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CLASSIFICATION OF SOLVABLE FEYNMAN PATH INTEGRALS

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ABSTRACT

A systematic classification of Feynman path integrals in quantum mechanics is presented and a table of solvable path integrals is given which reflects the progress made during the last ten years or so, including, of course, the main contributions since the invention of the path integral by Feynman in 1942. An outline of the general theory is given. Explicit formulæ for the so-called basic path integrals are presented on which our general scheme to classify and calculate path integrals in quantum mechanics is based.

1. Introduction

During this conference (May 1992) we are celebrating the fiftieth anniversary of the invention of path integrals in quantum mechanics, which appear for the first time on page 35 of Feynman's thesis [8] dated May 4, 1942. By means of his path integral [9] Feynman gave a new formulation of quantum mechanics "in which the central mathematical concept is the analogue of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some sort of least action principle be available" [8]. A few years later, Feynman generalized the path integral to quantum electrodynamics and derived from it for the first time the "Feynman rules" providing an extremely effective method for performing calculations in perturbation theory.

In this contribution we restrict ourselves to path integrals in quantum mechanics. Until fairly recently, only a few examples of exactly solvable path integrals were known; see the books by Feynman and Hibbs [10] and by Schulman [31], which give a good account of the state of art at the time of 1965 and 1981, respectively. However, the situation has drastically changed during the last decade or so, and it is no exaggeration to say that we are able to solve today essentially all path integrals in quantum mechanics which correspond to problems for which the corresponding Schrödinger equation can be solved exactly. (This, of course, excludes all classically chaotic systems). It thus appears to us that the time has come to look for a systematic classification of path integrals in quantum mechanics. A comprehensive "Table of Feynman Path Integrals" will appear soon [21]. In the present short contribution we are only able to give the main idea how our classification scheme works and which classes of path integrals are exactly solvable. Due to lack of space, we also restrict ourselves to purely bosonic degrees of freedom. For fermionic path integrals we have to refer to the literature [25, 33]. In the following we are not able to give a complete list of references. A very extensive list on the literature on path integrals comprising more than 1400 papers will be given in our monography [22] which is in preparation.

2. Formulation of the Path Integral

Let us set up the definition of the Feynman path integral. We first consider the simple case of a classical Lagrangian $\mathcal{L}(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$ in D dimensions. Then the integral kernel ($x \in \mathbb{R}^D$)

$$K(x'',x';t'',t') = \left\langle x'' \right| \mathrm{e}^{-\mathrm{i}\,H(t''-t')/\hbar} \left| x' \right\rangle \Theta(t''-t') \tag{1}$$

of the time-evolution equation

$$\Psi(x'',t'') = \int_{\mathbb{R}^D} K(x'',x';t'',t')\Psi(x',t')dx' \quad , \tag{2}$$

is represented in the form (Feynman path integral [8-10])

$$K(x'',x';t'',t') = \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int_{\mathbb{R}^D} dx_{(j)} \times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(x_{(j)}-x_{(j-1)})^2 - \epsilon V(x_{(j)})\right]\right\} \tag{3a}$$

$$\equiv \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]dt\right\} \quad . \tag{3b}$$

Here we have used the abbreviations $\epsilon = (t''-t')/N \equiv T/N$, $x_{(j)} = x(t_{(j)})$ ($t_{(j)} = t' + \epsilon j,\ j = 0,\ldots,N$), and we interpret the limit $N\to\infty$ as equivalent to $\epsilon\to 0$, T fixed.

The next step is to consider a generic classical Lagrangian of the form $\mathcal{L}(q,\dot{q}) = \frac{m}{2}g_{ab}(q)\dot{q}^a\dot{q}^b - V(q)$ in some D-dimensional Riemannian space $\mathbb{M}$ with line element $ds^2 = g_{ab}(q)dq^a dq^b$. This case, as first systematically discussed by DeWitt [4], requires a careful treatment. The Feynman path integral is most conveniently constructed by considering the Weyl-ordering prescription (e.g. [20] and references therein) in the corresponding quantum Hamiltonian. The result then is

$$K(x'',x';t'',t') = \left[g(q')g(q'')\right]^{-1/4}\lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int_{\mathbb{M}} dq_{(j)}\cdot\prod_{j=1}^{N}\sqrt{g(\bar{q}_{(j)})} \times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}g_{ab}(\bar{q}_{(j)})\Delta q^a_{(j)}\Delta q^b_{(j)} - \epsilon V(\bar{q}_{(j)}) - \epsilon\Delta V(\bar{q}_{(j)})\right]\right\} \quad . \tag{4}$$

