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Quantum Symmetry in Quantum Theory

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Abstract

Symmetry concepts have always been of great importance for physical problems like explicit calculations, classification or model building. More recently, new "quantum symmetries" ((quasi) quantum groups) attracted much interest in quantum theory. It is shown that all these quantum symmetries permit a conventional formulation as symmetry in quantum mechanics. Symmetry transformations can act on the Hilbert space $\mathcal{H}$ of physical states such that the ground state is invariant and field operators transform covariantly. Models show that one must allow for "truncation" in the tensor product of representations of a quantum symmetry. This means that the dimension of the tensor product of two representations of dimension δ_1 and δ_2 may be strictly smaller than $\delta_1\delta_2$. Consistency of the transformation law of field operators local braid relations leads us to expect, that *(weak) quasi quantum groups* are the most general symmetries in local quantum theory. The elements of the $\mathcal{R}$–matrix which appears in these local braid relations turn out to be operators on $\mathcal{H}$ in general. It will be explained in detail how examples of field algebras with weak quasi quantum group symmetry can be obtained. Given a set of observable field with a finite number of superselection sectors, a quantum symmetry together with a complete set of covariant field operators which obey local braid relations are constructed. A covariant transformation law for adjoint fields is not automatic but will follow when the existence of an appropriate antipode is assumed. At the example of the chiral critical Ising model, non-uniqueness of the quantum symmetry will be demonstrated. Generalized quantum symmetries yield examples of gauge symmetries in non-commutative geometry. In the case of (weak) quasi quantum groups which are only quasi-co-associative rather than co-associative, a quasi-associative generalization of non-commutative differential geometry appears naturally. Quasi-quantum planes are introduced as the simplest examples of quasi- associative differential geometry. (Weak) quasi quantum groups can act on them by generalized derivations much as quantum groups do in non-commutative (differential-) geometry.

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1 Introductory Remarks

In classical mechanics, symmetries are described by *groups* of transformations acting on a phase space. In view of the predominant role group symmetries played in classical mechanics, it was natural to introduce them also into quantum theory. In fact, this was done soon after the discovery of quantum mechanics and group symmetries turned out to be an important tool to obtain predictions of quantum theory [81].

Since the late seventies much work was done to investigate quantum mechanical models on a two ore three dimensional space-time. Solving models of 1+1 dimensional quantum field theory, Sklyanin, Takhtadzhyan and Faddeev revealed a new algebraic structure that was called quantum algebra [20, 21]. In the more axiomatic treatment of Drinfel'd and Jimbo [18, 45] it appeared as a special class of Hopf algebras [1] and consequently as a generalization of groups. The new name *quantum group* was used henceforth. To cut a long story short we just mention that signs of quantum group symmetries have been found in many quantum mechanical models such as integrable spin chains (e.g. [68, 73]), rational conformal quantum field theories (e.g. [3, 67, 54, 43]), and massive integrable models (e.g. [71]). The process of generalization continued. In 1989 Drinfel'd introduced quasi quantum groups to be able to "twist" quantum groups. Non-trivial examples of quasi quantum groups were constructed and shown to be related to orbifold models [14]. For reasons that remain to be discussed later, all models which exhibit generalized *quantum symmetries* are defined on a low dimensional space time.

From another point of view, generalized symmetries had to be expected in quantum theory. In classical mechanics, the observables form a commutative algebra $\mathcal{A}$ which consists of functions f on phase space. In quantum mechanics, the points of phase space have gone away, but a (non-commutative) algebra of observables or field operators persists. Groups are topological spaces G whose points are the group elements a. When the points of phase space on which the group elements act have gone away, one does not need the group elements a any more either. It suffices to have an algebra of "functions on the group G". This algebra may be non-commutative and even non-associative and it can describe the symmetry of non-commutative spaces (e.g. super-groups describe symmetries of super-manifolds). This reasoning brings us closer to two other approaches to quantum groups developed by Woronowicz [82, 83] and Manin [63]. Although quite different in detail, they are both inspired by the ideas of non-commutative geometry [13].

Part I

On the Construction of Field Algebras with Quantum Symmetry

To begin with, let us list the most important algebraic structures which are common to all known examples of quantum symmetries. The basic structure is an associative *-algebra $\mathcal{G}^*$ (generalization of the group algebra). Representations of $\mathcal{G}^*$ give rise to an action of the quantum symmetry on vector spaces. Unitarity of the representation τ on a Hilbert space V means that $\tau(\xi)^* = \tau(\xi^*)$ for all $\xi \in \mathcal{G}^*$. Among the representations of a quantum symmetry, one can always find a "trivial" one–dimensional representation ϵ. Moreover, tensor products of representations can be performed. The tensor product of two representations τ, τ' on V, V' is a representation $\tau \boxtimes \tau'$ on $V \otimes V'$. It is furnished by a homomorphism $\Delta : \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$ (a "co-product" of $\mathcal{G}^*$) according to the formula

$$(\tau \boxtimes \tau')(\xi) = (\tau \otimes \tau')\Delta(\xi) \ .$$

This tensor product need not be associative and commutative and it may involve truncation, i.e. the representation $\tau \boxtimes \tau'$ possibly vanishes on a non-trivial subspace of $V \otimes V'$

We adopt the framework of second quantized quantum mechanics, so that there is a Hilbert space $\mathcal{H}$ of physical states which is generated from a unique ground state $|0\rangle$ by application of a set of field operators $\Psi^I_i(x,t)$. Superscripts $I, J, K, \dots$ distinguish between field multiplets while subscripts label members of the multiplets. A quantum mechanical system is said to possess a quantum symmetry $\mathcal{G}^*$, if the Hilbert space $\mathcal{H}$ carries an unitary representation $\mathcal{U}$ of $\mathcal{G}^*$, such that the ground state $|0\rangle$ is invariant and field operators $\Psi^I_i(x,t)$ transform covariantly. Invariance of the ground state means that $|0\rangle$ transforms according to the trivial one–dimensional representation ϵ of $\mathcal{G}^*$. The formulation of the transformation law of field operators involves the tensor product of representations of $\mathcal{G}^*$. More precisely, a field multiplet $\Psi^I_i(x,t)$ is said to transform covariantly according to the finite dimensional representation τ^I of $\mathcal{G}^*$, if

$$U(\xi)\Psi^I_i(x,t) = \Psi^I_j(x,t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi)$$

holds for all $\xi \in \mathcal{G}^*$ [12]. The symmetry transformations considered here, do not act on the space time argument of the field operators, in other words: they are internal symmetries. We give a more comprehensive explanation of the notion of quantum symmetry in the next section.

In second quantized quantum theory, Bose and Fermi statistics are implemented through local commutation or anticommutation relations of field operators which create particles,

$$\Psi_i^I(x,t)\Psi_j^J(y,t) = \pm\Psi_j^J(y,t)\Psi_i^I(x,t) \quad \text{for } x \neq y \quad . \tag{0.1}$$

Consistency of a symmetry with Bose/Fermi statistics requires that this relation should be preserved by a symmetry transformation. This is indeed true for internal symmetry groups.

In two and less space dimensions, Bose/Fermi statistics is not the most general possibility, but braid group statistics can also occur. Fröhlich proposed that local (anti-)commutation relations of fields should be be replaced by *local braid relations.*

$$\Psi_i^I(x,t)\Psi_j^J(y,t) = \omega^{IJ}\Psi_l^J(y,t)\Psi_k^I(x,t)\hat{\mathcal{R}}_{kl,ij}^{IJ>} \tag{0.2}$$

if $x > y$ for some ordering of space coordinates and ω^{IJ} are phase factors. Originally, $R_{kl,ij}^{IJ>}$ were proposed to be complex numbers.

The main question is, whether such local braid relations can be consistent with the transformation law under some non-trivial quantum symmetry. The answer is affirmative. It will be seen, however, that braid matrices $\hat{\mathcal{R}}$ with entries in $\mathcal{G}^*$ should be admitted.

In general, local braid relations of this more general type can be consistent with non-trivial quantum symmetries, provided the tensor product of representations is associative and commutative at least up to equivalence.

$$(\tau^I \boxtimes \tau^J) \;\cong\; (\tau^J \boxtimes \tau^I) \tag{0.3}$$

$$(\tau^I \boxtimes \tau^J)\boxtimes \tau^K \;\cong\; \tau^I \boxtimes (\tau^J \boxtimes \tau^K) \quad . \tag{0.4}$$

The precise formulation of this requirement will involve an element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ and a "re-associator" $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ with certain properties. They will furnish invertible intertwiners $R^{IJ} = (\tau^I \otimes \tau^J)(R)$ and $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$ between the representations on the right and left hand side of eq. (0.3,0.4). When $\varphi = \sum_\sigma \varphi_\sigma^1 \otimes \varphi_\sigma^2 \otimes \varphi_\sigma^3$ one introduces $\varphi_{213} = \sum_\sigma \varphi_\sigma^2 \otimes \varphi_\sigma^1 \otimes \varphi_\sigma^3$. In this notation, consistent braid matrices are given by

$$\hat{\mathcal{R}}_{kl,ij}^{IJ>} = (\tau_{ki}^I \otimes \tau_{lj}^J \otimes \mathcal{U})(\varphi_{213}(R \otimes e)\varphi^{-1}) \in \mathcal{U}(\mathcal{G}^*) \quad . \tag{0.5}$$

Quantum symmetries with re-associator φ and R-element R are called *weak quasi quantum groups.* They were introduced in [56] as a generalization of Drinfeld's quasi quantum groups [19]. A precise definition will be given in section 3. Consistency of weak quasi quantum group symmetries with local braid relations is discussed in more detail in section 4.1.

It will be shown that examples of quantum field theories with quantum symmetry can be constructed within the framework of algebraic quantum field theory [41] with locally generated sectors. For quantum field theories with a finite number of superselection sectors, a symmetry algebra $\mathcal{G}^*$ is constructed together with an algebra of field operators Ψ_m^J which generate the whole Hilbert space $\mathcal{H}$ of states from the vacuum. In detail they will have the following properties.

The Hilbert space $\mathcal{H}$ carries a unitary representation of $\mathcal{G}^*$ such that the vacuum is invariant. The field operators Ψ_m^J transform covariantly under $\mathcal{G}^*$. The quantum symmetry $\mathcal{G}^*$ acts as a gauge symmetry (of first kind), i.e. observables commute with the symmetry transformations.

A tensor product of representations of $\mathcal{G}^*$ is defined through a co-product Δ. It is commutative and associative up to equivalence. The equivalence is implemented by an R-element R and a re-associator φ. They are constructed from the fusion structure of the superselection sectors. This idea was suggested in [57] to obtain the quantum symmetry of the chiral critical Ising model.

Field operators Ψ_m^J satisfy local braid relations (0.2) with an $\mathcal{R}$ matrix furnished by φ and $\mathcal{R}$ according to the formula (0.5).

If the quantum symmetry $\mathcal{G}^*$ admits in addition an antipode, it is a weak quasi quantum group. The existence of an antipode will imply a covariant transformation law of adjoints of field operators.

Some basic ideas from the algebraic theory of superselection sectors are reviewed in section 2. Covariant field operators will be constructed in section 2.3.. Local braid relations are established in section 4.2. To discuss the locality properties of field operators some mathematical results on symmetry algebras with re-associator φ and R-element R are needed. In order not to clutter the presentation in section 4, this mathematical background will be anticipated in section 3. Special examples of weak quasi quantum groups with non-trivial element φ are associated with the quantum group algebra $\mathcal{U}_q(sl_2)$ at the roots of unity $q^p = 1$. They are called "truncated quantum algebras" because they are obtained from the quantum group algebra $U_q(sl_2)$ by "truncation". Truncation is not the only way to construct weak quasi quantum groups. We will give at least one further example. It has the same set of irreducible representations and the same selection rules as $U_q^T(sl_2), q = i$.

It is shown in [57] that the chiral Ising model - i.e. the conformal field theory with central charge $c = \frac{1}{2}$-provides an example, with the truncated quantum group algebra $U_q^T(sl_2)$ with $q = \pm i$ as a symmetry. To the best of our knowledge, this was the first time that the consistency of non-abelian local braid relations (0.2) has been demonstrated through the construction of a model. Originally it had been proposed that minimal conformal models have quantum groups as symmetries [3, 68], but this identification is not quite satisfactory, because the local braid relations, which should come with the symmetry, are not satisfied [54]. It will result from our discussion that the quantum symmetry of the chiral Ising model is non-unique.

Internal symmetries recently led to a classification of quantum field theories with permutation group statistics [17]. Doplicher and Roberts proved that a unique compact symmetry group G can be associated with every higher dimensional quantum field theory. Moreover, the quantum field on which the symmetry transformations act, commute or anticommute for spacelike separations. For quantum field theories with braid group statistics, analogous results are not known, but there are several attempts in this direction. For related work see Fröhlich and Kerler [50, 37], Majid [62], Rehren [69, 70] and Todorov et. al. [44, 43].

1 The Notion of Quantum Symmetry

Let us begin with a short review on group symmetries in quantum mechanics. Consider some quantum mechanical system $(\mathcal{H}, \{\Psi\}, |0\rangle, H)$ within a second quantized formalism. The Hilbert space $\mathcal{H}$ of physical states should contain a unique ground state $|0\rangle$ with respect to the Hamiltonian H. We assume that $\mathcal{H}$ is generated from $|0\rangle$ by multiplets of field operators $\Psi_i^I(x,t)$ (I labels multiplets, i labels fields in the multiplet I) which create particles or excitations.

A compact group G is called (internal) symmetry of this system if there is a unitary representation $\mathcal{U} : G \to \mathcal{B}(\mathcal{H})$ such that the ground state $|0\rangle$ and the Hamiltonian H are invariant and field operators Ψ^I_i transform covariantly according to the representation τ^I of G. To state these requirements in mathematical terms, let us recall two notions from the representation theory of groups. Every group has a trivial representation $\epsilon_G : G \to \mathbb{C}$ defined by $\epsilon_G(\xi) = 1$ for all $\xi \in G$. The tensor product $\boxtimes$ of representations τ, τ' is given by

$$(\tau \boxtimes \tau')_{kl,ij}(\xi) = \tau_{ki}(\xi)\tau'_{lj}(\xi) \quad \text{for all} \quad \xi \in G \ . \tag{1.1}$$

If we set $\xi^* = \xi^{-1}$, unitarity of $\mathcal{U}$ asserts $\mathcal{U}(\xi)^* = \mathcal{U}(\xi)^{-1} = \mathcal{U}(\xi^{-1}) = \mathcal{U}(\xi^*)$. Invariance of the ground state $|0\rangle$ can be expressed as

$$\mathcal{U}(\xi)|0\rangle = |0\rangle = |0\rangle\epsilon_G(\xi) \quad \text{for all } \xi \in G \ . \tag{1.2}$$

We say that $\Psi^I_i(x,t)$ transforms covariantly according to the representation τ^I of G, if for all $\xi \in G$

$$\mathcal{U}(\xi)\Psi^I_i(x,t) = \Psi^I_j(x,t)\tau^I_{ji}(\xi)\mathcal{U}(\xi) = \Psi^I_j(x,t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi) \ . \tag{1.3}$$

Since we concentrate on internal symmetries, there is no action on the space-time variable of the field. For this reason we will often neglect to write arguments (x,t) explicitly.

In conclusion, the formulation of symmetry in quantum theory involves a conjugation $*$ to express unitarity, a trivial representation ϵ to state invariance and a tensor product $\boxtimes$ of representations to write down the covariance law. The mathematical structure behind these notions is known as "(not necessarily co-associative) bi-*-algebra" $(\mathcal{G}^*, \Delta, \epsilon, *)$. In detail this means that $\mathcal{G}^*$ is a *-algebra with unit e and $\Delta : \mathcal{G}^* \to \mathcal{G}^* \otimes \mathcal{G}^*$ (*co-product*), $\epsilon : \mathcal{G}^* \to \mathbb{C}$ (*co-unit*) are *-homomorphisms. For Δ the notion of a *-homomorphism involves a definition of $*$ on $\mathcal{G}^* \otimes \mathcal{G}^*$, which is not unique (cf. [56]) since there are two possibilities to define a *-operation on $\mathcal{G}^{*\otimes_{N+1}}$

$$\begin{aligned} \text{(I)} \quad (\xi \otimes \eta)^* &= \xi^* \otimes \eta^* & (1.4)\\ \text{(II)} \quad (\xi \otimes \eta)^* &= \eta^* \otimes \xi^* & (1.5) \end{aligned}$$

for all $\xi \in \mathcal{G}^*, \eta \in \mathcal{G}^{*\otimes_N}$. We will refer to them as (I),(II) in the following. Finally, the co-product Δ and the co-unit ϵ satisfy

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta \ . \tag{1.6}$$

Note that $\Delta(e) = e \otimes e$ is not assumed. For us, the the homomorphism property of Δ means $\Delta(\eta\xi) = \Delta(\eta)\Delta(\xi)$ and implies only that $\Delta(e)$ is a projector, i.e. $\Delta(e)\Delta(e) = \Delta(e)$. When we want to stress that Δ is not unit preserving, $(\mathcal{G}^*, \Delta, \epsilon, *)$ will be called "weak" bi-*-algebra.

The co-product Δ determines a tensor product $\tau \boxtimes \tau'$ for representations τ, τ' of $\mathcal{G}^*$

$$(\tau \boxtimes \tau')(\xi) = (\tau \otimes \tau')(\Delta(\xi)) \quad \text{for all} \quad \xi \in \mathcal{G}^* \ . \tag{1.7}$$

With respect to this tensor product of representations, the co-unit ϵ furnishes a trivial one-dimensional representation. Triviality refers to the property $\epsilon \boxtimes \tau = \tau = \tau \boxtimes \epsilon$ for all representations τ of $\mathcal{G}^*$, which follows from (1.6). If $\Delta(e) \neq e \otimes e$ the tensor product of representations is truncated. This means that it is zero on a non-trivial subspace of the tensor product of representation spaces.

Let us explain how to abstract a bi-*-algebra from the representation theory of the compact group G. In this case, $\mathcal{G}^*$ should denote the group algebra of the compact gauge group G, i.e. a space of "linear combinations" of elements in G. All homomorphisms of the group G can be uniquely extended to algebra-homomorphisms of the group algebra $\mathcal{G}^*$. Consequently it suffices to fix $\Delta_G, \epsilon_G, *$ on elements ξ in the group G. $\epsilon_G, *$ have been defined above and comparison of (1.7) with (1.1) yields

$$\Delta_G(\xi) = \xi \otimes \xi \quad \text{for all } \xi \in G \ . \tag{1.8}$$

Since $\Delta_G(e) = e \otimes e$, this co-product is unit preserving. Assuming that the action of $*$ on $\mathcal{G}^* \otimes \mathcal{G}^*$ is specified by $(\xi \otimes \eta)^* = \xi^* \otimes \eta^*$, $(\mathcal{G}^*, \Delta_G, \epsilon_G, *)$ is easily shown to satisfy all assumptions listed above. In this sense, group algebras are only special examples of bi-*-algebras.

