DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 92-132
September 1992

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ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

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ISSN 0418 – 9833

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Abstract

We discuss the quantization of strongly chaotic systems and apply several quantization rules to a model system given by the unconstrained motion of a particle on a compact surface of constant negative Gaussian curvature. We study the periodic–orbit theory for distinct symmetry classes corresponding to a parity operation which is always present when such a surface has genus two. Recently, several quantization rules based on periodic orbit theory have been introduced. We compare quantizations using the dynamical zeta function $Z(s)$ with the quantization condition

$$\cos(\pi \mathcal{N}(E)) = 0 ,$$

where a periodic–orbit expression for the spectral staircase $\mathcal{N}(E)$ is used. A general discussion of the efficiency of periodic–orbit quantization then allows us to compare the different methods. The system dependence of the efficiency, which is determined by the topological entropy τ and the mean level density $\bar{d}(E)$, is emphasized.

[1] Supported by Deutsche Forschungsgemeinschaft under Contract No. DFG–Ste 241/4–6

I Introduction

One of the main goals in the study of chaotic systems is to find a *semiclassical quantization rule* for the discrete quantal energy spectrum. In the case of *integrable systems* this is achieved by the so-called Einstein–Brillouin–Keller (EBK) quantization. Due to the N constants of motion an integrable system with N degrees of freedom possesses, the classical Hamiltonian $H(\vec{q}, \vec{p})$ can be canonically transformed to angle and action variables such that the new Hamiltonian $H(\vec{I})$ depends only on the conserved action variables $\vec{I}$. Then the semiclassical approximations to the quantal energies are given by the EBK quantization condition $E_{\vec{m}} = H(\vec{I}_{\vec{m}})$ with $\vec{I}_{\vec{m}} = (\vec{m} + \frac{1}{4}\vec{\alpha})\hbar$, where $\vec{m}$ is the vector of quantum numbers and $\vec{\alpha}$ is the vector of Maslov indices. This simple quantization rule yields approximations to the quantal energies which are the more accurate the higher the quantal energies are. To find such a simple semiclassical quantization condition which can by applied to chaotic systems is one of the fundamental issues of quantum chaology. However, no such formula has been found up to now.

About 1970 Gutzwiller [1] derived his *periodic–orbit theory* from the Feynman path integral formalism. Semiclassically it expresses the trace of the energy–dependent Green's function (resolvent) of the quantum Hamiltonian in terms of the actions along the periodic orbits of the classical system. Since the poles of the trace of the Green's function are located at the positions of the quantal energies, the periodic–orbit theory can be considered as a semiclassical quantization condition. A drawback, however, is that the periodic–orbit theory has to take into account periodic orbits with increasing actions for computing higher energies. This contrasts with the EBK quantization rule, where the computational effort is independent of the energy. Since all chaotic systems have the property that the number $N(S)$ of periodic orbits with actions smaller than S increases exponentially with S, the computational effort even increases exponentially. Thus the semiclassical periodic–orbit theory cannot be applied for numerical computations in the genuine semiclassical regime. However, at present, nearly all quantization conditions are based on the periodic–orbit theory, apart from those numerical schemes which directly solve the corresponding Schrödinger equation and the new quantization condition of Bogomolny, which is derived from a quantum version of a classical Poincaré map [2]. The two quantization methods which we want to discuss here are also based on the periodic–orbit theory. The efficiency of the periodic–orbit theory is system–dependent since it depends on the topological entropy τ, which determines the exponential proliferation of periodic orbits, and on the mean level density $\bar{d}(E)$ of the quantal energies. The spectrum ranges from benign chaotic systems, where a few periodic orbits suffice to determine a lot of quantal energies, to systems, where many million periodic orbits are necessary for the determination of just the first ten quantal energies.

Apart from the efficiency of the periodic–orbit theory the exponential proliferation of periodic orbits causes further problems, since it leads to a periodic–orbit series for the trace of the Green's function which is not absolutely convergent. Hence the periodic–orbit theory can only be applied if the series is at least conditionally convergent in the domain where the poles due to the quantal energies lie. However, whether this is the case is a priori unknown. The convergence properties can be improved if one considers *smoothed* periodic–orbit sum rules, which replace the singularities by sufficiently large maxima [3,4,5], instead of the trace of the Green's function.

The first calculation of quantal energies on the basis of the periodic–orbit theory was carried out for the anisotropic Kepler model [6,7]. Further applications followed thereafter, e.g. the Hadamard–Gutzwiller model [3,4], the hydrogen atom in a magnetic field [8] and the hyperbola billiard [9]. For other examples see Gutzwiller's book [10] and references therein. An additional example where the periodic–orbit theory has been used is the Riemann zeta function $\zeta(s)$ whose non–trivial zeros on the critical line $\mathrm{Re}\, s = \frac{1}{2}$ are thought to be connected with the quantal energies of a chaotic system, which is, however, unknown. Starting from the Euler–product formula over primes p, $\zeta(s) = \prod_{\{p\}}(1 - p^{-s})^{-1}$, Berry derived in [11] a (divergent) expression for the oscillatory part of the spectral density $d_{osc}(E)$ which is analogous to a periodic–orbit formula, if one identifies the set $\{\ln p\}$ with the length spectrum $\{l_n\}$ of the primitive periodic orbits of the unknown system. For the application of the periodic–orbit theory it is thus not necessary to know the physical system.

Analogous zeta functions can be defined for general chaotic systems whose non–trivial zeros are connected with the quantal energies of the quantum mechanical counterpart. Such zeta functions are called *dynamical zeta functions*. They have been studied for a lot of classically chaotic systems in order to determine quantal energies or quantum resonances. The computation of quantum resonances of the three discs scattering system is carried out in [12] and in [13], where in the latter reference the dynamical zeta function is computed by the cycle expansion [14]. Instead of using the complex valued dynamical zeta function one can construct from it a new real function, if an appropriate functional relation is available. This method, which facilitates the calculation of zeros, was applied to the hyperbola billiard [15], to Artin's billiard [16], to the anisotropic Kepler model and to the closed three discs billiard [17]. Recently it has been shown that in the semiclassical limit the dynamical zeta function can be obtained from the boundary–element method in the case of chaotic billiard problems [18]. As an alternative quantization function, whose zeros also yield the quantal energies, the function $\cos(\pi \mathcal{N}(E))$ has recently been proposed using the spectral staircase $\mathcal{N}(E)$ expressed in terms of periodic orbits in [19], where it was applied to hyperbolic octagons. In addition, it was shown in [20] that the new quantization condition works well also for the hyperbola billiard and for Artin's billiard. In the following we want to discuss the quantization by the dynamical zeta function in comparison to the quantization using the function $\cos(\pi \mathcal{N}(E))$.

Here we study *hyperbolic octagons* [10,21] which have the advantage that the periodic–orbit theory is exact [6], since it is identical to the Selberg trace formula [22] which represents a deep theorem in the mathematics of harmonic analysis and hyperbolic geometry. This model is a conservative Hamiltonian system with two degrees of freedom, which is strongly chaotic (K–system). It consists classically of a point particle sliding freely on a closed surface of constant negative curvature. One is led to study such a system because it can be viewed as a simple case of a large class of general potential problems. In general, the motion of a particle which is solely determined by a given potential $V(\vec{q})$ can be considered, using the Jacobian metric, as the geodesic motion on a curved manifold, where the curvature is determined by the potential $V(\vec{q})$. Since Hopf proved that negative curvature necessaryly leads to ergodicity [23], it is natural to study the simplest case, i. e. the case of constant negative Gaussian curvature $K = -1$. Furthermore, Gelfand and Fomin showed that constant negative curvature forced a system to be chaotic in general [24].

The surfaces to be considered are compact Riemann surfaces $\mathcal{F}$ of constant negative curvature $K = -1$ with genus $g = 2$, i. e. they have the topology of a sphere with two handles. Due to the Gauß–Bonnet theorem, $\mathrm{Area}(\mathcal{F}) = 4\pi(g-1)$, the area of such a surface is $\mathrm{Area}(\mathcal{F}) = 4\pi$. The sphere with two handles can be cut so that one obtains an octagon

with geodesic edges, where opposite sides must be identified leading to ***periodic boundary conditions***. A given octagon is mapped into the Poincaré disc $\mathcal{D}$, which consists of the interior of the unit circle in the complex z–plane ($z = x_1 + ix_2$) endowed with the hyperbolic metric

$$g_{ij} = \frac{4}{(1 - x_1^2 - x_2^2)^2}\delta_{ij} \ , \ i,j = 1,2 \tag{1}$$

corresponding to constant negative Gaussian curvature $K = -1$. (This fixes the length scale.) The periodic boundary conditions are realized by identifying the points z and $z' \equiv b(z)$,

$$b(z) := \frac{\alpha z + \beta}{\beta^* z + \alpha^*} \ , \ |\alpha|^2 - |\beta|^2 = 1 \ , \tag{2}$$

where the "boosts"

$$b = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \in \mathrm{SU}(1,1)/\{\pm 1\}$$

are chosen such that they map a given edge onto the opposite edge. These boosts can be computed by the method outlined in [25]. Four boosts $b_1, \ldots, b_4 \in \mathrm{SU}(1,1)/\{\pm 1\}$ are sufficient for the description of a given octagon. These four boosts and their inverses are the generators of a discrete subgroup Γ of $\mathrm{SU}(1,1)/\{\pm 1\}$ which is also called a Fuchsian group or "octagon group". The octagon then is a fundamental domain $\mathcal{F}$ for Γ, i.e. under the action (2) of Γ on $\mathcal{F}$ the whole disc $\mathcal{D}$ is tessellated by infinitely many copies of $\mathcal{F}$.

The *classical motion* is determined by the Hamiltonian

$$H = \frac{1}{2m} p_i g^{ij} p_j \ , \ p_i = m g_{ij} dx^j/dt \ . \tag{3}$$

The energy $E = H$ is the only constant of motion, and no invariant tori exist in phase space. The geodesics are circles intersecting the boundary of the Poincaré disc $\mathcal{D}$ perpendicularly. They are all unstable and neighbouring trajectories diverge with time at a rate $e^{\omega t}$, where $\omega = \sqrt{2E/m}$ is the Lyapunov exponent. Pesin's equality [26] $h = \omega$, relates ω to the Kolmogorov–Sinai entropy h, which determines the number of periodic orbits with a period shorter than T asymptotically by e^{hT}/hT. With the length $\ell = hT$ one obtains Huber's law [27]

$$N(\ell) \sim \frac{e^\ell}{\ell} \quad \text{for} \quad \ell \to \infty \ . \tag{4}$$

The *quantum mechanical system* is governed by the Schrödinger equation

$$-\Delta\Psi_n(z) = E_n\Psi_n(z) \ , \ \text{with} \ \Delta = \frac{1}{4}(1 - x_1^2 - x_2^2)^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \ , \tag{5}$$

where we used $\hbar = 2m = 1$. The solutions of the Schrödinger equation (5) have to obey the periodic boundary conditions

$$\Psi_n(z) = \Psi_n(b(z)) \quad \text{for all} \quad b \in \Gamma \ , \tag{6}$$

and are normalized according to the metric (1). The quantal energy spectrum $\{E_n\}$ is discrete with a non–degenerate zero mode: $0 = E_0 < E_1 \leq E_2 \leq \ldots$.

Hyperbolic octagons always possess *parity symmetry*. With the method outlined in [25] every hyperbolic octagon can be constructed in such a way that the edges of the octagon are mapped onto the opposite ones by the parity operation $z \to -z$. Since the Schrödinger equation (5) is also parity invariant, the eigenstates can be classified by their parity $\Psi_n(z) = \pm\Psi_n(-z)$. Quantization rules based on the periodic–orbit theory are more efficient when they are applied only to a separate parity class because of the smaller mean level density $\bar{d}(E)$. However, the original Selberg trace formula does not apply to a single parity class. In the next section, a generalization of the Selberg trace formula is derived, which allows a desymmetrization with respect to parity symmetry.

