it is equal to zero at $k^2 = -M^2$;

$$X(-M^2,g^2) = 0. \tag{2.3}$$

The above equation will reappaer, when we want to describe the gauge invariance of the S-matrix near the Z^0 resonance and of the complex pole order by order in perturbation theory. It is precisely for this reason that we give this one loop calculation in considerable detail; once this is understood, the rest will follow.

In section 2.2 we define the process $e^+e^- \rightarrow \mu^+\mu^-$ at the one loop level. We consider an infinitesimal change of the gauge breaking term and based on the gauge invariance of the S-matrix element at one loop, we derive the above equation for $X(-M^2,g^2)$. We then proceed by doing an explicit one loop calculation, thereby confirming the results obtained in section 2.2. In section 2.3, starting from the Feynman-'t Hooft gauge, we will first derive the modified Feynman rules resulting from an ϵ change of gauge. Using these new Feynman rules, we calculate in section 2.4 the vector boson self energy at one loop and determine the expression for $X(k^2,g^2)$, thereby confirming eq.(2.3). In section 2.5 the remaining one loop corrections to the process $e^+e^- \to \mu^+\mu^-$ are evaluated. In section 2.6 we discuss the results. In section 2.7 we briefly consider the amplitude at the two loop level.

2.2 The one loop amplitude for $e^+e^- \to \mu^+\mu^-$.

Our model corresponds to the Standard Model Lagrangian where the weak mixing angle θ_w has been set equal to zero, and we thus fully neglect the photon contribution. It follows that there is only one diagram in lowest non-zero order for the process $e^+e^- \to \mu^+\mu^-$, which is displayed in fig.1. The corresponding expression is given by

$$A^0(e^+e^- \to \mu^+\mu^-) = \left(-\frac{ig}{4}\right)^2 \cdot \bar{u}(p_2)\gamma^\alpha(1+\gamma^5)u(p_1)\cdot\frac{1}{k^2+M^2-i\epsilon}\cdot\bar{u}(q_2)\gamma^\alpha(1+\gamma^5)u(q_1) \quad (2.4)$$

The momenta p_1, p_2, q_1, q_2 are all ingoing and are defined in fig.1. The external fermions are assumed to be massless, thus $p_1^2 = p_2^2 = q_1^2 = q_2^2 = 0$, and $-k^2 = s$ is the center of mass energy squared. M is the vector boson mass and g is the coupling constant. For the discussion in this section it is not necessary to specify the Lorentz indices explicitly, and we write the amplitude in compact notation as follows

$$A^0(e^+e^- \to \mu^+\mu^-) = V(g)\frac{1}{k^2+M^2-i\epsilon}V(g), \quad (2.5)$$

where $V(g)$ is the tree level $Zf\bar{f}$ vertex. For the initial vertex $f = e$, and for the final vertex $f = \mu$.

Fig.2 displays the contributions to the one loop correction of the matrix element of this process. Fig.2a shows the vector boson self energy contribution, and the corresponding expression may be written as

$$A^1(\mathrm{Selfw}) = V(g)\frac{1}{k^2+M^2-i\epsilon}\Pi(k^2,g^2)\frac{1}{k^2+M^2-i\epsilon}V(g). \quad (2.6)$$

Here $\Pi(k^2,g^2)$ is the $\delta_{\alpha\beta}$ part of the self energy of the vector boson at one loop (see eq.(2.1)). For the vertex corrections of fig.2b we have

$$A^1(\mathrm{Vertex}) = V(k^2,g^3)\frac{1}{k^2+M^2-i\epsilon}V(g) + V(g)\frac{1}{k^2+M^2-i\epsilon}V(k^2,g^3). \quad (2.7)$$

Here $V(k^2,g^3)$ denotes the vertex correction, which includes the wave function renormalization of the external fermions. For the box diagram of fig.2c we write

$$A^1(\mathrm{Box}) = B(k^2,g^4), \quad (2.8)$$

and the one loop amplitude for $e^+e^- \to \mu^+\mu^-$ is given by

$$A^1(e^+e^- \to \mu^+\mu^-) = A^1(\mathrm{Selfw}) + A^1(\mathrm{Vertex}) + A^1(\mathrm{Box}). \quad (2.9)$$

Next, we consider an infinitesimal change of the gauge breaking term, and $\Pi(k^2,g^2)$, $V(k^2,g^2)$ and $B(k^2,g^2)$ are changed by an infinitesimal amount as follows

$$\begin{aligned}\Pi(k^2,g^2) &\to \Pi(k^2,g^2) + \epsilon X(k^2,g^2),\\ V(k^2,g^3) &\to V(k^2,g^3) + \epsilon\dot{Y}(k^2,g^3),\\ B(k^2,g^4) &\to B(k^2,g^4) + \epsilon Z(k^2,g^4).\end{aligned} \quad (2.10)$$

The infinitesimal quantity ϵ parametrizes the change of the gauge breaking term.

We know that the S-matrix in each order in g has to be gauge invariant. Therefore $A^1(e^+e^- \to \mu^+\mu^-)$ is invariant for the above transformations of eq.(2.10) and must be independent of ϵ. We thus require that

$$\begin{aligned}&V(g)\frac{1}{k^2+M^2-i\epsilon}X(k^2,g^2)\frac{1}{k^2+M^2-i\epsilon}V(g)\\ &+Y(k^2,g^3)\frac{1}{k^2+M^2-i\epsilon}V(g) + V(g)\frac{1}{k^2+M^2-i\epsilon}Y(k^2,g^3)\\ &+Z(k^2,g^4) = 0.\end{aligned} \quad (2.11)$$

We may expand X, Y and Z about $k^2 + M^2 = 0$ as follows;

$$\begin{aligned}X(k^2,g^2) =& X(-M^2,g^2) + (k^2+M^2)X'(-M^2,g^2)\\ &+\frac{1}{2}(k^2+M^2)^2X''(-M^2,g^2) + (k^2+M^2)^3X_R(k^2,g^2),\end{aligned} \quad (2.12)$$

$$Y(k^2,g^2) = Y(-M^2,g^2) + (k^2+M^2)Y'(-M^2,g^2) + (k^2+M^2)^2Y_R(k^2,g^2), \quad (2.13)$$

$$Z(k^2,g^4) = Z(-M^2,g^4) + (k^2+M^2)Z_R(k^2,g^4). \quad (2.14)$$

The notations used in the above equations should be clear; for example $X'(-M^2,g^2)$ is the derivative of $X(k^2,g^2)$ with respect to k^2, subsequently evaluated at $k^2 = -M^2$. Substituting these expressions into eq.(2.11) we obtain

$$\begin{aligned}&\frac{1}{(k^2+M^2)^2}\cdot V(g)X(-M^2,g^2)V(g)\\ &+\frac{1}{k^2+M^2}\cdot\{V(g)X'(-M^2,g^2)V(g) + Y(-M^2,g^3)V(g) + V(g)Y(-M^2,g^3)\}\\ &+\frac{1}{2}V(g)X''(-M^2,g^2)V(g) + Y'(-M^2,g^3)V(g) + V(g)Y'(-M^2,g^3) + Z(-M^2,g^4)\\ &+(k^2+M^2)\{V(g)X_R(k^2,g^2)V(g) + Y_R(k^2,g^2)V(g) + V(g)Y_R(k^2,g^2)\\ &+ Z_R(k^2,g^4)\} = 0.\end{aligned} \quad (2.15)$$

For each given order in $k^2 + M^2$ the argument has to be zero and it follows immediately that we must have

$$X(-M^2, g^2) = 0. \tag{2.16}$$

Similarily we obtain for example

$$V(g)X'(-M^2, g^2)V(g) + Y(-M^2, g^3)V(g) + V(g)Y(-M^2, g^3) = 0, \tag{2.17}$$

and also

$$\begin{aligned} &Y'(-M^2, g^3)V(g) + V(g)Y'(-M^2, g^3) \\ &+\frac{1}{2}V(g)X''(-M^2, g^2)V(g) + Z(-M^2, g^4) = 0. \end{aligned} \tag{2.18}$$

In the next sections we are going to derive the expressions for X, Y and Z as a function of k^2 and verify that at $k^2 = -M^2$ the above results are reproduced.

2.3 An infinitesimal change of the gauge breaking term.

A calculation is done by using Feynman rules, which in turn are derived from the Lagrangian. First consider the Lagrangian in a two parameter gauge:

$$\mathcal{L} = \mathcal{L}_{\rm inv} + \mathcal{L}(\kappa, \lambda)_{\rm gf} + \mathcal{L}(\kappa, \lambda)_{\rm FP}. \tag{2.19}$$

Here $\mathcal{L}_{\rm inv}$ is the part of the Lagrangian that is invariant under the gauge transformations of the fields, $\mathcal{L}_{\rm gf}$ is the gauge fixing term and $\mathcal{L}_{\rm FP}$ is the corresponding Faddeev-Popov ghost Lagrangian. The explicit form of the gauge-fixing term is given by

$$\mathcal{L}(\kappa, \lambda)_{\rm gf} = -\frac{1}{2}(-\kappa\partial^\alpha W^a_\alpha + \lambda M\phi^a)^2. \tag{2.20}$$

Here ϕ is the Higgs ghost, $a = 1, 2, 3$ is the isospin index and the neutral vector boson Z^0 in this notation corresponds to W^3. We have a renormalizable gauge if $\kappa \neq 0$. In order to avoid $W\phi$ mixing at the tree level, make the choice $\kappa = \frac{1}{\lambda}$;

$$\mathcal{L}(\lambda)_{\rm gf} = -\frac{1}{2}\left(-\frac{1}{\lambda}\partial_\alpha W^a_\alpha + \lambda M\phi^a\right)^2, \tag{2.21}$$

and the corresponding Faddeev-Popov ghost Lagrangian is given by

$$\begin{aligned} \mathcal{L}(\lambda)_{\rm FP} =& \psi^{a*}(\frac{1}{\lambda}\partial^2 - \lambda M^2)\psi^a + \frac{1}{\lambda}g\epsilon^{abc}\partial_\mu\psi^{a*}\psi^b W^c_\mu \\ & \frac{1}{2}g\lambda M\epsilon^{abc}\psi^{a*}\psi^b\phi^c - \frac{1}{2}g\lambda M\psi^{a*}\psi^a H, \end{aligned} \tag{2.22}$$

where ψ is the Faddeev-Popov ghost and H is the Higgs field. We have the Feynman-'t Hooft gauge for $\lambda = 1$. Starting from the Feynman-'t Hooft gauge, the infinitesimal change of gauge is:

$$\lambda = 1 \to \lambda = 1 + \epsilon. \tag{2.23}$$

Some Feynman rules change. Up to first order in ϵ, the Feynman rules that are modified are:

(1) The vector boson propagator

$$\Delta^{ab}_{\alpha\beta}(\lambda = 1 + \epsilon) = \delta^{ab}\left\{\frac{\delta_{\alpha\beta}}{k^2 + M^2} + 2\epsilon\frac{k_\alpha k_\beta}{(k^2 + M^2)^2}\right\}. \tag{2.24}$$

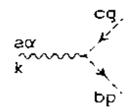
(2) The Higgs ghost propagator

$$\Delta^{ab}(\lambda = 1 + \epsilon) = \delta^{ab}\left\{\frac{1}{k^2 + M^2} - 2\epsilon\frac{M^2}{(k^2 + M^2)^2}\right\}. \tag{2.25}$$

(3) The Faddeev-Popov ghost propagator

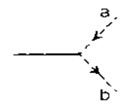
$$\Delta^{ab}(\lambda = 1 + \epsilon) = \delta^{ab}\left\{\frac{1}{k^2 + M^2} + \epsilon\frac{(k^2 - M^2)}{(k^2 + M^2)^2}\right\}. \tag{2.26}$$

(4) The $W^a\bar{\psi}^b\psi^c$ vertex



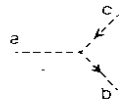
$$i(1 - \epsilon)g\epsilon^{abc}p_\alpha. \tag{2.27}$$

(5) The $H\bar{\psi}^b\psi^a$ vertex



$$-\frac{1}{2}(1 + \epsilon)Mg\delta^{ab}. \tag{2.28}$$

(6) The $\phi^a\bar{\psi}^b\psi^c$ vertex



$$\frac{1}{2}(1 + \epsilon)Mg\epsilon^{abc}. \tag{2.29}$$

Setting $\epsilon = 0$, we recover the Feynman rules in the Feynman-'t Hooft gauge.