Definition 1 (quantum symmetry) [56] *A *-algebra $\mathcal{G}^*$ with co-product Δ and co-unit ϵ is called* quantum symmetry *of the system $(\mathcal{H}, \{\Psi\}, |0\rangle, H)$, if there exits a representation*

$$\mathcal{U} : \mathcal{G}^* \mapsto \mathcal{B}(\mathcal{H}) \qquad \textit{such that}$$

i) $\mathcal{U}$ is unitary in the sense that $\mathcal{U}(\xi^*) = \mathcal{U}(\xi)^*$ for all $\xi \in \mathcal{G}^*$.
ii) the Hamiltonian H and the ground state $|0\rangle$ are invariant, i.e.

$$[H, \mathcal{U}(\xi)] = 0 \quad , \quad \mathcal{U}(\xi)|0\rangle = |0\rangle\epsilon(\xi) \quad \textit{for all} \quad \xi \in \mathcal{G}^* \ . \tag{1.9}$$

iii) the field operators $\Psi^I_i(x,t)$ transform covariantly with respect to the representation τ^I of $\mathcal{G}^*$, i.e.

$$\begin{aligned} \mathcal{U}(\xi)\Psi^I_i(x,t) &= \sum_p \Psi^I_j(x,t)(\tau^I \boxtimes \mathcal{U})_{ji}(\xi) & (1.10)\\ &= \sum_p \Psi^I_j(x,t)\tau^I_{ji}(\xi^1_p)\mathcal{U}(\xi^2_p) \ , \ \textit{if } \Delta(\xi) = \sum_p \xi^1_p \otimes \xi^2_p \ . \end{aligned}$$

We mention that the transformation law of adjoint fields Ψ^* involves a *-anti-automorphism $\mathcal{S} : \mathcal{G}^* \to \mathcal{G}^*$ (*antipode*). Consequently, the existence of $\mathcal{S}$ should also be stated among the defining features of a quantum symmetry [56]. Details about antipodes $\mathcal{S}$ in the context of (not necessarily co-associative) weak bi-*-algebras can be found in Appendix A. In comparison to [56] the discussion has been improved, since the original definition was not fully motivated by physics and it turned out to be too restrictive in the presence of truncation.

Bi-*-algebras appeared in the mathematics literature long before they entered physics (cf. [64] for an early comprehensive treatment). Unlike most texts, we do not assume co-associativity of the co-product [1]. Nowadays, bi-*-algebras with antipode $\mathcal{S}$ are usually called *Hopf-*-algebras* [1]. We will use this term to denote (not necessarily co-associative) weak Hopf-*-algebras.

The covariant transformation law (1.10) tells us how to shift representation operators $\mathcal{U}(\xi)$ through fields from left to right. Together with the invariance of the ground state $|0\rangle$ it determines the transformation properties of states. We demonstrate this for the 1-excitation states.

$$\mathcal{U}(\xi)\Psi^I_i|0\rangle = \sum \Psi^I_j\tau^I_{ji}(\xi^1_p)\mathcal{U}(\xi^2_p)|0\rangle = \sum \Psi^I_j\tau^I_{ji}(\xi^1_p)\epsilon(\xi^2_p)|0\rangle = \Psi^I_j|0\rangle\tau^I_{ji}(\xi) \ .$$

[1] A co-product Δ is co-associative, if $(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)$ for all $\xi \in \mathcal{G}^*$ (see below).

The transformation law of higher excitations can be found along the same lines. As a result one finds that they transform according to some tensor product of representations τ^I.

$$\mathcal{U}(\xi)\Psi^{I_1}_{i_1}\dots\Psi^{I_n}_{i_n}|0\rangle = \Psi^{I_1}_{j_1}\dots\Psi^{I_n}_{j_n}|0\rangle(\tau^{I_1}\boxtimes(\tau^{I_2}\boxtimes(\dots\boxtimes\tau^{I_n})))_{j_1\dots j_n,i_1\dots i_n}(\xi) \quad .$$

The brackets in the tensor product of representations on the right hand side are necessary since the tensor product (1.7) need not be associative. We will come back to this point later.

2 Algebraic Methods for Field Construction

According to the laws of local relativistic quantum mechanics, observables are selfadjoint operators acting in a Hilbert space $\mathcal{H}$ of physical states. The Hilbert space of physical states $\mathcal{H}$ may decompose into orthogonal subspaces $\mathcal{H}^J$, called *superselection sectors*, such that observables A do not make transitions between different subspaces $\mathcal{H}^J$ [80]. Different sectors $\mathcal{H}^J$ carry inequivalent irreducible positive energy representations of the algebra $\mathcal{A}$ of observables A, possibly with some multiplicity [41]. Among the sectors is the unique vacuum sector $\mathcal{H}^0$ which contains the vacuum $|0\rangle$ and appears with multiplicity 1.

When several superselection sectors exist, it is of interest to construct *additional field operators* which make transitions between superselection sectors so that the whole Hilbert space of physical states is generated from the vacuum by application of field operators. These fields operators should commute with all observables when their space–time arguments are spacelike localized.

By definition, observables have to commute with the generators of an internal gauge symmetry. A field operator which transforms according to some non-trivial representation of an internal gauge symmetry is necessarily non-observable, i.e. it maps states in different superselection sectors into each other. This implies that states in different sectors possibly transform according to inequivalent representations of the internal symmetry.

According to the algebraic theory of superselection sectors [15, 16], non-observable fields are constructed by adjoining localized endomorphisms ρ to the algebra $\mathcal{A}$ of observables. These ideas will be explained in some detail below. For a comprehensive introduction we recommend the recent book of Haag [42] and lectures by Roberts in [49].

2.1 Observables and superselection sectors in local quantum field theory

In this section, M_d is a d–dimensional space-time manifold with a global causal structure. $\mathcal{K}$ will denote the set of all double cones $\mathcal{O}$ (non void intersections of forward and backward light cones) in M_d. To be specific let us choose M_d to be the d-dimensional Minkowski space. We will consider another possibility at the end of this subsection. Two subsets S_1, S_2 of M_d are called spacelike separated, if any two points $x_i \in S_i$ are relatively spacelike. The causal complement S' of a $S \subset M_d$ is the set of all points spacelike separated from S. If d=2 and $\mathcal{O}$ a double cone, $\mathcal{O}'$ decomposes into two disconnected components $\mathcal{O}_{\lessgtr}$. We use the notation $\mathcal{O}_1 \lessgtr \mathcal{O}$ whenever $\mathcal{O}_1 \subset \mathcal{O}_{\lessgtr}$. In the following, G is a group of global transformations of M_d which respect the causal structure. It is supposed to contain space-time translations. Elements $g \in G$ transform regions $\mathcal{O} \in \mathcal{K}$ to $g\mathcal{O}$.

Consider a set of observable local quantum fields $\phi_b(x)$ on the space-time M_d. One might for example think of an energy–momentum tensor $T(x)$ or a family of real currents $J_a(x)$. They are described by a commutator which vanishes for relatively spacelike arguments. According to the Wightman theory [39, 76], fields are operator valued distributions. So they should be evaluated on real test functions $f : M_d \to \mathbf{R}$ to obtain (unbounded) operators on the Hilbert space of physical states.

$$\phi_b(f) = \int_{M_d} dx \phi_b(x) f(x) \quad .$$

Regarded as operators on the vacuum sector $\mathcal{H}^0$, these smeared out fields "generate" the algebra of observables. We define an algebra $\mathcal{A}(\mathcal{O})$ of observables localized in $\mathcal{O}$ to be the von Neumann (weakly closed *–) algebra generated by all bounded functions of the operators $\phi_a(f)$ with $supp(f) \in \mathcal{O}$. Properties of the set of observable fields $\phi_b(x)$ can be translated into properties of the family $\mathcal{A}(\mathcal{O})$ they generate. We would like to mention that in this step one has to be aware of certain subtleties which might be overlooked at a first glance. In general, domain problems of the unbounded operators $\phi_a(f)$ spoil local commutativity $[\Phi(f_1), \Phi(f_2)] = 0$ of bounded functions Φ in $\phi(f_i)$ for spacelike separated support of f_i. For a review on the status of these problems see [85] and references therein.

At least in many applications to concrete models, we are led (cf. [42] for a thorough discussion) to consider a family of von Neumann algebras $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}} \subset \mathcal{B}(\mathcal{H}^0)$ in a Hilbert space $\mathcal{H}^0$ which satisfies the following basic (Haag-Kastler) axioms [41].

1. It is isotonic, i.e. $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.
2. Einstein causality (or locality) is satisfied, which means that $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)'$ if $\mathcal{O}_1 \subset \mathcal{O}_2'$. Here $\mathcal{A}(\mathcal{O}_2)'$ is the set of all bounded operators in $\mathcal{H}^0$ which commute with $\mathcal{A}(\mathcal{O}_2)$.
3. There is a strongly continuous unitary representation U_0 of G in $\mathcal{H}^0$ which implements automorphisms $\alpha_g : \mathcal{A}(\mathcal{O}) \to \mathcal{A}(g(\mathcal{O}))$ defined by
$$\alpha_g(A) = U_0(g) A U_0(g^{-1}) \quad \text{for every } g \in G \quad . \tag{2.1}$$
The generators of the translation subgroup should have their spectrum in the closed forward light cone. We will refer to these properties as covariance and spectrum condition respectively.
4. There is an (up to a phase) unique vector $|0\rangle \in \mathcal{H}^0$ which is invariant under the action of G, i.e. $U_0(g)|0\rangle = |0\rangle$. $|0\rangle$ is cyclic for each $\mathcal{A}(\mathcal{O})$.

Algebraic quantum field theory starts from this algebraic structure. Specific properties of underlying Wightman fields (which often exist in the applications but possibly not in general) are not needed in the analysis. So we might take the family $\mathcal{A}$ of local observables instead of a set of Wightman fields to define (the observable content of) the model.

All the properties of the family $\mathcal{A}$ stated above, reflect deep physical principles (Einstein causality, covariance, etc.). Later we will often need another assumption on the structure of $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$, which cannot be justified on the same footing. The family $\mathcal{A}$ is said to satisfy *Haag duality* if

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{O}')' \quad \text{for all} \quad \mathcal{O} \in \mathcal{K} \quad . \tag{2.2}$$

Here $\mathcal{A}(\mathcal{O}')$ is defined to be the C^*-algebra generated by the algebras $\mathcal{A}(\mathcal{O}_1)$ for $\mathcal{O}_1 \subset \mathcal{O}'$, $\mathcal{O}_1 \in \mathcal{K}$. Haag duality can be regarded as a strong version of Einstein causality, which asserts that $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')'$. Generally speaking, breakdown of this duality indicates spontaneous breakdown of symmetry [72]. Bisognano and Wichmann have shown that for families $\mathcal{A}$ generated by Wightman fields one can always pass to the bi-dual $\mathcal{B}(\mathcal{O}) := \mathcal{A}(\mathcal{O})')'$ which then satisfies duality [5, 6]. We shall make some more specific remarks on validity of (2.2) in conformal quantum field theories below.

According to our introductory remarks, superselection sectors carry irreducible representations of the observables algebras $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$. A representation of $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$ is a family of representations $\pi^{\mathcal{O}}, \mathcal{O} \in \mathcal{K}$ of $\mathcal{A}(\mathcal{O})$ on some Hilbert space $\mathcal{H}_\pi$ together with a strongly continuous representation U_π of the group G such that

1) $\pi^{\mathcal{O}_1}|_{\mathcal{A}(\mathcal{O}_2)} = \pi^{\mathcal{O}_2}$ if $\mathcal{O}_2 \subset \mathcal{O}_1$.

2) $Ad_{U_\pi(g)} \circ \pi^{\mathcal{O}} = \pi^{g\mathcal{O}} \circ \alpha_g|_{\mathcal{A}(\mathcal{O})}$,

where $Ad_{U_\pi(g)}$ is the adjoint action. For the defining representation on the vacuum sector $\mathcal{H}^0$ we will use the symbol π_0. One often prefers to work with representations of one C^*- algebra instead of representations of the family $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$. This is possible since every representation of $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$ defines a representation of the C^*–inductive limit

$$\mathcal{A} = \overline{\cup_{\mathcal{O}\in\mathcal{K}}\mathcal{A}(\mathcal{O})} \ . \tag{2.3}$$

Here the bar denotes closure with respect to the operator norm. The *algebra $\mathcal{A}$ of quasi-local observables* contains all the local algebras $\mathcal{A}(\mathcal{O})$. To perform the inductive limit it is essential that the sets $\mathcal{O} \in \mathcal{K}$ in the Minkowski space form a directed set.

We will be concerned only with a small subset of representations which have been singled out because of their relevance for elementary particle physics [7, 8]. A representation π is said to be locally normal if

$$\pi|_{\mathcal{A}(\mathcal{O})} \cong \pi_0|_{\mathcal{A}(\mathcal{O})} \quad \text{for all} \quad \mathcal{O} \in \mathcal{K} \ . \tag{2.4}$$

This has a direct physical interpretation. Elements in $\mathcal{A}(\mathcal{O})$ describe measurements which can be performed in $\mathcal{O}$. Typically, different superselection sectors can be distinguished only by their global properties ("total charge"). In other words, representations π of the algebra $\mathcal{A}$ of quasi-local observables become equivalent when they are restricted to the local subalgebras $\mathcal{A}(\mathcal{O})$. Local normality (2.4) will be tacitly assumed throughout this text. By definition, the Hilbert space $\mathcal{H}_\pi$ carries a strongly continuous representation $\mathcal{U}_\pi$ of the space time translations. When the spectrum of the corresponding generators is contained in the closed forward light cone, π is a *positive energy representation.*

In [15] Doplicher, Haag and Roberts introduced yet another criterion which selects in general an interesting subset of positive energy representations. They called π *locally generated with respect to* π_0, if

$$\pi|_{\mathcal{A}(\mathcal{O}')} \cong \pi_0|_{\mathcal{A}(\mathcal{O}')} \ . \tag{2.5}$$

This criterion looks similar to local normality (2.4). However it is much stronger, since sectors which satisfy (2.5) cannot be distinguished as long as measurements inside a given region $\mathcal{O}$ are forbidden. One might take quantum electro dynamics as a counterexample for this situation, because by Gauss law the charge inside $\mathcal{O}$ can be calculated from he flux through the surface of $\mathcal{O}$. To measure the latter, one need not enter the region $\mathcal{O}$. Since criterion (2.5) is obviously too restrictive one had to look for generalizations. Under some additional assumptions, Buchholz

and Fredenhagen have been able to establish a similar criterion (localization in spacelike cones ("strings")) which allows to consider all positive energy representations with an isolated mass shell in $d \geq 3$-dimensional quantum field theories [10]. To exploit this stringlike localization, Haag duality for spacelike cones $\mathcal{C}$ instead of double cones $\mathcal{O}$ should be supposed.

Conformal quantum field theories live on the tube $\tilde{M}_d = \mathbf{S}^1 \times \mathbf{R}$ [52, 53]. This causes minor changes in the standard framework described so far. For our purposes it will suffice to give some details concerning two–dimensional conformal quantum field theories. We parametrize the space time manifold $\tilde{M}_2$ by (τ, σ) , $\tau = -\infty \ldots \infty$, $\sigma = 0 \ldots 2\pi$. $\tilde{M}_2$ contains Minkowski spaces M_ζ as subspaces (see figure 1).

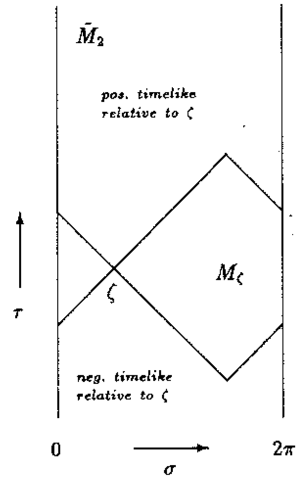

Figure 1: Möbius invariant global causal structure on the tube $\tilde{M}_2 = \mathbf{R} \times \mathbf{S}^1$. M_ζ is the Minkowski space with point ζ at "spacelike infinity". It consists of all points of $\tilde{M}_2$ which are relatively spacelike to ζ.

Their positions are fixed by the unique point $\zeta \in \tilde{M}_2$ at spacelike infinity of M_2. Manifold $\tilde{M}_2$ inherits from M_2 a global causal structure — i.e. a notion of positive timelike, spacelike, and negative timelike — which is invariant under the action of a covering of a direct product of two Möbius groups on the light cone coordinates $\sigma_\pm = \frac{1}{2}(\tau \mp \sigma)$. As usual, the set $\mathcal{K}$ should contain all double cones $\mathcal{O}$ in $\tilde{M}_2$. Double cones $\mathcal{O} \in \mathcal{K}$ have the obvious property that their causal complement $\mathcal{O}'$ in again in $\mathcal{K}$. This implies the selection criterion (2.5) becomes equivalent to local normality (2.4). Consequently, *every (locally normal) representation in conformal quantum field theory is locally generated with respect to the vacuum representations* π_0 [11].

In two–dimensional conformal quantum field theory observable fields often split into mutually commuting chiral components $\phi_\pm(\sigma_\pm)$ which depend only on one light cone coordinate. For such chiral observables, periodicity in the coordinate σ which parametrizes the space $\mathbf{S}^1$ implies periodicity in time τ. This means that $\phi_\pm$ can be regarded as one-valued field on the circle $|z_\pm| = 1$, where $z_\pm = e^{i\sigma_\pm}$. We restrict attention to one chiral component of the observables and drop the suffix + or –. Since spacelike separated double cones $\mathcal{O} \in \tilde{M}_2$ project onto disjoint intervals I on the circle $|z| = 1$, *chiral observables $\phi(z_1)$ and $\phi(z_2)$ commute when* $z_1 \neq z_2, (z_i \in \mathbf{S}^1)$. A set of chiral observables $\phi(z)$ generates a family of von Neumann algebras $\mathcal{A}(I)$ indexed by intervals $I \in \mathbf{S}^1$. To formulate Haag Kastler axioms for such families, double cones $\mathcal{O}$ should be replaced by intervals I. The complement $I' = \mathbf{S}^1 \setminus I$ substitutes for the causal complement $\mathcal{O}'$ and one of the Möbius groups $SL(2,\mathbf{R})/\mathbf{Z}_2$ acts as symmetry group G on the circle $\mathbf{S}^1$. In a positive energy representation π the generator L_0 of "rotations" has positive spectrum. Recently, Brunetti et al. and Fröhlich et al. proved Haag duality for chiral conformal quantum field theories [9, 36].