Here $\bar{q}_{(j)} = \frac{1}{2}(q_{(j)} + q_{(j-1)})$ denotes the midpoint coordinate, $\Delta q_{(j)} = (q_{(j)} - q_{(j-1)})$, and $\Delta V(q)$ is a well-defined "quantum potential" of order $\hbar^2$ having the form ($\Gamma_a = \partial_a \ln\sqrt{g},\ g = \det(g_{ab})$)

$$\Delta V(q) = \frac{\hbar^2}{8m}\left[g^{ab}\Gamma_a\Gamma_b + 2(g^{ab}\Gamma_a)_{,b} + g^{ab}{}_{,ab}\right] \quad . \tag{5}$$

The midpoint prescription together with ΔV appears in a completely natural way as an unavoidable consequence of the Weyl-ordering prescription in the corresponding quantum Hamiltonian

$$H = -\frac{\hbar^2}{2m}g^{-1/2}\partial_a g^{1/2}g^{ab}\partial_b + V(q)$$

$$= \frac{1}{8m}\Big(g^{ab}p_a p_b + 2p_a g^{ab}p_b + p_a p_b g^{ab}\Big) + V(q) + \Delta V(q) \ , \qquad (6)$$

with $p_a = -\,\mathrm{i}\,\hbar(\partial_a + \frac{1}{2}\Gamma_a)$, the momentum operator conjugate to the coordinate q_a in $\mathbb{M}$. Of course, choosing another prescription leads to a different lattice definition in Eq. (4) and a different quantum potential $\widetilde{\Delta V}$. However, every consistent lattice definition of Eq. (4) can be transformed into another one by carefully expanding the relevant metric terms (integration measure- and kinetic energy term).

Indispensable tools in path integral techniques are transformation rules. In order to avoid cumbersome notation, we restrict ourselves to the one-dimensional case. For the general case we refer to Refs. [18-20] and references therein. We consider the path integral (3) and perform the coordinate transformation $x = F(q)$. Implementing this transformation, one has to keep all terms of $O(\epsilon)$ in (3). Expanding about midpoints, the result is

$$K\big(F(q''),F(q');T\big) = \Big[F'(q'')F'(q')\Big]^{-1/2} \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{1/2}\prod_{j=1}^{N-1}\int dq_{(j)}\cdot\prod_{j=1}^{N}F'(\bar{q}_{(j)})$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}F'^2(\bar{q}_{(j)})(\Delta q_{(j)})^2 - \epsilon V(F(\bar{q}_{(j)})) - \frac{\epsilon\hbar^2}{8m}\frac{F''^2(\bar{q}_{(j)})}{F'^4(\bar{q}_{(j)})}\right]\right\} \ . \qquad (7)$$

Note that the path integral (7) has the canonical form of the path integral (4). It is not difficult to incorporate the explicitly time-dependent coordinate transformation $x = F(q,t)$ [21, 22]. Then

$$K\big(F(q'',t''),F(q',t');t'',t'\big) = A(q'',q';t'',t')\widetilde{K}(q'',q';t'',t') \ , \qquad (8)$$

with the prefactor

$$A(q'',q';t'',t') = \Big[F'(q'',t'')F'(q',t')\Big]^{-1/2}$$

$$\times\exp\left\{\frac{\mathrm{i}\,m}{\hbar}\left[\int^{q''}F'(z,t'')\dot{F}(z,t'')dz - \int^{q'}F'(z,t')\dot{F}(z,t')dz\right]\right\} \ , \qquad (9)$$

and the path integral representation for the kernel $\widetilde{K}$ given by ($\bar{F}_{(j)} = F(\bar{q}_{(j)},\bar{t}_{(j)})$, $\bar{t}_{(j)} = \frac{1}{2}(t_{(j)} + t_{(j-1)})$)