We note that in ref.[7] a one loop calculation has been performed for four fermion processes in the general $\kappa = \frac{1}{\lambda}$ gauge. Also in there the external fermions are considered massless.

2.4 The vector boson self energy

At one loop the vector boson self energy in the $\lambda = 1+\epsilon$ gauge is given by

$$\Pi(\lambda = 1+\epsilon, k^2, g^2) = \Pi(\lambda = 1, k^2, g^2) + \epsilon X(k^2, g^2). \tag{2.30}$$

Using dimensional regularization, we will now calculate $X(k^2, g^2)$, according to the modified Feynman rules derived in section 2.3. The one loop diagrams are displayed in fig.3. Fig.3a shows the diagrams that give a non-zero contribution after a change of gauge. The sum of the two diagrams of fig.3b is zero and thus no Higgs mass dependence remains. The gauge invariant diagrams are shown in fig.3c, and they therefore do not contribute to X.

Let us work out in detail the first diagram of fig.3a, i.e. $\Pi(\lambda = 1+\epsilon, k^2, g^2)^{\phi\phi}_{\alpha\beta}$. It is only because of the modified propagator of the Higgs ghost ϕ that this diagram gives a non-zero contribution to X, since the $W\phi\phi$ vertex is gauge invariant. Using eq.(2.25), we can immediately write down the corresponding expression:

$$\begin{aligned}\Pi(\lambda = 1+\epsilon, k^2, g^2)^{\phi\phi}_{\alpha\beta} =& \frac{g^2}{4}\int d_nq \frac{(2q+k)_\alpha(2q+k)_\beta}{(q^2+M^2)\{(q+k)^2+M^2\}}\cdot \\ & \left\{1 - \frac{2\epsilon M^2}{(q^2+M^2)}\right\}\cdot\left\{1 - \frac{2\epsilon M^2}{\{(q+k)^2+M^2\}}\right\}.\end{aligned} \tag{2.31}$$

The above expression will contain a $\delta_{\alpha\beta}$ piece and a $k_\alpha k_\beta$ piece. But remember that in the end we consider massless external fermions and only the $\delta_{\alpha\beta}$ contribution needs to be retained, which in this case comes from the $q_\alpha q_\beta$ term after integration. Keeping only this $q_\alpha q_\beta$ term, we have up to first order in ϵ

$$\begin{aligned}\Pi(\lambda = 1+\epsilon, k^2, g^2)^{\phi\phi}_{\alpha\beta} =& \Pi(\lambda = 1, k^2, g^2)^{\phi\phi}_{\alpha\beta} - 2g^2M^2\epsilon\int d_nq\cdot(q_\alpha q_\beta)\cdot \\ & \left\{\frac{1}{(q^2+M^2)^2\{(q+k)^2+M^2\}} + \frac{1}{(q^2+M^2)\{(q+k)^2+M^2\}^2}\right\}.\end{aligned} \tag{2.32}$$

The other diagrams of fig.3a are evaluated in exactly the same fashion. After having done this and after adding up the contributions of all the diagrams of fig.3a, we arrive at the result

$$\begin{aligned}\Pi(\lambda = 1+\epsilon, k^2, g^2)_{\alpha\beta} =& \Pi(\lambda = 1, k^2, g^2)_{\alpha\beta} + X(k^2, g^2)_{\alpha\beta} \\ =& \Pi(\lambda = 1, k^2, g^2)_{\alpha\beta} \\ & + \epsilon(k^2+M^2)F_1(k^2, g^2)_{\alpha\beta} + \epsilon(k^2+M^2)^2F_2(k^2, g^2)_{\alpha\beta},\end{aligned} \tag{2.33}$$

where

$$F_1(k^2, g^2)_{\alpha\beta} = g^2\int d_nq\left\{\frac{4q_\alpha q_\beta}{P_1^2P_2} + \frac{4q_\alpha q_\beta}{P_1P_2^2} - \frac{4\delta_{\alpha\beta}}{P_1^2}\right\} \tag{2.34}$$

and

$$F_2(k^2, g^2)_{\alpha\beta} = g^2\int d_nq\left\{\frac{2\delta_{\alpha\beta}}{P_1^2P_2} + \frac{2\delta_{\alpha\beta}}{P_1P_2^2}\right\}. \tag{2.35}$$

In the above equations we have defined

$$\begin{aligned}P_1 =& q^2 + M^2, \\ \text{and}\quad P_2 =& (q+k)^2 + M^2.\end{aligned} \tag{2.36}$$

Note that we do not bother to do the integration over q. We only have to keep in mind that $X(k^2, g^2)$ is given by the $\delta_{\alpha\beta}$ part of $X(k^2, g^2)_{\alpha\beta}$.

We see that $X(k^2, g^2)$ is proportional to $k^2 + M^2$ and as in agreement with eq.(2.16), it is zero at $k^2 + M^2 = 0$. This we had previously derived in section 2.2, based on the gauge invariance of the S-matrix element at one loop.

2.5 The vertex correction, wave function renormalization and the box diagrams

The diagrams contributing to the one loop correction of the lowest order $Z^0e^+e^-$ vertex are shown in fig.4. After an infinitesimal change of gauge, this vertex correction will change accordingly as follows (compare with eq.(2.10))

$$V^V(\lambda = 1+\epsilon, k^2, g^3)_\alpha = V^V(\lambda = 1, k^2, g^3)_\alpha + \epsilon Y^V(k^2, g^3)_\alpha. \tag{2.37}$$

The expression for Y^V_α may be derived by evaluating the diagrams of fig.4, using the modified Feynman rules of section 2.3. The result simplifies to

$$Y^V(k^2, g^3)_\alpha = V(g)_\beta\cdot\{F_3(k^2, g^2)_{\alpha\beta} + (k^2+M^2)F_4(k^2, g^2)_{\alpha\beta}\}, \tag{2.38}$$

where

$$F_3(k^2, g^2)_{\alpha\beta} = g^2\int d_nq\cdot\left\{\delta_{\alpha\beta}\left(\frac{4}{P_1^2} - \frac{1}{2P_1^2}\right) + q_\alpha q_\beta\left(-\frac{2}{P_1^2P_2} - \frac{2}{P_1P_2^2}\right)\right\} \tag{2.39}$$

and

$$F_4(k^2, g^2)_{\alpha\beta} = g^2\int d_nq\cdot\delta_{\alpha\beta}\left(-\frac{2}{P_1^2P_2} - \frac{2}{P_1P_2^2}\right). \tag{2.40}$$

P_1 and P_2 are defined in eq.(2.36). There are only two diagrams corresponding to the electron self energy at one loop, and they are shown in fig.5. After wave function renormalization, the external fermion lines are multiplied by the following factors;

$$\text{Incoming electron}:\ u \to \frac{1}{2}(A + B\gamma^5)u, \tag{2.41}$$

$$\text{Incoming positron}:\ \bar{u} \to \bar{u}\frac{1}{2}(A - B\gamma^5), \tag{2.42}$$

and for the corresponding correction to the vertex we may write

$$V_\alpha^{\rm WF} = V(g)_\alpha (A+B). \tag{2.43}$$

An infinitesimal change of the gauge breaking term leads to the following change of $V_\alpha^{\rm WF}$;

$$V^{\rm WF}(\lambda = 1+\epsilon, k^2, g^3)_\alpha = V^{\rm WF}(\lambda = 1, k^2, g^3)_\alpha + \epsilon Y^{\rm WF}(k^2, g^3)_\alpha, \tag{2.44}$$

where

$$Y^{\rm WF}(k^2, g^3)_\alpha = -\frac{3g^2}{2} V(g)_\beta \int d_n q \frac{\delta_{\alpha\beta}}{P_1^2} = V(g)_\beta F_5(k^2, g^2)_{\alpha\beta}. \tag{2.45}$$

Defining $Y = Y^{\rm V} + Y^{\rm WF}$, we have

$$Y(k^2, g^3)_{\alpha\beta} = V(g)_\beta \cdot \{F_3(k^2, g^2)_{\alpha\beta} + F_5(k^2, g^2)_{\alpha\beta} + (k^2+M^2) F_4(k^2, g^2)_{\alpha\beta}\}. \tag{2.46}$$

The box diagrams are shown in fig.6. Defining

$$B(\lambda = 1+\epsilon, k^2, g^4) = B(\lambda = 1, k^2, g^4) + \epsilon Z(k^2, g^4), \tag{2.47}$$

we find

$$Z(k^2, g^4) = V(g)_\alpha V(g)_\beta \cdot g^2 \int d_n q \delta_{\alpha\beta} \left(\frac{2}{P_1^2 P_2} + \frac{2}{P_1 P_2^2} \right) = V(g)_\alpha V(g)_\beta F_6(k^2, g^2)_{\alpha\beta}. \tag{2.48}$$

2.6 Discussion of the results

Using the results obtained in sections 2.4 and 2.5, we are now going to verify eq.(2.11), which states that in first order in ϵ, the resulting change in the S-matrix element at one loop is zero. Putting back the Lorentz indices, eq.(2.11) reads

$$\begin{aligned} & V(g)_\alpha \frac{1}{k^2+M^2} X(k^2, g^2)_{\alpha\beta} \frac{1}{k^2+M^2} V(g)_\beta \\ & + Y(k^2, g^3)_\alpha \frac{1}{k^2+M^2} V(g)_\alpha + V(g)_\alpha \frac{1}{k^2+M^2} Y(k^2, g^3)_\alpha \\ & + Z(k^2, g^4) = 0. \end{aligned} \tag{2.49}$$

Substituting the expressions of eqs.(2.33), (2.46) and (2.47) into eq.(2.49), we obtain the following equation, expressed in terms of the F_i's;

$$\begin{aligned} & \frac{1}{k^2+M^2} \{F_1(k^2, g^2)_{\alpha\beta} + 2F_3(k^2, g^2)_{\alpha\beta} + 2F_5(k^2, g^2)_{\alpha\beta}\} \\ & + F_2(k^2, g^2)_{\alpha\beta} + 2F_4(k^2, g^2)_{\alpha\beta} + F_6(k^2, g^2)_{\alpha\beta} = 0. \end{aligned} \tag{2.50}$$

We find that indeed this equation is satisfied. It may verified that the following equations hold separately, i.e.

$$F_1(k^2, g^2)_{\alpha\beta} + 2F_3(k^2, g^2)_{\alpha\beta} + 2F_5(k^2, g^2)_{\alpha\beta} = 0, \tag{2.51}$$

$$F_2(k^2, g^2)_{\alpha\beta} + 2F_4(k^2, g^2)_{\alpha\beta} + F_6(k^2, g^2)_{\alpha\beta} = 0. \tag{2.52}$$

Eqs.(2.17) and (2.18) may now be readily verified.