The construction of a C^*-algebra of "quasi local observables" is not straight forward since intervals I in the circle $\mathbf{S}^1$ do not form a directed set. To define an inductive limit of local algebras, one has to remove a point $\zeta \in \mathbf{S}^1$ ("point at infinity").

$$\mathcal{A}_\zeta = \overline{\cup_{\mathcal{I}\not\ni\zeta}\mathcal{A}(\mathcal{I})} \ . \tag{2.6}$$

Note that after the choice of $\zeta \in \mathbf{S}^1$, the complement of an interval $I \not\ni \zeta$ in $\mathbf{S}^1 \setminus \zeta$ decomposes into left and right components $I_{\lessgtr}$. Disjoint intervals $I_1, I_2 \subset \mathbf{S}^1 \setminus \zeta$ can be ordered like double cones in two–dimensional Minkowski space M_2.

Even though $\mathcal{A}_\zeta$ will suffice for all model independent studies, it is often inconvenient in the applications. It is an obvious disadvantage of $\mathcal{A}_\zeta$ that local algebras $\mathcal{A}(I)$ are not embedded into $\mathcal{A}_\zeta$ if $\zeta \in I$. This motivates to look for a C^*-algebra $\mathcal{A}_{\text{univ}}$ such that

1. every local algebra $\mathcal{A}(I)$ can be embedded into $\mathcal{A}_{\text{univ}}$ by a unital map i^I such that

$$i^{I_1}|_{\mathcal{A}(I_2)} = i^{I_2} \quad \text{for all} \quad I_2 \subset I_1$$

   and $\mathcal{A}_{\text{univ}}$ is generated by the algebras $i^I(\mathcal{A}(I)), I \subset \mathbf{S}^1$.

2. for every representation $\{\pi^I\}_{I\subset\mathbf{S}^1}$ of the family $\mathcal{A}(I)_{I\subset\mathbf{S}^1}$ there is a unique representation π of $\mathcal{A}_{\text{univ}}$ which satisfies $\pi \circ i^I = \pi^I$.

The "universal algebra" $\mathcal{A}_{\text{univ}}$ does exist and is unique [27, 25], but its explicit construction is subtle. Unlike the algebras $\mathcal{A}$ resp. $\mathcal{A}_\zeta$ obtained from the inductive limits (2.3 resp. 2.6), the center of the universal algebra $\mathcal{A}_{\text{univ}}$ is in general non-trivial. This means that the vacuum representation π_0 of $\mathcal{A}_{\text{univ}}$ may not be faithful. Indeed it is possible that two charge operators localized in domains $I_1, I_2, I_1 \cup I_2 = \mathbf{S}^1$ add up to to a global quantity which commutes with all elements in $\mathcal{A}_{\text{univ}}$. These charges may have different values in different superselection sectors. They must not be identified with multiples of the identity.

The setup for theories with charges localized along strings in three–dimensional Minkowski space is very similar to the situation in chiral conformal quantum field theories (cf. [25] and references therein). Points ζ on the circle $\mathbf{S}^1$ are substituted by directions in the two–dimensional space and one uses spacelike cones C instead of intervals $I \subset \mathbf{S}^1$.

2.2 Localized endomorphisms and fusion structure

A detailed analysis of the structure of superselection sectors was first performed by Doplicher, Haag and Roberts [15, 16]. It was restricted to sectors which are locally generated with respect to the vacuum sector and formulated for theories on the four-dimensional Minkowski space. The generalization to string-like localized sectors [10] gives essentially the same structure. If the dimension of the space time manifold is decreased, specific new features appear. In d=2 (resp. d=3) dimensional space-times, double cones (resp. spacelike cones) can be ordered and – as we will explain below – this gives rise to representations of the braid group. Such situations were considered more recently by Fredenhagen, Rehren and Schroer [24, 25]. Our

short exposition will concentrate on representations localized in double cones $\mathcal{O}$ in Minkowski space. We included some remarks relevant to treat chiral conformal theories and string-like localized sectors in three–dimensional quantum field theory. For many further results and discussions the reader is refered to the original papers – especially to [24, 25] – and the reviews of Fredenhagen [26] or by Kastler, Mebkhout and Rehren (in [49]). Theories with charges localized in spacelike cones in M_3 have been considered explicitly by Fröhlich et al. [33, 34, 35]. Applications to models in two-dimensional conformal quantum field theory can be found in [11, 54, 55, 38].

From now on, families $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$ are always supposed to satisfy Haag-Kastler axioms and Haag duality (2.2). Except from some remarks, representations are assumed to be localized on double cones (2.5) $\mathcal{O}$ in Minkowski space M_d. Equivalence classes of locally generated positive energy representations form a set denoted by $\mathcal{R}ep$.

The notion of *localized endomorphisms* will provide the key to all further analysis. By definition, an endomorphism ρ of the C^* algebra $\mathcal{A}$ is a linear map $\rho : \mathcal{A} \to \mathcal{A}$ with the properties

$$\begin{aligned} \rho(AB) &= \rho(A)\rho(B) \ , \\ \rho(A^*) &= \rho(A)^* \ , \\ \rho(\mathbf{1}) &= \mathbf{1} \ . \end{aligned}$$

It is called an *automorphism* if it has an inverse. Endomorphisms of $\mathcal{A}$ fall into equivalence classes $[\rho]$ with respect to inner automorphisms, i.e. conjugation by unitaries $U \in \mathcal{A}$. ρ is said to be *localized* in $\mathcal{O} \in \mathcal{K}$ if

$$\rho(A) = A \quad \text{for all} \quad A \in \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A} \ , \quad \mathcal{O}_1 \subset \mathcal{O}' \ .$$

An endomorphism ρ localized in $\mathcal{O}$ is called *transportable* whenever equivalent morphisms $\sigma \in [\rho]$ localized in the transformed region $g\mathcal{O}$ exist for all $g \in G$. Transportable endomorphisms localized in spacelike separated regions commute [15].

Endomorphisms of $\mathcal{A}$ can be used to obtain positive energy representations $\pi_0 \circ \rho$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}^0$. For every locally generated positive energy representation π of $\mathcal{A}$ there is a localized transportable endomorphism ρ such that

$$\pi \cong \pi_0 \circ \rho \quad . \tag{2.7}$$

This can be seen as follows. According to the criterion (2.5), $\pi|_{\mathcal{A}(\mathcal{O}')}$ is unitary equivalent to $\pi_0|_{\mathcal{A}(\mathcal{O}')}$, i.e. for each double cone $\mathcal{O}$ there is a unitary $V : \mathcal{H}_\pi \to \mathcal{H}^0$ such that

$$V\pi(A) = \pi_0(A)V \quad \text{for all } A \in \mathcal{A}(\mathcal{O}') \ .$$

By Haag duality, the map

$$\rho(A) = V\pi(A)V^*$$

defines an endomorphism of $\mathcal{A}$ localized in $\mathcal{O}$. It is transportable and has the desired property (2.7). Two localized endomorphisms $\rho_i, i = 1, 2$ are equivalent if and only if the representations $\pi_0 \circ \rho_i$ are equivalent. We conclude that elements of $\mathcal{R}ep$, i.e. equivalence classes of locally generated positive energy representations of $\mathcal{A}$, correspond one by one to equivalence classes $[\rho]$ of localized transportable endomorphisms ρ. This observation will be used to identify both objects.

Let us pause for a moment to comment on (chiral) conformal quantum field theory. In this case superselection sectors carry irreducible positive energy representations of the C^*–algebra $\mathcal{A}_\zeta$ introduced in the last section. By the results in [11, 9, 36], every positive energy representation is obtained as a composition $\pi_0 \circ \rho$ of the vacuum representation π_0 with a endomorphism ρ of $\mathcal{A}_\zeta$. We can assume ρ to be localized and transportable in an appropriate sense. The general considerations below can be established on $\mathcal{A}_\zeta$ without modifications. In practice it will be more convenient to work with endomorphisms ρ of the universal algebra $\mathcal{A}_{univ}$. Fredenhagen has shown [27] that the localized and locally transportable endomorphisms of $\mathcal{A}_{univ}$ exist for all elements of $\mathcal{R}ep$. They restrict to localized transportable endomorphisms of $\mathcal{A}_\zeta \subset \mathcal{A}_{univ}$ when ζ lies outside the domain of localization. Basically, these remarks apply also to stringlike localized sectors in 2+1 dimensional quantum field theory.

All this is much more than a technicality. Endomorphisms of $\mathcal{A}$ can be composed and thus lead to a proper definition of a product of sectors. Given two representations $\pi_i = \pi_0 \circ \rho_i, i = 1,2$ on the Hilbert space $\mathcal{H}^0$, their product $\pi_1 \times \pi_2$ is defined by

$$\pi_1 \times \pi_2 = \pi_0 \circ \rho_1 \circ \rho_2 \quad .$$

The equivalence class of $\pi_1 \times \pi_2$ is an element of $\mathcal{R}ep$ if equivalence classes of $\pi_i, i = 1,2$, are. In other words a product in $\mathcal{R}ep$ is defined by $[\rho_1] \times [\rho_2] \equiv [\rho_1 \circ \rho_2]$. These assertions do not depend on the type of localization.

Two representations $\pi_0 \circ \rho_i, i = 1,2$, in $\mathcal{H}^0$ have equivalent subrepresentations if an isometry $U \in \mathcal{B}(\mathcal{H}^0)$ intertwines between them, i.e. $U\pi_0(\rho_1(A)) = \pi_0(\rho_2(A))U$ for all local observables $A \in \mathcal{A}$. When restricted with the "source" projection U^*U of U, the representation $\pi_0 \circ \rho_1$ is equivalent to the restriction of $\pi_0 \circ \rho_2$ to the range of U. By Haag duality, such intertwining operators U can be obtained as image of a local intertwiner $T \in \mathcal{A}$

$$T\rho_1(A) = \rho_2(A)T \quad \text{for all} \ \ A \in \mathcal{A} \ .$$

Local observables $T \in \mathcal{A}$ which satisfy this equation span a complex linear space $\mathcal{T}(\rho_1, \rho_2)$. If $S, T \in \mathcal{T}(\rho_1,\rho_2)$ then TS^* is a local observable in $\mathcal{T}(\rho_2,\rho_2)$. Schurs lemma asserts, that irreducibility of ρ implies $\mathcal{T}(\rho,\rho) \cong \mathbf{C}$. In conclusion, $< T, S > \equiv TS^*$ defines a scalar product on $\mathcal{T}(\rho_1,\rho_2)$ if ρ_2 is irreducible. Therefore $\mathcal{T}(\rho_1,\rho_2)$ is a Hilbert space in this case.

The product of sectors $[\rho_i], i = 1,2$, is commutative in the sense that $[\rho_1 \circ \rho_2] = [\rho_2 \circ \rho_1]$. To see this we pick two endomorphisms σ_i from the equivalence classes $[\rho_i]$ which are localized in spacelike separated double cones. As we remarked above, their action on the observables commutes and so the assertion follows. For every pair of localized and transportable endomorphisms $\rho_i, i = 1,2$ there is a unitary local intertwiner $\epsilon(\rho_1,\rho_2) \in \mathcal{T}(\rho_1 \circ \rho_2, \rho_2 \circ \rho_1)$, the *statistics operator*. The collection of statistics operators in uniquely determined by the following equations (a detailed proof can be found in the contribution of Mebkhout et al. in [49]).

$$\epsilon(\rho_1,\rho_2)\rho_1(T_2)T_1 \;=\; T_2\sigma_2(T_1)\epsilon(\sigma_1,\sigma_2) \qquad \text{for all} \quad T_i \in \mathcal{T}(\sigma_i,\rho_i) \ ,$$

$$\epsilon(\rho_1 \circ \rho_2, \sigma) = \epsilon(\rho_1,\sigma)\rho_1(\epsilon(\rho_2,\sigma)) \qquad \epsilon(\sigma,\rho_1 \circ \rho_2) = \rho_1(\epsilon(\sigma,\rho_2))\epsilon(\sigma,\rho_1) \ , \tag{2.8}$$

$$\epsilon(\rho_1,\rho_2) \;=\; 1 \quad \text{whenever} \quad \rho_1 > \rho_2 \ .$$

In the last row, $\rho_1 > \rho_2$ refers to the order of localization regions, provided they can be ordered. For localization in double cones contained in a two–dimensional Minkowski space this is the

case and $\rho_1 > \rho_2$ means that ρ_1 is localized on a domain $\mathcal{O}_1$ left from the localization region $\mathcal{O}_2$ of ρ_2, i.e. $\mathcal{O}_1 > \mathcal{O}_2$. Trivialization for $\rho_1 < \rho_2$ would give rise to the opposite statistics operator $\epsilon(\rho_2,\rho_1)^*$. In higher dimensional quantum field theories there is no invariant distinction between left and right so that trivialization is possible for all pairs of spacelike separated endomorphisms. For two–dimensional light cone theories [stringlike localized sectors in theories on M_3], the notion $\lessgtr$ refers to the point ζ at infinity [the direction in the two–dimensional space] which we have to single out to define the inductive limit in (2.6).

It follows from these relations that the vacuum sector carries a representation of the colored braid group. In particular, $\sigma_i = \rho^{i-1}(\epsilon(\rho,\rho))$ satisfy Artin relations ([4]).

$$\sigma_i\sigma_k = \sigma_k\sigma_i \quad \text{if } |k-i| \geq 2 \quad , \quad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ , \tag{2.9}$$

$$\sigma_i\sigma_i^{-1} = \mathbf{1} = \sigma_i^{-1}\sigma_i \ , \tag{2.10}$$

i.e. the elements σ_i and σ_i^{-1} $(i = 1 \ldots n-1)$ generate the braid group B_n. In theories without an invariant distinction between left and right, $\sigma_i = \sigma_i^{-1}$ so that we obtain a representation of the permutation group. Accordingly the statistics operators give rise to an intrinsic notion of statistics of a superselection sector.

At this stage we would like to restrict to sectors of *finite statistics*. An irreducible transportable endomorphism has finite statistics whenever it possesses a left inverse [2] Φ with $\Phi(\epsilon(\rho,\rho)) \neq 0$. It has been shown by Longo [51] that this is equivalent to a finite Jones index [46] of the inclusion $\rho(\mathcal{A}(\mathcal{O})) \subset \mathcal{A}(\mathcal{O})$. We do not plan to justify this restriction (see however [23, 10]) but list the consequences required later. Whenever two sectors of finite statistics are composed, their product decomposes into a finite direct sum of sectors which have finite statistics again. This means that the subset $\mathcal{R}ep' \subset \mathcal{R}ep$ of finite direct sums of sectors with finite statistics is closed under products. Moreover, every sector with finite statistics has a unique conjugate, i.e. for every irreducible endomorphism ρ of $\mathcal{A}$ there exists a unique irreducible endomorphism $\bar{\rho}$ such that $\bar{\rho} \circ \rho$ contains the vacuum sector. One can show that $dim\mathcal{T}(\bar{\rho} \circ \rho, id) = 1$.

Equivalence classes of irreducible representations in $\mathcal{R}ep'$ will be labelled by elements $I, J, K, \ldots$ of an index set $\mathcal{I}_\mathcal{A}$. We reserve $0 \in \mathcal{I}_\mathcal{A}$ for the vacuum sector. Let us fix a set of representative endomorphisms ρ_I, one for each superselection sector in $\mathcal{R}ep'$. According to the general remarks on sectors with finite statistics, the product of two endomorphisms $\rho_I \circ \rho_J$ is equivalent to a finite direct sum of irreducibles ρ_K. As we saw above, the space $\mathcal{T}(\rho_I \circ \rho_J | \rho_K) \equiv \mathcal{T}(IJ|K)$ is a Hilbert space. The dimension $dim(\mathcal{T}(IJ|K))$ defines the fusion rules N_K^{IJ} which appear in the decomposition

$$[\rho_I \circ \rho_J] = \bigoplus_K [\rho_K] N_K^{IJ} \ .$$

It follows from the associativity of the composition of endomorphisms and the commutativity we found above that the fusion rules are associative and commutative in the sense

$$N_K^{IJ} \;=\; N_K^{JI} \quad \text{and} \tag{2.11}$$

$$\sum_M N_L^{IM} N_M^{JK} \;=\; \sum_M N_L^{MK} N_M^{IJ} \ . \tag{2.12}$$

[2] By definition, a left inverse is a positive linear map $\Phi : \mathcal{A} \mapsto \mathcal{A}$ such that $\Phi(\rho(A)B\rho(C)) = A\Phi(B)C$ for all $A, B, C \in \mathcal{A}$ and $\Phi(1) = 1$.

The endomorphism $\bar\rho_J$ should be the unique conjugate $\bar\rho_J$ of ρ_J. Since the vacuum sector $[\rho_0]$ appears with multiplicity 1 in the decomposition of $[\rho_J \circ \rho_J]$ we have

$$N_0^{IJ} = \delta_{I,J} \ .$$

In the Hilbert space $\mathcal{T}(IJ|K)$ we choose an orthonormal basis $T_a({}_K{}^J{}_I) \in \mathcal{A}$. In detail this means that the operators $T_a({}_K{}^J{}_I), a = 1 \dots N_K^{IJ}$, satisfy

$$T_a({}_K{}^J{}_I)\rho_I \circ \rho_J(A) \ = \ \rho_K(A)T_a({}_K{}^J{}_I) \quad \text{for all } A \in \mathcal{A} \ , \tag{2.13}$$

$$T_a({}_K{}^J{}_I)T_b({}_L{}^J{}_I)^* = \delta_{a,b}\delta_{L,K} \quad , \quad \sum_{K,a} T_a({}_K{}^J{}_I)^* T_a({}_K{}^J{}_I) = 1 \ . \tag{2.14}$$

The *fusion- and braiding- matrices* are complex valued matrices defined by

$$T_a({}_L{}^K{}_M)T_b({}_M{}^J{}_I) \ = \ \sum_N T_c({}_L{}^N{}_I)\rho_I(T_d({}_N{}^K{}_J))F_{MN}\begin{bmatrix} K & J \\ L & I \end{bmatrix}_{ab}^{cd} \ , \tag{2.15}$$

$$T_a({}_L{}^J{}_M)T_b({}_M{}^K{}_I)\rho_I(\epsilon(\rho_J,\rho_K)) \ = \ \sum_N T_c({}_L{}^K{}_N)T_d({}_N{}^J{}_I)B_{MN}\begin{bmatrix} J & K \\ L & I \end{bmatrix}_{ab}^{cd} \ . \tag{2.16}$$

As a special case of eq. (2.16) for $I = 0$ we introduce the matrix $\Omega({}_L{}^J{}_M)$.

$$T_a({}_L{}^J{}_K)\epsilon(\rho_J,\rho_K) \equiv T_b({}_L{}^K{}_J)\Omega_a^b({}_L{}^J{}_K) \ . \tag{2.17}$$

By Schurs' lemma, the coefficients F, B, Ω are certain complex matrices. They are determined by the model and depend only on the equivalence classes $[\rho_I]$ but not on ρ_I itself. The braiding matrices B, Ω are often denoted by $\Omega(+), B(+)$ to emphasize their dependence on the choice of trivialization of the statistics operators $\epsilon(\rho_J,\rho_K)$ (2.8). The corresponding matrices for the opposite statistics operators $\epsilon(\rho_K,\rho_J)^*$ are called $B(-), \Omega(-)$. We restrict attention to one trivialization and neglect to write $(\pm)$. A short calculation reveals the following proposition.