II Selberg Trace Formula for Distinct Parity Classes

Gutzwiller's trace formula [1] which replaces the semiclassical EBK quantization in the case of classically chaotic systems has been successfully applied in many cases among which *euclidean billiards* have been one of the first and most intensively studied ones. Their treatment seems to be particularly transparent so that we choose them as an example to introduce the periodic–orbit theory. A billiard consists of a single particle sliding freely on a "billiard table" D, which is some domain in $\mathbf{R}^2$. The particle shall be elastically reflected on the boundary ∂D of the domain D. We now consider completely chaotic billiards, where all classical periodic orbits are unstable (characterized by positive Lyapunov exponents) and are isolated in phase space. The quantum version of such a billiard system is governed by the Schrödinger equation $H\Psi = E\Psi$, where $H = -(\partial_x^2 + \partial_y^2)$ is the Hamiltonian where $\hbar = 2m = 1$. The wave functions Ψ are required to vanish on the boundary ∂D. We only discuss billiards with purely discrete energy spectra $0 < E_1 < E_2 \le E_3 \le \ldots$. Then $E_n = p_n^2$, where p_n denotes the momentum of the particle.

To obtain a quantization rule one can study the trace of the resolvent $(H-E)^{-1}$ and try to detect its poles on the real energy axis. For two-dimensional billiards, however, the number $\mathcal{N}(E)$ of quantal energies up to E is asymptotically given by Weyl's law

$$\mathcal{N}(E) \sim \frac{1}{4\pi}\text{Area}(D)\; E \;, \quad E \to \infty \;. \tag{7}$$

Therefore $E_n \sim 4\pi\,\text{Area}(D)^{-1}\, n$, $n \to \infty$. Thus $\text{tr}(H-E)^{-1} = \sum_{n=1}^{\infty}(E_n - E)^{-1}$ diverges. To avoid this divergence one can smear out the resolvent using some regularization. In terms of a general regularization Gutzwiller's trace formula for a chaotic billiard system looks like [5]

$$\sum_{n=1}^{\infty} h(p_n) \sim 2\int_0^{\infty} dp\, p\, h(p)\, \bar{d}(p) + \sum_{\{l_n\}}\sum_{k=1}^{\infty} \frac{\chi_n^k l_n g(kl_n)}{e^{kl_n\lambda_n/2} - \sigma_n^k e^{-kl_n\lambda_n/2}} \;. \tag{8}$$

This trace formula relates a sum over the quantal energy spectrum to a sum over the classical periodic orbits. In it there appears the smearing function $h(p)$, which has to be even and holomorphic in a strip $|\text{Im}\, p| \le \tau - \frac{\bar{\lambda}}{2} + \varepsilon$, $\varepsilon > 0$. Furthermore, $h(p)$ has to decrease faster than $|p|^{-2}$ for $|p| \to \infty$, but otherwise is arbitrary. Under these assumptions all sums and integrals involved are absolutely convergent. $g(x) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ipx} h(p)$ is the Fourier-transform of $h(p)$ and $\bar{d}(p)$ denotes the mean energy density expressed as a function of momentum p. The sum on the r.h.s. of (8) runs over all classical periodic orbits with l_n denoting the length of the orbit with label n. χ_n is a character attached to the n-th orbit, which is determined by $\chi_n = (-1)^{j_n}$, where j_n is the number of reflections from the boundary. The sign of the trace of the monodromy matrix for an orbit is denoted by σ_n. λ_n is the Lyapunov exponent of an

orbit defined as $\lambda_n = u_n/l_n$, where u_n is the stability exponent. $\bar{\lambda}$ denotes the asymptotic average of the λ_n (the metric entropy).

In the sequel we want to study an equivalent of an euclidean billiard in hyperbolic geometry. This has the advantage that the relevant trace formula is exact rather than a semiclassical approximation. It has been known as *Selberg's trace formula* [22] in mathematics long before Gutzwiller derived his more general formula. A simplification of the matter under study is also given by the fact that in hyperbolic geometry all Lyapunov exponents λ_n (and hence also the metric entropy $\bar{\lambda}$) and the topological entropy τ share the same value one.

As mentioned in the introduction we choose the unconstrained motion of a particle on a compact surface of genus $g = 2$ as our model. The surface will be endowed with the hyperbolic metric of constant negative curvature introduced above. From the theory of Riemann surfaces (see e.g. [28]) it is known that every compact surface of genus two is hyperelliptic, i.e. it may be viewed as a two–sheeted covering of the sphere branched at six points ($2g+2$ points for a general hyperelliptic surface of genus g). There always exists a symmetry operation that interchanges the two sheets of the covering and hence fixes the $2g+2$ branch points. We call this symmetry *parity operation*. It is now possible to choose the fundamental domain $\mathcal{F}$ in a special manner, see also [25]. It may be placed in $\mathcal{D}$ such that the generating boosts of Γ identify opposite edges and the parity operation is realized as the mapping $z \to -z$. One corner, z_1, is chosen to be on the positive real axis, and $z_2, \ldots, z_{2g}$ denote the corners going counterclockwise around the fundamental domain (see fig.1 for genus $g = 2$). These are then located in the upper half of $\mathcal{D}$ ($\text{Im}\, z_i > 0$). The remaining $2g$ corners are given by the images of $z_1, \ldots, z_{2g}$ under the parity operation. The $2g-1$ points $z_2, \ldots, z_{2g}$ may be chosen arbitrarily in the upper half of the disc. The constraint $\text{Area}(\mathcal{F}) = 4\pi(g-1)$ then fixes the location of z_1 on the real axis. This surface now possesses generically no symmetry besides parity. We illustrate the construction of our octagon ($g = 2$) in fig.1 where we also indicate the location of the $2g+2 = 6$ branching points $w_1, \ldots, w_{2g+2}$: $w_1 = 0$ is clearly fixed under $z \to -z$. Also w_2, which represents the corners of $\mathcal{F}$ that are all being identified under Γ, is obviously a fixed point of the parity operation. Since Γ identifies opposite edges of $\mathcal{F}$ the remaining $2g$ points $w_3, \ldots, w_{2g+2}$ are given by the mid-points (as determined by the metric (1)) of each pair of edges.

We now denote one generating boost, say the one identifying the edge joining z_1 and z_2, with the one joining $-z_1$ and $-z_2$, by b_1; $b_2, \ldots, b_{4g}$ will be the boosts identifying the following pairs of edges when going counterclockwise around the octagon. Thus $b_{2g+1} = b_1^{-1}, \ldots, b_{4g} = b_{2g}^{-1}$. These generating boosts obey the relation

$$b_1 b_2^{-1} b_3 b_4^{-1} \ldots b_{2g-1} b_{2g}^{-1} \; b_1^{-1} b_2 \ldots b_{2g-3}^{-1} b_{2g-2} b_{2g-1}^{-1} b_{2g} = \mathbf{1} \;. \tag{9}$$

The group Γ then is the group generated by $b_1, \ldots, b_{4g}$ subject to this one relation.

To apply the periodic–orbit theory to this system one has to determine the classical periodic orbits. These are the closed geodesics on the octagon and are known to be in a one to one correspondence to the conjugacy classes $\{b\}$ of boosts b in the octagon group Γ. The length $l(b)$ of the periodic orbit given by $\{b\}$ is determined by $2\cosh(l(b)/2) = |\text{tr}\, b| = |2\text{Re}\,a|$. The Selberg trace formula for a compact surface with hyperbolic metric now reads [22]

$$\sum_{n=0}^{\infty} h(p_n) = \frac{\text{Area}(\mathcal{F})}{4\pi} \int_{-\infty}^{+\infty} dp\, p\, h(p)\, \tanh(\pi p) \; + \; \sum_{\{b\}}\sum_{k=1}^{\infty} \frac{l(b) g(kl(b))}{2\sinh(kl(b)/2)} \;. \tag{10}$$

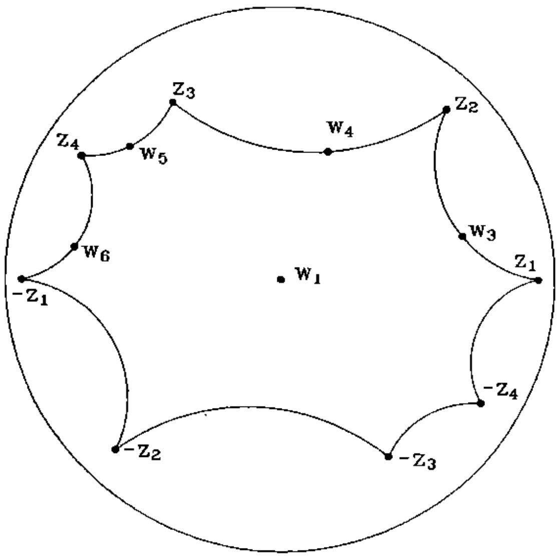

Figure 1: The asymmetric hyperbolic octagon is shown in the Poincaré disc, whose boundary is indicated by the circle, together with the special points discussed in the text. This hyperbolic octagon is uniquely defined by the following 4 corner–points $z_k = r_k e^{i\varphi_k}$: $r_1 = 0.9405185836$, $\varphi_1 = 0$, $r_2 = 0.8701653$, $\varphi_2 = 0.8023654$, $r_3 = 0.7609273$, $\varphi_3 = 2.1175027$, $r_4 = 0.8575482$, $\varphi_4 = 2.5846103$.

The sums and integrals involved in this formula are absolutely convergent provided the smearing function satisfies the requirements stated for the smeared Gutzwiller formula (8). The momenta p_n in (10) are related to the energies E_n through $E_n = p_n^2 + \frac{1}{4}$. (Notice that a zero–mode $E_0 = 0$ appears belonging to the constant eigenfunction $\Psi_0(z) = (4\pi)^{-1/2}$ having positive parity.) The trace formula (10), however, incorporates both parity classes $\Psi(-z) = \pm\Psi(z)$ of eigenfunctions and eigenvalues. To obtain a "generic" chaotic system we have to further desymmetrize our model with respect to the parity operation.

We will now derive the appropriate trace formula for the desymmetrized system, which was not available before. To this end we will study a modification of the group Γ by adding the parity operation to it. Γ consists of boosts b with $|\mathrm{tr}\, b| > 2$. These are called *hyperbolic* elements of $\mathrm{SU}(1,1)/\{\pm 1\}$. Besides these boosts there are also other elements in $\mathrm{SU}(1,1)/\{\pm 1\}$, which do not fulfill the trace condition. One of these is $S = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \mathrm{SU}(1,1)$, with $\mathrm{tr}\, S = 0$. S is an example of an *elliptic* element of $\mathrm{SU}(1,1)$ (a rotation in hyperbolic geometry). The corresponding transformation $z \to S(z) = \frac{-iz}{i} = -z$ is a realization of the parity operation. If we add S as a new generator to Γ and keep in mind the relation $S^2 = \mathbf{1}$ (as an identity in $\mathrm{SU}(1,1)/\{\pm 1\}$, since the matrix S^2 yields minus the unit matrix) we end up with a new discrete subgroup Γ' of $\mathrm{SU}(1,1)/\{\pm 1\}$ that also contains elliptic elements besides hyperbolic ones already present in Γ. Γ' therefore consists of all words in the generators $b_1, \ldots, b_{4g}, S$ subject to the relation (9) and to $b_{2g+1} = b_1^{-1}, \ldots, b_{4g} = b_{2g}^{-1}$, $S^2 = \mathbf{1}$. Since the b_i's identify opposite edges of the octagon, one notices that $b_i^{-1} = Sb_iS$. Using this rule it is always possible to reduce an arbitrary word in the generators of Γ' to a word of the form $b_{i_1} \ldots b_{i_n} S^\varepsilon$,

$\varepsilon \in \{0,1\}$. Thus Γ' may be decomposed as

$$\Gamma' = \Gamma \cup \Gamma S\ , \tag{11}$$

where $\Gamma \cap \Gamma S$ is empty. We now form conjugacy classes $\{b'\}$ of elements $b' \in \Gamma'$. These can either be hyperbolic or elliptic ones. But for the following reason it is possible to determine all elliptic classes: an elliptic $b' \in \Gamma'$ has one fixed point in $\mathcal{D}$ (a hyperbolic one has two fixed points on $\partial\mathcal{D}$), and its conjugacy class $\{b'\}$ fixes the class of points in $\mathcal{D}$ that are identified under Γ'. On the surface $\mathcal{F}'$ (a fundamental domain for Γ') therefore each elliptic conjugacy class has one fixed point, and there are only the $w_1, \ldots, w_{2g+2}$ as possible candidates. Thus there are exactly $2g+2$ elliptic conjugacy classes, and one can easily give representatives for each of them:

- w_1 is fixed by S,
- w_2 is fixed by $b_1^{-1} b_2 b_3^{-1} b_4 \ldots b_{2g-1}^{-1} b_{2g} S$,
- w_i is fixed by $b_{i-2} S$ for $i = 3, \ldots, 2g+2$.