2.7 An infinitesimal change of gauge at two loops

Based on the gauge invariance of the S-matrix, we were able to derive in section 2.2 that at one loop

$$X(-M^2, g^2) = 0.$$

We now derive in a similar fashion what the constraint is for $X(k^2, g^4)$ at two loops. Thus let

$$\Pi(\lambda = 1+\epsilon, k^2, g^4) = \Pi(\lambda = 1, k^2, g^4) + \epsilon X(k^2, g^4), \tag{2.53}$$

where Π denotes the *irreducible* self energy contribution. Furthermore

$$\begin{aligned} & V(\lambda = 1+\epsilon, k^2, g^5) = V(\lambda = 1, k^2, g^5) + \epsilon Y(k^2, g^5), \\ & B(\lambda = 1+\epsilon, k^2, g^6) = B(\lambda = 1, k^2, g^6) + \epsilon Z(k^2, g^6). \end{aligned} \tag{2.54}$$

The external line corrections are absorbed in the above expressions, as they do not change the argument of our derivation and do not need to be specified separately. The matrix element at the two loop order for the process $e^+e^- \to \mu^+\mu^-$ is displayed in fig.7. Just like was done in section 2.2, we now consider the above infinitesimal change of gauge of eqs.(2.53) and (2.54) and impose that the resulting S-matrix element is gauge invariant. Retaining only the terms up to first order in ϵ, we arrive at the following equation

$$\begin{aligned} & \frac{1}{(k^2+M^2)^3} \cdot 2V(g) X(k^2, g^2) \Pi(k^2, g^2) V(g) \\ & + \frac{1}{(k^2+M^2)^2} \{V(g) X(k^2, g^4) V(g) + Y(k^2, g^3) \Pi(k^2, g^2) V(g) \\ & + V(g) \Pi(k^2, g^2) Y(k^2, g^3) + V(k^2, g^3) X(k^2, g^2) V(g) + V(g) X(k^2, g^2) V(k^2, g^3)\} \\ & + \frac{1}{k^2+M^2} \{Y(k^2, g^3) V(k^2, g^3) + V(k^2, g^3) Y(k^2, g^3) \\ & + Y(k^2, g^5) V(g) + V(g) Y(k^2, g^5)\} \\ & + Z(k^2, g^6) = 0. \end{aligned} \tag{2.55}$$

Again, expand all functions that are dependent on k^2 about $k^2 + M^2 = 0$. The above equation becomes

$$\begin{aligned}
&\frac{1}{(k^2+M^2)^3}\cdot 2V(g)X(-M^2,g^2)\Pi(-M^2,g^2)V(g)\\
&+\frac{1}{(k^2+M^2)^2}\cdot\{2V(g)X'(-M^2,g^2)\Pi(-M^2,g^2)V(g)\\
&+2V(g)X(-M^2,g^2)\Pi'(-M^2,g^2)V(g)+V(g)X(-M^2,g^4)V(g)\\
&+Y(-M^2,g^3)\Pi(-M^2,g^2)V(g)+V(g)\Pi(-M^2,g^2)Y(-M^2,g^3)\\
&+V(-M^2,g^3)X(-M^2,g^2)V(g)+V(g)X(-M^2,g^2)V(-M^2,g^3)\}\\
&+\frac{1}{k^2+M^2}\{\ldots\}+\ldots=0.
\end{aligned} \tag{2.56}$$

We have written down explicitly only the coefficients of the two lowest orders in $k^2 + M^2$. The coefficient for each given order in $k^2 + M^2$ must be equal to zero and we once again rederive eq.(2.16);

$$X(-M^2,g^2)=0.$$

Consider now the coefficient of the $1/(k^2+M^2)^2$ term. Substituting the equation $X(-M^2,g^2)=0$, we are left with

$$\begin{aligned}
&2V(g)X'(-M^2,g^2)\Pi(-M^2,g^2)V(g)+V(g)X(-M^2,g^4)V(g)\\
&+Y(-M^2,g^3)\Pi(-M^2,g^2)V(g)+V(g)\Pi(-M^2,g^2)Y(-M^2,g^3)=0.
\end{aligned} \tag{2.57}$$

In section 2.2 we derived (see eq.(2.17))

$$V(g)X'(-M^2,g^2)V(g)+Y(-M^2,g^3)V(g)+V(g)Y(-M^2,g^3)=0. \tag{2.58}$$

After multiplying the above equation with $\Pi(-M^2,g^2)$ and substituting into eq.(2.57), we obtain

$$X(-M^2,g^4)+X'(-M^2,g^2)\Pi(-M^2,g^2)=0. \tag{2.59}$$

It is clear, that when we consider a change of gauge at the i^{th} loop order, this will give us a constraint for $X(-M^2,g^{2i})$.

For completeness, we also write the coefficient of the $1/(k^2+M^2)$ term;

$$\begin{aligned}
&V(g)Y(-M^2,g^5)+Y(-M^2,g^5)V(g)\\
&+Y(-M^2,g^3)V(-M^2,g^3)+V(-M^2,g^3)Y(-M^2,g^3)\\
&+V(g)X'(-M^2,g^4)V(g)+Y(-M^2,g^3)\Pi'(-M^2,g^2)V(g)\\
&+V(-M^2,g^3)X'(-M^2,g^2)V(g)+V(g)X'(-M^2,g^2)V(-M^2,g^3)\\
&+V(g)\Pi'(-M^2,g^2)Y(-M^2,g^3)+V(g)\Pi(-M^2,g^2)X''(-M^2,g^2)V(g)\\
&+V(g)\Pi(-M^2,g^2)Y'(-M^2,g^3)+Y'(-M^2,g^3)\Pi(-M^2,g^2)V(g)\\
&+2V(g)X'(-M^2,g^2)\Pi'(-M^2,g^2)V(g)=0.
\end{aligned} \tag{2.60}$$

3. The process $e^+e^- \to \mu^+\mu^-$ near the Z^0 resonance

3.1 Introduction

Consider the lowest non-zero order matrix element of fig.1, for the process $e^+e^- \to \mu^+\mu^-$, as given by eq.(2.5);

$$A^0(e^+e^- \to \mu^+\mu^-) = V(g)\frac{1}{k^2+M^2-i\epsilon}V(g). \tag{3.1}$$

For energy values for which $k^2 + M^2$ is large, this diagram is of the order g^2. In this perturbative energy region we know unambiguously the order in g of any diagram. For instance the diagrams of fig.2 are $\mathcal{O}(g^4)$ and for this set of diagrams Ward identities hold and gauge invariance is exact. The same can be said for example for the $\mathcal{O}(g^6)$ set of diagrams displayed in fig.7.

But what happens when we approach the Z^0 resonance, thus when k^2 approaches $-M^2$? The propagator, given in eq.(3.1), blows up and in this non-perturbative region things are not well-defined.

As is well-known [3], the trick is to consider the dressed propagator, given by the summation of bubble graphs. In order to find the lowest order matrix element for $e^+e^- \to \mu^+\mu^-$ in the resonance region, we only need to do a partial summation of the one loop self energy bubbles, to obtain

$$A^0(e^+e^- \to \mu^+\mu^-) = V(g)\frac{1}{k^2+M^2-\Pi(k^2,g^2)}V(g), \tag{3.2}$$

where $\Pi(k^2,g^2)$ is the $\delta_{\alpha\beta}$ part of the one loop self energy of the vector boson. Near the resonance $k^2+M^2-\Pi(k^2,g^2)=\mathcal{O}(g^2)$, and in lowest non-zero order the S-matrix element is $\mathcal{O}(1)$. If we want to calculate the matrix element up to $\mathcal{O}(g^2)$, then we need to include the partial summation of the one and two loop self energy bubbles. The vertex diagrams of fig.2b need to be included as well, however in this case the box diagram of fig.2c will not contribute, being of $\mathcal{O}(g^4)$.

It is obvious what is happening; due to the fact that $k^2+M^2-\Pi(k^2,g^2)$ is of order $\mathcal{O}(g^2)$, we are mixing up the orders of the diagrams. Therefore the question arises, whether we can still talk about gauge invariance order by order in perturbation theory. This we investigate in this chapter and in section 3.2 we start by showing the gauge invariance of the complex pole as derived from the gauge invariance of the S-matrix. In section 3.3 we proceed by discussing the gauge invariance of the complex pole order by order in perturbation theory.

3.2 The S-matrix, Part I

We begin by defining the complex pole, M_c. After this we proceed by writing down the complete S-matrix element for the process $e^+e^- \to \mu^+\mu^-$. We know that the S-matrix is gauge invariant everywhere, and thus also in the Z^0 resonance energy region. From this fact we will deduce the gauge invariance of M_c, by considering an infinitesimal change of the gauge breaking term, done in the same way as described in section 2.2.

The expression for the complete dressed propagator of the vector boson is given by

$$\frac{1}{k^2+M^2}\sum_{j=0}^{\infty}\left(\frac{\Pi(k^2)}{k^2+M^2}\right)^j = \frac{1}{k^2+M^2-\Pi(k^2)}. \tag{3.3}$$

Here $\Pi(k^2)$ is the $\delta_{\alpha\beta}$ part of the collection of all *irreducible* self energy graphs in all orders of perturbation theory. Thus

$$\Pi(k^2) = \sum_{j=1}^{\infty}\Pi(k^2, g^{2j}). \tag{3.4}$$

The series on the left hand side of eq.(3.3) is clearly divergent in the neighbourhood $k^2 = -M^2$. The right hand side is perfectly well behaved and must be used in the region around the resonance. We remark that a long time ago it has been shown that the S-matrix, with the prescription that the propagator of the unstable particle is given by the right hand side of eq.(3.3), satisfies the requirements of unitarity and causality [3].

The mass and width are defined by the location of the zero of the denominator. Let M_c be the (complex) mass for which this zero occurs, i.e.

$$-M_c^2 + M^2 - \Pi(-M_c^2) = 0. \tag{3.5}$$

This zero is located on the second Riemann sheet. The mass M_z and width Γ_z of the unstable particle are then given by the real and imaginary part of M_c^2, i.e.

$$M_c^2 = M_z^2 - iM_z\Gamma_z. \tag{3.6}$$

As argued by Stuart [5] on the basis of analytic continuation, the location of the complex pole M_c is gauge invariant.

The expression for the S-matrix element for $e^+e^- \to \mu^+\mu^-$ may be written as

$$A(e^+e^- \to \mu^+\mu^-) = V(k^2)\frac{1}{k^2+M^2-\Pi(k^2)}V(k^2) + B(k^2). \tag{3.7}$$

Here $V(k^2)$ represents the collection of the $Vf\bar{f}$ vertices (f(initial) $= e$, f(final) $= \mu$), including the external fermion line corrections, in all orders in g. B represents box type diagrams that do not contain a pole. The S-matrix is gauge invariant, and because of analytic continuation it is also gauge invariant in the complex plane. It follows that the location of the pole, i.e. M_c, is gauge invariant. Just like we did in section 2.2, we will now verify this by considering an infinitesimal change of gauge. Thus, following eq.(2.10),

$$\begin{aligned}\Pi(k^2) &\to \Pi(k^2) + \epsilon X(k^2),\\ V(k^2) &\to V(k^2) + \epsilon Y(k^2),\\ B(k^2) &\to B(k^2) + \epsilon Z(k^2).\end{aligned} \tag{3.8}$$

By requiring that the S-matrix of eq.(3.7) is invariant for the above transformations, we obtain the following equation for X, Y and Z;

$$\frac{V(k^2)X(k^2)V(k^2)}{\{k^2+M^2-\Pi(k^2)\}^2} + \frac{Y(k^2)V(k^2)+V(k^2)Y(k^2)}{\{k^2+M^2-\Pi(k^2)\}} + Z(k^2) = 0. \tag{3.9}$$

By expanding about $k^2 = -M_c^2$, eq.(3.9) may also be written as

$$\frac{C_{-2}}{(k^2+M_c^2)^2} + \frac{C_{-1}}{k^2+M_c^2} + C_0 + (k^2+M_c^2)C_1 + \ldots = 0, \tag{3.10}$$

and for each given order in $(k^2+M_c^2)$ the argument has to be zero, i.e.