Proposition 2 (Polynomial equations) [24] *The fusion- and braiding matrices F, Ω defined by (2.16,2.15) solve the polynomial equations*

$$\sum_N F_{NP}\begin{bmatrix} K & J \\ L & I \end{bmatrix}(\Omega({}_L{}^N{}_K)\otimes 1)F_{MN}\begin{bmatrix} J & I \\ L & K \end{bmatrix} \ = \ (1\otimes\Omega({}_P{}^J{}_K))F_{MP}\begin{bmatrix} J & K \\ L & I \end{bmatrix}(1\otimes\Omega({}_M{}^I{}_K)) \tag{2.18}$$

$$\sum_Q F_{QS}\begin{bmatrix} L & K \\ R & J \end{bmatrix}_{23}F_{NR}\begin{bmatrix} L & Q \\ P & I \end{bmatrix}_{12}F_{MQ}\begin{bmatrix} K & J \\ N & I \end{bmatrix}_{23} \ = \ P_{23}F_{MR}\begin{bmatrix} S & J \\ P & I \end{bmatrix}_{13}F_{NS}\begin{bmatrix} L & K \\ P & M \end{bmatrix}_{12} \tag{2.19}$$

$$\sum_P F_{PQ}\begin{bmatrix} I & J \\ K & L \end{bmatrix} F_{PR}\begin{bmatrix} I & J \\ K & L \end{bmatrix}^* = \mathbf{1}\delta_{Q,R} \qquad \sum_Q F_{PQ}\begin{bmatrix} I & J \\ K & L \end{bmatrix}^* F_{RQ}\begin{bmatrix} I & J \\ K & L \end{bmatrix} = \mathbf{1}\delta_{P,R} \tag{2.20}$$

$$\Omega({}_L{}^I{}_J)\Omega({}_L{}^I{}_J)^* = \mathbf{1} \qquad \Omega({}_L{}^I{}_J)^*\Omega({}_L{}^I{}_J) = \mathbf{1} \tag{2.21}$$

$$F_{IJ}\begin{bmatrix} J & 0 \\ K & I \end{bmatrix} \ = \ \mathbf{1} \tag{2.22}$$

The braiding-matrix B can be calculated from F, Ω.

$$B_{MN}\begin{bmatrix} J & K \\ L & I \end{bmatrix} = \sum_P F_{NP}\begin{bmatrix} K & J \\ L & I \end{bmatrix}^*(1\otimes\Omega({}_P{}^J{}_K))F_{MP}\begin{bmatrix} J & K \\ L & I \end{bmatrix} \ . \tag{2.23}$$

We used an obvious matrix notation and **1** *denotes an appropriate unit matrix. F_{12} is defined on threefold tensor products $u \otimes v \otimes w$ by $F_{12}(u\otimes v\otimes w) = F(u\otimes v)\otimes w$ etc. P_{23} acts as permutation of the second and third component.*

The first two relations (2.18,2.19) are the famous Moore-Seiberg "hexagon" and "pentagon" identities known from conformal quantum field theory [65, 66]. In our context they reflect deep properties of the fusion structure of superselection sectors. In particular, conformal symmetry was not assumed.

PROOF: We do not want to prove all the relations but just demonstrate the type of calculations to be done at the example of eq. (2.23). The product of operators which appears on the left hand side of eq. (2.16) can be manipulated in two different ways. One is just the step from the left to the right hand side of (2.16). For the other we apply the definition (2.15) of the fusion matrix, the endomorphism property of ρ_I, the definition (2.17) and relation (2.15) in this order.

$$\begin{aligned} T_a({}_L{}^J{}_M)T_b({}_M{}^K{}_I)\rho_I(\epsilon(\rho_J,\rho_K)) \ &= \ \sum_P T_c({}_L{}^P{}_I)\rho_I(T_d({}_P{}^J{}_K))\rho_I(\epsilon(\rho_J,\rho_K))F_{MP}\begin{bmatrix} J & K \\ L & I \end{bmatrix}_{ab}^{cd} \\ &= \ \sum_P T_c({}_L{}^P{}_I)\rho_I(T_e({}_P{}^K{}_J))\Omega({}_P{}^J{}_K)_d^e F_{MP}\begin{bmatrix} J & K \\ L & I \end{bmatrix}_{ab}^{cd} \\ &= \ \sum_{NP} T_f({}_L{}^K{}_N)T_g({}_N{}^J{}_I)(F_{NP}\begin{bmatrix} K & J \\ L & I \end{bmatrix}^*)_{ce}^{fg}\Omega({}_P{}^J{}_K)_d^e F_{MP}\begin{bmatrix} J & K \\ L & I \end{bmatrix}_{ab}^{cd} \ . \end{aligned}$$

We used the first relation in (2.20) for the last equality. If we compare the result with the right hand side of (2.16) we find that the same operator has been expressed by two linear combinations of the same basis elements. Consequently, the coefficients have to agree and this gives eq. (2.23).

2.3 Covariant field operators and the field algebra $\mathcal{F}$

Now we are prepared to construct field operators Ψ which make transitions between different superselection sectors. We want them to transform non-trivially under the action of elements ξ from an "appropriate" symmetry algebra $\mathcal{G}^*$. A simple assumption on the bi-*-algebra structure of $\mathcal{G}^*$ will turn out to ensure the existence of such covariant field operators. They will be constructed as a sum of "vertex operators".

$\mathcal{G}^*$ is assumed to be a semisimple bi-*-algebra. Equivalence classes $[\tau]$ of finite dimensional irreducible representations τ of $\mathcal{G}^*$ are labelled by elements of an index set $\mathcal{I}_{\mathcal{G}^*}$. Because of semisimplicity [3], tensor products of two finite dimensional irreducible representations τ, τ' on vector spaces V, V' can be decomposed into irreducibles τ^α on V^α. The corresponding "Clebsch Gordon" intertwiners $C(\tau \boxtimes \tau'|\tau^\alpha) : V \otimes V' \mapsto V^\alpha$ form complex vector spaces $\mathcal{C}(\tau \boxtimes \tau'|\tau^\alpha)$.

$$C(\tau\boxtimes\tau'|\tau^\alpha)(\tau\boxtimes\tau')(\xi) = \tau^\alpha(\xi)C(\tau\boxtimes\tau'|\tau^\alpha) \ .$$

Recall that a representation τ of $\mathcal{G}^*$ on a Hilbert space V is unitary, if $\tau(\xi)^* = \tau(\xi^*)$ for all $\xi \in \mathcal{G}^*$

Assumption: *Let I_A denote the set of superselection sectors with finite statistics as in the preceding subsection. We assume that there is a bijection $\theta : \mathcal{I}_A \mapsto \mathcal{I}_{\mathcal{G}^*}$ and a set of unitary representatives $\tau^{\theta(I)}$ from the equivalence classes of irreducible representations of $\mathcal{G}^*$ such that*

$$dim(\mathcal{C}(\tau^{\theta(I)}\boxtimes\tau^{\theta(J)}|\tau^{\theta(K)})) = N_K^{IJ} \ . \tag{2.24}$$

[3] Note that every finite dimensional representation of a semisimple algebra is a direct sum of irreducible representations.

In other words: There is a unique equivalence class of irreducible representations of $\mathcal{G}^*$ associated with every superselection sector and fusion rules of the sectors are in agreement with the selection rules of the prospective symmetry. By uniqueness of the conjugate, θ will automatically map the vacuum sector to the equivalence class of the one–dimensional trivial representation ϵ of $\mathcal{G}^*$. Since ϵ is unitary we can always choose

$$\tau^{\theta(0)} = \epsilon \ .$$

We will not distinguish between I and $\theta(I)$ in the following.

If a consistent transformation law of adjoint field operators is required, one has to start with a Hopf-*-algebra $\mathcal{G}^*$ which satisfies the above assumption. The existence of appropriate Hopf-*-algebras will be studied elsewhere. Some partial results on the construction of Hopf-*-algebras are collected in Appendix B. Actually, the only non-trivial part of this construction is connected with the antipode $\mathcal{S}$. *Suitable bi-*-algebras do always exist (cf. Appendix B)*, if $\mathcal{I}_{\mathcal{G}^*}$ is finite.

Given the required bi-*-algebra (or Hopf-*-algebra) $\mathcal{G}^*$, we can start to build covariant field operators. They will act on a Hilbert space $\mathcal{H}$ of physical states which is a direct sum of irreducible representation spaces $\mathcal{H}^J$ for the algebra of observables $\mathcal{A}$, each with multiplicity δ_J determined by the dimension of the representation τ^J of $\mathcal{G}^*$, i.e. $\delta_J \equiv dim(\tau^J)$.

$$\mathcal{H} = \bigoplus_{J\in\mathcal{I}} \bigoplus_{1}^{\delta_J} \mathcal{H}^J_i \ . \qquad (2.25)$$

$\mathcal{H}^0$ carries the vacuum representation π_0 of $\mathcal{A}$ and it occures with multiplicity one (since $dim(\tau^0) = dim(\epsilon) = 1$).

Next we define a representation π of the observables algebra $\mathcal{A}$ on all of $\mathcal{H}$ by its restrictions to the subspaces $\mathcal{H}^J_m$,

$$\pi(A) = \pi_J(A) \text{ on } \mathcal{H}^J_m \ (m = 1...\delta_J) \ . \qquad (2.26)$$

According to our general discussion, the representation π_J on $\mathcal{H}^J_m$ is equivalent to $\pi_0 \circ \rho_J$ which is realized on the vacuum Hilbert space $\mathcal{H}^0$. This equivalence can be expressed by isometries $i^*_{Jm} : \mathcal{H}^0 \mapsto \mathcal{H}^J_m$ with the intertwining property

$$\pi_J(A) i^*_{Jm} = i^*_{Jm} \pi_0(\rho_J(A)) \ . \qquad (2.27)$$

To specify the action of elements $\xi \in \mathcal{G}^*$ on $\mathcal{H}$, we choose an orthonormal basis e^J_m in the finite dimensional representation space V^J. When the corresponding matrix elements of $\tau^J(\xi)$ are denoted by $\tau^J_{km}(\xi)$, an unitary representation $\mathcal{U}$ of $\mathcal{G}^*$ on $\mathcal{H}$ is obtained according to

$$\mathcal{U}(\xi) i^*_{Jm} |\psi> = i^*_{Jk} |\psi> \tau^J_{km}(\xi) \qquad (2.28)$$

for arbitrary $|\psi> \in \mathcal{H}^0$. The symmetry acts as a *gauge symmetry (of first kind)*, i.e. all observables are invariant

$$[\mathcal{U}(\xi), \pi(A)] = 0 \ \text{ for all } A \in \mathcal{A},\ \xi \in \mathcal{G}^* \ . \qquad (2.29)$$

Our main task is to construct field operators $\Psi^J_m(\rho_J)$ which make transitions between the sectors $\mathcal{H}^I_i$ with different I. They will have the following properties

1. Intertwining property for representations of $\mathcal{A}$,

$$\pi(A)\Psi^J_m(\rho_J) = \Psi^J_m(\rho_J)\pi(\rho_J(A)) \ . \qquad (2.30)$$

As a consequence, $\Psi^J_m(\rho_J)$ commutes with observables localized in the causal complement $\mathcal{O}'$ of the localization region $\mathcal{O}$ of the endomorphism ρ_J. This property reflects locality of the field operators with respect to observables.

2. Field operators $\Psi^J_m(\rho_J)$ transform covariantly according to the representation τ^J of $\mathcal{G}^*$.

$$\mathcal{U}(\xi)\Psi^J_m(\rho_J) = \Psi^J_k(\rho_J)(\tau^J_{km} \otimes \mathcal{U})(\Delta(\xi)) \ . \qquad (2.31)$$

The field operators $\Psi^J_m(\rho_J)$ are determined by these properties up to a phase factor. They will be build up from the following "vertex operators".

$$_k\Psi_l({_K}{^J}{_L})_a(\rho_J) \ : \ \mathcal{H}^L_l \mapsto \mathcal{H}^K_k \quad \text{for } a = 1\ldots N^{JL}_K \ , \qquad (2.32)$$

$$_k\Psi_l({_K}{^J}{_L})_a(\rho_J) \ = \ i^*_{Kk}\pi_0(T_a({_K}{^J}{_L}))i_{Ll} \ . \qquad (2.33)$$

Combining the intertwining relations (2.27) and (2.13) we find that these operators satisfy the intertwining property *1.* for representations of $\mathcal{A}$. We extend the vertex operators $_k\Psi_l({_K}{^J}{_L})_a$ to all of $\mathcal{H}$ such that they vanish on states in $\mathcal{H}^I, i$ for all $i \neq l, I \neq L$. The extended operators are denoted by the same symbol. Their intertwining properties with observables is not affected by this extension.

To obtain a covariant field operator $\Psi^J_j(\rho_J)$ on the whole Hilbert space $\mathcal{H}$, we fix an orthonormal basis $C^a(JL|K), a = 1\ldots N^{JL}_K$ in the Hilbert spaces $\mathcal{C}(JL|K) \equiv \mathcal{C}(\tau^J \boxtimes \tau^L|\tau^K)$ of Clebsch Gordon intertwiners. By assumption, the fusion rules and tensor product decomposition match so that the index a assumes the same values as for the vertex operators. With this knowledge we can form the following linear combination of vertex operators.

$$\Psi^J_m(\rho_J) = \sum_{K,k,L,l} {_k\Psi_l}({_K}{^J}{_L})_a(\rho_J) \begin{bmatrix} J & L & K \\ m & l & k \end{bmatrix}^a \ . \qquad (2.34)$$

The complex coefficients $[\because]^a$ are matrix elements of the Clebsch Gordon map $C^a(JL|K)$ in the basis $e^J_j \otimes e^L_l$ resp. e^K_k. The field operators $\Psi^J_m(\rho_J)$ meet all the requirements stated before.

Theorem 3 *Let $\mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}} \subset \mathcal{B}(\mathcal{H}^0)$ be a family of local observable algebras with properties as before. Suppose that $\mathcal{G}^*$ is a (weak) bi-*-algebra which satisfies assumption (2.24). Then there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, a unitary representation $\mathcal{U}$ of $\mathcal{G}^*$ on $\mathcal{H}$ and a family $\mathcal{F}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}} \subset \mathcal{B}(\mathcal{H})$ of *-algebras generated by field operators such that*

1. *the vacuum representation π_0 is a subrepresentation of π on $\mathcal{H}^0 \subset \mathcal{H}$ and appears with multiplicity 1. States in $\mathcal{H}^0$ are invariant with respect to the action of $\mathcal{G}^*$. In particular,*

$$\mathcal{U}(\xi)|0\rangle = |0\rangle\epsilon(\xi) \quad \textit{for all} \quad \xi \in \mathcal{G}^* \ .$$

The Hilbert space $\mathcal{H}$ is generated from $|0\rangle$ by local algebras $\mathcal{F}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$.

2. *the algebras $\mathcal{F}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H})$ are generated by elements of $\mathcal{A}(\mathcal{O})$ together with operators $\Psi^J_m(\rho_J)$, where ρ_J are endomorphisms of $\mathcal{A}$ localized in $\mathcal{O}$. The field operators $\Psi^J_m(\rho_J)$ are local relative to observables and transform covariantly according to the representation τ^J of $\mathcal{G}^*$. Explicitly this means*
$$\Psi^J_m(\rho_J)\pi(A) = \pi(A)\Psi^J_m(\rho_J) \quad \text{for all} \quad A \in \mathcal{A}(O') , \tag{2.35}$$
$$\mathcal{U}(\xi)\Psi^J_m(\rho_J) = \Psi^J_k(\rho_J)(\tau^J_{km} \otimes \mathcal{U})(\Delta(\xi)) . \tag{2.36}$$
A consistent transformation law for adjoint field operators holds, if $\mathcal{G}^*$ is a Hopf--algebra, i.e.if $\mathcal{G}^*$ has also an antipode S (cf. Appendix A).*
3. *the algebra $\pi(\mathcal{A}(\mathcal{O}))$ consists of all invariant elements of $\mathcal{F}(\mathcal{O})$, i.e.*
$$\pi(\mathcal{A}(\mathcal{O})) = \{\Psi \in \mathcal{F}(\mathcal{O}) | [\Psi, U(\xi)] = 0 \text{ for all } \xi \in \mathcal{G}^*\} .$$
4. *each equivalence class of irreducible representations in $\mathcal{R}ep'$ is realized as a subrepresentation of π.*

The tupel $(\mathcal{H}, \pi, \mathcal{G}^*, \mathcal{F})$ is called a complete field system with quantum symmetry *$\mathcal{G}^*$ of the algebra $\mathcal{A}$.*

For chiral conformal quantum field theories, field operators can be obtained in the same way. It was explained above that the corresponding endomorphisms ρ_J act on an algebra $\mathcal{A}_\zeta$ which depends on the choice of the "point at infinity" $\zeta \in \mathbf{S}^1$. This dependence on ζ does also show up in the field operators $\Psi^{\zeta J}_m(\rho_J)$ and the field algebras $\mathcal{F}^\zeta(I)$. If ζ is changed, field operators are multiplied with a unitary element from the center of the universal algebra $\mathcal{A}_{\text{univ}}$. This means that fields do not "live" on the circle $\mathbf{S}^1$ but on a covering thereof. The same behaviour is found for field operators which create charges localized along strings in a three-dimensional Minkowski space. For an enlightening discussion of these points the reader is refered to [25].