$$\widetilde{K}(q'',q';t'',t') = \lim_{N\to\infty}\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{1/2}\prod_{j=1}^{N-1}\int dq_{(j)}\cdot\prod_{j=1}^{N}\bar{F}'_{(j)}$$

$$\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\bar{F}'^2_{(j)}(\Delta q_{(j)})^2 - \epsilon V(\bar{F}_{(j)})\right.\right.$$

$$\left.\left. - \frac{\epsilon\hbar^2}{8m}\frac{\bar{F}''^2_{(j)}}{\bar{F}'^4_{(j)}} - \epsilon m\int^{\bar{q}_{(j)}}F'(z,t)\ddot{F}(z,t)dz\right]\right\} \ . \qquad (10)$$

It is obvious that the path integral representation (10) is not completely satisfactory. Whereas the transformed potential $V(F(q,t))$ may have a convenient form when expressed in the new coordinate q, the kinetic term $\frac{m}{2}F'^2\,\dot{q}^2$ is in general nasty. Here the so-called "time-transformation" comes into play which leads in combination with the "space-transformation" already carried out to general "space-time transformations" in path integrals. The time-transformation is implemented [7] by introducing a new "pseudo-time" s'' by means of

$$s'' = \int_{t'}^{t''} \frac{ds}{F'^2(q(s),s)} \quad . \tag{11}$$

A rigorous lattice derivation is far from being trivial and has been discussed by many authors. Recent attempts to put it on a sound footing can be found in Refs. [3, 11]. A convenient way to derive the corresponding transformation formulæ uses the energy dependent Green's function $G(E)$ of the kernel $K(T)$ defined by

$$G(q'',q';E) = \left\langle q'' \left| \frac{\hbar}{H-E-\mathrm{i}\,\epsilon} \right| q' \right\rangle = \mathrm{i}\int_0^\infty dT\,\mathrm{e}^{\mathrm{i}(E+i\epsilon)T/\hbar}\,K(q'',q';T) \quad . \tag{12}$$

For the path integral (7) one obtains the following transformation formula (here we consider the time-independent case only)

$$K(x'',x';T) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\,\mathrm{i}\,\hbar}\,\mathrm{e}^{-\,\mathrm{i}\,ET/\hbar}\,G(q'',q';E) \quad , \tag{13}$$

$$G(q'',q';E) = \mathrm{i}\left[F'(q'')F'(q')\right]^{1/2}\int_0^\infty ds''\widehat{K}(q'',q';s'') \quad , \tag{14}$$

with the transformed path integral $\widehat{K}$ given by

$$\begin{aligned}\widehat{K}(q'',q';s'') = \lim_{N\to\infty}&\left(\frac{m}{2\pi\,\mathrm{i}\,\epsilon\hbar}\right)^{1/2}\prod_{j=1}^{N-1}\int dq_{(j)} \\ &\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(\Delta q_{(j)})^2-\epsilon F'^2(\bar{q}_{(j)})\Big(V(F(\bar{q}_{(j)}))-E\Big)\right.\right. \\ &\left.\left.-\frac{\epsilon\hbar^2}{8m}\left(3\frac{F''^2(\bar{q}_{(j)})}{F'^2(\bar{q}_{(j)})}-2\frac{F'''(\bar{q}_{(j)})}{F'(\bar{q}_{(j)})}\right)\right]\right\} \quad .\end{aligned} \tag{15}$$

Further refinements are possible and general formulæ of practical interest and importance can be derived. Let us note that also an explicitly time-dependent "space-time transformation" can be formulated similarly to the formulæ (13-15),

c.f. Refs. [21, 22]. By the same technique also the separation of variables in path integrals can be stated, c.f. Ref. [16]. But this will not be discussed here any further.

3. Basic Path Integrals

In this Section we present the path integrals which we consider as the Basic Path Integrals.