$$C_i = 0, \text{ for all } i. \tag{3.11}$$

The gauge invariance of M_c follows from eq.(3.11) for the case $i = -2$. In order to find the precise expressions for the C_i's we must expand all functions appearing in eq.(3.9) about $k^2 = -M_c^2$. Thus for example

$$X(k^2) = X(-M_c^2) + (k^2+M_c^2)X_R(k^2), \tag{3.12}$$

and also

$$\Pi(k^2) = \Pi(M_c^2) + (k^2+M_c^2)\Pi'(M_c^2) + (k^2+M_c^2)^2\Pi_R(k^2)\ldots\text{etc.} \tag{3.13}$$

The expression for the denominator of the dressed propagator becomes

$$\begin{aligned}k^2+M^2-\Pi(k^2) =& k^2+M^2-\Pi(M_c^2)-(k^2+M_c^2)\Pi'(M_c^2)-(k^2+M_c^2)^2\Pi_R(k^2)\\ =&(k^2+M_c^2)\{1-\Pi'(-M_c^2)-(k^2+M_c^2)\Pi_R(k^2)\},\end{aligned} \tag{3.14}$$

where we used eq.(3.5) to eliminate M^2. As such it may be easily derived that for example

$$C_{-2} = \frac{V(-M_c^2)X(-M_c^2)V(-M_c^2)}{\{1-\Pi'(-M_c^2)\}^2} = 0. \tag{3.15}$$

Now, going back to the S-matrix element of eq.(3.7), we can also expand here about $k^2 = -M_c^2$ and we note that $V(-M_c^2)$ cannot be zero, since otherwise we do not have a pole. Thus we find

$$X(-M_c^2) = 0. \tag{3.16}$$

Similarly it may be derived that

$$C_{-1} = \frac{V(-M_c^2)X'(-M_c^2)V(-M_c^2)}{\{1-\Pi'(-M_c^2)\}^2} + \frac{Y(-M_c^2)V(-M_c^2)+V(-M_c^2)Y(-M_c^2)}{1-\Pi'(-M_c^2)} = 0, \tag{3.17}$$

etc.

3.3 The complex pole

In this section we investigate the gauge invariance of M_c in each order in perturbation theory. We note that M_c can only be considered a useful parameter of the theory, if we are able to calculate it up to any given order in g. Simultaneously we would like it to be exactly gauge invariant. M_c is given by the solution of the equation

$$-M_c^2 + M^2 - \Pi(-M_c^2) = 0, \tag{3.20}$$

After an infinitesimal change of gauge

$$\Pi(k^2) \to \Pi(k^2) + \epsilon X(k^2). \tag{3.21}$$

$\Pi(-M_c^2)$ is gauge invariant, i.e. $X(-M_c^2) = 0$. Knowing that M_c is also a function of g, we introduce the functions $\mathcal{X}$ as follows

$$X(-M_c^2) = \sum_{j=1}^{\infty} \mathcal{X}(-M^2, g^{2j}) = 0. \tag{3.22}$$

It follows that

$$\mathcal{X}(-M^2, g^{2j}) = 0, \text{ for all } j, \tag{3.23}$$

and the object is thus to find the expressions for the functions $\mathcal{X}$. This is done by simply expanding the function $X(k^2)$ about $k^2 = -M^2$ and then considering the special case $k^2 = -M_c^2$. Starting with the expansion about $k^2 = -M^2$, we have

$$X(k^2) = X(-M^2) + (k^2 + M^2)X'(-M^2) + \frac{1}{2}(k^2 + M^2)^2 X''(-M^2) + \dots. \tag{3.24}$$

Here

$$X(-M^2) = \sum_{j=1}^{\infty} X(-M^2, g^{2j}), \tag{3.25}$$

where the $X(-M^2, g^{2j})$ are well-defined functions in each order in g. We now take the specific value $k^2 = -M_c^2$, and obtain

$$X(-M_c^2) = X(-M^2) + (-M_c^2 + M^2)X'(-M^2) + \frac{1}{2}(-M_c^2 + M^2)^2 X''(-M^2) + \dots = 0, \tag{3.26}$$

However we are not done yet, because M_c^2 is also a funcion of g. In fact we have

$$-M_c^2 + M^2 = \Pi(-M_c^2)$$

and eq.(3.26) becomes

$$\begin{aligned} X(-M_c^2) &= X(-M^2) + \Pi(-M_c^2)X'(-M^2) + \frac{1}{2}\Pi(-M_c^2)^2 X''(-M^2) + \dots = 0 \\ &= \sum_{j=0}^{\infty} \mathcal{X}(-M^2, g^{2j}). \end{aligned} \tag{3.27}$$

Here

$$\Pi(-M_c^2) = \sum_{j=1}^{\infty} \mathcal{P}(-M^2, g^{2j}), \tag{3.28}$$

and our problem is now reduced to finding the functions $\mathcal{P}$. From eq.(3.27) we already know the expression for the function $\mathcal{X}(-M^2, g^2)$, since we know that $\Pi(-M_c^2)$ is at least of the order g^2. The first term on the right hand side of eq.(3.27) contains an order g^2 term, namely $X(-M^2, g^2)$, while the second term is at least of the order g^4. Similarly the third term is at least of the order g^6,...etc. We conclude that

$$\mathcal{X}(-M^2, g^2) = X(-M^2, g^2) = 0.$$

In order to find the higher order terms, we need to find the functions $\mathcal{P}$ of eq.(3.28). Thus expand $\Pi(k^2)$ about $k^2 = -M^2$, just like was done for $X(k^2)$, i.e.

$$\Pi(k^2) = \Pi(-M^2) + (k^2 + M^2)\Pi'(-M^2) + \frac{1}{2}(k^2 + M^2)^2 \Pi''(-M^2) + \dots. \tag{3.29}$$

and again take the value $k^2 = -M_c^2$;

$$\Pi(-M_c^2) = \Pi(-M^2) + (-M_c^2 + M^2)\Pi'(-M^2) + \frac{1}{2}(-M_c^2 + M^2)^2 \Pi''(-M^2) + \dots. \tag{3.30}$$

Substituting $-M_c^2 + M^2 = \Pi(-M_c^2)$, we obtain

$$\begin{aligned} \Pi(-M_c^2) &= \Pi(-M^2) + \Pi(-M_c^2)\Pi'(-M^2) + \frac{1}{2}\Pi(-M_c^2)^2 \Pi''(-M^2) + \dots \\ &= \sum_{j=0}^{\infty} \mathcal{P}(-M^2, g^{2j}). \end{aligned} \tag{3.31}$$

It is now very easy to see that

$$\mathcal{P}(-M^2, g^2) = \Pi(-M^2, g^2), \tag{3.32}$$

and

$$\mathcal{P}(-M^2, g^4) = \Pi(-M^2, g^4) + \Pi(-M^2, g^2)\Pi'(-M^2, g^2), \tag{3.33}$$

etc. Now go back to eq.(3.27) and in order to express it as a function specifying each order in g, we substitute eqs.(3.25) and (3.28), i.e.

$$\begin{aligned} X(-M_c^2) &= \sum_{j=1}^{\infty} X(-M^2, g^{2j}) + \sum_{k=1}^{\infty} \mathcal{P}(-M^2, g^{2k}) \cdot \sum_{j=1}^{\infty} X'(-M^2, g^{2j}) \\ &+ \frac{1}{2}\left\{\sum_{k=1}^{\infty} \mathcal{P}(-M^2, g^{2k})\right\}^2 \cdot \sum_{j=1}^{\infty} X''(-M^2, g^{2j}) + \dots = 0. \end{aligned} \tag{3.34}$$

Using eqs.(3.32) and (3.33) we work out the above equation up to $\mathcal{O}(g^8)$;

$$\begin{aligned}
X(-M_c^2) &= X(-M^2,g^2) + X(-M^2,g^4) + X(-M^2,g^6) \\
&+ \Pi(-M^2,g^2)X'(-M^2,g^2) + \Pi(-M^2,g^4)X'(-M^2,g^2) + \Pi(-M^2,g^2)X'(-M^2,g^4) \\
&+ \Pi(-M^2,g^2)\Pi'(-M^2,g^2)X'(-M^2,g^2) + \frac{1}{2}\Pi(-M^2,g^2)^2X''(-M^2,g^2) \\
&+ \mathcal{O}(g^8) = 0. \qquad (3.35)
\end{aligned}$$

If eq.(3.22) is to hold in each order of perturbation theory, then from eq.(3.35) we see that the following equations separately must hold;

$$\mathcal{X}(-M^2,g^2) = X(-M^2,g^2) = 0. \qquad (3.36)$$

$$\mathcal{X}(-M^2,g^4) = X(-M^2,g^4) + \Pi(-M^2,g^2)X'(-M^2,g^2) = 0. \qquad (3.37)$$

$$\begin{aligned}
\mathcal{X}(-M^2,g^6) &= X(-M^2,g^6) + \Pi(-M^2,g^4)X'(-M^2,g^2) + \Pi(-M^2,g^2)X'(-M^2,g^4) \\
&+ \Pi(-M^2,g^2)\Pi'(-M^2,g^2)X'(-M^2,g^2) + \frac{1}{2}\Pi(-M^2,g^2)^2X''(-M^2,g^2) = 0. \qquad (3.38)
\end{aligned}$$

At this point we would like to make a few remarks. Within the description of the physics in the non-perturbative energy region, thus in the neighbourhood of the Z^0 resonance, we have defined the complex pole M_c. Here we have investigated the gauge invariance of M_c in each order of perturbation theory. For example, eqs. (3.36) through (3.38) state the condition for the gauge invariance of M_c to hold in the three lowest orders in g. But in that case these equations must be derivable from the physics far away from the Z^0, where perturbation theory is valid. Indeed, going back to chapter 2, we find that eq.(3.36) corresponds to eq.(2.16) and eq.(3.37) corresponds to eq.(2.59). In chapter 4, we will show that eq.(3.23) is in fact a consequence of a Ward identity, thereby showing that M_c^2 is exactly gauge invariant in each order of perturbation theory.

3.4 The S-matrix, Part II

When we consider the S-matrix around the Z^0 resonance, we employ the Dyson summed propagator to describe the physics in this energy region. As explained in section 3.1, because $k^2+M^2-\Pi(k^2,g^2)=\mathcal{O}(g^2)$, we are mixing up the order of the diagrams and we no longer know if gauge invariance holds in each order in g, although we know that the complete S-matrix is gauge invariant. Moreover, we seem to have lost the *precise* definition of the S-matrix up to a certain order in g, since we do not know precisely the order of the diagrams, that contain a pole. The motivation to define the Z^0 mass as the real part of the complex pole M_c, is that it is exactly gauge invariant in each order in g. But the experimentalist measures the S-matrix, from which M_c is be derived, and therefore

we must be able to formulate the S-matrix in an exactly gauge invariant manner order by order in g, otherwise the usefulness of this definition is lost. This is the issue that we address in this section.