The field operators $\Psi^J_m(\rho_J)$ are localized in the localization domain of ρ_J. One may construct operators $\Psi^J_m(x,t)$ associated with a point (x,t) by taking appropriate limits. If ρ^α_J is a sequence of endomorphisms from the equivalence class of ρ_J such that the localization region shrinks to a point (x,t) in the limit $\alpha \to \infty$, $\Psi^J_m(x,t)$ is obtained formally as
$$\Psi^J_m(x,t) = \lim_{\alpha \to \infty} \mathcal{N}_{\rho^\alpha_J} \Psi^J_m(\rho^\alpha_J) ,$$
where $\mathcal{N}_{\rho^\alpha_J}$ are suitable normalization factors. This procedure has been successfully applied to charged fields of the $U(1)$-current algebra on the circle $\mathbf{S}^1$ [11] and there is much hope to develop a general technique for chiral conformal quantum field theories [47, 29].

3 Weak Quasi Quantum Groups

The formulation of quantum symmetry in section 1 involved only a bi-*-algebra structure. One cannot expect that every bi-*-algebra is actually realized as a quantum symmetry of a quantum mechanical system. At this point it should suffice to remark that only group symmetries seem to be realized in higher dimensional quantum field theory. We will elucidate the reasons later.

To describe distinguished algebraic structures within the class of bi-algebras we introduce and discuss some relevant notions. Weak quasi quantum groups will be defined. They were introduced in [56] as a generalization of Drinfeld's quasi quantum groups [19]. References to our physical framework have been avoided to emphasize the purely mathematical nature of the arguments. Our presentation is restricted to those parts of the theory which are needed later.

3.1 Fundamental definitions and results

In a bi-algebra tensor products of representations are defined with the help of the co-product $\Delta : \mathcal{G}^* \mapsto \mathcal{G}^* \otimes \mathcal{G}^*$. Properties of the tensor product (1.7) of representations can be traced back to properties of the co-product. As an example consider the group algebra associated with a compact group G. In this case, the tensor product (1.1) of representations is well known to be associative and commutative. This corresponds to a co-commutative and co-associative co-product Δ_G. A co-product Δ is called co-associative, if
$$(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi) \tag{3.1}$$
and co-commutativity means that
$$\Delta(\xi) = \Delta'(\xi) . \tag{3.2}$$
Given the expansion $\Delta(\xi) = \sum \xi^1_\sigma \otimes \xi^2_\sigma$, Δ' is defined by $\Delta'(\xi) = \sum \xi^2_\sigma \otimes \xi^1_\sigma$. For Δ_G both properties can be verified from the explicit action (1.8) of Δ_G on elements in G. In the following we introduce a special class of bi-algebras for which tensor products of representations are at least commutative and associative *up to equivalence.*

Definition 4 (quasi-co-associativity) [19, 56] *The co-product Δ of a bi-algebra $(\mathcal{G}^*, \Delta, \epsilon)$ is called* quasi-co-associative, *if an element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ exists, such that*

1. *φ has a quasi-inverse $\varphi^{-1} \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ such that*
$$\varphi\varphi^{-1} = (id \otimes \Delta)\Delta(e) \quad , \quad \varphi^{-1}\varphi = (\Delta \otimes id)\Delta(e) . \tag{3.3}$$
2. *φ satisfies the "intertwining" relation*
$$\varphi(\Delta \otimes id)\Delta(\xi) = (id \otimes \Delta)\Delta(\xi)\varphi \quad \text{for all} \quad \xi \in \mathcal{G}^* . \tag{3.4}$$
3. *the following* pentagon equation *holds*
$$(id \otimes id \otimes \Delta)(\varphi)(\Delta \otimes id \otimes id)(\varphi) = (e \otimes \varphi)(id \otimes \Delta \otimes id)(\varphi)(\varphi \otimes e) . \tag{3.5}$$
4. *φ and the co-unit ϵ obey $(id \otimes \epsilon \otimes id)(\varphi) = \Delta(e)$.*

An element $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ with these properties is called re-associator.

Drinfel'd introduced the notion of quasi-co-associativity in [19] for the case without truncation, viz. $\Delta(e) = e \otimes e$. Without truncation, φ^{-1} is a true inverse of φ. Let us discuss the meaning of this definition in terms of representation theory. Consider tensor products of three representations $\tau^i, i = 1,2,3$. Due to the freedom in placing the brackets, there exist two different ways to perform threefold tensor products, $(\tau^1 \boxtimes \tau^2) \boxtimes \tau^3$ *and* $\tau^1 \boxtimes (\tau^2 \boxtimes \tau^3)$. They are constructed from the two different combinations of the co-product, $(\Delta \otimes id)\Delta$ resp. $(id \otimes \Delta)\Delta$. If Δ is quasi-co-associative in the sense of the above definition, the two threefold tensor products are equivalent. Indeed it follows from equations (3.3,3.4) that φ furnishes an invertible intertwiner $(\tau^1 \otimes \tau^2 \otimes \tau^3)(\varphi)$. Definition 4.3 expresses equality of two intertwiners between fourfold tensor products of representations. The name derives from the fact that the equation describes commutativity of a pentagon shaped diagram, in which the edges are indexed with the five factors of the equation. The relation in 4. is consistent since $(\tau^1 \boxtimes \epsilon) \boxtimes \tau^2 = \tau^1 \boxtimes \tau^2 = \tau^1 \boxtimes (\epsilon \boxtimes \tau^2)$ by triviality of ϵ. Corresponding equations for the other components of φ follow with the help of the pentagon equation. In particular we will need the relation
$$(id \otimes id \otimes \epsilon)(\varphi) = \Delta(e) . \tag{3.6}$$

Definition 5 (quasi-triangularity) [19, 56] *A bi-algebra $(\mathcal{G}^*, \Delta, \epsilon)$ with quasi-co-associative co-product Δ and re-associator φ is called quasi-triangular, if there exists $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that*

1. *R has a quasi-inverse $R^{-1} \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that*

$$RR^{-1} = \Delta'(e) \quad , \quad R^{-1}R = \Delta(e) \ . \tag{3.7}$$

2. *R satisfies the "intertwining relation"*

$$R\Delta(\xi) = \Delta'(\xi)R \quad \text{for all} \quad \xi \in \mathcal{G}^* \ . \tag{3.8}$$

3. *the following* hexagon equations *are fulfilled*

$$\begin{aligned}(id \otimes \Delta)(R) &= \varphi_{231}^{-1} R_{13} \varphi_{213} R_{12} \varphi^{-1} \ , \\ (\Delta \otimes id)(R) &= \varphi_{312} R_{13} \varphi_{132}^{-1} R_{23} \varphi \ .\end{aligned} \tag{3.9}$$

We used the standard notation. If $R = \sum r_a^1 \otimes r_a^2$ then $R_{13} = \sum r_a^1 \otimes e \otimes r_a^2$ etc. Given the expansion $\varphi = \sum \varphi_\sigma^1 \otimes \varphi_\sigma^2 \otimes \varphi_\sigma^3$ and any permutation s of 123 we set $\varphi_{s(1)s(2)s(3)} = \sum_\sigma \varphi_\sigma^{s^{-1}(1)} \otimes \varphi_\sigma^{s^{-1}(2)} \otimes \varphi_\sigma^{s^{-1}(3)}$.

The discussion of this definition parallels the one given for definition 4. Quasi-triangularity implies that the two representations $\tau^1 \boxtimes \tau^2$ and $\tau^2 \boxtimes \tau^1$ are equivalent. The intertwiner is furnished by $(\tau^1 \boxtimes \tau^2)(R)$. Equations $(\epsilon \otimes id)R = 1, (id \otimes \epsilon)R = 1$ follow with the help of the hexagon equation.

After these definitions we are prepared to explain the title of this section. A (not necessarily co-associative weak) Hopf-*–algebra with re-associator φ and R-element R is called *weak quasi quantum group* [56]. They are generalizations of Drinfeld's quasi quantum groups [19] in which truncation is not allowed, i.e. $\Delta(e) = e \otimes e$. For a quantum group, the re-associator φ is trivial, i.e. $\varphi = e \otimes e \otimes e$. In this framework group algebras appear as special examples of quantum groups when $R = e \otimes e$.

Let us discuss some properties of bi-algebras with re-associator φ and R-element R. The relations stated in definition 4,5 imply validity of quasi Yang Baxter equations,

$$R_{12}\varphi_{312}R_{13}\varphi_{132}^{-1}R_{23}\varphi = \varphi_{321}R_{23}\varphi_{231}^{-1}R_{13}\varphi_{213}R_{12} \ , \tag{3.10}$$

and this guarantees that R together with φ determines a representation of the braid group [59]. To state this result we introduce some notations. Write

$$e^n = e \otimes \ldots \otimes e \qquad \text{(n factors)} \tag{3.11}$$

and similarly for id^n. In addition we abbreviate $\mathcal{G}^{*\otimes n} = \mathcal{G}^* \otimes \ldots \otimes \mathcal{G}^*$ (n factors), and

$$\Delta^n = (id^{n-1} \otimes \Delta) \cdots (id \otimes \Delta)\Delta \quad \text{for} \quad n \geq 2 \ , \tag{3.12}$$

$$\Delta^1 = \Delta \ , \ \Delta^0 = id \ , \ \Delta^{-1} = \epsilon \ . \tag{3.13}$$

Furthermore we introduce the following permutation maps $\mathbf{P}_k^n : \mathcal{G}^{*\otimes n} \mapsto \mathcal{G}^{*\otimes n}$ defined by

$$\mathbf{P}_k^n(\xi_n \otimes \ldots \xi_k \otimes \xi_{k-1} \ldots \otimes \xi_1) = (\xi_n \otimes \ldots \xi_{k-1} \otimes \xi_k \ldots \otimes \xi_1) \ . \tag{3.14}$$

Theorem 6 (Artin relations) [58, 59] *Let $R^+ = R$ and $R^- = R'^{-1}$ where ' interchanges factors in $\mathcal{G}^* \otimes \mathcal{G}^*$. For $k = 1, \ldots, n-1$ define maps $\sigma_k^{n\pm} : \mathcal{G}^{*\otimes n} \mapsto \mathcal{G}^{*\otimes n}$ by*

$$\sigma_k^{n\pm} = \Delta^{n-1}(e)\mathbf{P}_k^n(id^{n-k+1} \otimes \Delta^{k-2})(e^{n-k-1} \otimes \varphi_{213}(R^\pm \otimes e)\varphi^{-1}) \ . \tag{3.15}$$

Then $\sigma_k^{n\pm}$ obey Artin relations (2.9) and σ_k^{n-} is the quasi-inverse of σ_k^{n+}, i.e.

$$\sigma_k^{n+}\sigma_k^{n-} = \Delta^{n-1}(e) \ .$$

The proof can be found in [59]. For the special case in which $\varphi = e \otimes e \otimes e$ and $\Delta(e) = e \otimes e$ the explicit relation to representations of the braid group prompted the discovery of quantum groups and can be used to construct them from solutions of the Yang Baxter equations (see e.g. [61]).

Let τ^I denote a complete set of representatives from the equivalence classes of irreducible representations of $\mathcal{G}^*$. τ^I are assumed to be finite dimensional and unitary representations on the Hilbert spaces V^I, i.e. $\tau^I(\xi^*) = (\tau^I(\xi))^*$. For semisimple $\mathcal{G}^*$, representations $\tau^I \boxtimes \tau^J$ can be decomposed into a direct sum of representations τ^K and this decomposition determines a Hilbert space $C(IJ,K)$ of Clebsch Gordon intertwiners $C(IJ|K) : V^I \otimes V^J \to V^K$ as in section 2.3.

When a re-associator φ and an element R exist, the tensor product of representations is associative and commutative up to equivalence so that the dimensions $\nu_K^{IJ} = dim C(IJ|K)$ furnish a solution of the eqs. (2.12). To describe the action of φ and R on the Clebsch Gordon intertwiners, we fix a set of complex phases $\omega^{IJ} = \overline{\omega^{JI}}$ such that

$$\omega^{IJ}\omega^{IK} = \omega^{IL} \quad , \quad \omega^{JI}\omega^{KI} = \omega^{LI} \tag{3.16}$$

whenever τ^L is a subrepresentation of $\tau^J \boxtimes \tau^K$. Note that one can always choose $\omega^{IJ} = 1$ for all I, J. Due to the intertwining properties (3.4,3.8) of φ, R and Schur's lemma, φ, R determine a set of complex matrices Φ, ω defined by

$$C(IP|L)C(JK|P)_{23}\varphi^{IJK} = \sum_Q \Phi_{PQ}[^I_L {}^J_K]^* C(QK|L)C(IJ|Q)_{12} \ , \tag{3.17}$$

$$C(IJ|K)\hat{R}^{+IJ} = \omega^{IJ}\omega(_K{}^J{}_I)C(JI|K) \ . \tag{3.18}$$

Here $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$, $\hat{R}^{+IJ} = P(\tau^I \otimes \tau^J)(R)$ with $P : V^I \otimes V^J \mapsto V^J \otimes V^I$ the permutation map. The properties of R, φ give rise to relations among the matrices Φ, ω. A short calculation shows that they satisfy the polynomial equations (in proposition 2) when Φ substitutes for F and the matrix ω appears in place of Ω.

Conversely we can start from a bi-*-algebra $\mathcal{G}^*$ with an associative and commutative set of dimensions $\nu_K^{IJ} \equiv dim C(IJ|K)$. Note that the dimensions of the matrices F, Ω are the only parameters in the polynomial equations. In proposition 2 these dimensions were given by the fusion rules N_K^{IJ}. However, it is realized immediately that every solution of (2.12) – in particular ν_K^{IJ} – determines a set of polynomial equations. Our aim is to construct elements φ and R from a known solution of these polynomial equations. We will succeed for semisimple algebras $\mathcal{G}^*$.

Theorem 7 (reconstruction theorem) *Let $\mathcal{G}^*$ be semisimple a semisimple bi-*-algebra. Suppose that the multiplicities ν_K^{IJ} which appear in the Clebsch Gordon decomposition*

$$\tau^I \boxtimes \tau^J \cong \bigoplus \nu_K^{IJ} \tau^K$$

are commutative and associative in the sense of eq. (2.12) and that a solution of eq. (3.16) has been fixed. Then every solution of the polynomial equations (proposition 2) associated with ν_K^{IJ} *determines a pair of elements* φ, R *with properties as in definition* 4, 5. *Their action on Clebsch Gordon maps is given by eq. (3.17,3.18).*

PROOF: The proof of this theorem consists of two parts. First one has to show that for a given set of $\varphi^{IJK}, \hat{R}^{+IJ}$ defined by (3.17,3.18) one can always find elements $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ and $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)\varphi$, $\hat{R}^{+IJ} = P(\tau^I \otimes \tau^J)(R)$. We do this for R. Let M_K be the full matrix algebra that consists of $(dim(\tau^K) \times dim(\tau^K))$-matrices. Since $\mathcal{G}^*$ is semisimple, $\tau^K(\mathcal{G}^*) = M_K$ for all irreducible representations τ^K. By definition $P\hat{R}^{+IJ} \in M_J \otimes M_I$ so that $P\hat{R}^{+IJ}$ is a sum of tensor products of matrices, $P\hat{R}^{+IJ} = \sum_\sigma m_\sigma^1 \otimes m_\sigma^2$, $m_\sigma^1 \in M_J, m_\sigma^2 \in M_I$. We can find elements $s_\sigma^i \in \mathcal{G}^*, i = 1,2$, such that $\tau^J(s_\sigma^1) = m_\sigma^1$ and $\tau^I(s_\sigma^2) = m_\sigma^2$. Take $S^{JI} \in \mathcal{G}^*$ to be the element $P^J \otimes P^I \sum s_\sigma^1 \otimes s_\sigma^2$ (P^I is the minimal central projection corresponding to the irreducible representation τ^I) and repeat this construction for every pair (I,J) of representations. Finally, $R = \sum_{IJ} S^{IJ}$ satisfies $\hat{R}^{+IJ} = P(\tau^I \otimes \tau^J)(R)$. The second part of the proof is to show that the elements φ, R satisfy all the relations in Definition (4,5). This is an immediate consequence of the polynomial equations (proposition 2). The quasi-inverses act according to

$$C(QK|L)C(IJ|Q)_{12}(\varphi^{-1})^{IJK} = \sum_P \Phi_{PQ}\begin{bmatrix} I & J \\ L & K \end{bmatrix} C(IP|L)C(JK|P)_{23} \ , \tag{3.19}$$

$$C(JI|K)\hat{R}^{-IJ} = \omega^{JI}\omega\begin{pmatrix} J \\ K \ I \end{pmatrix}^* C(IJ|K) \ , \tag{3.20}$$

with $(\varphi^{-1})^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi^{-1})$, $\hat{R}^{-IJ} = (\tau^J \otimes \tau^I)(R^{-1})P$ and $P : V^I \otimes V^J \mapsto V^J \otimes V^I$ the permutation map.

The idea to reconstruct elements φ and R from a solution of the polynomial equations appeared first in [57]. There is was done in a concrete example. In the language of categories, a similar observation was formulated by Majid [62] and extended to cases with truncation by Kerler in [50].

3.2 Examples of weak quasi quantum groups

The most prominent quantum group algebras are the one parameter deformations $U_q(sl_2)$ of the universal enveloping of $su(2)$. The quantum group algebras $U_q(sl_2), q \in \mathbf{C}$, is generated by elements $q^{\pm H/2}$, 1 and $S_\pm$ subject to the relations

$$q^{H/2}q^{-H/2} = q^{-H/2}q^{H/2} = 1 \ , \tag{3.21}$$

$$q^{H/2}S_\pm = q^{\pm\frac{1}{2}}S_\pm q^{H/2} \ , \tag{3.22}$$

$$[S_+, S_-] = \frac{q^H - q^{-H}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \ . \tag{3.23}$$

Here and in the following we write 1 for the unit element in $U_q(sl_2)$.