All other conjugacy classes in Γ' have to be hyperbolic ones and hence correspond to periodic orbits on $\mathcal{F}'$. To compute the length spectrum of periodic orbits for the octagon illustrated in fig.1 numerically, we determined, using symbolic algebra, for each hyperbolic conjugacy class $\{b\}$ exactly one representative $b = b_{i_1} b_{i_2} \ldots b_{i_n} S^\varepsilon$ where all products in the generators $b_1, \ldots, b_8$ up to $n = 12$ are considered. To ensure that only one representative is generated by the computer program, it must be checked that the products are not conjugated to each other, and furthermore that the products cannot be transformed into each other by using the relations (9) and $S^2 = \mathbf{1}$. We compared the computed length spectrum $\{l_n\}$ with the asymptotic distribution given by Huber's law and found that up to the length $\mathcal{L} = 12$ nearly all periodic orbits have been found for $n = 12$. The few missing periodic orbits belong to generator products with length n larger than $n = 12$. In the following numerical applications we will thus use the length spectrum only up to the length $\mathcal{L} = 12$.

We now introduce two representations χ^+ and χ^- of Γ'. χ^+ will be the trivial representation $\chi^+(b') = 1$ for all $b' \in \Gamma'$. The representation χ^-, however, will be defined as

$$\chi^-(b') = \begin{cases} +1\ , & b' \in \Gamma \\ -1\ , & b' \in \Gamma S \end{cases} . \tag{12}$$

The wavefunctions Ψ are now required to transform under Γ' according to these representations:

$$\Psi^\pm(b'(z)) = \chi^\pm(b')\Psi^\pm(z)\ . \tag{13}$$

This means that Ψ^+ is invariant under Γ'; Ψ^-, however, satisfies $\Psi^-(b(z)) = \Psi^-(z)$ for $b \in \Gamma$, and $\Psi^-(b'(z)) = \chi^-(b')\Psi^-(z) = -\Psi^-(z)$ for $b' \in \Gamma S$. Thus $\Psi^\pm$ are wavefunctions obeying $\Psi^\pm(b(z)) = \Psi^\pm(z)$ for $b \in \Gamma$ and $\Psi^\pm(-z) = \pm\Psi^\pm(z)$. They are therefore just the wavefunctions on the original surface having definite parity.

From this observation we conclude that the Schrödinger equation on $\mathcal{F}$ (periodic boundary conditions with respect to Γ) for parity $+1$ and -1 is equivalent to the Schrödinger equation on $\mathcal{F}'$, where the representations χ^+ and χ^- have to be taken into account according to eq.(13), respectively. The Selberg trace formula for Γ' must include the contribution coming from the elliptic conjugacy classes. In general this is given by [22]

$$\sum_{\{R\}} \sum_{k=1}^{m(R)-1} \frac{\chi^{\pm}(R)^k}{2m(R)\sin\frac{k\pi}{m(R)}} \int_{-\infty}^{+\infty} dp\, h(p) \frac{e^{-2\pi p \frac{k}{m(R)}}}{1+e^{-2\pi p}} , \tag{14}$$

where $\{R\}$ denotes the elliptic conjugacy classes, and $m(R)$ is the order of R, i.e. the minimal positive integer such that $R^{m(R)} = \pm 1$. In our case there are $2g+2$ elliptic classes which are all of order $m(R) = 2$. Using $\text{Area}(\mathcal{F}') = \frac{1}{2}\text{Area}(\mathcal{F})$ we find that

$$\sum_{n=0}^{\infty} h(p_n^{\pm}) = \frac{\text{Area}(\mathcal{F})}{8\pi} \int_{-\infty}^{\infty} dp\, p\, \tanh(\pi p)\, h(p) \pm \frac{g+1}{4} \int_{-\infty}^{\infty} dp \frac{h(p)}{\cosh \pi p} + \sum_{\{b'\}} \sum_{k=1}^{\infty} \frac{\chi^{\pm}(b')^k\, l(b')}{2\sinh\frac{k\, l(b')}{2}}\, g(k\, l(b')) . \tag{15}$$

Here $p_n^{\pm}$ denotes the momenta corresponding to the energy eigenvalues $E_n^{\pm} = p_n^{\pm 2} + \frac{1}{4}$ of the two parity classes ± 1. The sum on the r.h.s. of (15) runs over all hyperbolic conjugacy classes in Γ'. Equivalently, it may be viewed as a sum over all periodic orbits on $\mathcal{F}'$. In this context we label their lengths as l_n.

The information on the spectrum of H contained in the r.h.s. of (15) can also be encoded in a single function $Z^{\pm}(s)$ of a complex variable $s = \frac{1}{2} - ip$ (therefore $E = s(1-s) = p^2 + \frac{1}{4}$), called the *Selberg zeta function*. To achieve this one chooses

$$h(p) = \frac{1}{p^2 + (s-1/2)^2} - \frac{1}{p^2 + (\sigma - 1/2)^2} \tag{16}$$

for $\text{Re}\, s > 1$, $\text{Re}\, \sigma > 1$, as a test function in (15). Its Fourier–transform is given by

$$g(x) = \frac{1}{2s-1}\, e^{-(s-1/2)|x|} - \frac{1}{2\sigma - 1}\, e^{-(\sigma-1/2)|x|} . \tag{17}$$

The trace formula (15) with this choice looks like

$$\begin{aligned} \sum_{n=0}^{\infty} \left[\frac{1}{p_n^{\pm 2} + (s-1/2)^2} - \frac{1}{p_n^{\pm 2} + (\sigma - 1/2)^2} \right] = \\ - (g-1)\left[\psi(s) - \psi(\sigma)\right] \\ \pm \frac{g+1}{2} \left\{ \frac{1}{2s-1} \left[\psi\left(\frac{s+1}{2}\right) - \psi\left(\frac{s}{2}\right) \right] - \frac{1}{2\sigma-1} \left[\psi\left(\frac{\sigma-1}{2}\right) - \psi\left(\frac{\sigma}{2}\right) \right] \right\} \\ + \frac{1}{2s-1} \frac{d}{ds} \ln Z^{\pm}(s) - \frac{1}{2\sigma - 1} \frac{d}{d\sigma} \ln Z^{\pm}(\sigma) . \end{aligned} \tag{18}$$

Here $\psi(z)$ denotes $\frac{d}{dz} \ln \Gamma(z)$ and

$$Z^{\pm}(s) = \prod_{\{l_n\}} \prod_{k=0}^{\infty} \left(1 - \chi_n^{\pm}\, e^{-(s+k)l_n} \right) . \tag{19}$$

This Euler product representation of $Z^{\pm}(s)$ converges absolutely for $\text{Re}\, s > 1$ and in this domain defines a holomorphic function without zeros. The trace formula (18) shows that

$\frac{d}{ds} \ln Z^{\pm}(s)$ can be continued to a meromorphic function for all $s \in \mathbf{C}$. One can further find that $Z^{\pm}(s)$ can be analytically continued to yield a holomorphic function for $\text{Re}\, s > 0$. In this domain its zeros are located at $s_n = \frac{1}{2} \pm ip_n^{\pm}$ and are of order $d_n^{\pm}$, where $d_n^{\pm}$ denotes the multiplicity of the quantal energy $E_n^{\pm} = s_n(1-s_n)$. This means in particular that $Z^{+}(s)$ has a simple zero at $s_0 = 1$ coming from the zero-mode $E_0^{+} = 0$. If no small eigenvalues ($0 < E_n < \frac{1}{4}$) are present, all of these zeros are located on the *critical line* $\text{Re}\, s = \frac{1}{2}$. In this case an analogue of the Riemann hypothesis is fulfilled for the Selberg zeta functions.

One can also derive a functional equation for the Selberg zeta functions. One substitutes $s \to 1 - s$ and subtracts the resulting formula from the original one. Integrating in s yields (using $s = \frac{1}{2} - ip$ and $1 - s = \frac{1}{2} + ip$)

$$Z^{\pm}\left(\frac{1}{2} + ip\right) = e^{-2\pi i \overline{\mathcal{N}^{\pm}}(E)}\, Z^{\pm}\left(\frac{1}{2} - ip\right) , \tag{20}$$

where the mean spectral staircase $\overline{\mathcal{N}^{\pm}}(E)$ is given by (see also (27))

$$\overline{\mathcal{N}^{\pm}}(E) = \int_0^p dp'\, p'\, \tanh \pi p' \pm \frac{3}{2\pi} \text{arc}\sin(\tanh \pi p) . \tag{21}$$

From the above consideration we conclude that any procedure to analytically continue the Euler product (19) to the critical line $\text{Re}\, s = \frac{1}{2}$ may serve as a quantization rule, since then the zeros s_n of the Selberg zeta function on this axis can be computed. These in turn give the quantal energies $E_n = s_n(1 - s_n)$. In the following sections we will discuss some methods that solve or somehow circumvent the problem to analytically continue the Euler product in more or less efficient ways.

III The Spectral Staircase $\mathcal{N}(E)$

For the quantization rules which will be discussed in the next sections, the spectral staircase $\mathcal{N}(E)$ is needed. To derive an exact formula for it from the trace formula (15) we choose the following Gaussian smoothing

$$h(p) = \frac{1}{\varepsilon\sqrt{\pi}} \left\{ e^{-(p'-p)^2/\varepsilon^2} + e^{-(p'+p)^2/\varepsilon^2} \right\} , \tag{22}$$

which was already used in [3,4]. The smoothing parameter $\varepsilon > 0$ determines the width of the peaks at the quantal energies. The three conditions stated after (8) are fulfilled for every $\varepsilon > 0$, and thus the integrals and the series in the periodic–orbit sum rule converge absolutely. With the choice (22) one obtains from (15) for genus $g = 2$

$$\begin{aligned} \frac{1}{\varepsilon\sqrt{\pi}} \sum_{n=0}^{\infty} \left[e^{-\frac{(p-p_n^{\pm})^2}{\varepsilon^2}} + e^{-\frac{(p+p_n^{\pm})^2}{\varepsilon^2}} \right] = \frac{1}{2\varepsilon\sqrt{\pi}} \int_{-\infty}^{\infty} dp'\, p'\, \tanh(\pi p') \left[e^{-\frac{(p-p')^2}{\varepsilon^2}} + e^{-\frac{(p+p')^2}{\varepsilon^2}} \right] \\ \pm \frac{3}{4\varepsilon\sqrt{\pi}} \int_{-\infty}^{\infty} dp' \frac{e^{-\frac{(p-p')^2}{\varepsilon^2}} + e^{-\frac{(p+p')^2}{\varepsilon^2}}}{\cosh \pi p'} \\ + \frac{1}{2\pi} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{\chi_n^{\pm k}\, l_n}{\sinh\frac{k l_n}{2}} \cos(p\, k\, l_n)\, e^{-\frac{\varepsilon^2}{4}(k l_n)^2} . \end{aligned} \tag{23}$$

To derive the expression for $\mathcal{N}(E)$ eq.(23) has to be integrated from $p_1 = 0$ to $p_2 = \sqrt{E - \frac{1}{4}}$ in the limit $\varepsilon \to 0^+$. Because of

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon\sqrt{\pi}} e^{-(x-y)^2/\varepsilon^2} = \delta(x-y) \tag{24}$$

one obtains from the l. h. s. of eq.(23) the exact spectral staircase

$$\mathcal{N}^{\pm}(E) = M^{\pm} + \int_0^{\sqrt{E-\frac{1}{4}}} dp \sum_{n=M^{\pm}}^{\infty} \left\{ \delta(p - p_n^{\pm}) + \delta(p + p_n^{\pm}) \right\} = \#\{ E_n^{\pm} \,|\, E_n^{\pm} \leq E \} \ , \tag{25}$$

where $M^{\pm}$ is the number of quantal energies $E_n^{\pm} < \frac{1}{4}$. Integration of the r. h. s. of eq.(23) in the same limit yields ($p = \sqrt{E - \frac{1}{4}}$)

$$\mathcal{N}^{\pm}(E) = \int_0^p dp'\, p' \tanh \pi p' \pm \frac{3}{2\pi} \arcsin(\tanh \pi p) + \frac{1}{2\pi} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{\chi_n^{\pm k} \sin(k\, l_n\, p)}{k \sinh \frac{k\, l_n}{2}} \ . \tag{26}$$

The first two terms on the r. h. s. of eq.(26) describe the mean behaviour of $\mathcal{N}^{\pm}(E)$,

$$\overline{\mathcal{N}^{\pm}}(E) = \int_0^p dp'\, p' \tanh \pi p' \pm \frac{3}{2\pi} \arcsin(\tanh \pi p) \tag{27}$$

$$= \frac{1}{2} E - \frac{1}{6} \pm \frac{3}{4} + O\left(e^{-\pi\sqrt{E}}\right) \ . \tag{28}$$

The first term in (28) is identical to Weyl's famous law (7).