The complete S-matrix for the process $e^+e^- \to \mu^+\mu^-$, given by eq.(3.7),

$$A(e^+e^- \to \mu^+\mu^-) = V(k^2)\frac{1}{k^2+M^2-\Pi(k^2)}V(k^2) + B(k^2), \qquad (3.39)$$

is gauge invariant. If we expand about $k^2=-M_c^2$, the S-matrix may also be written as

$$A(e^+e^- \to \mu^+\mu^-) = \frac{R}{k^2+M_c^2} + S(k^2), \qquad (3.40)$$

where, as argued by Stuart [5], necessarily M_c, R and $S(k^2)$ are all separately gauge invariant. Here

$$\begin{aligned}
S(k^2) &= S(-M_c^2) + (k^2+M_c^2)S'(-M_c^2) + \ldots \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}(k^2+M_c^2)^n S^{(n)}(-M_c^2). \qquad (3.41)
\end{aligned}$$

If, in addition to M_c, we can find expressions for R and $S(k^2)$ up to arbitrary order in g, then we can calculate the S-matrix near the Z^0 resonance in a gauge invariant way to any accuracy. In order to find R and $S(k^2)$ we must expand all functions appearing in eq.(3.39) about $k^2=-M_c^2$ and from eq.(3.14) we have for example

$$k^2+M^2-\Pi(k^2) = (k^2+M_c^2)\{1-\Pi'(-M_c^2)-(k^2+M_c^2)\Pi_R(k^2)\}. \qquad (3.42)$$

As such it may be found that

$$R = \frac{V(-M_c^2)V(-M_c^2)}{1-\Pi'(-M_c^2)} = \sum_{j=1}^{\infty}\mathcal{R}(-M^2,g^{2j}), \qquad (3.43)$$

and for example

$$\begin{aligned}
S(-M_c^2) &= \frac{V'(-M_c^2)V(-M_c^2)+V(-M_c^2)V'(-M_c^2)}{1-\Pi'(-M_c^2)} \\
&+ \frac{1}{2}\frac{\Pi''(-M_c^2)V(-M_c^2)V(-M_c^2)}{\{1-\Pi'(-M_c^2)\}^2} + B(-M_c^2). \qquad (3.44)
\end{aligned}$$

Considering now the infinitesimal change of gauge of eq.(3.8), we have

$$R \to R + \epsilon C_{-1}. \qquad (3.45)$$

$$S(-M_c^2) \to S(-M_c^2) + \epsilon C_0. \qquad (3.46)$$

$$\vdots$$

$$S^{(n)}(-M_c^2) \to S^{(n)}(-M_c^2) + \epsilon C_n. \tag{3.47}$$

Eq.(3.11) states that $C_i = 0$ for all i, and thus R and $S(k^2)$ are gauge invariant.

We want to calculate the S-matrix up to a certain order in g, and we therefore introduce the functions $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{S}^{(n)}$ as follows,

$$M_c^2 = M^2 - \Pi(-M_c^2) = M^2 - \sum_{j=1}^{\infty} \mathcal{P}(-M^2, g^{2j}), \tag{3.48}$$

$$R = \sum_{j=1}^{\infty} \mathcal{R}(-M^2, g^{2j}), \tag{3.49}$$

$$S^{(n)} = \sum_{j=2}^{\infty} \mathcal{S}^{(n)}(-M^2, g^{2j}), \tag{3.50}$$

where the two lowest orders $\mathcal{P}(-M^2, g^2)$ and $\mathcal{P}(-M^2, g^4)$ are given by eqs.(3.32) and (3.33). The S-matrix may now be written as

$$\begin{aligned} A(e^+e^- \to \mu^+\mu^-) &= \frac{\sum_{j=1}^{\infty} \mathcal{R}(-M^2, g^{2j})}{k^2 + M^2 - \sum_{k=1}^{\infty} \mathcal{P}(-M^2, g^{2k})} \\ &+ \sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \cdot \left[k^2 + M^2 - \sum_{j=1}^{\infty} \mathcal{P}(-M^2, g^{2j}) \right]^n \cdot \sum_{j=2}^{\infty} \mathcal{S}^{(n)}(-M^2, g^{2j}) \right\}, \end{aligned} \tag{3.51}$$

displaying the order in g explicitly, and which is still exact. The next step is to make the approximation by only calculating the first few terms in the expansion. Given that $k^2 + M^2 - \Pi(-M^2, g^2)$ is of $\mathcal{O}(g^2)$, the lowest order is given by

$$A^0(e^+e^- \to \mu^+\mu^-) = \frac{\mathcal{R}(-M^2, g^2)}{k^2 + M^2 - \mathcal{P}(-M^2, g^2)} + \mathcal{O}(g^2), \tag{3.52}$$

where $\mathcal{R}(-M^2, g^2) = V(g)V(g)$ and $\mathcal{P}(-M^2, g^2) = \Pi(-M^2, g^2)$, as given by eq.(3.32). We note that the above expression is exactly gauge invariant, while eq.(3.2) is not. Eq.(3.2) breaks the gauge invariance at $\mathcal{O}(g^2)$, compared to the lowest order considered. Including the next to leading order, we have according to eq.(3.51),

$$A^{0+1}(e^+e^- \to \mu^+\mu^-) = \frac{\mathcal{R}(-M^2, g^2) + \mathcal{R}(-M^2, g^4)}{k^2 + M^2 - \mathcal{P}(-M^2, g^2) - \mathcal{P}(-M^2, g^4)} + \mathcal{O}(g^4). \tag{3.53}$$

Here $\mathcal{P}(-M^2, g^4)$ is given by eq.(3.33), and from eq.(3.43) it may be derived that

$$\mathcal{R}(-M^2, g^4) = V(-M^2, g^3)V(g) + V(g)V(-M^2, g^3) + V(g)\Pi'(-M^2, g^2)V(g). \tag{3.54}$$

As will be shown in section 5, when the photon contribution is included, there will be a non-pole term of $\mathcal{O}(g^2)$ as well. After an infinitesimal change of gauge, we have

$$\mathcal{R}(-M^2, g^4) \to \mathcal{R}(-M^2, g^4) + \epsilon \mathcal{C}_{-1}(-M^2, g^4), \tag{3.55}$$

where

$$\begin{aligned} \mathcal{C}_{-1}(-M^2, g^4) &= Y(-M^2, g^3)V(g) + V(g)Y(-M^2, g^3) \\ &+ V(g)X'(-M^2, g^2)V(g) = 0. \end{aligned} \tag{3.56}$$

We note that eq.(3.56) corresponds to eq.(2.17), demonstrating that the above results are indeed derivable from perturbation theory.

It is clear however, that the expression of eq.(3.53) contains terms that are of higher order, because

$$\begin{aligned} &\frac{1}{k^2 + M^2 - \mathcal{P}(-M^2, g^2) - \mathcal{P}(-M^2, g^4)} \\ &= \frac{1}{k^2 + M^2 - \mathcal{P}(-M^2, g^2)} \left\{ 1 + \frac{\mathcal{P}(-M^2, g^4)}{k^2 + M^2 - \mathcal{P}(-M^2, g^2)} \right. \\ &+ \left. \left[\frac{\mathcal{P}(-M^2, g^4)}{k^2 + M^2 - \mathcal{P}(-M^2, g^2)} \right]^2 + \ldots \right\} \\ &= \frac{1}{k^2 + M^2 - \mathcal{P}(-M^2, g^2)} \left\{ 1 + \mathcal{O}(g^2) + \mathcal{O}(g^4) + \ldots \right\}. \end{aligned} \tag{3.57}$$

The arbitrariness of the S-matrix up to a certain order in g, may thus be formulated as follows. When we calculate the amplitude up to $\mathcal{O}(g^{2k})$, we introduce an arbitrariness starting at $\mathcal{O}(g^{2k+2})$, which however is gauge invariant as well. Therefore, as long as the $\mathcal{O}(g^{2k+2})$ term is beyond the experimental accuracy, this arbitrariness is of no concern to us.

Including the two next to leading orders, we obtain

$$\begin{aligned} A^{0+1+2}(e^+e^- \to \mu^+\mu^-) &= \frac{\mathcal{R}(-M^2, g^2) + \mathcal{R}(-M^2, g^4) + \mathcal{R}(-M^2, g^6)}{k^2 + M^2 - \mathcal{P}(-M^2, g^2) - \mathcal{P}(-M^2, g^4) - \mathcal{P}(-M^2, g^6)} \\ &+ \mathcal{S}(-M^2, g^4) + \mathcal{O}(g^6), \end{aligned} \tag{3.58}$$

where,

$$\begin{aligned} \mathcal{P}(-M^2, g^6) &= \Pi(-M^2, g^6) + \Pi(-M^2, g^4)\Pi'(-M^2, g^2) + \Pi(-M^2, g^2)\Pi'(-M^2, g^4) \\ &+ \Pi(-M^2, g^2)\Pi'(-M^2, g^2)^2 + \frac{1}{2}\Pi(-M^2, g^2)^2\Pi''(-M^2, g^2), \end{aligned} \tag{3.59}$$

$$\begin{aligned} \mathcal{R}(-M^2, g^6) &= V(g)V(-M^2, g^5) + V(-M^2, g^5)V(g) + V(-M^2, g^3)V(-M^2, g^3) \\ &+ V(g)\Pi'(-M^2, g^4)V(g) + V(-M^2, g^3)\Pi'(-M^2, g^2)V(g) \\ &+ V(g)\Pi'(-M^2, g^2)V(-M^2, g^3) + V(g)\Pi(-M^2, g^2)\Pi''(-M^2, g^2)V(g) \\ &+ V(g)\Pi'(-M^2, g^2)^2V(g) + V(g)\Pi(-M^2, g^2)V'(-M^2, g^3) \\ &+ V'(-M^2, g^3)\Pi(-M^2, g^2)V(g), \end{aligned} \tag{3.60}$$

$$S(-M^2, g^4) = V'(-M^2, g^3)V(g) + V(g)V'(-M^2, g^3) + \frac{1}{2}V(g)\Pi''(-M^2, g^2)V(g) + B(-M^2, g^4). \quad (3.61)$$

Eq.(3.59) is derived from eq.(3.31). Similarly, eqs.(3.60) and (3.61) are derived from eqs.(3.43) and (3.44) respectively. After an infinitesimal change of gauge, we have

$$\mathcal{R}(-M^2, g^6) \to \mathcal{R}(-M^2, g^6) + \epsilon \mathcal{C}_{-1}(-M^2, g^6), \quad (3.62)$$

and

$$S(-M^2, g^4) \to S(-M^2, g^4) + \epsilon \mathcal{C}_0(-M^2, g^4), \quad (3.63)$$

where $\mathcal{C}_{-1}(-M^2, g^6) = 0$ is given by eq.(2.60), and $\mathcal{C}_0(-M^2, g^4) = 0$ is given by eq.(2.18).

We conclude that we can formulate the S-matrix up to $\mathcal{O}(g^{2k})$, which is gauge invariant. Although this S-matrix element does contain terms of $\mathcal{O}(g^{2k+2})$ and higher, these terms are gauge invariant by themselves. Since we only have $\mathcal{O}(g^{2k})$ accuracy to begin with, we conclude that we do not need to worry about this arbitrariness. As such the definition of the mass of the Z^0 as the real part of the complex pole M_c may be considered useful.

4. Ward identity

4.1 Introduction

We have verified the gauge invariance of the complex pole M_c in the few lowest orders in g, and showed this to be derivable from perturbation theory. If there is a gauge invariant property valid order by order in perturbation theory, we know that this must be derivable from a Ward identity. This we will now demonstrate for M_c, using the diagrammatic method. We follow closely the methods of ref.[8].

In section 4.2, we start by defining the Ward identity, which is of course well known. It is only written down here for the purpose of completeness and to introduce the diagrammatic notations of ref.[8]. In section 4.3 we consider an infinitesimal change of gauge. There will be new Feynman rules, and for a given Greens function there will be correspondingly new diagrams. These new diagrams are in fact part of a Ward identity and after some manipulations it can be derived that M_c is gauge invariant.