We denote the co-multiplication in $U_q(sl_2)$ by Δ_q in order to distinguish it from the co-multiplication Δ for the truncated quantum group algebra which we are going to introduce. It is given by

$$\Delta_q(q^{\pm H/2}) = q^{\pm H/2} \otimes q^{\pm H/2} \ ,$$

$$\Delta_q(S_\pm) = S_\pm \otimes q^{H/2} + q^{-H/2} \otimes S_\pm \ .$$

To define a *-operation we distinguish two different ranges of the parameter q.

$$q \text{ real}: \quad S_\pm^* = S_\mp \ , \quad (q^{\pm H/2})^* = q^{\pm H/2} \ , \tag{3.24}$$

$$|q| = 1: \quad S_\pm^* = S_\mp \ , \quad (q^{\pm H/2})^* = q^{\mp H/2} \ . \tag{3.25}$$

In both cases the *-operation satisfies $\Delta_q(\xi^*) = \Delta_q(\xi)^*$ provided we adopt the appropriate convention for the adjoint on $\mathcal{G}^* \otimes \mathcal{G}^*$, viz. (1.4) if q is real, and (1.5) if $|q| = 1$. The co-unit and the antipode act as

$$\epsilon(q^{\pm H/2}) = 1 \quad , \quad \epsilon(S_\pm) = 0 \ ,$$

$$S(q^{\pm H/2}) = q^{\mp H/2} \quad , \quad S(S_\pm) = -q^{\pm 1/2}S_\pm \ .$$

One verifies that the tupel $(U_q(sl_2), \Delta_q, \epsilon, *, S)$ defines a Hopf-*-algebra. The co-product Δ_q is co-associative but not co-commutative. However there exists a canonical R-element for $U_q(sl_2)$ which we denote by R_q. It is given by

$$\mathcal{R}_q = q^{H \otimes H} \sum_{n \geq 0} \frac{(1 - q^{-1})^n}{[n]!} q^{-\frac{1}{4}n(n-1)} q^{nH/2} S_+^n \otimes q^{-nH/2} S_-^n \ . \tag{3.26}$$

To prove that this R-element satisfies all the properties stated in definition 5 with $\varphi = e \otimes e \otimes e$ is non-trivial. The best (but not straight forward) way is to use the quantum double construction [18]. If $|q| = 1$ and if we adopt the choice (1.5) of the *-operation, viz. $(a \otimes b)^* = b^* \otimes a^*$ we get that R_q is *unitary,*

$$R_q^* = R_q^{-1} = (S \otimes id)(R_q) \ .$$

The "quantum dimension" d_τ of a representation τ is defined by

$$d_\tau = tr\tau(q^H) \ . \tag{3.27}$$

If q is a (primitive p-th) root of unity, then $U_q(sl_2)$ is not semisimple, and tensor products of its irreducible representations are in general not fully reducible. Its irreducible representations τ^J with non-zero quantum dimension are called "physical" representations. They are labelled by $J = 0, \frac{1}{2}, \ldots \frac{1}{2}(p-2)$ and have dimension $2J + 1$.

We denote the tensor product of $U_q(sl_2)$-representations by $\boxtimes_q$. The tensor product of $\tau^I \boxtimes_q \tau^J$ of two physical representations decomposes in general into physical representations, plus unphysical representations with quantum dimension 0. If we multiply $(\tau^I \boxtimes_q \tau^J)(\xi)$ with a projection operator P_{IJ} which cuts away the contribution with zero quantum dimension, one obtains what is known as the truncated tensor product of physical representations of $U_q(sl_2)$.

A weak quasi quantum group algebra $U_q^T(sl_2)$ is canonically associated with $U_q(sl_2)$ for q a root of unity. As an algebra $U_q^T(sl_2) = U_q(sl_2)/\mathcal{J}$, where $\mathcal{J}$ is the ideal which is annihilated by all the physical representations $\tau^I, 2I = 0 \ldots p-2$, of $U_q(sl_2)$. $U_q^T(sl_2)$ is semisimple, its representations are fully reducible, and the irreducible ones are precisely the physical representations of $U_q(sl_2)$. Let

$$u(I,J) = \min\{|I + J|, p - 2 - I - J\} \tag{3.28}$$

and let P_{IJ} be the projector on the physical subrepresentations K, $|I - J| \leq K \leq u(I,J)$ of the tensor product $\tau^I \boxtimes_q \tau^J$ of $U_q(sl_2)$ representations. There exists $P \in U_q^T(sl_2) \otimes U_q^T(sl_2)$ such that $P_{IJ} = (\tau^I \otimes \tau^J)(P)$. The co-product in $U_q^T(sl_2)$ is determined in terms of the co-product Δ_q in $U_q(sl_2)$ as

$$\Delta(\xi) = P\Delta_q(\xi) \ , \tag{3.29}$$

hence $\Delta(e) = P \neq e \otimes e$. This co-product specifies a tensor product $\boxtimes$ which is equal to the truncated tensor product of physical $U_q(sl_2)$ representations. Thus

$$\tau^I \boxtimes \tau^J = \bigoplus_{|I-J| \leq K \leq u(IJ)} \tau^K . \tag{3.30}$$

There exists an element φ such that $\varphi^{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\varphi)$ implements the well known unitary equivalence of the truncated tensor products $\tau^I \boxtimes (\tau^J \boxtimes \tau^K)$ and $(\tau^I \boxtimes \tau^J) \boxtimes \tau^K$. The map φ^{IJK} can be specified by its action on Clebsch Gordon intertwiners

$$C(IP|L)C(JK|P)_{23}\varphi^{IJK} = \sum_Q \{{}^{K\ J\ P}_{I\ L\ Q}\}_q C(IJ|Q)C(QK|L)_{12} , \tag{3.31}$$

where $\{\because\}_q$ denote the 6J–symbols of the quantum group algebra $U_q(sl_2)$ and $C(..|.)$ are the Clebsch Gordon maps for $U_q(sl_2)$ and $U_q^T(sl_2)$ at the same time.

The R–element of $U_q^T(sl_2)$ is given in terms of the R–element R_q for $U_q(sl_2)$ by

$$R = R_q\Delta(e) = \Delta'(e)R_q , \tag{3.32}$$

while antipode and co-unit are the same as in $U_q(sl_2)$. One can show that the defining properties of a weak quasi quantum group algebra are satisfied. Details are spelled out in [56]. Since $\Delta(e)^* = \Delta(e^*) = \Delta(e)$, the unitarity of R_q implies that

$$R^* = R^{-1} . \tag{3.33}$$

Let us give at least one example of a weak quasi quantum group which is not obtained from a quantum group algebra by a process of truncation. One can construct (cf. Appendix B) a semisimple Hopf-*-algebra $(\mathcal{G}^*, \Delta, \epsilon, *, \mathcal{S})$ with irreducible unitary representations τ^J, ($J = 0, \frac{1}{2}, 1$), of dimensions $1, 2, 1$. Their Clebsch Gordon decomposition is given by the formula (3.30) as for $U_q(sl_2), q = i$. The general theory discussed in the preceding section can be applied to define a re-associator φ and a R-element R from the solution of the polynomial equations furnished by the truncated quantum group algebra $U_q^T(sl_2)$. This means that there are at least two non-isomorphic weak quasi quantum group algebras with the same selection rules and associated with the same solution of the polynomial equations.

4 Quantum Symmetry, Statistics and Locality

In quantum field theory, permutation group statistics is implemented through quadratic relations among the field operators, viz. canonical (anti–)commutation relations for Bosons (Fermions). The spin statistics theorem states that Fermions have spin $s = \frac{1}{2}, \frac{3}{2}, \ldots$, whereas Bosons have integer spin. More general values for the spin (remember that the spin labels representations of the rotation group, e.g. SO(2) in 2 space dimensions) are possible in low dimensional quantum field theory. They are associated with braid group statistics. It has been proposed to implement braid group statistics through local braid relations [31].

$$\Psi_i^I(x,t)\Psi_j^J(y,t) = \omega^{IJ}\Psi_l^J(y,t)\Psi_k^I(x,t)\mathcal{R}^{IJ>}_{kl,ij} \quad \text{for} \quad x > y . \tag{4.1}$$

Here, ω^{IJ} are complex phase factors. As in section 2, the meaning of $x > y$ depends on the dimension of the space-time on which the theory lives. In contrast with [31] we do not restrict the $\mathcal{R}$-matrix to have C–number entries, but the matrix elements may take values in $\mathcal{U}(\mathcal{G}^*)$ instead. For the order $x < y$ a similar relation follows with $\mathcal{R}^{JI<}$ and $\omega^{JI} = \overline{\omega^{IJ}}$. Denoting with $\hat{\mathcal{R}}$ the matrix obtained from $\mathcal{R}$ by interchange of the first indices k, l, the two matrices $\mathcal{R}^{IJ>}$ and $\mathcal{R}^{JI<}$ obey

$$\hat{\mathcal{R}}^{IJ<} = (\hat{\mathcal{R}}^{JI>})^{-1} .$$

Note that for $\mathcal{R}^{IJ>}_{kl,ij} = 1$ and $\omega^{IJ} = \pm$ we recover Bose/Fermi–commutation relations as a special case of eq. (4.1).

Consistency of local braid relations with the transformation law (1.10) distinguishes weak quasi quantum groups from arbitrary Hopf algebra symmetries [56]. We will review these arguments here. In the second subsection we determine a re-associator φ and a R-element R for the quantum symmetry of the field algebra $\mathcal{F}$ constructed in section 2.3. Local braid relations of the field operators (2.34) will be established with $\mathcal{R}^{IJ>}_{kl,ij}$ furnished by the elements φ, R.

4.1 Weak quasi quantum group symmetry and local braid relations

In a quantum theory with quantum symmetry the matrix-elements $\mathcal{R}^{IJ>}_{kl,ij}$ are not free to take any values. Consistency of (4.1) with the transformation law of field operators and associativity of the product of operators constrain $\mathcal{R}$. If the quantum symmetry is quasi-co-associative and quasi-triangular with re-associator φ and R-element R, a solution of these constraints is given by

$$\begin{aligned} \mathcal{R}^{IJ>}_{kl,ij} &= (\tau^I \otimes \tau^J \otimes \mathcal{U})_{kl,ij}(\varphi_{213}(R \otimes e)\varphi^{-1}) \\ &= \sum_{\tau,a,\sigma} \tau^I_{ki}(\varphi^2 r^1_a \xi^1_\tau)\tau^J_{lj}(\varphi^1 r^2_a \xi^2_\tau)U(\varphi^3_\sigma \xi^3_\tau) , \end{aligned} \tag{4.2}$$

where we use the same notations as in the preceding section and $\xi = \varphi^{-1}$. From the second line we read off that the matrix elements $\mathcal{R}^{IJ>}_{kl,ij}$ are not numbers in general but operators in the Hilbert space. The expression is a linear combination of representation operators $U(\varphi^3_\sigma \xi^3_\tau)$ and is therefore equal to a representation operator $U(\eta)$ of some element $\eta \in \mathcal{G}^*$. A numerical matrix $\mathcal{R}$ is obtained if and only if the re-associator φ is trivial, i.e. if $\varphi = e \otimes e \otimes e$.

To motivate eq. (4.2) we demonstrate that *local braid relations* (4.1) *with* $\mathcal{R}^{IJ>}$ *given by* (4.2) *are consistent with the transformation law of fields*, i.e. that both sides of the equation transform in the same way. The products of covariant fields which appear in (4.1) are in general not covariant, if the co-product Δ is not co-associative. Suppose now that there exists a re-associator φ satisfying (3.4). Then one can construct a "covariant product" $\times$ of field operators [56, 58].

$$(\Psi^I \times \Psi^J)_{ij} \equiv \sum \Psi^I_m \Psi^J_n \tau^I_{mi}(\varphi^1_\sigma)\tau^J_{nj}(\varphi^2_\sigma)\mathcal{U}(\varphi^3_\sigma) . \tag{4.3}$$

By (3.4), $\Psi^I \times \Psi^J$ *transforms covariantly according to the tensor product representation* $\tau^I \boxtimes \tau^J$. Ordinary products of field operators can be recovered from covariant ones,

$$\Psi^I_i \Psi^J_j = \sum_\sigma (\Psi^I \times \Psi^J)_{mn}\tau^I_{mi}(\xi^1_\sigma)\tau^J_{nj}(\xi^2_\sigma)U(\xi^3_\sigma) . \tag{4.4}$$

Using this covariant product, local braid relations (4.1,4.2) may be rewritten as

$$(\Psi^I \times \Psi^J)_{ij} = \omega^{IJ}(\Psi^J \times \Psi^I)_{lk}(\tau^I_{ki} \otimes \tau^J_{lj})(R) \quad . \tag{4.5}$$

The arguments of the field operators, which we neglect to write, satisfy conditions as in eq.(4.1). Because of the covariance properties of field operators, both sides transform in the same way if the element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ obeys intertwining property (3.8).

To prepare for our second consistency check one should notice that associativity of the product of field operators is equivalent to the following *quasi- associativity of the covariant product.*

$$((\Psi^I \times \Psi^J) \times \Psi^K)_{ijk} = (\Psi^I \times (\Psi^J \times \Psi^K))_{i'j'k'}(\tau^I \otimes \tau^J \otimes \tau^K)_{i'j'k',ijk}(\varphi) \quad . \tag{4.6}$$

This relation is a consequence of eq.(3.5). Consistency requires that the matrix $\mathcal{R}^{IJ>}$ satisfies constraints which come from the possibility of interchanging triples $\Psi^I_i(x,t)\Psi^J_j(y,t)\Psi^K_k(z,t)$, $x > y > z$, of fields in two different ways (see figure 2), leading to the same result.

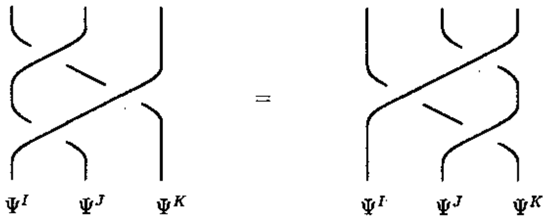

Figure 2: Different possibilities to interchange triples of field operators. The crossings in the picture represent application of eq. (4.5).

The constraints are exploited most easily if one starts from the threefold ×–product $\Psi^I \times (\Psi^J \times \Psi^K)$. By an alternating application of eq. (4.6) and eq. (4.5) we end up with $(\Psi^K \times \Psi^J) \times \Psi^I$. There are two ways to perform these manipulations which give the same result, provided that R, φ satisfy the quasi Yang Baxter equations (3.10).

In our present context eqs. (3.9) imply validity of local braid relations for composite operators, i.e. when the product $\Psi^{I_1} \times \Psi^{I_2}$ is inserted in place of Ψ^I. More precisely suppose that local braid relations (4.5) hold for the products $\Psi^{I_i}(x_i,t) \times \Psi^J(y,t), i = 1,2$ and $x_i > y$. Then by (3.9) we have

$$((\Psi^{I_1} \times \Psi^{I_2}) \times \Psi^J)_{i_1 i_2 j} = \omega^{I_1 J}\omega^{I_2 J}(\Psi^J \times (\Psi^{I_1} \times \Psi^{I_2}))_{lk_1k_2}((\tau^{I_1} \boxtimes \tau^{I_2})_{k_1k_2i_1i_2} \otimes \tau^J_{lj})(R) \quad . \tag{4.7}$$

Similar considerations for the field operator Ψ^J lead to

$$(\Psi^I \times (\Psi^{J_1} \times \Psi^{J_2}))_{ij_1j_2} = \omega^{IJ_1}\omega^{IJ_2}((\Psi^{J_1} \times \Psi^{J_2}) \times \Psi^I)_{l_1l_2k}(\tau^I_{ki} \otimes (\tau^{J_1} \boxtimes \tau^{J_2})_{l_1l_2j_1j_2})(R) \quad . \tag{4.8}$$

For the proof one uses the quasi–associativity (4.6) of the × product and local braid relations (4.5) as in the preceding paragraph.

We mentioned that local braid relations are expected to give rise to representations of the braid group B_n on the space of n–particle excitations. When $\mathcal{R}^{IJ>}$ is given by eq. (4.2), this is indeed the case. Consider states which are created from the ground state $|0\rangle$ by application of a n-fold product of field operators $\Psi^{I_k}_{i_k}(x_k,t)$. Suppose that $x_1 < x_2 < ... < x_n$, for instance.

Operators $\varsigma_k, (k = 1 \dots n-1)$ should act on such states by interchange of field operators $\Psi^{I_k}_{i_k}$ and $\Psi^{I_{k+1}}_{i_{k+1}}$.

$$\varsigma^n_k \Psi^{I_1}_{i_1} \dots \Psi^{I_k}_{i_k}\Psi^{I_{k+1}}_{i_{k+1}} \dots \Psi^{I_n}_{i_n}|0\rangle = \Psi^{I_1}_{i_1} \dots \Psi^{I_{k+1}}_{i_{k+1}}\Psi^{I_k}_{i_k} \dots \Psi^{I_n}_{i_n}|0\rangle \quad . \tag{4.9}$$

Using (4.1) and (4.2) on verifies by a short calculation that the action of ς^n_k is obtained from the maps $\sigma^n_k : \mathcal{G}^{*\otimes n} \mapsto \mathcal{G}^{*\otimes n}$ introduced in theorem 6. Since the latter obey Artin relations (2.9) and the additional factors ω^{IJ} cancel in the calculations, eq. (4.9) defines a representation of the braid group as it was announced.

4.2 Quantum symmetry and local braid relations in the field algebra $\mathcal{F}$

When the field algebra $\mathcal{F}$ with quantum symmetry $\mathcal{G}^*$ was discussed in section 2.3, we did not consider locality properties of field operators. We will demonstrate now that the field operators $\Psi^J_m(\rho_J)$ constructed in section 2.3 satisfy local braid relations. We need no extra conditions on the the quantum symmetry $\mathcal{G}^*$ than those stated there, namely that the fusion rules of the quantum field theory are in agreement with the selection rules of the bi-*-algebra $\mathcal{G}^*$. A re-associator φ and a R-element R are constructed from the fusion- and braiding matrix of the quantum field theory. In agreement with the general discussion, they determine the matrix $\mathcal{R}^{IJ}_{kl,ij}$ according to formula (4.2).

Our first aim is to construct appropriate elements φ, R. We use the same notations as above. In section 2.3 we supposed that the dimensions $dim\mathcal{C}(IJ|K)$ coincide with the fusion rules N^{IJ}_K defined by the superselection structure. The latter are commutative and associative in the sense of eq. (2.12) so that we are now in the position to apply theorem 7. It asserts that every solution of the polynomial equations (proposition 2) associated with the fusion rules N^{IJ}_K yields elements φ, R which satisfy the relations in definition 4, 5. In the present situation the desired solution is furnished by the fusion- and braiding- matrices F, Ω of the quantum field theory (proposition 2). Formulae (3.17) for φ and (3.18) for R take the following explicit form.

$$C(IP|L)C(JK|P)_{23}\varphi^{IJK} = \sum_Q F_{PQ}[^I_L {}^J_K]^* C(QK|L)C(IJ|Q)_{12} \quad , \tag{4.10}$$

$$C(IJ|K)\hat{R}^{+IJ} = \omega^{IJ}\Omega(_K{}^J{}_I)C(JI|K) \quad , \tag{4.11}$$

where $\omega^{IJ} = \overline{\omega^{JI}}$ denotes an arbitrary solution of (3.16) by complex phase factors. The bi-*-algebra $(\mathcal{G}^*, \Delta, \epsilon, *)$ with re-associator φ and R-element R is dual to the quantum field theory in the sense of Fröhlich [32].