Since the limit $\varepsilon \to 0^+$ has been taken the convergence of the periodic–orbit sum is no longer guarantied by the Selberg trace formula. In the case of negative parity it numerically turns out that for the characters χ_n^- both signs ± 1 occur with equal probability which seems to cause conditional convergence of the periodic–orbit sum, since neighbouring "length–terms" nearly cancel each other. Thus the periodic–orbit sum can simply be truncated at a sufficiently large cut–off length $\mathcal{L}$ to obtain an approximation to the oscillatory part of $\mathcal{N}^-(E)$. This contrasts with the case of positive parity where all characters χ_n^+ are positive and no cancellation of neighbouring length–terms can occur. In this case a remainder term $R^+(p,\mathcal{L})$ must be taken into account if the periodic–orbit sum is truncated for numerical applications. To derive this remainder term the omitted part of the periodic–orbit sum is approximated by the following integral,

$$\int_{\mathcal{L}}^{\infty} dl\, \frac{d\,N(l)}{dl} \frac{l\, g(l)}{2 \sinh \frac{l}{2}} \simeq \int_{\mathcal{L}}^{\infty} dl\, e^{l/2} g(l) \equiv R^+(p,\mathcal{L}) \ , \tag{29}$$

where Huber's law (4) has been used. Here only the asymptotically leading $k = 1$ contribution of the k–summation of the periodic–orbit sum is taken into account, since the terms with $k \geq 2$ are exponentially suppressed. With

$$g(l) = \frac{1}{\pi} \frac{\sin(l\,p)}{l} e^{-\frac{\varepsilon^2}{4} l^2} \ , \tag{30}$$

i. e. the term obtained before the limit $\varepsilon \to 0^+$ has been taken, one gets

$$R^+(p,\mathcal{L}) = \frac{1}{\pi} \int_{\mathcal{L}}^{\infty} dl\, \frac{\sin(l\,p)}{l} e^{-\frac{\varepsilon^2}{4} l^2 + \frac{1}{2} l} = \frac{1}{\pi} \,\mathrm{Im} \left[\int_0^{\infty} dt \int_{\mathcal{L}}^{\infty} dl\, e^{(ip + \frac{1}{2} - t)l - \frac{\varepsilon^2}{4} l^2} \right] \ . \tag{31}$$

In the last step the identity

$$\frac{1}{l} = \int_0^{\infty} dt\, e^{-lt}$$

was used. The l–integration can be carried out leading to

$$R^+(p,\mathcal{L}) = \frac{1}{\sqrt{\pi}\,\varepsilon} \,\mathrm{Im} \left[\int_0^{\infty} dt\, e^{(t - \frac{1}{2} - ip)^2/\varepsilon^2} \,\mathrm{erfc}\left(\frac{t - \frac{1}{2} - ip}{\varepsilon} + \frac{\varepsilon \mathcal{L}}{2} \right) \right] \ . \tag{32}$$

Since we are interested in the remainder $R^+(p,\mathcal{L})$ in the limit $\varepsilon \to 0^+$, the asymptotics

$$\mathrm{erfc}(z) \sim \frac{1}{\sqrt{\pi}} \frac{e^{-z^2}}{z} \quad , \quad |z| \to \infty \quad , \quad |\arg z| < \pi \ ,$$

can be used ($|z| \geq \frac{p}{\varepsilon}, p > 0$). Then the t–integration can be carried out leading to

$$R^+(p,\mathcal{L}) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \,\mathrm{Im} \left[e^{\frac{\varepsilon^2 \mathcal{L}^2}{4}} \mathrm{E}_1\left(\left(-\frac{1}{2} - ip \right) \mathcal{L} + \frac{\varepsilon^2 \mathcal{L}^2}{2} \right) \right]$$

$$= \frac{1}{\pi} \,\mathrm{Im} \left[\mathrm{E}_1\left(\left(-\frac{1}{2} - ip \right) \mathcal{L} \right) \right] \ . \tag{33}$$

Here $\mathrm{E}_1(z)$ denotes the exponential integral defined by

$$\mathrm{E}_1(z) = \int_z^{\infty} dt\, \frac{e^{-t}}{t} \ , \tag{34}$$

where it is assumed that the path of integration excludes the origin and does not cross the negative real axis. $R^+(p,\mathcal{L})$ is the remainder term which has to be taken into account if the periodic–orbit sum is truncated at the cut–off length $\mathcal{L}$ in the case of positive parity. It is worthwhile to remark that this remainder term "knows" nothing about the quantal energy spectrum since it is valid for every system for which Huber's law is valid (i. e. the topological entropie is $\tau = 1$) and all characters are $\chi_n = 1$. This is the case for every hyperbolic fundamental domain with genus $g \geq 2$ [27]. Furthermore, it is valid for every arbitrarily chosen compact surface of constant negative curvature $K = -1$, where the boundary conditions cause the characters to be $\chi_n = 1$. A successful test of the remainder term (33) has already been carried out in [19,20]. A non–compact billiard, where (33) has been successfully used, is Artin's billiard with Neumann boundary conditions.

IV Zeta Function Quantization

As mentioned in the previous sections the Selberg zeta function serves as a model function where the properties of the dynamical zeta functions of general strongly chaotic systems can be studied. One tries to detect the non–trivial zeros on the critical line to obtain the quantal energies by using the Euler product formula over the length spectrum $\{l_n\}$ which would then give a quantum mechanical item by a classical one. The main obstacle is that the Euler product converges absolutely only to the right of the "entropy barrier" (e. g. [29]), i. e. for $\mathrm{Re}\, s > \tau$. In the case of the hyperbolic octagons one gets for the topological entropy $\tau = 1$. Since the non–trivial zeros lie at $\mathrm{Re}\, s = \frac{1}{2}$, the Euler product does not converge absolutely in the domain of interest. This is a property which occurs for dynamical zeta functions in general. There is, however, nothing known in general about a possible conditional convergence of the Euler product for Re s smaller than τ. In a convergent product a zero can only occur, when at least one of the factors vanishes at this point. Since this would mean that a single length l_k would determine a quantal energy E_n, a convergence of the Euler product on the critical line cannot be expected. An alternative to using the Euler product is, however, given by the possibility to transform the product into a Dirichlet series. Following [30,31] the k–product in eq.(19) is rewritten by Euler's identity

$$\prod_{k=0}^{\infty} \left(1 - y\, x^k \right) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m\, y^m\, x^{\frac{1}{2}m(m-1)}}{\prod_{r=1}^{m} (1 - x^r)} \quad , \quad |x| < 1 \quad , \quad y \in \mathbf{C} \quad , \tag{35}$$

which leads to

$$Z(s) = \prod_{\{l_n\}} \left(1 + \sum_{m=1}^{\infty} \frac{a_{mn}}{N_n^{s\,m}} \right) \tag{36}$$

with

$$a_{mn} \equiv \frac{(-1)^m\, \chi_n^m\, e^{-\frac{1}{2}m(m-1)l_n}}{\prod_{r=1}^{m} (1 - e^{-r\, l_n})} \quad \text{and} \quad N_n \equiv e^{l_n} \quad . \tag{37}$$

(In this section the indices $\pm$ are suppressed since the discussion is valid for every dynamical zeta function of type (19).) Expanding the remaining product over the periodic orbits one obtains with the definitions

$$A_\rho \equiv \prod_i a_{m_i n_i} \quad , \quad N_\rho \equiv \prod_i N_{n_i}^{m_i} = \exp\Big(\sum_i m_i l_{n_i} \Big) \tag{38}$$

a Dirichlet series

$$Z(s) = 1 + \sum_\rho \frac{A_\rho}{N_\rho^s} \quad , \quad \text{Re}\, s > 1 \quad , \tag{39}$$

over so–called "pseudo orbits" [31] with lengths $L_\rho \equiv \sum_i m_i l_{n_i}$. If the pseudo–length spectrum is ordered as $L_1 \leq L_2 \leq \ldots$, the convergence properties of the Dirichlet series (39) are determined by

$$\sigma_a = \limsup_{n\to\infty} \frac{1}{\ln N_n} \ln \sum_{\rho=1}^{n} | A_\rho | \tag{40}$$

and

$$\sigma_c = \limsup_{n\to\infty} \frac{1}{\ln N_n} \ln \Big| \sum_{\rho=1}^{n} A_\rho \Big| \quad . \tag{41}$$

Here σ_a denotes the abscissa of absolute convergence, and σ_c denotes the abscissa of conditional convergence. This means that (39) converges for Re $s > \sigma_c$ and converges absolutely for Re $s > \sigma_a$. (Notice that $\sigma_a \geq \sigma_c$.) If one obtains $\sigma_c < \frac{1}{2}$, the Dirichlet series (39) can be used for the determination of quantal energies.

To determine the abscissa of absolute convergence from (40) one has to know how $\ln N_n$ increases with n. In order to find this dependence one has to count the number $N^p(L)$ of pseudo–orbits with pseudo–lengths not exceeding L. This is suggested to be

$$N^p(L) \sim a e^{\tau L} \quad , \quad L \to \infty \; , \tag{42}$$

with a system–dependent constant a. Up to now, there are only two cases where this form is rigorously known to be true. The first is the Riemann case, where one trivially obtains

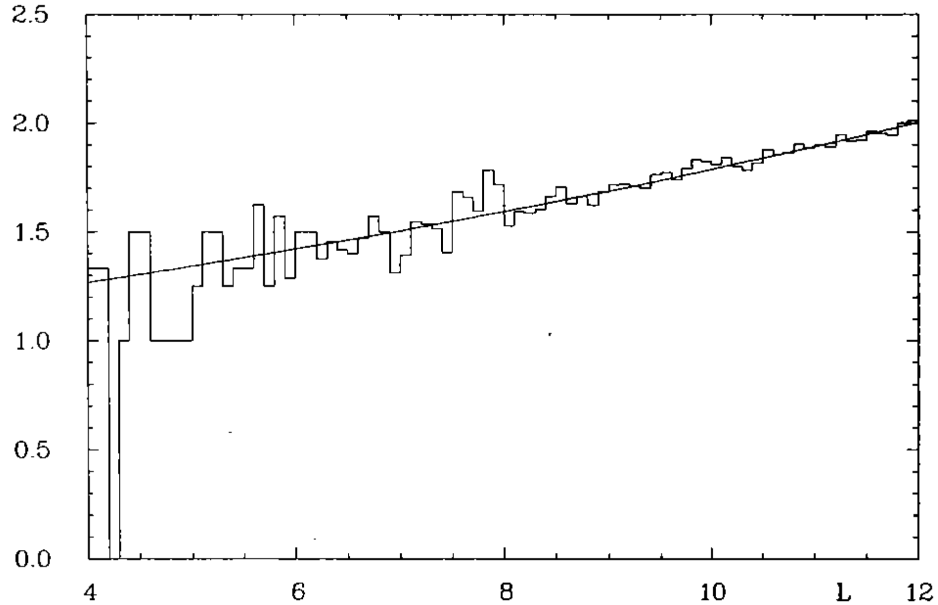

Figure 2: The increase of the multiplicities of the pseudo–orbits on the hyperbolic octagon defined in figure 1 is shown in comparison with the fit (44).