4.2 The Ward identity

Consider the Lagrangian in the general $\kappa = \frac{1}{\lambda}$ gauge, i.e.

$$\mathcal{L} = \mathcal{L}_{\rm inv} - \frac{1}{2}\mathcal{C}^2 + \mathcal{L}_{\rm FP}, \quad (4.1)$$

where $-\frac{1}{2}\mathcal{C}^2$ is the gauge fixing term, and for example in the Feynman-'t Hooft gauge it is given by (see eq.2.21)

$$-\frac{1}{2}\mathcal{C}(\lambda = 1)^2 = -\frac{1}{2}(-\partial_\alpha W^a_\alpha + M\phi^a)^2. \quad (4.2)$$

$\mathcal{L}_{\rm inv}$ is invariant for the following infinitesimal transformations of the fields;

$$\begin{aligned} W^a_\alpha &\to W^a_\alpha + g\epsilon^{abc}\Lambda^b W^c_\alpha - \partial_\alpha \Lambda^a, \\ \phi^a &\to \phi^a + \frac{1}{2}g\epsilon^{abc}\Lambda^b\phi^c - \frac{1}{2}g\Lambda^a H - M\Lambda^a, \\ H &\to H + \frac{1}{2}g\Lambda^a\phi^a, \\ l^+ &\to (1 - \frac{1}{2}ig\Lambda^a\tau^a)l^+, \quad l^- \to l^-, \end{aligned} \quad (4.3)$$

where H is the Higgs, ϕ^a is the Higgs ghost with isopin index $a = 1,2,3$, and W^a_α is the vector boson. l^+ is a left-handed fermion doublet and l^- is a right-handed fermion singlet, with $l^\pm = \frac{1}{2}(1 \pm \gamma^5)l$. The gauge fixing term $\mathcal{C}$ transforms under (4.3) as follows

$$\mathcal{C}^a \to \mathcal{C}^a + (\hat{m}^{ab} + g\hat{l}^{ab})\Lambda^b. \quad (4.4)$$

The caret on both m^{ab} and l^{ab} indicates that derivatives may occur. Furthermore $\hat{m}^{ab}$ is field independent, while $\hat{l}^{ab}$ may depend on the fields. The corresponding Faddeev-Popov Lagrangian is then given by

$$\mathcal{L}_{\rm FP} = \psi^{a*}(\hat{m}^{ab} + g\hat{l}^{ab})\psi^b, \tag{4.5}$$

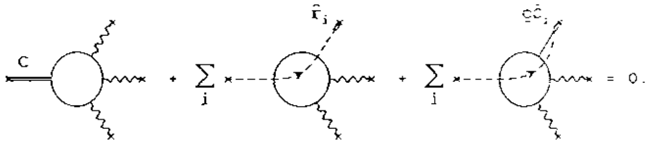
where we have for $\lambda = 1$, from eq.(2.22),

$$\hat{m}^{ab} = \delta^{ab}(\partial^2 - M^2),$$
$$g\hat{l}^{ab} = -g\epsilon^{abc}\partial_\alpha W^c_\alpha - g\epsilon^{abc}W^c_\alpha\partial_\alpha + \frac{1}{2}Mg\epsilon^{abc}\phi^c - \frac{1}{2}MgH\delta^{ab}. \tag{4.6}$$

We now consider any function $\mathcal{F}$ of the various fields, and apply the transformations of eq.(4.3):

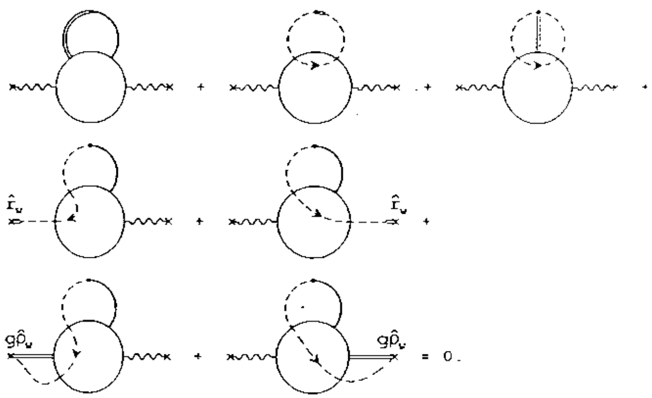
$$\mathcal{F} \to \mathcal{F} + (\hat{r} + g\hat{\rho})\Lambda, \tag{4.7}$$

the Ward identity for the Greens functions between sources may be written as



(4.8)

Here C is the Fourier transform of the gauge fixing term and the Faddeev-Popov field is denoted by the dashed line with an arrow. The Ward identity, needed to derive the gauge invariance of M_c, is for 3 outgoing $\mathcal{F}$ lines, where $\mathcal{F}_2$ and $\mathcal{F}_3$ correspond to vector bosons fields. $\mathcal{F}_1 = \mathcal{F}$ remains unspecified. Furthermore assuming that the momentum going into the C line is q and the momentum going into the $\mathcal{F}$ line is $-q$, then these two lines may be connected and the desired Ward identity is given by



(4.9)

Here $\hat{r}_w$ and $g\hat{\rho}_w$ are defined by

$$W \to W + (\hat{r}_w + g\hat{\rho}_w)\Lambda, \tag{4.10}$$

and from eq.(4.3) we see that $\hat{r}^{ab}_{w\alpha} = -\partial_\alpha\delta^{ab}$ and $\hat{\rho}^{ab}_{w\alpha} = g\epsilon^{abc}W^c_\alpha$.

4.3 An infinitesimal change of gauge.

Consider an infinitesimal change of gauge. The gauge fixing term C is changed to C', i.e.

$$C \to C' = C + \epsilon\mathcal{F}, \tag{4.11}$$

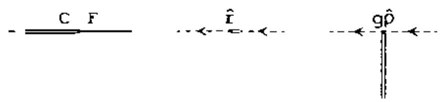
If we had started out with for example the Feynman-'t Hooft gauge, then $\mathcal{F}^a = \partial_\alpha W^a_\alpha + \phi^a$. The new gauge fixing term C' transforms under (4.3) as follows

$$C' \to C' + (\hat{m} + g\hat{l})\Lambda + \epsilon(\hat{r} + g\hat{\rho})\Lambda, \tag{4.12}$$

and the Lagrangian $\mathcal{L}$ of eq.(4.1) is changed to $\mathcal{L}'$ accordingly, i.e.

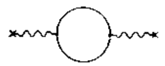
$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} - \epsilon C\mathcal{F} + \epsilon\psi^*(\hat{r} + g\hat{\rho})\psi. \tag{4.13}$$

In compact notation, the corresponding new Feynman rules are given by



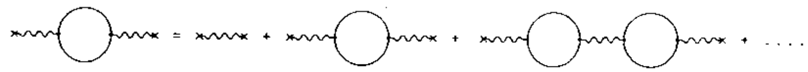
(4.14)

Consider the vector boson self energy, summed to all orders in perturbation theory, between two physical sources, i.e.



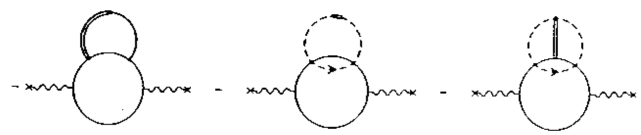
$$= J_\mu \cdot \frac{\delta_{\mu\nu}}{k^2 + M^2 - \Pi(k^2)} \cdot J_\nu. \tag{4.15}$$

We remark that for a physical source J_μ we have $J_\mu \cdot k_\mu = 0$, and therefore only the $\delta_{\mu\nu}$ contribution of the vector boson self energy survives. Furthermore, the self energy blob also contains the identity, thus



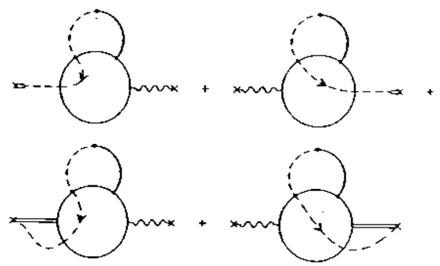
(4.16)

After the infinitesimal change of gauge, the difference is to first order in ϵ,



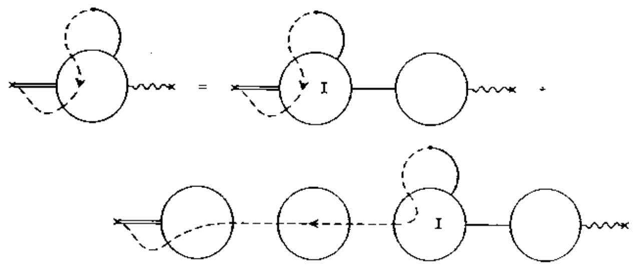
(4.17)

Using the Ward identity of eq.(4.9), this is found to be equal to.



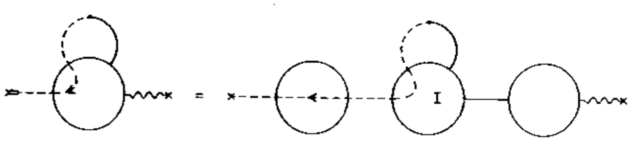
(4.18)

The blobs in the above equation include everything, i.e. reducible and irreducible graphs. The next step is to seperate out the irreducible graphs.



(4.19)

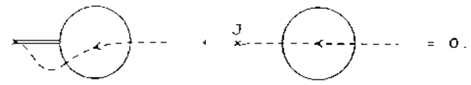
and



(4.20)

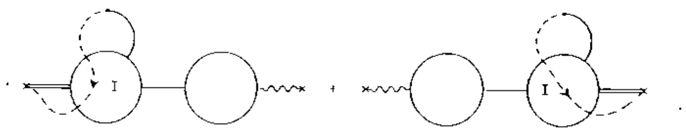
where the I denotes the irreducible blobs. The self energy blobs also contain the identity, just like the self energy blob for the vector boson as shown in eq.(4.16), and thus represent

complete propagators. The condition for a source J to be physical is that



(4.21)

and we see that the second diagram of eq.(4.19) cancels against the diagram of eq.(4.20). Therefore eq.(4.17) reduces to



(4.22)

and the corresponding expression may be written as

$$J_\mu(k) \cdot f_{\mu\alpha}(k) \cdot \frac{\delta_{\alpha\nu}}{k^2 + M^2 - \Pi(k^2)} \cdot J_\nu(k). \tag{4.23}$$

We have $f_{\mu\alpha}(k) = \delta_{\mu\alpha} f_1(k^2) + k_\mu k_\alpha f_2(k^2)$, and since $J_\mu(k) \cdot k_\mu = 0$, eq.(4.23) becomes

$$J_\mu(k) \cdot f_1(k^2) \cdot \frac{\delta_{\mu\nu}}{k^2 + M^2 - \Pi(k^2)} \cdot J_\nu(k). \tag{4.24}$$

Let us now go back to eq.(4.15). After an infinitesimal change of gauge we have $\Pi(k^2) \to \Pi(k^2) + \epsilon X(k^2)$, and we conclude that

$$\frac{f_1(k^2)}{k^2 + M^2 - \Pi(k^2)} = \frac{X(k^2)}{\{k^2 + M^2 - \Pi(k^2)\}^2}. \tag{4.25}$$

Expanding $X(k^2)$ about $k^2 = -M_c^2$, thus $X(k^2) = X(-M_c^2) + (k^2 + M_c^2) X_R(k^2)$, we see immediately that

$$X(-M_c^2) = 0, \tag{4.26}$$

since $k^2 + M^2 - \Pi(k^2) = (k^2 + M_c^2)\{1 - \Pi_R(k^2)\}$ (see eq.(3.14)).