Let us now turn to the discussion of local braid relations among the field operators $\Psi^J_m(\rho_J)$. The statistics operator $\epsilon(\rho_J, \rho_K)$ was introduced in (2.8) as an element of the observables algebra $\mathcal{A}$. Thus, the action $\pi(\epsilon(\rho_J, \rho_K))$ on the Hilbert space $\mathcal{H}$ is well defined. Local braid relations will follow from

$$\Psi^J_j(\rho_J)\Psi^K_k(\rho_K)\pi(\epsilon(\rho_J, \rho_K)) = \omega^{JK}\Psi^K_m(\rho_K)\Psi^J_n(\rho_J)\mathcal{R}^{JK>}_{nm,jk} \quad . \tag{4.12}$$

with $\mathcal{R}$ given by (4.2). To prove this identity we insert definition (2.34) for the field operators $\Psi^I_m(\rho_I)$ into the left hand side and use the intertwining relation (2.27) and the definition (2.23) of the braiding-matrix B.

$$\Psi^J_j(\rho_J)\Psi^K_k(\rho_K)\pi(\epsilon(\rho_J, \rho_K)) = \sum i^*_{Ll}T_c(_L{}^K{}_N)T_d(_N{}^J{}_I)i_{Ii}B_{MN}[^J_L {}^K_I]^{cd}_{ab}[^J_j {}^M_m {}^L_l]^a[^K_k {}^I_i {}^M_m]^b \quad .$$

With the help of (2.23) the braiding matrix B can be expressed in terms of the matrices F, Ω so that it is possible to apply the definition (4.10,4.11) of φ and $\mathcal{R}$. The result is

$$\Psi_j^J(\rho_J)\Psi_k^K(\rho_K)\pi(\epsilon(\rho_J,\rho_K)) = \omega^{JK} \sum i^*_{Ll} T_c({}_L{}^K{}_N) T_d({}_N{}^J{}_I) i_{Ii'} \begin{bmatrix} K & N & L \\ m & \nu & l \end{bmatrix}^c \begin{bmatrix} J & I & N \\ n & i & \nu \end{bmatrix}^d \tau^I_{i'i}(\mathcal{R}^{JK>}_{nm,jk}) \ .$$

This equals the right hand side of the equation (4.12). Since the statistics operators were normalized to 1 for $\rho_J > \rho_K$ (cf. (2.8)), local braid relations follow from eq. (4.12). We collect these results in a theorem.

Theorem 8 (Local braid relations in $\mathcal{F}$) *Suppose that the fusion rules N_K^{IJ} defined by the superselection structure of $\mathcal{A}$ coincide with the multiplicities in the Clebsch Gordon decomposition of an (otherwise arbitrary) semisimple bi-*-algebra $\mathcal{G}^*$ (in the sense of (2.24)). Then the complete field system $(\mathcal{H}, \pi, \mathcal{G}^*, \mathcal{F})$ with quantum symmetry $\mathcal{G}^*$ constructed in section 2.3 has the following properties.*

1. *The field operators (2.34) which generate the local algebras $\mathcal{F}(\mathcal{O})$ satisfy local braid relations*
$$\Psi_i^I(\rho_I)\Psi_j^J(\rho_J) = \omega^{IJ}\Psi_l^J(\rho_J)\Psi_k^I(\rho_I)\mathcal{R}^{IJ>}_{kl,ij} \quad \text{for} \quad \rho_I > \rho_J \ . \tag{4.13}$$
2. *The operators $\mathcal{R}^{IJ>}_{lk,ij}$ are determined by elements $\varphi \in \mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$ and $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ which satisfy all axioms in definition 4, 5*
$$\mathcal{R}^{IJ>}_{lk,ij} = (\tau^I \otimes \tau^J \otimes \mathcal{U})_{lk,ij}(\varphi_{213} R_{12} \varphi^{-1}) \ . \tag{4.14}$$

If $\mathcal{G}^*$ is a Hopf--algebra, the quantum symmetry of the field system is a weak quasi quantum group with re-associator φ and R-element R.*

Even though this theorem was formulated for quantum field theories on the Minkowski space and for sectors which satisfy criterion (2.5), analogous results hold for stringlike localized sectors or quantum fields in chiral conformal quantum field theory.

In the case of quantum field theories with permutation group statistics, Doplicher and Roberts have established the existence of a group algebra $\mathcal{G}^*$ which satisfies the assumption of theorem 8. Moreover, they found out that the elements R,φ determined by (4.10,4.11) become trivial, when the phases ω^{IJ} are fixed to be -1 if I,J label sectors with para-Fermi statistics and $+1$ otherwise. Triviality of R,φ means $R = e \otimes e$ and $\varphi = e \otimes e \otimes e$ so that we recover ordinary Bose-/Fermi (anti-) commutation relations from equation (4.13).

5 Discussion and Outlook

For the construction of the field algebra $\mathcal{F}$ with weak quasi quantum group symmetry, every Hopf-algebra $\mathcal{G}^*$ can be admitted, provided the multiplicities in the Clebsch Gordon decomposition coincide with the fusion rules of quantum field theory. This selection criterion for $\mathcal{G}^*$ as well as the explicit expressions for the re-associator and the R-matrix were completely determined by the observables $\mathcal{A}$ of the model. Of course, one should try to construct appropriate Hopf-*-algebras $\mathcal{G}^*$ systematically. It is indicated in Appendix B how this can be performed by studying a relatively simple algebraic problem.

In [54] we used algebraic methods to construct a field algebra with quantum symmetry for the critical chiral Ising model. Instead of C^*-algebras we used the Virasoro algebra $\mathrm{Vir}_{c=1/2}$ as

a chiral Lie algebra Lie$\mathcal{A}$ of observables. This Lie algebra admits three inequivalent unitary irreducible positive energy representations π_J in the Hilbert spaces $\mathcal{H}^J, J = 0, \frac{1}{2}, 1$. To construct suitable endomorphisms, we had to enlarge $\mathrm{Vir}_{c=1/2}$ to a Lie algebra Lie$\tilde{\mathcal{A}}$. All unitary irreducible positive energy representations π_J of the Virasoro algebra $\mathrm{Vir}_{c=1/2}$ on $\mathcal{H}^J$ extend to representations of Lie$\tilde{\mathcal{A}}$ in the same Hilbert space $\mathcal{H}^J$. A complete set of endomorphisms ρ_J of Lie$\tilde{\mathcal{A}}$ to reach all the sectors was found explicitly. The fusion rules were calculated from the explicit expressions.

$$\begin{aligned} [\rho_{1/2} \circ \rho_{1/2}] &= [\rho_0] + [\rho_1] \ , \\ [\rho_{1/2} \circ \rho_1] &= [\rho_1 \circ \rho_{1/2}] = [\rho_{1/2}] \ , \\ [\rho_1 \circ \rho_1] &= [\rho_0] \end{aligned} \tag{5.1}$$

and $[\rho_0 \circ \rho_J] = [\rho_J \circ \rho_0] = [\rho_J]$ for all $J = 1, \frac{1}{2}, 1$. The matrices N_K^{IJ} can be read off.

In case of the Ising model we already know two admissible symmetry algebras $\mathcal{G}^*$. One comes from the truncated quantum group algebra $U_q^T(sl_2)$ associated with $U_q(sl_2), q = i$. The three inequivalent unitary irreducible representations $\tau^J, J = 0, \frac{1}{2}, 1$, have dimensions $\delta_0 = 1, \delta_{1/2} = 2, \delta_1 = 3$. Comparison of (5.1) with the formula (3.30) for the decomposition of tensor products $\tau^I \boxtimes \tau^J$ reveals that the algebra $\mathcal{G}^*_{(1)} = U_q^T(sl_2)$ can be admitted as symmetry algebra of the Ising model. The corresponding field algebra $\mathcal{F}^{(1)}$ was constructed explicitly. Local braid relations of covariant field operators were established with elements R and φ as in section 3.2 [57]. A second possible Hopf-*-algebra $\mathcal{G}^*_{(2)}$ is described in Appendix B. Since its (unitary) irreducible representations have dimensions $\delta_0 = 1, \delta_{1/2} = 2, \delta_1 = 1$, it gives rise to an inequivalent field algebra $\mathcal{F}^{(2)}$. The non-uniqueness of the quantum symmetry is in sharp contrast to the uniqueness of the symmetry group found by Doplicher and Roberts. This freedom deserves further investigation.

Even though it may be possible to construct further quantum symmetries of the chiral critical Ising model, they will all be truncated (in particular they cannot be group symmetries). Without truncation the dimensions δ_J of irreducible representations τ^J have to satisfy

$$\delta_I \delta_J = \sum_K N_K^{IJ} \delta_K \ . \tag{5.2}$$

When N_K^{IJ} are the fusion rules of a quantum field theory with permutation group statistics, an integer solution of this equation is furnished by the "statistics dimensions" d_J of the superselection sectors [16]. The set of statistics dimensions d_J gives at the same time the dimensions of representations of the symmetry groups constructed by Doplicher and Roberts. On the simple level of eq. (5.2) differences with the situation in low dimensional quantum field theory show up already. As one can easily check for the fusion rules of the chiral critical Ising model, positive integer solutions of (5.2) do not exist. Whenever this happens it excludes the possibility of non-truncated quantum symmetries. If one admits for truncation, the condition (5.2) on the dimensions δ_J of irreducible representations becomes an inequality, which has an infinite number of solutions so that there is a priori an enormous freedom in the construction of quantum symmetries.

It should be mentioned at the end that soliton sectors in massive two-dimensional quantum field theory do not fit into the present theory of superselection structure. The usual analysis applies only to a special class of models which was discussed by Fröhlich [30]. Inspired by the properties of classical multi-soliton solutions, Fredenhagen proposed that in generic situations, soliton sectors can only be composed if they "fit together" [27, 28]. The structure of possible "quantum symmetries" is not known for these more general cases, but will probably be quite different from the quantum symmetries treated here.

6 Appendix A: The Antipode

The aim of this appendix is to explain the notion of antipode in the general context of a (not necessarily co-associative) weak bi-*-algebra. The antipode $\mathcal{S}$ is a linear *-anti-automorphism of $\mathcal{G}^*$. As for the co-unit there is a compatibility condition between antipode $\mathcal{S}$ and the co-product. To state it let us introduce the following notations.

$$\Delta^{(2)} \equiv (id \otimes \Delta)\Delta, \quad \mathcal{S}^{(2)} \equiv \mathcal{S} \otimes id \ .$$

The multiplication map $m : \mathcal{G}^* \otimes \mathcal{G}^* \to \mathcal{G}^*$ is defined as usual and

$$m^{(2)} \equiv m \otimes id \ .$$

Definition 9 (antipode) *Let $(\mathcal{G}^*, \Delta, \epsilon, *)$ be a (not necessarily co-associative) weak bi-*-algebra. A *-anti-automorphism $\mathcal{S} : \mathcal{G}^* \to \mathcal{G}^*$ is called antipode if*

$$m^{(2)}(\mathcal{S}^{(2)} \otimes id)\Delta^{(2)}(\xi) = (e \otimes \xi)\Delta(e) \ , \ m^{(2)}(id \otimes \mathcal{S}^{(2)})\Delta^{(2)}(\xi) = \mathcal{S}^{(2)}(\Delta(e))(e \otimes \xi) \ . \quad (6.1)$$

Both sides of these equations are elements of $\mathcal{G}^* \otimes \mathcal{G}^* \otimes \mathcal{G}^*$. If we apply the co-unit ϵ to the third factor in equations (6.1) and use equation (1.6) we deduce

$$m(\mathcal{S} \otimes id)\Delta(\xi) = \epsilon(\xi) \ , \ m(id \otimes \mathcal{S})\Delta(\xi) = \epsilon(\xi) \ . \quad (6.2)$$

Let us explore what this means in terms of representation theory. The *-anti-automorphism $\mathcal{S}$ enables us to define the notion of contragredient representation. If τ is a representation of $\mathcal{G}^*$ then the contragredient representation $\tilde{\tau}$ is given by $\tilde{\tau} \equiv {}^t\tau \circ \mathcal{S}^{-1}$. Here ^{t} denotes the transpose and appearance of $\mathcal{S}^{-1}$ instead of $\mathcal{S}$ is a matter of convention. The relations (6.2) which followed from the compatibility equations (6.1) imply that the trivial representations appears as a subrepresentation of the tensor products $\tau \boxtimes \tilde{\tau}, \tilde{\tau} \boxtimes \tau$ as it should be for a "good" contragredient representation.

It remains to discuss the use of the antipode for quantum theory. To come to a transformation law of an adjoint field multiplet Ψ_i^{I*} we demand that the transformation law (1.10) is consistent with

$$(\tilde{\tau}^I \otimes \mathcal{U})_{ij}\Delta(\xi)\Psi_j^I = \Psi_i^I \mathcal{U}(\xi) \ . \quad (6.3)$$

Acting with $*$ on this relation gives the transformation law for adjoint fields Ψ_i^{I*}

$$\begin{aligned} \text{(I)} \quad \mathcal{U}(\xi)\Psi_i^{I*} &= \Psi_j^{I*}\tilde{\tau}_{ji}^I(\xi_\sigma^1)\mathcal{U}(\xi_\sigma^2) \ , \quad &(6.4)\\ \text{(II)} \quad \mathcal{U}(\xi)\Psi_i^{I*} &= \Psi_j^{I*}\tilde{\tau}_{ji}^I(\xi_\sigma^2)\mathcal{U}(\xi_\sigma^1) \ . \quad &(6.5)\end{aligned}$$

The numbers (I), (II) again refer to the different choices of $*$ operation on $\mathcal{G}^* \otimes \mathcal{G}^*$. We see that in case (I), adjoint fields transform covariantly according to the contragredient representation while this is not true for (II). The components ξ_σ^1 and ξ_σ^2 are interchanged in (II) so that this transformation law involves a second co-product $\Delta'(\xi) = \sum \xi_\sigma^2 \otimes \xi_\sigma^1$ which differs from

$\Delta(\xi) = \sum \xi_\sigma^1 \otimes \xi_\sigma^2$ in general. Nevertheless, a covariant adjoint $\bar{\Psi}_i^I$ can be defined also in case (II), if there is an element $R \in \mathcal{G}^* \otimes \mathcal{G}^*$ such that the following intertwining relation holds.

$$R\Delta(\xi) = \Delta'(\xi)R \ .$$

With the help of R, a covariant operator $\bar{\Psi}_m^J$ is obtained by

$$\bar{\Psi}_i^I = \Psi_j^{I*}(\tilde{\tau}^I \boxtimes \mathcal{U})^{ji}(R) \ . \quad (6.6)$$

Consistency of (1.10) with (6.3) requires the relations (6.1). If we apply (1.10) to the left hand side (6.3) to move the representation operators $\mathcal{U}(\eta)$ from left to right we obtain

$$\begin{aligned} &= \Psi_k^I \tau_{kj}^I(\xi_{\sigma\tau}^{21})\tilde{\tau}_{ij}^I(\xi_\sigma^1)\mathcal{U}(\xi_{\sigma\tau}^{22})\\ &= \Psi_k^I \tau_{ki}^I(\xi_{\sigma\tau}^{21}\mathcal{S}^{-1}(\xi_\sigma^1))\mathcal{U}(\xi_{\sigma\tau}^{22}) \ . \quad (6.7)\end{aligned}$$

Before we compare this with the right hand side of (6.3) we note that the transformation law (1.10) implies $\Psi_i^I = \mathcal{U}(e)\Psi_i^I = \Psi_k^I(\tau_{ki}^I \otimes \mathcal{U})(\Delta(e))$. Consequently, we find

$$\Psi_i^I \mathcal{U}(\xi) = \Psi_k^I(\tau_{ki}^I \otimes \mathcal{U})(\Delta(e))\mathcal{U}(\xi) \ . \quad (6.8)$$

Together with (6.7) we arrive at the equality

$$\xi_{\sigma\tau}^{21}\mathcal{S}^{-1}(\xi_\sigma^1) \otimes \xi_{\sigma\tau}^{22} = \Delta(e)(e \otimes \xi)$$

which is equivalent to the second relation in (6.1). The first relation can be motivated in the same way.

7 Appendix B: Construction of Bi- and Hopf-*-algebras

This appendix contains some material on the construction of semisimple bi- and Hopf-*-algebras $\mathcal{G}^*$ which solve the assumption (2.24) in section 2.3. We start from the "fusion rules" N_K^{IJ}. They associative and commutative in the sense of eq. (2.12) and there is a distinguished label 0 such that $N_J^{I0} = \delta_{I,J}$. With respect to 0, every label J has a conjugate $\bar{J}$ which is uniquely determined by the property $N_0^{IJ} = \delta_{I\bar{J}}$.

In order to start our construction we choose a set of finite dimensions $\delta_J \geq 1$ such that $\delta_0 = 1$ and $\delta_J = \delta_{\bar{J}}$. V^J should denote a δ_J-dimensional Hilbert space and e_J the unit operator on V^J. V^0 is spanned by one normalized vector $|e_0\rangle$ with dual $\langle e_0|$.

For certain choices of the dimensions δ_J we can hope to find a family of linear maps $C^a(IJ|K) : V^I \otimes V^J \mapsto V^K (a = 1 \dots N_K^{IJ})$ which satisfies the following equations.

$$\begin{aligned} C^a(IJ|K)C^b(IJ|L)^* &= \delta_{K,L}\delta_{a,b}e_K \ , \quad &(7.1)\\ C(I0|I) = e_I = C(0I|I) \quad &, \quad C(I\bar{I}|0) = C(\bar{I}I|0)P \ , \quad &(7.2)\\ (C(I\bar{I}|0) \otimes e_K)(e_I \otimes C^a(\bar{I}K|J))^* &= C^a(IJ|K) \ . \quad &(7.3)\end{aligned}$$

where $P : V^I \otimes V^I \mapsto V^I \otimes V^I$ it the permutation map. We introduce the Hilbert space $V = \bigoplus_J V^J$ and extend e_J and $C^a(IJ|K)$ to linear maps on V resp. $V \otimes V$ by $e_I V^J = e_I \delta_{I,J}$ etc..