$a = \tau = 1$, since the pseudo–length spectrum is simply $\{\ln n\}$, $n \in \mathbf{N}$. The second is just provided by the compact surfaces of negative curvature considered here, where the asymptotic distribution $N^p(L) \sim \frac{Z(2)}{Z'(1)} e^L$ is proved in [32]. (In this zeta function all characters χ_n must be $\chi_n = 1$.)

The index m of the coefficient a_{mn} from (37) labels the number of traversals of the primitive orbit l_n. Asymptotically, for $l \to \infty$, the number of orbits traversed only once dominate the number of all orbits; contributions from $m \geq 2$ are exponentially suppressed. Thus most (in the just mentioned sense) a_{mn} have $m = 1$, and hence most of the coefficients A_ρ appearing in the Dirichlet series (39) are of the form $A_\rho = \prod_i a_{1n_i} \sim \prod_i(-\chi_{n_i})$, for $L_\rho \to \infty$. Using now in (40) $|A_\rho| \sim 1$ and the proliferation (42) of pseudo–orbits, which yields $\ln N_n \sim \frac{1}{\tau} \ln n$, leads to $\sigma_a = \tau$.

The abscissa of conditional convergence is in general unknown and will presumably differ from system to system; but it can be computed numerically from (41). Inspecting (37) and (38) shows that A_ρ will in general have no definite sign so that one expects $\sigma_c < \sigma_a$. Assuming a random distribution of these signs one can use a statistical model to find a prediction for σ_c (see [29] where this has been done for the Ruelle–type zeta function $R(s)$ to be introduced below in eq.(51)). The asymptotic form $A_\rho \sim \prod_i(-\chi_{n_i})$ for the coefficients in the Dirichlet series for $Z(s)$ now is just the exact expression for the coefficients in the Dirichlet series for $R(s)$ that has been used in [29]. We can thus apply the statistical model of [29] also for $Z(s)$.

All pseudo–orbits ρ sharing the same pseudo–length L_n also have identical coefficients A_n. Denoting the multiplicity of L_n by $g_p(L_n)$ thus allows to write

$$Z(s) = 1 + \sum_{n=1}^{\infty} g_p(L_n) A_n\, e^{-sL_n} \quad . \tag{43}$$

We now replace the multiplicity $g_p(L)$ by its mean behaviour $\overline{g_p}(L_n)$ for which the following asymptotic behaviour is assumed [29]

$$\overline{g_p}(L) \sim d\, e^{\alpha L}\ , \quad L \to \infty\ . \tag{44}$$

Using the available pseudo–length spectrum we fitted (44) to the numerical data and obtained for the two parameters $d = 1.009$ and $\alpha = 0.0572$. In figure 2 a comparison between this fit and the mean multiplicities is shown, where the mean multiplicities are the averages of the multiplicities of the pseudo–orbits over length intervals of $\Delta L = 0.1$. The replacement $g_p \to \overline{g_p}$ may be translated into a shift in the argument of $Z(s)$ through

$$Z(s) = 1 + \sum_{n=1}^{\infty} A_n d\, \frac{g_p(L_n)}{d\, e^{\alpha L_n}}\, e^{-(s-\alpha)L_n}\ . \tag{45}$$

Because of (42) and (44) the distinct pseudo–lengths L_n grow asymptotically like $L_n \sim \frac{1}{\tau-\alpha}\ln n$, which has to be used in (41). Now, $|A_n| \sim 1$, and $g_p(L_n)/d\, e^{\alpha L_n}$ fluctuates about one. Thus the coefficients appearing in (45) fluctuate about d. The model introduced in [29] now supposes that the signs of A_n are randomly distributed so that $S_n \equiv \sum_{k=1}^{n} dA_k g_p(L_k)/d\, e^{\alpha L_k}$ can be viewed as a random variable with zero mean. Replacing $|S_n|$ by its mean spread $\sqrt{\overline{S_n^2}}$, whose value ($\propto \sqrt{n}$) can be obtained from a random walk model, then leads to the observation that the abscissa of conditional convergence of (45) in the variable $s - \alpha$ is given by $\frac{1}{2}(\tau - \alpha)$. Shifting the variable $s - \alpha$ back to s thus yields

$$\sigma_c = \frac{1}{2}(\tau + \alpha)\ . \tag{46}$$

In fig.3 the ~~numerical evaluations of (40) and (41) are shown in comparison with the~~ prediction (46) (dashed line) of the statistical model. The expressions (40) and (41) are plotted as a function of L, i.e. (40) and (41) are computed from the part of the pseudo–length spectrum with $L_n < L$. The limes superior $n \to \infty$ is then determined by the maxima occuring at large values of L. The theoretical prediction (46) lies only slightly above the critical line shown as the full line in figure 3. For both parity classes one observes agreement with (46) since for the case of negative parity (fig.3b) the curve representing the conditional convergence lies always below the line corresponding to (46), whereas for the case of positive parity (fig.3a) this is true for $L > 7.5$. Remarkably, on large L–intervals the curve for the conditional convergence lies also below the critical line which suggests that the zeta–function $Z(s)$ may be numerically computed on the critical line by its Dirichlet series since σ_c at least seems to be very close to this line.

The detection of the zeros on the critical line is simplified, if one considers a function, which is real on that line. Since the phase of $Z(s)$ along $s = \frac{1}{2} - ip$ can be read off from the functional equation (20), it is advantageous to use the function [15,16,33]

$$\xi(p) \equiv \text{Re}\left[Z\left(\frac{1}{2} + ip\right) e^{i\pi\overline{N}(E)} \right]\ . \tag{47}$$

The expression in the brackets is real because of the functional equation (20). Thus, taking the real part on the r.h.s. of eq.(47), one obtains by using the Dirichlet series (39) and by assuming $\sigma_c < \frac{1}{2}$,

$$\xi(p) = \cos(\pi\overline{N}(E)) + \sum_{\rho} \frac{A_\rho}{\sqrt{N_\rho}} \cos(\pi\overline{N} - p\ln N_\rho)\ . \tag{48}$$

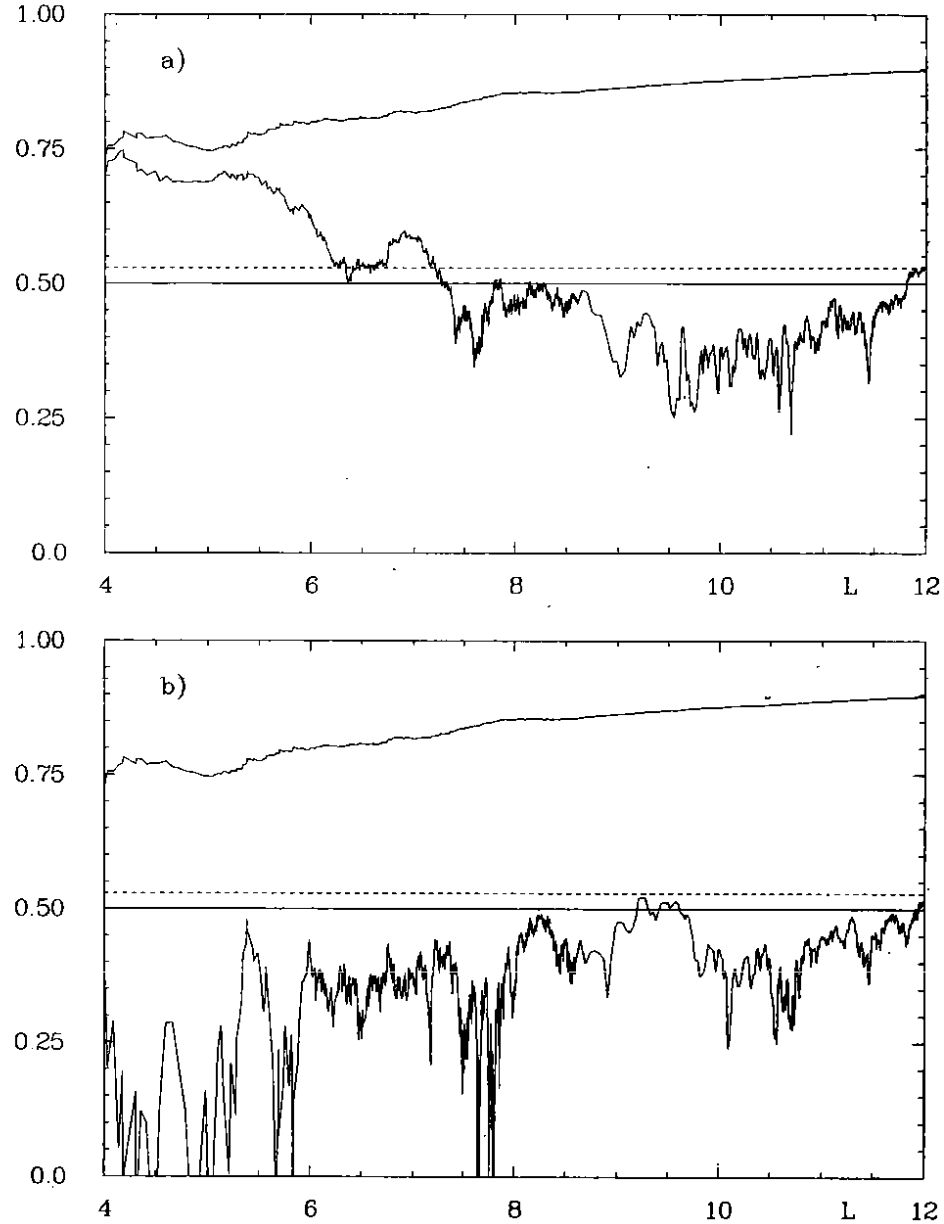

Figure 3: The abscissae of absolute convergence σ_a (upper curve) and of conditional convergence σ_c (lower curve) are shown together with the critical line (full line) and the theoretical prediction (46) for σ_c (dashed line), as expected from the mean increase of the multiplicities. Figure a) represents the case of positive and figure b) the case of negative parity.

Together with the *quantization condition*

$$\xi(p) = 0 \ , \qquad (49)$$

eq.(48) allows the determination of the quantal energies in terms of pseudo-orbits.

Figure 4 shows the numerical results obtained from the zeta function quantization approach for the positive parity class (fig.4a) and for the negative one (fig.4b). The quantal energies as computed by the finite-element method [25] are depicted as circles. The full curves belong to an evaluation of $\xi(p)$ where the cut-off length $\mathcal{L}$ is energy-independently chosen to be simply $\mathcal{L} = 12$ up to which the pseudo-orbit spectrum is known. One observes agreement with the zeros of $\xi(p)$ for energies up to $E \simeq 5$. For such small energies E the truncated series (48) has obviously already reached its limit. For higher energies much more pseudo-orbits seem to be necessary. In [31] it is suggested in analogy to the Riemann-Siegel formula for the Riemann zeta function on its critical line that the Dirichlet series (39) truncated at

$$\mathcal{L} = \pi \bar{d}(E) \frac{dE}{dp} \qquad (50)$$

is the complex conjugate of the omitted part of the series (see also section V). If this suggestion holds, then $\xi(p)$ should yield the correct zeros by truncating (48) according to (50). Since the pseudo-length spectrum is known up to $\mathcal{L} = 12$, the truncation condition (50) allows a computation up to $E \simeq 15$. The dashed curves in figure 4 show the evaluation of (48) truncated according to (50). There are unresolved quantal energies, and thus additional remainder terms seem to be necessary.

The zeta function approach as outlined above requires the computation of the pseudo-orbits with lengths $L_\rho = \sum_i m_i l_{n_i}$, i.e. all integer linear combinations of the original length spectrum $\{l_n\}$. One can reduce the combinatoric problem by using the Ruelle-type zeta function [29,32]

$$R(s) = \prod_{\{l_n\}} \left(1 - \chi_n e^{-s\, l_n} \right) \quad , \quad \mathrm{Re}\, s > 1 \ . \qquad (51)$$

Here the k-product, which occurs in (19), is absent and no use of Euler's identity is necessary. By expanding the product over the length spectrum in (51) one obtains "Ruelle-type" pseudo-orbits with lengths $L_\rho^R = \sum_i l_{n_i}$, i.e. all combinations of the original lengths l_n without any multiple traversals. Thus the combinatoric problem is much reduced and, furthermore, the Dirichlet coefficients are simply $A_\rho^R = \pm 1$. The zeta function $Z(s)$ can then be obtained from

$$Z(s) = \prod_{k=0}^{\infty} R(s+k) \ . \qquad (52)$$

The Ruelle-type zeta function $R(s)$ has the same zeros as $Z(s)$ on the critical line $\mathrm{Re}\, s = \frac{1}{2}$, since $R(s) \neq 0$ for $\mathrm{Re}\, s > 1$, as can be read off from (51) and (52). Using the function $R(s)$ suffers from the disadvantage that its phase $\arg R(s)$ has on the critical line no such simple form as has the phase of $Z(s)$. This is, however, necessary to construct a "quantization function" being real on the critical line. However, given $R(s)$, one can use (52) in (47).