3.1. Path Integral for the Harmonic Oscillator

The first elementary example is the path integral for the harmonic oscillator. It has been first evaluated by Feynman [9]. We have the identity ($x \in \mathbb{R}$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\int_{t'}^{t''}\left(\dot{x}^2 - \omega^2 x^2\right)dt\right]$$
$$= \sqrt{\frac{m\omega}{2\pi\,\mathrm{i}\,\hbar\sin\omega T}}\exp\left\{\frac{\mathrm{i}\,m\omega}{2\hbar}\left[(x'^2 + x''^2)\cot\omega T - \frac{2x'x''}{\sin\omega T}\right]\right\} \quad . \quad (16)$$

We do not state the expansion into wave-functions ($\propto$ Hermite polynomials) which can be done by means of the Mehler formula, nor the corresponding Green's function.

The path integral for quadratic Lagrangians can also be stated exactly ($x \in \mathbb{R}^D$)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathcal{L}(x,\dot{x})dt\right)$$
$$= \left(\frac{1}{2\pi\,\mathrm{i}\,\hbar}\right)^{D/2}\sqrt{\det\left(-\frac{\partial^2 S_{Cl}[x'',x']}{\partial x''_a \partial x'_b}\right)}\exp\left(\frac{\mathrm{i}}{\hbar}S_{Cl}[x'',x']\right) \quad . \quad (17)$$

Here $\mathcal{L}(x,\dot{x})$ denotes any classical Lagrangian at most quadratic in x and $\dot{x}$, and $S_{Cl}[x'',x'] = \int_{t'}^{t''}\mathcal{L}(x_{Cl},\dot{x}_{Cl})dt$ the corresponding classical action evaluated along the classical solution x_{Cl} satisfying the boundary conditions $x_{Cl}(t') = x'$, $x_{Cl}(t'') = x''$. The determinant appearing in Eq. (17) is known as the van Vleck-Pauli-Morette determinant (see e.g. Refs. [4, 28] and references therein). The explicit evaluation of $S_{Cl}[x'',x']$ may have any degree of complexity due to complicated classical solutions of the Euler-Lagrange equations as the classical equations of motion.

3.2. Path Integral for the Radial Harmonic Oscillator

In order to evaluate the path integral for the radial harmonic oscillator, one has to perform a separation of the angular variables, see Refs. [14, 29]. Here we are

not going into the subtleties of the Besselian functional measure due to the Bessel functions which appear in the lattice approach [11, 14, 20, 29, 34], which is actually necessary for the explicit evaluation of the radial harmonic oscillator path integral. One obtains (modulo the above mentioned subtleties) ($r > 0$)

$$\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2 - \hbar^2\frac{\lambda^2-\frac{1}{4}}{2mr^2} - \frac{m}{2}\omega^2 r^2\right)dt\right]$$
$$= \sqrt{r'r''}\frac{m\omega}{\mathrm{i}\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2\,\mathrm{i}\,\hbar}(r'^2+r''^2)\cot\omega T\right] I_\lambda\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right) \quad , \qquad (18)$$

where $I_\lambda(z)$ denotes the modified Bessel function.

3.3. Path Integral for the Pöschl-Teller Potential

There are two further basic path integral solutions based on the SU(2) [2, 24] and SU(1,1) [2] group path integration, respectively. The first yields the path integral identity for the solution of the Pöschl-Teller potential according to ($0 < x < \pi/2$)

$$\mathrm{i}\int_0^\infty dT\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\kappa^2-\frac{1}{4}}{\sin^2 x}+\frac{\lambda^2-\frac{1}{4}}{\cos^2 x}\right)\right]dt\right\}$$
$$= \frac{m}{\hbar}\sqrt{\sin 2x'\sin 2x''}\,\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\left(\frac{1-\cos 2x_<}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos 2x_<}{2}\right)^{(m_1+m_2)/2}$$
$$\times\left(\frac{1-\cos 2x_>}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos 2x_>}{2}\right)^{(m_1+m_2)/2}$$
$$\times\,{}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1-m_2+1; \frac{1-\cos 2x_<}{2}\right)$$
$$\times\,{}_2F_1\left(-L_E+m_1, L_E+m_1+1; m_1+m_2+1; \frac{1+\cos 2x_>}{2}\right) \qquad (19)$$

with $m_{1/2} = \frac{1}{2}(\lambda\pm\kappa)$, $L_E = -\frac{1}{2}+\frac{1}{2}\sqrt{2mE}\,/\hbar$, and $x_{<,>}$ the larger, smaller of x', x'', respectively. ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Here we have used the fact that it is possible to state closed expressions for the (energy dependent) Green's functions for the Pöschl-Teller and modified Pöschl-Teller potential (see below), respectively, by summing up the spectral expansion [27].