5. The photon contribution

It is very easy to extent the results of the preceeding sections, to the case where the photon contribution is included as well. The bare propagators for the vector boson and the photon are given by

$$\Delta_{zz}(k^2) = \frac{1}{k^2 + M^2 - i\epsilon}, \quad \Delta_{\gamma\gamma}(k^2) = \frac{1}{k^2 - i\epsilon}. \tag{5.1}$$

Next, we must perform the summation of the vector boson self energy , the mixed vector boson photon and the photon self energy bubbles, to obtain the ZZ, $\gamma\gamma$ and mixed $Z\gamma$ dressed propagators. We have

$$\bar{\Delta}_{zz} = \frac{1}{k^2 + M^2 - f(k^2)}, \tag{5.2}$$

$$\bar{\Delta}_{z\gamma} = \frac{1}{k^2 + M^2 - f(k^2)} \cdot \frac{\Pi_{z\gamma}(k^2)}{k^2 - \Pi_{\gamma\gamma}(k^2)}, \tag{5.3}$$

$$\bar{\Delta}_{\gamma\gamma} = \frac{1}{k^2 + M^2 - f(k^2)} \cdot \frac{k^2 + M^2 - \Pi_{zz}(k^2)}{k^2 - \Pi_{\gamma\gamma}(k^2)}, \tag{5.4}$$

where

$$f(k^2) = \Pi_{zz}(k^2) + \frac{\Pi_{z\gamma}(k^2)^2}{k^2 - \Pi_{\gamma\gamma}(k^2)}. \tag{5.5}$$

As before, the external fermions are taken to be massless, and therefore only the $\delta_{\alpha\beta}$ part of the propagators need to be taken into account. Π_{zz}, $\Pi_{z\gamma}$ and $\Pi_{\gamma\gamma}$ are the $\delta_{\alpha\beta}$ parts of the collections of all irreducible ZZ, $Z\gamma$ and $\gamma\gamma$ self energy graphs in all orders of perturbation theory. The complex pole M_c is defined by

$$-M_c^2 + M^2 - f(-M_c^2) = 0. \tag{5.6}$$

Thus

$$\begin{aligned} M_c^2 =& \frac{1}{2}[M^2 - \Pi_{zz}(-M_c^2) - \Pi_{\gamma\gamma}(-M_c^2)] \\ &+ \frac{1}{2}\sqrt{[M^2 - \Pi_{zz}(-M_c^2) + \Pi_{\gamma\gamma}(-M_c^2)]^2 + 4\Pi_{z\gamma}(-M_c^2)^2}. \end{aligned} \tag{5.7}$$

M_c is gauge invariant, and we may write

$$M_c^2 = M^2 - \sum_{j=1}^{\infty} \mathcal{P}(-M^2, g^{2j}). \tag{5.8}$$

The functions $\mathcal{P}(-M^2, g^{2j})$ are gauge invariant, and the corresponding expressions may be found according to the method described in section 3.3.

The complete S-matrix element for the process $e^+e^- \to \mu^+\mu^-$ may be written as

$$\begin{aligned} A(e^+e^- \to \mu^+\mu^-) &= V_z(k^2)\bar{\Delta}_{zz}(k^2)V_z(k^2) + V_z(k^2)\bar{\Delta}_{z\gamma}(k^2)V_\gamma(k^2) \\ &\quad + V_\gamma(k^2)\bar{\Delta}_{\gamma z}(k^2)V_z(k^2) + V_\gamma(k^2)\bar{\Delta}_{\gamma\gamma}(k^2)V_\gamma(k^2) + B(k^2) \\ &= \frac{R}{k^2 + M_c^2} + S(k^2), \end{aligned} \tag{5.9}$$

where V_z and V_γ are the $Zf\bar{f}$ and $\gamma f\bar{f}$ vertices. For the initial vertex we have $f = e$ and for the final vertex $f = \mu$. B represents box type diagrams that do not contain a pole. M_c, R and $S(k^2)$ are separately gauge invariant, where the expression for R is given by

$$\begin{aligned} R =& \frac{V_z(-M_c^2)V_z(-M_c^2)}{1 - f'(-M_c^2)} \\ &+ \frac{V_z(-M_c^2)\Pi_{z\gamma}(-M_c^2)V_\gamma(-M_c^2) + V_\gamma(-M_c^2)\Pi_{\gamma z}(-M_c^2)V_z(-M_c^2)}{\{1 - f'(-M_c^2)\}\{-M_c^2 - \Pi_{\gamma\gamma}(-M_c^2)\}} \\ &+ \frac{V_\gamma(-M_c^2)\Pi_{\gamma z}(-M_c^2)\Pi_{z\gamma}(-M_c^2)V_\gamma(-M_c^2)}{\{1 - f'(-M_c^2)\}\{-M_c^2 - \Pi_{\gamma\gamma}(-M_c^2)\}^2} \\ =& \sum_{j=1}^{\infty} \mathcal{R}(-M^2, g^{2j}). \end{aligned} \tag{5.10}$$

When the photon contribution is neglected, the above equation reduces to eq.(3.43). For $S(k^2)$ we have

$$S(k^2) = \sum_{n=0}^{\infty} \frac{1}{n!}(k^2 + M_c^2)^n S^{(n)}(-M_c^2) = S(-M_c^2) + (k^2 + M_c^2)S'(-M_c^2) + \ldots, \tag{5.11}$$

where M_c^2 is given by eq.(5.8), and

$$S^{(n)}(-M_c^2) = \sum_{j=1}^{\infty} S^{(n)}(-M^2, g^{2j}). \tag{5.12}$$

Writing

$$\begin{aligned} S(k^2) &= S_{zz}(k^2) + S_{z\gamma}(k^2) + S_{\gamma\gamma}(k^2) + B(k^2) \\ &= S_{zz}(-M^2, g^4) + S_{z\gamma}(-M^2, g^4) + S_{\gamma\gamma}(-M^2, g^2) + S_{\gamma\gamma}(-M^2, g^4) \\ &\quad + B(-M^2, g^4) + (k^2 + M^2 - \mathcal{P}(-M^2, g^2))S_{\gamma\gamma}(-M^2, g^2) + \mathcal{O}(g^6), \end{aligned} \tag{5.13}$$

we derive

$$S_{zz}(-M^2, g^4) = V_z(g)V_z{}'(-M^2, g^3) + V_z{}'(-M^2, g^3)V_z(g) + \frac{1}{2}V_z(g)V_z(g)f''(-M^2, g^2), \tag{5.14}$$

where $f''(-M^2,g^2) = \Pi_{zz}''(-M^2,g^2)$. Furthermore

$$\mathcal{S}_{z\gamma}(-M^2,g^4) = -\frac{1}{M^2}\cdot V_z(g)\left\{\Pi_{z\gamma}'(-M^2,g^2) + \frac{1}{M^2}\Pi_{z\gamma}(-M^2,g^2)\right\} V_\gamma(g)$$
$$-\frac{1}{M^2}\cdot V_\gamma(g)\left\{\Pi_{\gamma z}'(-M^2,g^2) + \frac{1}{M^2}\Pi_{\gamma z}(-M^2,g^2)\right\} V_z(g), \quad (5.15)$$

$$\mathcal{S}_{\gamma\gamma}(-M^2,g^2) + \mathcal{S}_{\gamma\gamma}(-M^2,g^4) =$$
$$-\frac{1}{M^2}V_\gamma(g)\left\{1 - \frac{\Pi_{\gamma\gamma}(-M^2,g^2)}{M^2} + \frac{\Pi_{zz}(-M^2,g^2)}{M^2}\right\} V_\gamma(g)$$
$$-\frac{1}{M^2}\{V_\gamma(-M^2,g^3)V_\gamma(g) + V_\gamma(g)V_\gamma(-M^2,g^3)\}, \quad (5.16)$$

$$\{k^2 + M^2 - \mathcal{P}(-M^2,g^2)\}\mathcal{S}_{\gamma\gamma}(-M^2,g^2) =$$
$$-\frac{1}{M^4}(k^2 + M^2 - \Pi_{zz}(-M^2,g^2))V_\gamma(g)V_\gamma(g). \quad (5.17)$$

6. Calculating the matrix element up to a given order

By considering the expansion about $k^2 + M_c^2 = 0$, the S-matrix element for the process $e^+e^- \to \mu^+\mu^-$ in the resonance energy region may be written as

$$A(e^+e^- \to \mu^+\mu^-) = \frac{R}{-s + M_c^2} + S(s), \quad (6.1)$$

where $-k^2 = s$ is the center of mass energy squared. The parameters M_c, R and $S(s)$ may be measured experimentally, which subsequently must be related to theoretical expressions. In the preceeding sections we have seen that these theoretical expressions can indeed be obtained as a function of the bare mass M, in a gauge invariant way up to any order in perturbation theory. Concerning renormalization, we remark that we may do this for the parameters M, g and the weak mixing angle θ_w, by considering low energy processes.

The mass M_z and the width Γ_z of the Z^0 vector boson may be defined as the real and the imaginairy part of the complex pole M_c, i.e.

$$M_c^2 = M_z^2 - iM_z\Gamma_z. \quad (6.2)$$

The theoretical expressions for M_c, R and $S(s)$ for the first few lowest orders in g, are listed below. There is agreement with the results obtained by Stuart [5].

a. <u>$\mathcal{O}(1)$ accuracy</u>

$$A(e^+e^- \to \mu^+\mu^-) = \frac{R}{-s + M_c^2} + \mathcal{O}(g^2), \quad (6.3)$$

where

$$M_c^2 = M_z^2 - iM_z\Gamma_z$$
$$= M^2 - \Pi_{zz}(-M^2,g^2), \quad (6.4)$$

and

$$R = V_z(g)V_z(g). \quad (6.5)$$

$\Pi_{zz}(-M^2,g^2)$ is the $\delta_{\alpha\beta}$ part of the vector boson self energy at order g^2, and is evaluated at $k^2 = -M^2$, where M is the bare vector boson mass. The initial $V_z(g)$ vertex is the tree level Z^0ee vertex. and the final $V_z(g)$ vertex is the tree level $Z^0\mu\mu$ vertex.

b. <u>$\mathcal{O}(g^2)$ accuracy</u>

$$A(e^+e^- \to \mu^+\mu^-) = \frac{R}{-s + M_c^2} + S + \mathcal{O}(g^4), \quad (6.6)$$

where

$$M_c^2 = M_z^2 - iM_z\Gamma_z$$
$$= M^2 - \Pi_{zz}(-M^2,g^2)$$
$$- \Pi_{zz}(-M^2,g^4) - \Pi_{zz}(-M^2,g^2)\Pi_{zz}'(-M^2,g^2) + \frac{1}{M^2}\Pi_{z\gamma}(-M^2,g^2)^2, \quad (6.7)$$

$$R = V_z(g)V_z(g) + V_z(-M^2,g^3)V_z(g) + V_z(g)V_z(-M^2,g^3) + V_z(g)\Pi_{zz}'(-M^2,g^2)V_z(g) - \frac{1}{M^2}\{V_z(g)\Pi_{z\gamma}(-M^2,g^2)V_\gamma(g) + V_\gamma(g)\Pi_{\gamma z}(-M^2,g^2)V_z(g)\}, \tag{6.8}$$

and

$$S = -\frac{1}{M^2}V_\gamma(g)V_\gamma(g). \tag{6.9}$$

In the above equations the initial $V_z(-M^2,g^3)$ vertex is the one loop Zee vertex, evaluated at $k^2=-M^2$, and $\Pi_{zz}'(-M^2,g^2)$ is the derivative of the $\delta_{\alpha\beta}$ part of the vector boson one loop self energy with respect to k^2, subsequently evaluated at $k^2=-M^2$.

c. $\mathcal{O}(g^4)$ accuracy

For completion, we also write down the expression for the amplitude accurate to $\mathcal{O}(g^4)$, although at the moment this is beyond the experimental accuracy.