V Efficiency of Periodic-Orbit Theory

Now we would like to discuss the efficiency of periodic-orbit theory in general which is determined by two crucial system-dependent items: the *mean level density* $\bar{d}(E)$ and the *topological*

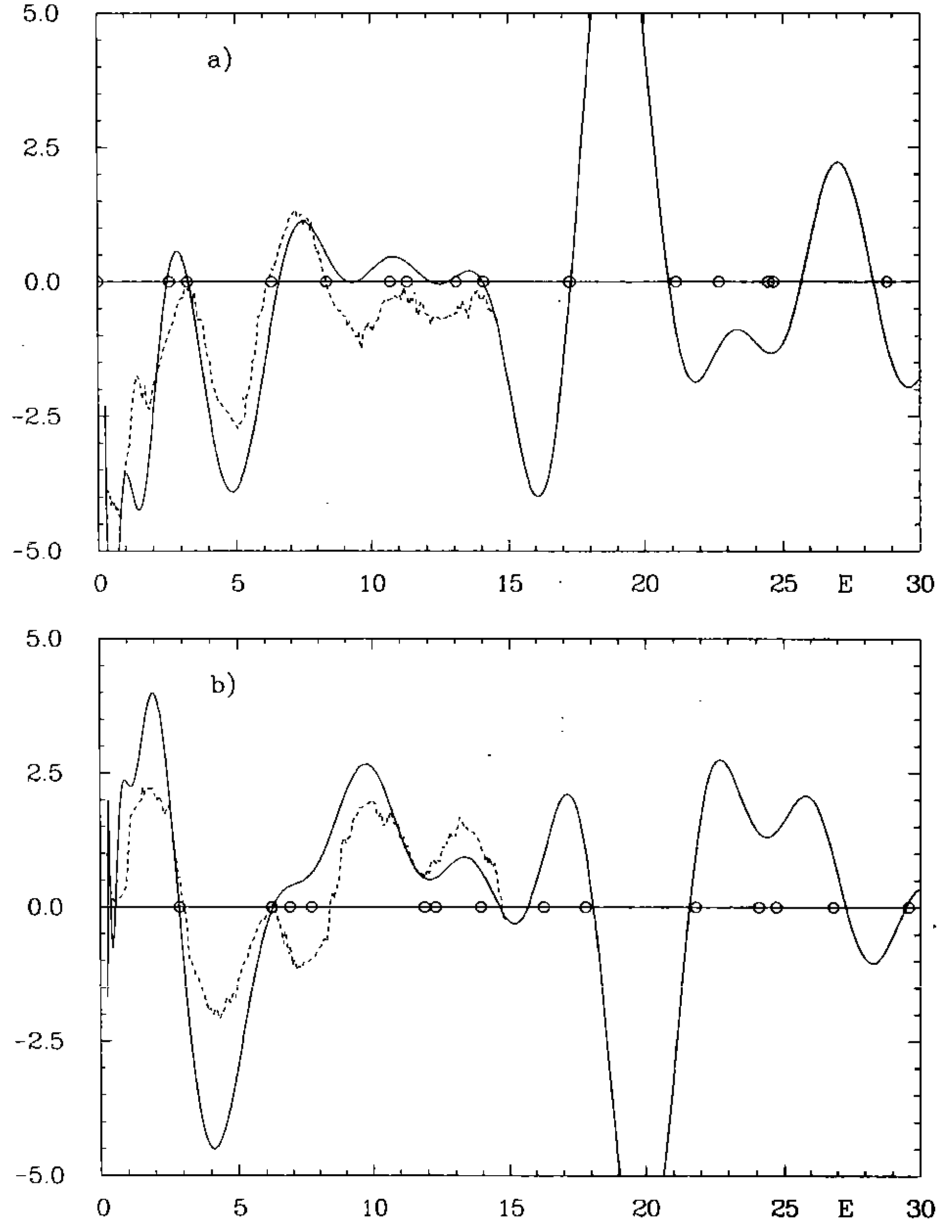

Figure 4: The quantization function $\xi(p)$ is shown for positive parity (a) and negative parity (b) as a function of $E = \frac{1}{4} + p^2$. The full curves show the evaluations where the Dirichlet series is truncated at $\mathcal{L} = 12$ independent of the energy. The truncation condition (50) is used in the dashed curves which show discontinuities when (50) leads to a jump in the number of pseudo-orbits.

entropy τ. To exemplify the system–dependent component, the efficiency is compared in the cases of chaotic billiards of finite area Area(D), the hyperbola billiard having infinite area, and the system being connected with the Riemann zeta function. For every numerical evaluation of periodic–orbit sums or of the pseudo–orbit sums in eqs.(39) and (48) the series must be truncated. Thereby two questions arise:

- What is the minimal possible cut–off length $\mathcal{L}$ to resolve a given quantal energy E ?
- How can one take into account the omitted part of the series ?

To address the first question one notes that in those periodic–orbit sum rules which are constructed to locate quantal energies, each periodic orbit with length l_n contributes an oscillation of "wavelength" $\Delta p = \frac{2\pi}{l_n}$ to the periodic–orbit sum (see eqs.(23) and (26) or the Breit–Wigner smearing in [4,33]). This is also valid for the Dirichlet series (39), if the length L_ρ of the pseudo–orbit is considered instead of the length l_n of a periodic orbit, since one derives from eq.(39) that every pseudo–orbit contributes to the Dirichlet series an oscillation proportional to e^{ipL_ρ}, i. e. a term of "wavelength" $\Delta p = \frac{2\pi}{L_\rho}$. The same behaviour is observed for $\xi(p)$ in eq.(48).

It is important to distinguish between two alternative possibilities for the determination of quantal energies. The first possibility is provided by periodic–orbit sum rules which produce large *maxima* at the location of quantal energies. These sum rules are naturally obtained by smoothing the poles occuring in the trace of the Green's function which constitutes the foundation of Gutzwiller's periodic–orbit theory. The other possibility is provided by a formulation of the periodic–orbit theory, where the quantal energies are determined by *zeros* of an appropriate function. This is the case for the zeta function quantization where the introduction of the Dirichlet series enforces the use of pseudo–orbits, and for the quantization condition we will discuss in the next section which relies directly on periodic orbits. The important fact is that quantizations relying on zeros need only a square root of the number of orbits compared to quantizations relying on maxima. To resolve two maxima which are separated by Δp, periodic orbits yielding a full wavelength Δp have to be taken into account in the periodic–orbit sum. This contrasts with the quantization conditions relying on zeros, where orbits yielding a wavelength $2\,\Delta p$ suffice, since zeros occur two times as frequently as maxima in the oscillations due to the orbits, of course. Furthermore, the number of orbits proliferate exponentially with length, and hence, the factor 2 reduces the number of orbits by a square root. This is the reason why a formulation using zeros for the detection of quantal energies is superior to a formulation based on maxima. In the following we discuss only formulations using zeros.

The shortest distance between two adjacent zeros with respect to p is determined by the orbit of largest length $\mathcal{L}$ which is taken into account in the sum, i. e. $\Delta_{\min} p = \frac{\pi}{\mathcal{L}}$. The cut–off length $\mathcal{L}$ needed at energy E is determined in turn by the mean level density $\bar{d}(E)$ which gives the mean distance ΔE between the zeros of adjacent quantal levels

$$\Delta E \;=\; \frac{1}{\bar{d}(E)} \;=\; \Delta p\,\frac{dE}{dp} \quad , \quad \bar{d}(E) \;\equiv\; \frac{d}{dE}\,\overline{N}(E) \quad ,$$

which must not be smaller than $\Delta_{\min} p$ to resolve the quantal energies. With $\mathcal{L} = \frac{\pi}{\Delta p}$ one arrives at

$$\mathcal{L} \;=\; \pi\,\bar{d}(E)\,\frac{dE}{dp} \quad . \tag{53}$$

This truncation condition (53), already used in the last section, is identical to the one which occurs in the Riemann–Siegel look–alike formula [31]. To see this one notes that (53) can be rewritten as

$$\frac{d}{dE}\,\left\{\,\pi\overline{N}(E) - p\mathcal{L}\,\right\} \;=\; 0 \quad . \tag{54}$$

Now, one introduces a new mean spectral staircase $\overline{N}\,'(p)$ which counts the eigenvalues in terms of the variable p occuring in $s = \frac{1}{2} - ip$, i. e. $\overline{N}(E) = \overline{N}\,'(p)$ with $E = p^2 + \frac{1}{4}$. Then condition (54) is equivalent to

$$\frac{d}{dp}\,\left\{\,\pi\overline{N}\,'(p) - p\mathcal{L}\,\right\} \;=\; 0 \quad . \tag{55}$$

(In [31] the authors use the notation $s = \frac{1}{2} - iE$, and thus their variable E corresponds to our variable p.)

Now we turn to the quantization using the Dirichlet series (39) truncated according to (53), i. e. a formulation of the periodic–orbit theory in terms of *pseudo–orbits*. A good measure for its efficiency is the increase of the number N^p of pseudo–orbits needed to resolve quantal energies with increasing p. As mentioned in the last section this number is suggested to grow according to eq.(42) as a function of the length L.

Let us now discuss three different examples where the periodic–orbit theory works with different efficiencies. The first one is given by the system associated with the Riemann zeta function, where the Riemann–Siegel formula, which truncates the Dirichlet series according to (53), works very efficiently. This is a consequence of the very slow increase of $\overline{N}\,'_R(p)$ behaving ~~asymptotically like~~

$$\overline{N}\,'_R(p) \;\sim\; \frac{p}{2\pi}\ln\frac{p}{2\pi} \;-\; \frac{p}{2\pi} \;-\; \frac{1}{8} \quad , \quad p \to \infty \quad . \tag{56}$$

From (53), together with the law $N^p(L) = e^L$, one obtains the asymptotic behaviour of the number N^p of pseudo–orbits needed to resolve a zero of $\zeta(s)$, as $N^p \sim \sqrt{\frac{p}{2\pi}}$. The crucial point is that N^p increases only like a square root of p, and not exponentially, as it is the case for other chaotic systems. The point may be emphasized by a comparison with the hyperbola billiard, which has also a logarithmic leading term in $\overline{N}\,'_H(p)$ due to its infinite area [34,35],

$$\overline{N}\,'_H(p) \;=\; \frac{p^2}{2\pi}\,\ln p \;+\; \frac{\gamma - \ln 2\pi}{2\pi}\,p^2 \;+\; O(\,p\,) \quad , \quad p \to \infty \quad . \tag{57}$$

By comparing (57) with (56) one is lead to the interpretation that in the Riemann case the first two leading terms are absent in $\overline{N}\,'_R(p)$, if one suggests that the unknown system is a billiard of infinite area which nevertheless has a discrete energy spectrum like the hyperbola billiard. In the case of the hyperbola billiard one obtains from (57) and (42) an exponentially growing number $N^p \sim \exp(\,\tau\,p\,\ln p\,)$ of pseudo–orbits needed in the pseudo–orbit sum. Since other numerical schemes, like collocation methods, boundary–element methods or the finite–element method, have an efficiency behaving as a power of p, one sees that for sufficiently large p the periodic–orbit theory is not suited for determining quantal energies. Nevertheless, it might be that the periodic–orbit theory behaves better at smaller values of p. The exponential increase of N^p, which is a typical feature of chaotic systems, can also be observed in chaotic billiards of finite area, for which one obtains from Weyl's law (7) the exponential behaviour

$N^p \sim a \exp(\frac{\tau}{2\pi} \mathrm{Area}(D)\, p)$. As already mentioned, in the case of hyperbolic octagons one has $a = \frac{Z(2)}{Z'(1)}$ and $\tau = 1$.