3.4. Path Integral for the Modified Pöschl-Teller Potential

Similarly one can derive a path integral identity for the modified Pöschl-Teller potential. One gets ($m_{1,2} = \frac{1}{2}(\eta \pm \sqrt{-2mE}\,/\hbar)$, $L_\nu = \frac{1}{2}(-1+\nu)$, $r > 0$)

$$\begin{aligned}
&\mathrm{i}\int_0^\infty dT\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 r} - \frac{\nu^2-\frac{1}{4}}{\cosh^2 r}\right)\right]dt\right\} \\
&\quad = \frac{m}{\hbar}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)} \\
&\qquad \times \left(\cosh r_<\right)^{-(m_1-m_2)} \left(\tanh r_<\right)^{m_1+m_2+\frac{1}{2}} \\
&\qquad \times \left(\cosh r_>\right)^{-(m_1-m_2)} \left(\tanh r_>\right)^{m_1+m_2+\frac{1}{2}} \\
&\qquad \times {}_2F_1\left(-L_\nu+m_1, L_\nu+m_1+1; m_1-m_2+1; \frac{1}{\cosh^2 r_<}\right) \\
&\qquad \times {}_2F_1\left(-L_\nu+m_1, L_\nu+m_1+1; m_1+m_2+1; \tanh^2 r_>\right) \ . \qquad (20)
\end{aligned}$$

3.5. General Formulæ

For the classification of solvable path integrals, one also requires a few additional formulæ which generalize the usual problems in quantum mechanics in a specific way. Here one has e.g.

i) Explicitly time-dependent problems according to e.g. $V(x) \mapsto V(x/\zeta(t))/\zeta^2(t)$,
ii) Incorporation of δ-function perturbation according to $V(x) \mapsto V(x) - \gamma\delta(x-a)$ (one dimension), and
iii) Boundary problems with impenetrable walls (half-space, infinite boxes) which can be derived from ii) by considering the limit $\gamma \to \infty$.

i) For the first class of problems, there is a general solution provided $\zeta(t)$ has a specific form. For $\zeta(t) = (at^2+2bt+c)^{1/2}$ one finds the general formula

$$\begin{aligned}
&\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - \frac{1}{\zeta^2(t)}V\left(\frac{x}{\zeta(t)}\right)\right]dt\right\} \\
&\quad = \left(\zeta''\zeta'\right)^{-D/2} \exp\left[\frac{\mathrm{i}\,m}{2\hbar}\left(x''^2\frac{\dot{\zeta}''}{\zeta''} - x'^2\frac{\dot{\zeta}'}{\zeta'}\right)\right] K_{\omega',V}\left(\frac{x''}{\zeta''}, \frac{x'}{\zeta'}; \int_{t'}^{t''}\frac{dt}{\zeta^2(t)}\right) \ , \qquad (21)
\end{aligned}$$

with $\zeta' = \zeta(t')$, $\zeta'' = \zeta(t'')$, etc. Here $\omega'^2 = ac - b^2$ and $K_{\omega',V}$ denotes the path integral

$$K_{\omega',V}(z'',z';s'') = \int\limits_{z(0)=z'}^{z(s'')=z''} \mathcal{D}z(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{z}^2 - \frac{m}{2}\omega'^2 z^2 - V(z)\right]ds\right\} \ . \qquad (22)$$

Another class of time-dependent problems has a time-dependence according to $V(x) \mapsto V(x - f(t))$. Here one gets [6] ($q' = x' - f'$, $f' = f(t')$, etc.)

$$\int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x-f(t))\right]dt\right\}$$

$$= \exp\left\{\frac{im}{\hbar}\left[\dot{f}''(x''-f'') - \dot{f}'(x'-f') + \frac{1}{2}\int_{t'}^{t''}\dot{f}^2(t)dt\right]\right\}$$

$$\times \int\limits_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2 - V(q) - m\ddot{f}(t)q\right]dt\right\} \quad , \tag{23}$$