$$A(e^+e^- \to \mu^+\mu^-) = \frac{R}{-s+M_c^2} + S + (-s+M_c^2)S' + \mathcal{O}(g^6), \tag{6.10}$$

where

$$\begin{aligned} M_c^2 &= M_z^2 - iM_z\Gamma_z \\ &= M^2 - \Pi_{zz}(-M^2,g^2) \\ &- \Pi_{zz}(-M^2,g^4) - \Pi_{zz}(-M^2,g^2)\Pi_{zz}'(-M^2,g^2) + \frac{1}{M^2}\Pi_{z\gamma}(-M^2,g^2)^2 \\ &- \Pi_{zz}(-M^2,g^6) - \Pi_{zz}(-M^2,g^2)\Pi_{zz}'(-M^2,g^4) - \Pi_{zz}(-M^2,g^4)\Pi_{zz}'(-M^2,g^2) \\ &- \Pi_{zz}(-M^2,g^2)\Pi_{zz}'(-M^2,g^2)^2 + \frac{1}{M^2}\Pi_{z\gamma}(-M^2,g^2)^2\Pi_{zz}'(-M^2,g^2) \\ &- \frac{1}{2}\Pi_{zz}(-M^2,g^2)^2\Pi_{zz}''(-M^2,g^2) + \frac{2}{M^2}\Pi_{z\gamma}(-M^2,g^2)\Pi_{z\gamma}(-M^2,g^4) \\ &+ \frac{2}{M^2}\Pi_{z\gamma}(-M^2,g^2)\Pi_{zz}(-M^2,g^2)\Pi_{z\gamma}'(-M^2,g^2) \\ &- \frac{1}{M^4}\Pi_{z\gamma}(-M^2,g^2)^2\{-\Pi_{zz}(-M^2,g^2) + \Pi_{\gamma\gamma}(-M^2,g^2)\}, \end{aligned} \tag{6.11}$$

$$\begin{aligned} R &= V_z(g)R_{zz}V_z(g) + V_z(g)R_{z\gamma}V_\gamma(g) + V_\gamma(g)R_{\gamma z}V_z(g) \\ &+ \{V_z(-M^2,g^3)V_z(g) + V_z(g)V_z(-M^2,g^3)\}\cdot\{1+\Pi_{zz}'(-M^2,g^2)\} \\ &+ V_z(g)V_z(-M^2,g^5) + V_z(-M^2,g^5)V_z(g) + V_z(-M^2,g^3)V_z(-M^2,g^3) \\ &+ \{V_z(g)V_z'(-M^2,g^3) + V_z'(-M^2,g^3)V_z(g)\}\cdot\Pi_{zz}(-M^2,g^2) \\ &- \frac{1}{M^2}\{V_z(-M^2,g^3)\Pi_{z\gamma}(-M^2,g^2)V_\gamma(g) + V_\gamma(g)\Pi_{\gamma z}(-M^2,g^2)V_z(-M^2,g^3) \\ &+ V_z(g)\Pi_{z\gamma}(-M^2,g^2)V_\gamma(-M^2,g^3) + V_\gamma(-M^2,g^3)\Pi_{\gamma z}(-M^2,g^2)V_z(g)\} \\ &+ \frac{1}{M^4}\left\{V_\gamma(g)\Pi_{\gamma z}(-M^2,g^2)\Pi_{z\gamma}(-M^2,g^2)V_\gamma(g)\right\}, \end{aligned} \tag{6.12}$$

with

$$\begin{aligned} R_{zz} &= 1 + \Pi_{zz}'(-M^2,g^2) + \Pi_{zz}'(-M^2,g^4) \\ &+ \Pi_{zz}'(-M^2,g^2)^2 + \Pi_{zz}(-M^2,g^2)\Pi_{zz}''(-M^2,g^2) \\ &- \frac{1}{M^2}\Pi_{z\gamma}(-M^2,g^2)\left\{2\Pi_{\gamma z}'(-M^2,g^2) + \frac{\Pi_{\gamma z}(-M^2,g^2)}{M^2}\right\}, \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} R_{z\gamma} &= -\frac{1}{M^2}\{\Pi_{z\gamma}(-M^2,g^2) + \Pi_{z\gamma}(-M^2,g^4) + \Pi_{zz}(-M^2,g^2)\Pi_{z\gamma}'(-M^2,g^2)\} \\ &- \frac{1}{M^2}\Pi_{z\gamma}(-M^2,g^2)\cdot\left\{\Pi_{zz}'(-M^2,g^2) + \frac{1}{M^2}\Pi_{zz}(-M^2,g^2) - \frac{1}{M^2}\Pi_{\gamma\gamma}(-M^2,g^2)\right\}. \end{aligned} \tag{6.14}$$

Finally

$$\begin{aligned} S + (-s+M_c^2)S' &= V_z(g)V_z'(-M^2,g^3) + V_z'(-M^2,g^3)V_z(g) \\ &+ \frac{1}{2}V_z(g)\Pi_{zz}''(-M^2,g^2)V_z(g) \\ &- \frac{1}{M^2}\{V_z(g)\Pi_{z\gamma}'(-M^2,g^2)V_\gamma(g) + V_\gamma(g)\Pi_{\gamma z}'(-M^2,g^2)V_z(g)\} \\ &- \frac{1}{M^4}\{V_z(g)\Pi_{z\gamma}(-M^2,g^2)V_\gamma(g) + V_\gamma(g)\Pi_{\gamma z}(-M^2,g^2)V_z(g)\} \\ &- \frac{1}{s}V_\gamma(g)V_\gamma(g) - \frac{1}{M^2}\{V_\gamma(g)V_\gamma(-M^2,g^3) + V_\gamma(-M^2,g^3)V_\gamma(g)\} \\ &+ \frac{1}{M^4}V_\gamma(g)\Pi_{\gamma\gamma}(-M^2,g^2)V_\gamma(g) + B(-M^2,g^4). \end{aligned} \tag{6.15}$$

In the above equation $-k^2 = s$ is the center of mass energy squared, where we approximated $1/s = [1 + (-s+M^2)/M^2 + \mathcal{O}(g^4)]/M^2$.

7. Concluding remarks

We have shown that the location of the complex pole M_c is exactly gauge invariant in each order of perturbation theory.

In defining the S-matrix that describes the physics in the resonance region, the bare vector boson propagator is replaced by the fully dressed propagator. We have shown that up to some order in g, the resulting S-matrix is indeed exactly gauge invariant. However this expression does contain higher order terms. This is not surprising, as the dressed propagator is a summation of all self energy graphs in all orders of perturbation theory. The reason that nevertheless everything can be formulated in a gauge invariant way up to some order, is because when we do the expansion about $k^2 = -M_c^2$, these higher order terms are, at $k^2 = -M_c^2$, gauge invariant by themselves.

The experiments measure a cross-section proportional to the matrix element squared and in order to derive the value of the vector boson mass we will have to compare the measured curves with our theoretical expression. This may be done in a gauge invariant way up to the desired accuracy, and we conclude that to define the Z^0 mass as the real part of M_c may be considered useful.

We have not discussed the case of massive external fermions, since we only looked at the $\delta_{\mu\nu}$ part of the vector boson propagator. In this respect our discussion is therefore not complete.

Acknowledgments

The author is grateful for the hospitality received at the University of Michigan, where part of this work was completed. The author thanks M. Veltman for many helpful discussions.

References

[1] The LEP collaboration, *Electroweak parameters of the Z^0 resonance and the Standard Model*, Phys. Lett. B276 (1992) 247.

[2] J. van der Bij and M. Veltman, Nucl. Phys. B231 (1984) 205.
J. van der Bij and F. Hoogeveen, Nucl. Phys. B283 (1987) 477.
A. Djouadi and C. Verzegnassi, Phys. Lett. B195 (1987) 265

[3] M. Veltman, Physica 29 (1963) 186.

[4] R.J. Eden, P.V. Landshoff, D.I. Olive and J.C. Polkinghorne, *The Analytic S − matrix*, Cambridge University Press, Cambridge (1966).

[5] R.G. Stuart, Phys. Lett. B262 (1991) 113; Phys. Lett. B272 (1991) 353.

[6] S. Willenbrock and G. Valencia, Phys. Lett. B259 (1991) 373.
G. López Castro, J.L. Lucio and J. Pestieau, Madrid preprint, FTUAM/91-08.
A. Sirlin, Phys. Lett. B267 (1991) 240.
A. Leike, T. Riemann and J. Rose, Phys. Lett. B273 (1991) 513.

[7] G. Degrassi and A. Sirlin, New York preprint, NYU-TR 92/01/01.

[8] G. 't Hooft and M. Veltman, Nucl. Phys. B50 (1972) 318.

Figure captions

Fig.1 Lowest order Feynman diagram for the process $e^+e^- \to \mu^+\mu^-$.

Fig.2 One loop Feynman diagrams for the process $e^+e^- \to \mu^+\mu^-$.
The '1' drawn in the blobs denotes that they are one loop order blobs.
(a) Vector boson self energy contribution.
(b) Vertex and external fermion line correction.
(c) Box diagram.

Fig.3 One loop diagrams contributing to the vector boson self energy.
(a) Non-zero contribution after a change of gauge.
(b) When added, gauge dependence drops out.
(c) Gauge independent.

Fig.4 One loop $Z^0e^+e^-$ vertex graphs.

Fig.5 One loop diagrams contributing to the electron self energy.

Fig.6 Box diagrams.

Fig.7 Two loop Feynman diagrams for the process $e^+e^- \to \mu^+\mu^-$.
The blobs labelled with a '1' are one loop blobs.
The blobs labelled with a '2' are two loop blobs.

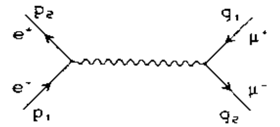


Fig. 1

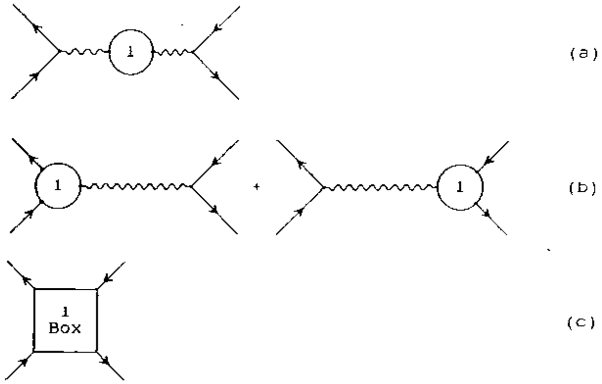


Fig. 2

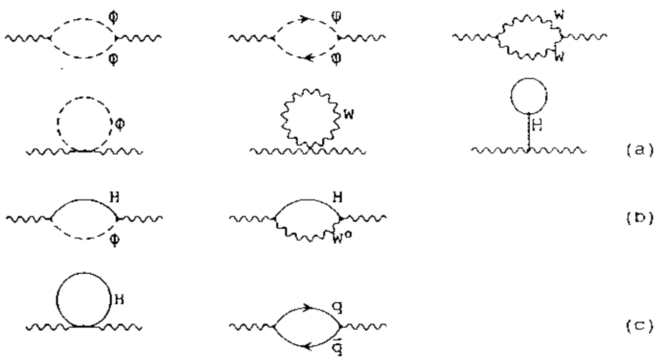


Fig. 3

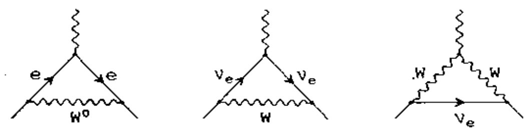


Fig. 4

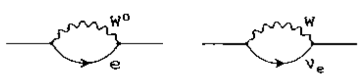


Fig. 5

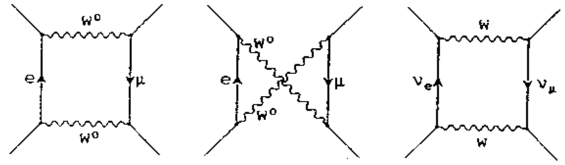


Fig. 6

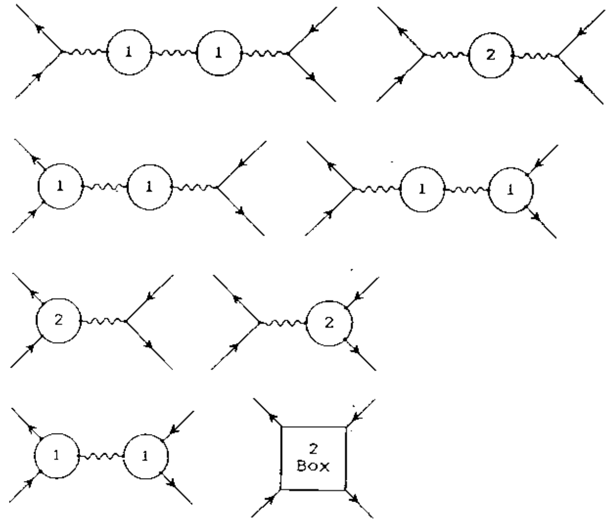


Fig. 7