We conclude that the ease with which the Riemann–Siegel formula allows a computation of non–trivial zeros is caused by the very slowly increasing $\overline{\mathcal{N}}\,'_R(p)$, and is not a miraculous property of the formula itself. It might be that the system lurking behind the Riemann zeta function has nothing to do with any kind of a chaotic system. For chaotic systems one typically observes an exponentially increasing number N^p of terms needed in the pseudo–orbit sum, and thus overwhelming difficulties appear in applications for high values of p.

Now let us come to the efficiency concerning *periodic–orbit sum rules*, where quantal energies are determined by zeros. They are more efficient since the truncation condition (53) is identical to the one for pseudo–orbits, and the computation of the pseudo–orbits is spared. Since the pseudo–orbits are combinations of the original primitive periodic orbits, one has $N^p(\ell) > N(\ell)$, where $N(\ell)$ is the number of primitive periodic orbits up to ℓ, which is asymptotically determined by the topological entropy τ

$$N(\ell) \sim \frac{e^{\tau\ell}}{\tau\ell} \quad , \quad \ell \to \infty \quad . \tag{58}$$

(Periodic orbits which arise by multiple traversal ($k \geq 2$) of primitive ones give a non leading contribution $\frac{e^{\tau\ell/k}}{\tau\ell/k}$ to (58).) Since $N^p(\ell)$ has no factor $\frac{1}{\ell}$, as $N(\ell)$ has, the number of the additional pseudo–orbits needed is ever faster increasing. Thus a quantization rule based directly on a periodic–orbit sum, like the one being discussed in the next section, seems to be preferable. Concerning billiard systems sharing a common topological entropy τ we would like to note that the periodic–orbit theory is the more efficient the smaller the area of a billiard is, since the cut–off length $\mathcal{L}$ is given by $\mathcal{L} = \frac{1}{2}\,\mathrm{Area}(D)\, p$. In the special case of billiards on a surface of constant negative curvature $K = -1$ all systems possess the same topological entropy $\tau = 1$, and thus the periodic orbits proliferate in the same way. In [36] a billiard on a surface of constant negative curvature $K = -1$ is considered which has an area of only $\frac{1}{96}$ of a hyperbolic octagon. Because of the small area much more quantal energies can be determined than for a hyperbolic octagon, emphasizing again the system–dependent efficiency of periodic–orbit theory.

The second question posed at the beginning of this section is much more difficult to answer, for it requires knowledge of the asymptotic behaviour of the omitted orbits and their characters. As already mentioned, in the case of the Dirichlet series of $Z(s)$ it was suggested in [31] that in analogy to the Riemann–Siegel formula the omitted part of the Dirichlet series truncated according to (50) should yield the complex conjugate of the truncated series. Thus the location of the zeros on the critical line is unaltered if the real part of the Dirichlet series, which is truncated by (50), is taken. There are hints that this is indeed the case if the Dirichlet series is at least conditionally convergent on the critical line [33]. Based on a smoothed cut–off, which was also used in [33], remainder terms are computed in [37] for the Dirichlet series. However, since a smoothed cut–off is used, much more additional pseudo–orbits are required which renders this method much more inefficient.

In the case of periodic–orbit sums one has to distinguish the case where all characters χ_n have the same sign from the one with stochastically alternating signs. In section III a remainder term (see eq.(33)) was derived for a sharp cut–off at $\mathcal{L}$ for the periodic–orbit sum rule describing $\mathcal{N}(E)$ which represents a case of equal signs. This procedure can also be carried out for other periodic–orbit sum rules (for the Gaussian and the Breit–Wigner

smoothing see [33]). In the case of stochastically alternating signs one has to split the omitted part of the periodic–orbit sum with respect to the two signs. Then one applies to the two terms corresponding to the two different signs the same procedure as in the case of equal signs yielding then two remainder terms. If both signs are stochastically distributed, the two remainder terms are identical in modulus but differ in sign, and thus cancel each other. Hence no remainder term occurs in that case.

At last we would like to emphasize the improvement of the efficiency of the periodic–orbit theory by considering completely desymmetrized systems. The cut–off length $\mathcal{L}$, see eq.(53), is proportional to the mean level density $\overline{d}(E)$, which is lowered by restricting energy spectra to definite symmetry classes. Since $\mathcal{L}$, and in turn $\overline{d}(E)$, enters the exponent of the formula describing the number of orbits, a decline of $\overline{d}(E)$ by a factor of, e.g., $\frac{1}{2}$ as for parity symmetry, reduces the number of required orbits to the square root of the original number. Thus one should always try to apply the periodic–orbit theory to completely desymmetrized systems. A general formulation of Gutzwiller's periodic–orbit theory for distinct symmetry classes in the case of discrete symmetries is outlined in [38]. This formulation can be used to write the dynamical zeta function as a product of zeta functions corresponding to the distinct symmetry classes [39]. In case the periodic–orbit sum is given by a Selberg trace formula, this desymmetrization procedure was already developed in [40] and [16].

VI The Quantization Condition $\cos(\,\pi\,\mathcal{N}(E)\,) = 0$

After having discussed efficiencies of different quantization methods based on periodic–orbit theory in some detail we are now going to employ the most preferable technique, which determines the quantal energies as zeros of a quantization function and uses primitive periodic orbits as its input. The starting point of this method is the spectral staircase $\mathcal{N}(E)$ introduced in section III. A direct quantization condition would be given by $\mathcal{N}^{\pm}(E_n^{\pm}) = n - \frac{1}{2}$, $n = 1, 2, 3, \ldots$. (Here we call E_1 the smallest quantal energy exceeding $\frac{1}{4}$, since smaller eigenvalues cannot be obtained by the method to be discussed below. These correspond to imaginary momenta $p = i\sqrt{\frac{1}{4} - E}$, whose possible occurrence is a special feature of hyperbolic geometry.) It is more convenient, however, to use $f^{\pm}(E) \equiv \cos(\pi\mathcal{N}^{\pm}(E)\,)$, which has zeros at the location of the quantal energies. The *quantization condition* then reads

$$\cos(\,\pi\,\mathcal{N}^{\pm}(E)\,) \;=\; 0 \quad . \tag{59}$$

If one would use in (59) only the mean spectral staircase $\overline{\mathcal{N}^{\pm}}(E)$ (see (27)) one would already obtain an approximate quantal spectrum with the correct mean density. The important fluctuations are now taken into account by using the periodic–orbit expression (26), where the periodic orbit sum will be truncated at $\mathcal{L}$. As discussed previously, the remainder term as introduced in section III depends on the characters χ_n. For positive parity all $\chi_n^+ = 1$, so that (see (33))

$$R^+(p,\mathcal{L}) = \frac{1}{\pi}\ \mathrm{Im}\ \left[\,\mathrm{E}_1\left(\,\left(-\frac{1}{2} - ip\right)\,\mathcal{L}\,\right)\,\right] \quad .$$

There is, however, no such remainder term for negative parity, $R^-(p,\mathcal{L}) = 0$, since both signs in $\chi_n^- = \pm 1$ occur with equal probabilities. We thus use the following periodic–orbit expression to compute the spectral staircase ($E = p^2 + \frac{1}{4} > \frac{1}{4}$):

$$\mathcal{N}^{\pm}(E) = \int_0^p dp'\, p' \tanh \pi p' \pm \frac{3}{2\pi} \arcsin(\tanh \pi p) + \frac{1}{2\pi} \sum_{\{l_n\}} \sum_{\substack{k=1 \\ k\, l_n \leq \mathcal{L}}}^{\infty} \frac{\chi_n^{\pm k} \sin(k\, l_n\, p)}{k \sinh \frac{k l_n}{2}} + R^{\pm}(p, \mathcal{L}) \quad . \tag{60}$$

This serves as the input for the quantization condition (59). In [19] this method has been first proposed and applied to a non-desymmetrized octagon and has been found to work quite well for the first quantal energies. Furthermore, the same procedure was applied to the hyperbola billiard and to Artin's Billiard in [20] where it was able to resolve a lot of quantal energies. In the case of the hyperbola billiard a semiclassical expression for $\mathcal{N}(E)$ was used based on Gutzwiller's periodic-orbit theory which then renders (59) as a semiclassical approximation. For hyperbolic octagons and for Artin's billiard the periodic-orbit theory is exact and errors are only due to the truncation.

Here we demonstrate the theory for the desymmetrized hyperbolic octagon introduced in section II. In fig. 5 we show a plot of $\cos(\pi\mathcal{N}(E))$ where the truncation condition (53) was used (the dashed curve). In addition, we present an evaluation of (59) where the total length spectrum up to $\mathcal{L} = 12$, which represents the limit up to which the length spectrum is nearly completely known, was used (the full curve). Fig. 5a is devoted to the case of positive parity, negative parity is dealt with in fig. 5b. One sees that this quantization condition works well to obtain the lower part of the quantal energy spectrum. It is considerably more accurate than the methods using the Dirichlet series for the zeta function. It even seems to resolve quantal energies beyond the value determined by the truncation condition (53). Like other methods the quantization (59) gets into trouble if two quantal energies are lying much closer than the mean level spacing. For such closely neighbouring quantal energies the truncation condition (53) prevents their resolution. It is thus favourable to use all available periodic orbits as it is done in the full curve in figure 5. The improvement can be seen at the two quantal energies lying close to $E \simeq 3$ in figure 5a. Using the maximal cut-off $\mathcal{L}$ does not much alter the efficiency, since the main computational effort arises in the calculation of the length spectrum itself. We thus recommend to consider (53) as a condition yielding the cut-off length for the highest desired quantal energy, having in mind that one has to pay attention in cases of near degeneracies.

VII Summary and Discussion

Quantization methods based on Gutzwiller's periodic-orbit theory for chaotic systems have been discussed. Because of the lack of an analogue to the celebrated EBK quantization of integrable systems the periodic-orbit theory is the only candidate quantization rule for chaotic systems, apart from numerical schemes for direct solutions of the corresponding Schrödinger equation. The periodic-orbit theory is applied to a strongly chaotic system consisting of a point particle sliding freely on a compact Riemann surface of constant negative curvature $K = -1$. The simplest realization having genus $g = 2$, the hyperbolic octagon which has the topology of a sphere with two handles, is chosen. Since such a system always possesses a parity symmetry the periodic-orbit theory is applied to distinct parity classes, which enhances