Eqs. (21,23) are special cases of Eq. (8) (note that $\dot{F}'(q,t) = 0$ in (23) and therefore an additional term in the prefactor $A(t'',t')$ appears).

ii) In the second class of general formulæ we consider the incorporation of δ-function perturbations, i.e. a δ-function as an additional potential located at $x = a$ with strength γ. However, here only a closed formula for the corresponding Green's function can be stated; an explicit result for the propagator can only be obtained in the simplest, or in some exceptional cases. One obtains [17]

$$\mathrm{i}\int_0^\infty dT\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x) + \gamma\delta(x-a)\right]dt\right\}$$

$$= G^{(V)}(x'',x';E) + \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{\frac{\hbar}{\gamma} - G^{(V)}(a,a;E)} \quad . \tag{24}$$

Here $G^{(V)}(E)$ denotes the Green's function for the unperturbed problem ($\gamma = 0$). Possible bound states are determined by the poles of $G(E)$, i.e. by the equation $G^{(V)}(a,a,E_n) = \hbar/\gamma$.

iii) The third class of general formulæ is obtained if we consider in Eq. (24) the limit $\gamma \to \infty$. This has the consequence that an impenetrable wall appears at $x = a$. The result then is for the motion in the half-space $x > a$

$$\mathrm{i}\int_0^\infty dT\, \mathrm{e}^{\mathrm{i}\,ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}_{half-space}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]dt\right\}$$

$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)} \quad . \tag{25}$$

Possible bound states are determined by the poles of $G(E)$, i.e. by the equation $G^{(V)}(a,a,E_n)=0$. Furthermore, for the motion inside a box with boundaries at $x=a$ and $x=b$ one obtains $(a<x<b)$

$$\mathrm{i}\int_0^\infty dT\,\mathrm{e}^{\mathrm{i}\,ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}_{box}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$

$$=\frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}(x'',b;E) & G^{(V)}(x'',a;E)\\ G^{(V)}(b,x';E) & G^{(V)}(b,b;E) & G^{(V)}(b,a;E)\\ G^{(V)}(a,x';E) & G^{(V)}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}{\begin{vmatrix} G^{(V)}(b,b;E) & G^{(V)}(b,a;E)\\ G^{(V)}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}\quad . \qquad (26)$$

4. A Table of Exactly Solvable Feynman Path Integrals

We are now in the position to present a systematic classification and a list of exactly solvable Feynman path integrals. Of course, due to lack of space, an actual table cannot be presented in this contribution. We therefore list the name of the potential, respectively the name of the quantum mechanical problem, and the basic path integrals to which the path integrals in question can be reduced.

In our table we order the quantum mechanical problems according to their underlying basic path integral. Of course, this classification is closely related to the classification scheme based on Schrödinger's factorization method as reviewed by Infeld and Hull [23], respectively the related classification scheme of Gendenshteĭn [12] based on supersymmetric quantum mechanics.

Our classification is according to the following classes

i) The general Lagrangian which is at most quadratic in x and $\dot{x}$ (the harmonic oscillator being the simplest and best known example),
ii) The radial harmonic oscillator,
iii) The Pöschl-Teller potential,
iv) The modified Pöschl-Teller potential,
v) Explicitly time-dependent problems,
vi) Path integrals with δ-function perturbation,
vii) Path integrals with infinite boundaries (infinite walls and boxes).

Because of limited space, our table includes only the classes i)-iv). A complete list will be given in [21].

Table of Exactly Solvable Feynman Path Integrals

Quadratic Lagrangian	Radial Harmonic Oscillator	Pöschl-Teller Potential	Modified Pöschl-Teller Potential
Infinite square well	Liouville mechanics	Scarf potential	Reflectionless potential
Linear potential	Morse potential	Symmetric top	Rosen-Morse potential
Repelling oscillator	Uniform magnetic field	Magnetic top	Wood-Saxon potential
Forced oscillator	Motion in a section	Spheres	Hultén potential
Saddle point potential	Calogero model	Bispherical coordinates	Manning-Rosen potential
Uniform magnetic field	Aharonov-Bohm problems		Hyperbolic Scarf potential
Driven coupled oscillators	Coulomb potential		Pseudospheres
Two-time action (Polaron)	Coulomb-like potentials in polar and parabolic coordinates		Pseudo-bispherical coordinates
Second derivative Lagrangians	Nonrelativistic monopoles		Poincaré disc
Semi-classical expansion	Kaluza-Klein monopole		Hyperbolic Strip
Anharmonic oscillator	Poincaré plane		Hyperbolic spaces of rank one
	Hyperbolic space + magnetic field + potentials		Kepler problem on spheres, and on pseuodspheres

Of course, in the case of general quantum mechanical problems, more than just one of the basic path integral solutions is required. However, such problems can be conveniently put into a hierarchy according to which of the basic path integral is

the most important one for its solution. For instance, in the path integral solution for the ring potential (an axially symmetric Coulomb-like potential), this hierarchy puts the radial harmonic oscillator path integral solution first, because it requires a space-time transformation to transform the Coulomb terms into oscillator terms.

It is obvious that all potential problems can be generalized to more complicated problems, i.e one can add an additional explicit time-dependence, implement a δ-function perturbation, respectively consider problems in half-spaces and infinite boxes, c.f. Eqs. (21-26). The construction of examples is left to the reader and can be found in [21, 22].

5. Summary

In this short contribution we have sketched our approach towards a "Table of Solvable Feynman Path Integrals"[21]. We do not claim completeness; however, we have done our best to gather as many information as possible. The last ten years or so have seen a lot of activity in solving path integrals and only a few problems seem to be left open to await an exact solution (for instance, the square well problem, all related problems with finite discrete potential steps, and an analysis with periodic potentials; see however Ref. [15], where periodic δ-functions are considered).

Since Feynman's beautiful paper [9] and his classic book written with Hibbs [10], several textbooks on path integration have been published [1, 5, 13, 26, 30-32, 35]. Now the time seems to be ripe for a comprehensive summary and critical review including a systematic classification and extensive bibliography which we are going to complete soon [21, 22].

It is our hope that such a compilation of our present knowledge will help to spread the results achieved into the physical and mathematical community, making them available for critical consideration and further progress, with the ultimate goal of a comprehensive and complete path integral description of quantum mechanics and quantum field theory, including quantum gravity and cosmology.

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7. References

[1] S.Albeverio, P.Blanchard and R.Høegh-Krohn: Some Applications of Functional Integration; *Lecture Notes in Physics* **153** (*Springer-Verlag*, Berlin, 1982).

[2] M.Böhm and G.Junker: Path Integration Over Compact and Noncompact Rotation Groups; *J.Math.Phys.* **28** (1987) 1978.

[3] D.P.L.Castrigiano and F.Stärk: New Aspects of the Path Integrational Treatment of the Coulomb Potential; *J.Math.Phys.* **30** (1989) 2785.

[4] B.S.DeWitt: Dynamical Theory in Curved Spaces. I. A Review of the Classical and Quantum Action Principles; *Rev.Mod.Phys.* **29** (1957) 377.

[5] W.Dittrich and M.Reuter: Classical and Quantum Dynamics. From Classical Paths to Path Integrals (*Springer-Verlag*, Berlin, 1992).

[6] I.H.Duru: Quantum Treatment of a Class of Time-Dependent Potentials; *J. Phys.A: Math.Gen.* **22** (1989) 4827.

[7] I.H.Duru and H.Kleinert: Solution of the Path Integral for the H-Atom; *Phys. Lett.* **B 84** (1979) 185; Quantum Mechanics of H-Atoms from Path Integrals; *Fortschr.Phys.* **30** (1982) 401.

[8] R.P.Feynman: The Principle of Least Action in Quantum Mechanics; Ph. D. Thesis, Princeton University, May 1942.

[9] R.P.Feynman: Space-Time Approach to Non-Relativistic Quantum Mechanics; *Rev.Mod.Phys.* **20** (1948) 367.

[10] R.P.Feynman and A.Hibbs: Quantum Mechanics and Path Integrals (*McGraw Hill, New York*, 1965)

[11] W.Fischer, H.Leschke and P.Müller: Changing Dimension and Time: Two Well-Founded and Practical Techniques for Path Integration in Quantum Physics; *J.Phys.A: Math.Gen.* **25** (1992) 3835.