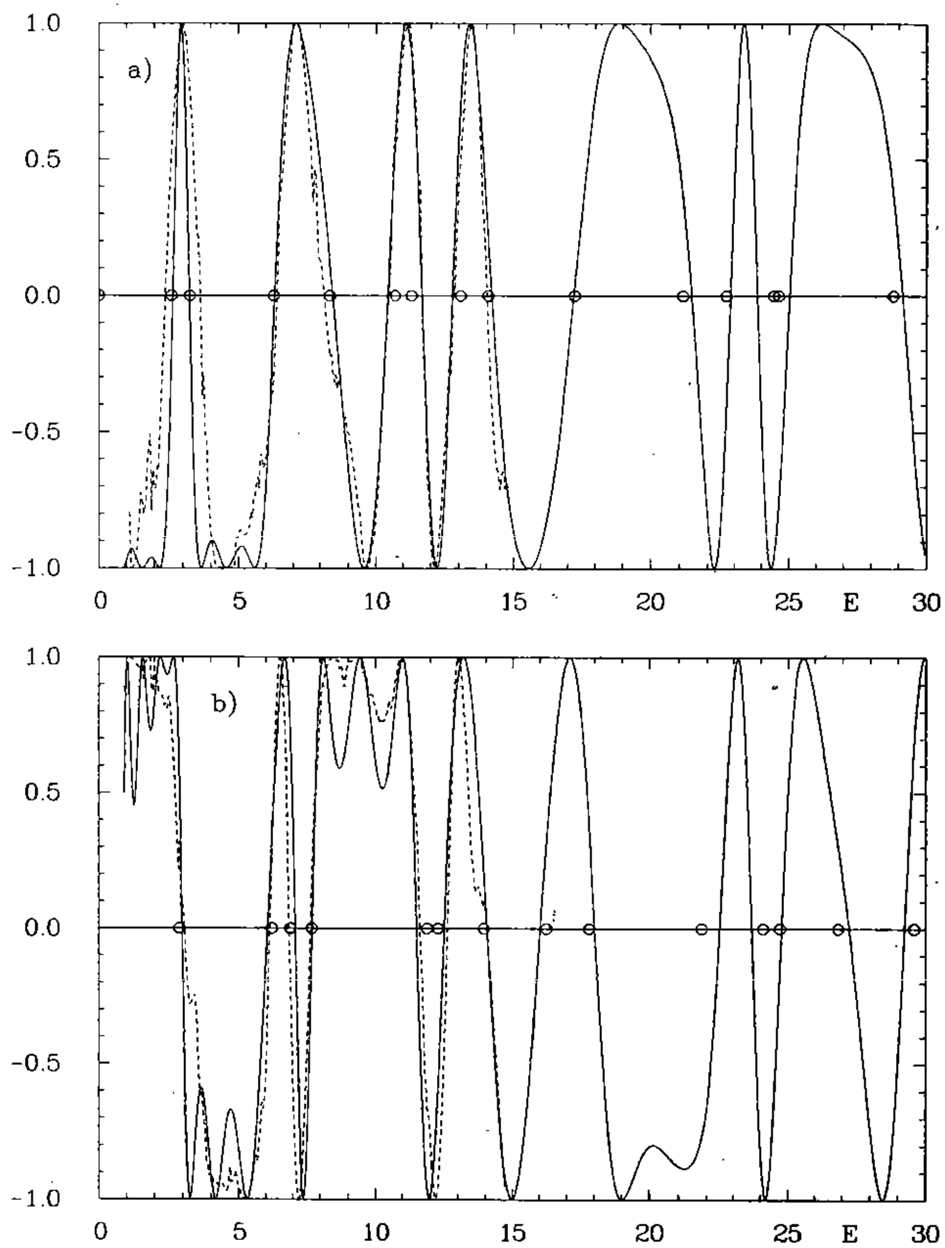

Figure 5: The quantization function $\cos(\pi\,\mathcal{N}(E))$ is shown for positive parity (a) and negative parity (b). The full curves show the evaluations where the Dirichlet series is truncated at $\mathcal{L} = 12$ independent of the energy, whereas the truncation condition (53) is used to obtain the dashed curves.

its efficiency. In contrast to general chaotic systems where Gutzwiller's periodic–orbit theory provides only a semiclassical approximation, it is exact for the hyperbolic octagon since it is identical to Selberg's trace formula.

In section II the Selberg trace formula is derived for distinct parity classes. For an arbitrarily chosen hyperbolic octagon defined in figure 1 the length spectrum of the periodic orbits is computed by considering all generator products containing up to 12 generators of the discrete group defining the octagon. This yields a length spectrum which is nearly complete up to the length $\mathcal{L} = 12$. The spectral staircase $\mathcal{N}^{\pm}(E)$ is computed in terms of the lengths of the periodic orbits for both parity classes in section III. $\mathcal{N}^{\pm}(E)$ consists of the exact mean spectral staircase $\overline{\mathcal{N}^{\pm}}(E)$ and the periodic–orbit sum, which describes the quantal fluctuations about this mean.

The dynamical zeta function $Z(s)$, which for the hyperbolic octagon is identical to the Selberg zeta function, is discussed in section IV. It has the important property that its non–trivial zeros lying on the critical line $\mathrm{Re}\, s = \frac{1}{2}$ are uniquely connected with the quantal energies. Thus it can be used to yield a quantization condition of the form $f(E) = 0$. Because a convergence of the Euler product (19) defining $Z(s)$ cannot be expected on the critical line, the product is rewritten as a Dirichlet series (39) in terms of so–called pseudo–orbits allowing a determination of the abscissa of convergence for this series by (40) and (41). The abscissa of conditional convergence σ_c is found to be in agreement with a statistical model which assumes that the signs of the Dirichlet coefficients A_ρ are stochastically shuffled by (37) and (38) with respect to a sorted pseudo–orbit spectrum. It turns out that σ_c lies very close to the critical line suggesting that a numerical evaluation of the Dirichlet series (39) yields meaningful results, since a possible divergence might not show up in a numerical approximation. The functional relation (20) allows to construct from $Z(s)$ a quantization function $\xi(p)$ which is real on the critical line. The function $\xi(p)$ is evaluated with two different cut–off lengths $\mathcal{L}$. The first one is an energy dependent cut–off according to (50) as proposed by Berry and Keating. Using this cut–off should not alter the locations of the zeros of $\xi(p)$, if the hypothesis that the truncated sum is the complex conjugate of the omitted part of the series, is correct. Because of the unsatisfactory result shown in figure 4 it is likely that there are some important remainder terms being omitted by this method, which must be taken into account. The other truncation used is simply given by the maximal length for which the pseudo–orbit spectrum is computed. Figure 4 reveals that for low energies E the convergence is fast enough to yield the correct zeros but at higher energies much more pseudo–orbits are required. However, this method can be employed for the determination of quantal energies, if enough pseudo–orbits are available.

Another quantization function, $\cos(\pi\mathcal{N}^{\pm}(E))$, which is based directly on periodic orbits, is introduced in section VI. It requires no computation of pseudo-orbits. and the remainder terms $R^{\pm}(p,\mathcal{L})$, which take into account the omitted part of the truncated series, can be computed. This quantization function yields for our system the best approximations to the quantal energies of all quantization conditions based on the periodic–orbit theory tested so far. To emphasize the quality of this quantization rule we have computed the length spectra for three further hyperbolic octagons, where all generator products are taken into account containing up to $n = 10$ generators. A comparison of the quantization function $\xi(p)$ truncated by (50) with $\cos(\pi\mathcal{N}^{\pm}(E))$ using the maximal possible cut-off $\mathcal{L}$ is presented in figure 6. One observes that the "cosine–quantization" (full curve) is superior to the quantization by $\xi(p)$ (dashed curve) together with (50). This fact is probably due to a better control of

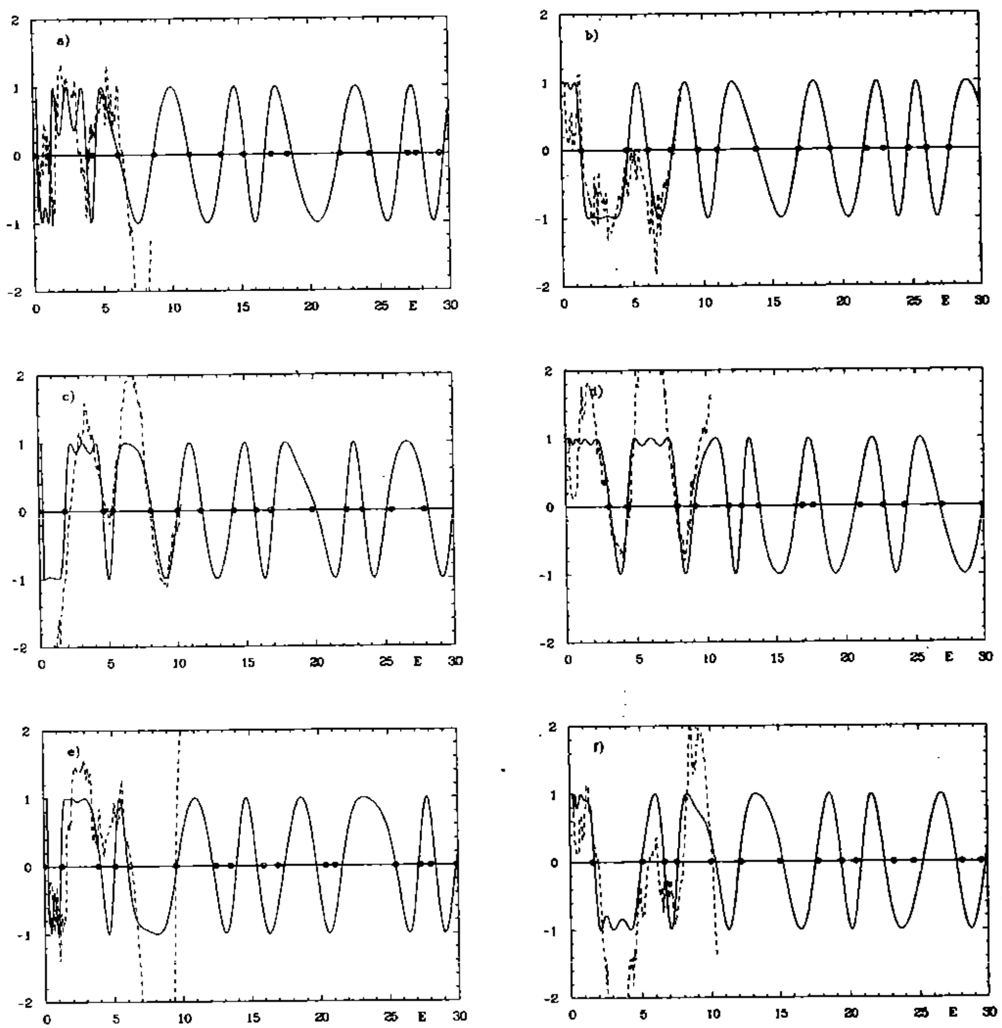

Figure 6: A comparison of the quantization function $\xi(p)$ (dashed curved) truncated according to (50) with the quantization function $\cos(\pi\mathcal{N}^{\pm}(E))$ (full curve) with an energy independent cut–off $\mathcal{L}$ is shown for three further hyperbolic octagons. Figures a), c) and e) show the results for positive parity, whereas figures b), d) and f) deal with negative parity. The cut–off length is $\mathcal{L} = 9$ for the octagon in figures a) and b), and $\mathcal{L} = 10$ for the other ones.

the remainder terms in the case of (59). Furthermore, much more quantal energies are well approximated beyond the limit which is enforced by (50) in the case of the quantization using $\xi(p)$. Only in cases where two quantal energies are lying much closer than the mean level spacing the quantization is inaccurate. But following the discussions in sections V and VI this comes at no surprise since the cut–off length $\mathcal{L}$ determines the smallest distance between the zeros of a single orbit contribution, and thus limits the resolution.

Apart from its accuracy the cosine–quantization is also more efficient since it is not based on pseudo–orbits. As discussed in section V the number of periodic orbits proliferates much weaker than the number of pseudo–orbits. To emphasize this point consider hyperbolic octagons where the asymptotic distribution of pseudo–orbits is known to be $N^p(L) \sim \frac{Z(2)}{Z'(1)} e^L$. This law depends on the special hyperbolic octagon, in contrast to the distribution of periodic orbits which is universal for all hyperbolic surfaces according to Huber's law $N(\ell) \sim \frac{e^\ell}{\ell}$. The factor $\frac{Z(2)}{Z'(1)}$ is the larger the smaller the shortest periodic orbit is, since then the more combinations of periodic orbits to yield pseudo–orbits of lengths up to a given L are possible. Whereas the quantization (59) can be carried out with the same numerical effort for each hyperbolic octagon, the quantization due to $\xi(p)$ requires the more pseudo–orbits the smaller the shortest periodic–orbit is. A further advantage of the quantization (59) is that its accuracy can easily be improved by simply enhancing the cut–off length $\mathcal{L}$. This is impossible in the case of the zeta function quantization which used the resummation of the long orbits being suggested to be valid only by a truncation according to (50). The other possibility given by a truncation of the series in (48) at higher values of $\mathcal{L}$, such that the sum has already reached its limit value, seems to be more inefficient since such a cut–off length must be much higher than the cut–off (53). This can be seen from figure 4 (full curves), as well as it is suggested from the hypothesis connected with the Riemann–Siegel look–alike formula that the omitted part of the series should yield the complex conjugate of the truncated series, thus being of the same modulus. The quantization condition $\cos(\pi \mathcal{N}(E)) = 0$ thus represents a progress towards a manageable quantization rule applicable to strongly chaotic systems. However, there remains the huge challenge to find a quantization rule whose numerical effort does not increase exponentially with energy.

Acknowledgement

We would like to thank Prof. F. Steiner for useful discussions. Furthermore we want to thank the Deutsche Forschungsgemeinschaft for financial support and the HLRZ at Jülich for the access to the CRAY Y–MP 832 computer.

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