[12] L.É.Gendenshteĭn: Derivation of Exact Spectra of the Schrödinger Equation by Means of Supersymmetry; *JETP Lett.* **38** (1983) 356;
L.É.Gendenshteĭn and I.V.Krive: Supersymmetry in Quantum Mechanics; *Sov. Phys.Usp.* **28** (1985) 645.

[13] J.Glimm and A.Jaffe: Quantum Physics: A Functional Point of View (*Springer-Verlag*, Berlin, 1981).

[14] M.J.Goovaerts: Path-Integral Evaluation of a Nonstationary Calogero Model; *J.Math.Phys.* **16** (1975) 720.

[15] M.J.Goovaerts and F.Broeckx: Analytic Treatment of a Periodic δ-Function Potential in the Path Integral Formalism; *SIAM J.Appl.Math.* **45** (1985) 479.

[16] C.Grosche: Separation of Variables in Path Integrals and Path Integral Solution of Two Potentials on the Poincaré Upper Half-Plane; *J.Phys.A: Math.Gen.* **23** (1990) 4885.

[17] C.Grosche: Path Integrals for Potential Problems With δ-Function Perturbation; *J.Phys.A: Math.Gen.* **23** (1990) 5205.

[18] C.Grosche: Coulomb Potentials by Path Integration; *Fortschr.Phys.* **40** (1992), 695.

[19] C.Grosche: An Introduction into the Feynman Path Integral; *Leipzig University preprint*, NTZ 29/92, November 1992, unpublished.

[20] C.Grosche and F.Steiner: Path Integrals on Curved Manifolds; *Zeitschr.Phys.* **C 36** (1987) 699.

[21] C.Grosche and F.Steiner: Table of Feynman Path Integrals; to appear in *Springer Tracts in Modern Physics.*

[22] C.Grosche and F.Steiner: Feynman Path Integrals; to appear in *Springer Lecture Notes in Physics*.
[23] L.Infeld and T.E.Hull: The Factorization Method; *Rev.Mod.Phys.* **23** (1951) 21.
[24] A.Inomata and R.Wilson: Path Integral Realization of a Dynamical Group; in *Lecture Notes in Physics* **261** (*Springer-Verlag*, Berlin, 1985), p.42.
[25] J.R.Klauder: The Action Option and a Feynman Quantization of Spinor Fields in Terms of Ordinary C-Numbers; *Ann.Phys.(N.Y.)* **11** (1960) 123.
[26] H.Kleinert: Path Integrals in Quantum Mechanics, Statistics and Polymer Physics (*World Scientific*, Singapore, 1990).
[27] H.Kleinert and I.Mustapic: Summing the Spectral Representations of Pöschl-Teller and Rosen-Morse Fixed-Energy Amplitudes; *J.Math.Phys.* **33** (1992) 643.
[28] C.Morette: On the Definition and Approximation of Feynman's Path Integrals; *Phys.Rev.* **81** (1951) 848.
[29] D.Peak and A.Inomata: Summation Over Feynman Histories in Polar Coordinates; *J.Math.Phys.* **10** (1969) 1422.
[30] G.Roepstorff: Pfadintegrale in der Quantenphysik (*Vieweg*, Braunschweig, 1992).
[31] L.S.Schulman: Techniques and Applications of Path Integration (*John Wiley & Sons*, New York, 1981).
[32] B.Simon: Functional Integration and Quantum Physics (*Academic Press*, New York, 1979).
[33] L.P.Singh and F.Steiner: Fermionic Path Integrals, the Nicolai Map and the Witten Index; *Phys.Lett.* **B 166** (1986) 155.
[34] F.Steiner: Path Integrals in Polar Coordinates from eV to GeV; in "Bielefeld Encounters in Physics and Mathematics VII; Path Integrals from meV to MeV", 1985, eds.: M.C.Gutzwiller et al. (*World Scientific*, Singapore, 1986), p.335.
[35] F.W.Wiegel: Introduction to Path-Integral Methods in Physics and Polymer Science (*World Scientific*, Singapore, 1986).