Proposition 10 *Suppose that equations (7.1, 7.2,7.3) can be solved by a family $C^a(IJ|K)$ Define a *-algebra $\mathcal{G}^*$ which consists of maps $\xi : V \mapsto V$ as follows*

$$\mathcal{G}^* = \{\xi : V \mapsto V | \xi : V^I \mapsto V^I \text{ for all } I \in \mathcal{I}\} .$$

*The *-operation on $\mathcal{G}^*$ is given by the usual adjoint of maps $\xi : V \mapsto V$. There exist a co-product Δ, a co-unit ϵ and an antipode $\mathcal{S}$ which enjoy the usual properties. Explicitly they act on elements $\xi \in \mathcal{G}^*$ as*

$$\Delta(\xi) = \sum C^a(IJ|K)^* \xi C^a(IJ|K) , \tag{7.4}$$
$$\epsilon(\xi) = \langle e_0|\xi|e_0\rangle , \tag{7.5}$$
$$\mathcal{S}(\xi) = \sum (e_I \otimes g_I)(e_I \otimes \xi \otimes e_I)(g_I^* \otimes e_I) . \tag{7.6}$$

In the last line $g_J \equiv C(J\bar{J}|0)$ has been introduced to simplify notations.

PROOF: Most of the properties are obvious. We shall give only some of the calculations and omit the others. Orthonormality (7.1) of $C^a(IJ|K)$ is needed for the co-product $\Delta : \mathcal{G}^* \mapsto \mathcal{G}^* \otimes \mathcal{G}^*$ to become a homomorphism.

$$\begin{aligned}\Delta(\xi)\Delta(\eta) &= \sum C^a(IJ|K)^* \xi C^a(IJ|K) \sum C^b(LM|N)^* \eta C^b(LM|N) \\ &= \sum C^a(IJ|K)^* \xi \delta_{a,b}\delta_{K,N} e_K \eta C^b(IJ|N) = \Delta(\xi\eta) .\end{aligned}$$

$\Delta(\xi)^* = \Delta(\xi^*)$ and the properties of ϵ are trivial. The normalization (7.2) is used to obtain

$$(\epsilon \otimes id)\Delta(\xi) = \sum C(0J|J)^* \xi C(0J|J) = \xi . \tag{7.7}$$

Things get slightly more complicated when we come to the antipode $\mathcal{S}$. To proof that $\mathcal{S} : \mathcal{G}^* \mapsto \mathcal{G}^*$ is an anti-homomorphism we note that

$$g_J(\eta \otimes \mathcal{S}(\xi)) = g_J(\xi\eta \otimes e_J) . \tag{7.8}$$

This relation follows from the definition of $\mathcal{S}$ together with $(g_I \otimes e_I)(e_I \otimes g_I^*) = e_I$ which is a special case of (7.3) and will be used frequently.

$$\begin{aligned}\mathcal{S}(\xi\eta) &= \sum (e_I \otimes g_I)(e_I \otimes \xi\eta \otimes e_I)(g_I^* \otimes e_I) \\ &= \sum (e_I \otimes g_I)(e_I \otimes \eta \otimes \mathcal{S}(\xi))(g_I^* \otimes e_I) \\ &= \sum (e_I \otimes g_I)(e_I \otimes \eta \otimes e_I)(g_I^* \otimes e_I)\mathcal{S}(\xi) \\ &= \mathcal{S}(\eta)\mathcal{S}(\xi) .\end{aligned}$$

The action of $\mathcal{S}$ commutes with the conjugation $*$ because $g_I P = g_I$ (eq. (7.2)).

$$\begin{aligned}\mathcal{S}(\xi)^* &= \sum (g_I \otimes e_I)(e_I \otimes \xi^* \otimes e_I)(e_I \otimes g_I^*) \\ &= \sum (g_I \otimes e_I)P_{13}(e_I \otimes \xi^* \otimes e_I)P_{13}(e_I \otimes g_I^*) \\ &= \sum (e_I \otimes g_I)(e_I \otimes \xi^* \otimes e_I)(g_I^* \otimes e_I) \\ &= \mathcal{S}(\xi^*) .\end{aligned}$$

$\mathcal{S}$ it actually bijective with inverse $\mathcal{S}^{-1} = \mathcal{S}$. It remains to check compatibility with the co-product. We proof only the first relation in (6.2). The following formula follows directly from the definition of the antipode $\mathcal{S}$.

$$m(\mathcal{S}(\xi) \otimes \eta) = \mathcal{S}(\xi)\eta = \sum (e_I \otimes g_I)(e_I \otimes \xi \otimes \eta)(g_I^* \otimes e_I) . \tag{7.9}$$

It is used to express $m_{12}(\mathcal{S}^{(2)} \otimes id)$ in terms of the maps $C^a(IJ|K)$.

$$\begin{aligned} & m_{12}(\mathcal{S} \otimes id \otimes id)(id \otimes \Delta)\Delta(\xi) \\ &= \sum (e_I \otimes g_I \otimes e_J) C^a(IJ|M)^*_{34} C^b(\bar{I}M|K)^*_{23}(e_I \otimes \xi) C^b(\bar{I}M|K)_{23} C^a(IJ|M)_{34}(g_I^* \otimes e_I \otimes e_J) \\ &= \sum C^a(\bar{I}M|J)_{23} C^b(\bar{I}M|K)^*_{23}(e_I \otimes \xi) C^b(IK|M)^* C^a(IJ|M) \\ &= \sum (e_I \otimes \xi) C^a(IJ|M)^* C^a(IJ|M) \\ &= (e \otimes \xi)\Delta(e) .\end{aligned}$$

In the step from the second to the third line we inserted the relation (7.3) twice. This concludes the proof of proposition 10.

Irreducible unitary representations τ^J of $\mathcal{G}^*$ are obtained by restriction to V^J.

$$\tau^J(\xi) \equiv \xi|_{V^J} .$$

The tensor product $\tau^I \boxtimes \tau^J$ acts on $V^I \otimes V^J$ by

$$(\tau^I \boxtimes \tau^J)(\xi) = \sum_{K,a} C^a(IJ|K)^* \tau^K(\xi) C^a(IJ|K)$$

so that $C^a(IJ|K)$ have been identified as Clebsch Gordon intertwiners. By construction, $dim(C(IJ|K)) = N_K^{IJ}$. We conclude that our initial problem is solved by the Hopf-*-algebra $(\mathcal{G}^*, \Delta, \epsilon, *, \mathcal{S})$ in proposition 10, provided it exits.

The above proposition reduces the problem of finding an appropriate Hopf-*-algebra to the solution of equations (7.1 f.). Of course such solutions will only exist for special choices of the dimensions δ_J. A necessary condition on δ_J can easily be derived from the orthonormality of $C^a(IJ|K)$ (7.1). It means that the $C^a(IJ|K)$ map vectors in the $\delta_I\delta_J$–dimensional Hilbert space $V^I \otimes V^J$ onto an orthogonal sum of δ_K–dimensional spaces which occur with a multiplicity N_K^{IJ} depending on K. The total dimension $\sum_K N_K^{IJ}\delta_K$ of the image cannot exceed the dimension $\delta_I\delta_J$ of $V^I \otimes V^J$,

$$\delta_I\delta_J \geq \sum_K N_K^{IJ}\delta_K . \tag{7.10}$$

This condition would also be sufficient if we were not interested in an antipode $\mathcal{S}$, i.e. if the relation (7.3) would be absent. Solutions of (7.1,7.2) can be obtained by the usual orthonormalization procedure. For finite number of equivalence classes of irreducible representations, a solution of (7.10) is furnished by $\delta_0 = 1$ and $\delta_J = \delta = max_{IJ}(\sum_K N_K^{IJ}), J \neq 0$, so that *an appropriate bi-*-algebra does always exist.* In presence of equation (7.3) the solution becomes less transparent and we are not prepared to study it systematically. Instead we solve the conditions on $C^a(IJ|K)$ in a simple example.

Example: Let us consider the fusion rules N_K^{IJ} of the chiral critical Ising model. They can be found in section 5. For these fusion rules and dimensions $\delta_0 = 1, \delta_{1/2} = 2, \delta_1 = 1$ a solution of eqs. (7.1,7.2,7.3) exists. In an appropriate basis it is given by (with 1 the two-dimensional unit matrix)

$$\begin{array}{ll} C(00|0) = 1 & C(\tfrac12\tfrac12|0) = \tfrac{1}{\sqrt2}(1\ 0\ 0\ 1) \\ C(0\tfrac12|\tfrac12) = 1 & C(11|0) = 1 \\ C(01|1) = 1 & C(\tfrac12\tfrac12|1) = \tfrac{1}{\sqrt2}(1\ 0\ 0\ -1) \\ C(\tfrac12 0|\tfrac12) = 1 & C(10|1) = 1 \\ C(\tfrac12 1|\tfrac12) = diag(1,-1) & C(1\tfrac12|\tfrac12) = diag(1,-1) \end{array}$$

Here $(1\ 0\ 0\ 0) = (1\ 0)\otimes(1\ 0)$, $(0\ 1\ 0\ 0) = (1\ 0)\otimes(0\ 1)$ etc..

Part II
Quasi–Quantum Planes as Examples of Quasi–Associative Differential Geometry

Groups can act as transformation groups on (differentiable) manifolds. Thus they are capable of describing the symmetry of (differential) geometric structures. Infinitesimal transformations determine derivations of an algebra $\mathcal{C}$ of functions on the manifold. In non-commutative (differential) geometry [13] manifolds are not defined. Instead one has to start from a non-commutative algebra $\mathcal{C}$ of "functions on the non-commutative manifold". Super-manifolds can be regarded as the simplest non-trivial examples. Their symmetries are not groups but super-groups. More sophisticated non-commutative manifolds have been constructed in the last years. Quantum symmetries – in particular quantum groups – might act on them as generalized derivations and should be regarded as symmetries in non-commutative geometry. We will discuss that (weak) quasi quantum groups [19, 56] play the same role in quasi-associative differential geometry [59].

The theory of complex quantum planes, as developed in [63, 79], provides the simplest examples of non-commutative differential geometries which admit true quantum group symmetries. Let us briefly describe the algebras $\mathcal{F}_q$ of "functions on the quasi quantum plane" associated with the quantum group algebra $U_q(sl_2)$. They are associative deformations (indexed by a complex parameter q) of the commutative algebra $\mathcal{F}_{cl}$ of polynomial functions on the complex plane $\mathbf{C}^2$. Generators $z_a, a = 1,2$ of the algebra $\mathcal{F}_q$ are subject to the relations

$$z_a \cdot z_b = c_1^{-1} z_c \cdot z_d R_{dc,ab}$$

where $R_{dc,ab} = (\tau^f_{da} \otimes \tau^f_{cb})(R)$ is furnished by the R-matrix R of $U_q(sl_2)$ evaluated in the fundamental two–dimensional representations τ^f, and $c_1 = q^{1/4}$.

The product in $\mathcal{F}_q$ determines a multiplication map $m : \mathcal{F}_q \otimes \mathcal{F}_q \mapsto \mathcal{F}_q$ defined by $m(f\otimes f') = f \cdot f'$. $\mathcal{F}_q$ is a representations space for $U_q(sl_2)$ and the action of $\xi \in U_q(sl_2)$ is a generalized derivation [75], i.e.

$$\xi(f\cdot f') = m(\Delta_q(\xi)(f\otimes f')) \ .$$

In addition to the elements in $U_q(sl_2)$, operators Z_a can act on $\mathcal{F}_q$ as multiplication with z_a.

The associative algebra $\mathcal{B}$ generated by multiplication operators Z_a and elements of $\mathcal{G}^* = U_q(sl_2)$ will serve as starting point for our generalization, when $\mathcal{G}^*$ becomes an arbitrary weak quasi quantum group. Operators Z_a transform covariantly under $\mathcal{G}^*$ and satisfy *braid relations*

$$Z_a Z_b = Z_c Z_d c_1^{-1} \hat{\mathcal{R}}_{dc,ab} \ . \tag{0.1}$$

$\mathcal{G}^*$ covariance of these relations turns out to require that $\hat{\mathcal{R}}_{dc,ab}$ are not numbers but elements of $\mathcal{G}^*$ in general.

From there we proceed to an algebra $\mathcal{F}$ of "functions on the quasi quantum plane" by forming cosets $\mathcal{F} = \mathcal{B}/\mathcal{G}^*$. The initial associative product in $\mathcal{B}$ will be destroyed in this process, but a new quasi-associative multiplication of elements in $\mathcal{F}$ can be defined. Quasi-associativity means that products with different positions of brackets are linear combinations of each other. Elements in $\mathcal{B}$ can be regarded as $\mathcal{G}^*$ valued functions on the quasi-quantum plane.

Next we will try to construct a differential calculus on $\mathcal{F}$, i.e. a pair $(\Lambda\mathcal{F}, d)$ of a graded quasi-associative algebra $\Lambda\mathcal{F} = \bigoplus_r \Lambda^r\mathcal{F}$ with $\Lambda^0\mathcal{F} = \mathcal{F}$ and an exterior derivative $d : \Lambda^r\mathcal{F} \mapsto \Lambda^{r+1}\mathcal{F}$ which enjoys the usual properties ($d^2 = 0$ and Leibniz rule) (cf. e.g. [84]). Nontrivial constraints are encountered. The differential calculus $(\Lambda\mathcal{F}, d)$ will exist if, the matrix $(\tau^f \otimes \tau^f)(R)$ has at most two distinct eigenvalues. Elements in $\Lambda\mathcal{F}$ will be obtained as equivalence classes $\Lambda\mathcal{B}/\mathcal{G}^*$ from an associative algebra $\Lambda\mathcal{B}$ of $\mathcal{G}^*$-valued differential forms on the quasi quantum plane.

1 Quasi Quantum Planes

1.1 The associative algebra $\mathcal{B}$

As in part I we consider an arbitrary weak quasi quantum group algebra $\mathcal{G}^*$ with unit element $e \in \mathcal{G}^*$. Co-product Δ, co-unit ϵ and antipode S are defined as usual. We reserve the letters R and φ to denote the R-matrix and re-associator respectively. Their properties can be found in part I. It should be emphasized that the co-product is not assumed to be unit preserving, i.e. $\Delta(e) \neq e \otimes e$ is allowed.

When dealing with associative algebras $\mathcal{A}$ which contain a weak quasi quantum group $\mathcal{G}^*$ as subalgebra one needs such notions as *covariance* and *covariant product.* They should be recalled before we define the algebra $\mathcal{B}$ of $\mathcal{G}^*$-valued functions on the quasi quantum plane.

Definition 1 ($\mathcal{G}^*$-covariance) *Let $\mathcal{A} \supset \mathcal{G}^*$ be an associative algebra and $\tau = (\tau_{\alpha\beta})_{\alpha,\beta\in I}$ be the representation matrix of a n–dimensional representation of $\mathcal{G}^*$. An n-tupel $F = (F_\alpha)_{\alpha\in I}$, $F_\alpha \in \mathcal{A}$, is said to transform covariantly according to the representation τ of $\mathcal{G}^*$ if*

$$\xi F_\alpha = F_\beta(\tau_{\beta\alpha} \otimes id)(\Delta(\xi)) \tag{1.1}$$

for all $\xi \in \mathcal{G}^*$ and summations over repeated indices are understood throughout. $F \in \mathcal{A}$ is called $\mathcal{G}^*$–invariant *if it transforms according to the one–dimensional representation ϵ of $\mathcal{G}^*$.*

If $\Delta(\xi) = \sum \xi^1_\sigma \otimes \xi^2_\sigma$ then relation (1.1) reads explicitly

$$\xi F_\alpha = \sum_\sigma F_\beta \tau_{\beta\alpha}(\xi^1_\sigma)\xi^2_\sigma \ . \tag{1.2}$$

This tells us how to shift elements ξ of $\mathcal{G}^*$ through factors F_α from left to right.

Because of lack of co-associativity in $\mathcal{G}^*$, products $F_\alpha F_\beta$ do not transform covariantly. This is the reason to introduce the following (covariant) $\times$-product. Suppose that $(F_\alpha)_{\alpha\in I}$ and $(F'_\beta)_{\beta\in I'}$ transform covariantly according to representations τ and τ' of $\mathcal{G}^*$ with dimensions n and n'. Define the nn'-tuple $F \times F'$ by

$$(F \times F')_{\alpha\beta} = \sum_{\gamma\in I}\sum_{\delta\in I'} F_\gamma F'_\delta(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes id)(\varphi) \ \in \mathcal{A} \ . \tag{1.3}$$

Note that $\times$ is in general a product of vectors whose entries are elements of $\mathcal{A}$, and not a product of individual elements of $\mathcal{A}$. Using the expansion $\varphi = \sum \varphi^1_\sigma \otimes \varphi^2_\sigma \otimes \varphi^3_\sigma$ the defining eq.(1.3) takes the form

$$(F \times F')_{\alpha\beta} = \sum_\sigma F_\gamma F'_\delta \tau_{\gamma\alpha}(\varphi^1_\sigma)\tau'_{\delta\beta}(\varphi^2_\sigma)\varphi^3_\sigma \ . \tag{1.4}$$

This exhibits the fact that the $(F \times F')_{\alpha\beta}$ are complex linear combinations of terms $F_\gamma F'_\delta \varphi^3_\sigma$ with coefficients $\varphi^3_\sigma \in \mathcal{G}^*$.

Proposition 2 (Properties of the $\times$-product) *Let $(F_\alpha), (F'_\beta)$ be specified as above. If $\mathcal{A} \supset \mathcal{G}^*$ is an associative algebra with unit $e \in \mathcal{G}^*$, the $\times$-product (1.3) has the following properties.*

1. *Eq.(1.3) can be inverted to recover ordinary products from covariant ones, viz.*

$$F_\alpha F'_\beta = \sum_{\gamma\in I}\sum_{\delta\in I'} (F \times F')_{\gamma\delta}(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes id)(\varphi^{-1}) \ . \tag{1.5}$$

2. *$F \times F'$ transforms covariantly according to the tensor product representation $\tau \boxtimes \tau'$ of $\mathcal{G}^*$. Hence we will often use the term* covariant product *instead of $\times$-product.*

3. *The $\times$-product is not associative. But it is quasi– associative in the following sense. If $F'' = (F''_\gamma)$ transforms covariantly according to representations and τ'' of $\mathcal{G}^*$ and F, F' as above, then*

$$\begin{aligned} (i) \quad ((F \times F') \times F'')_{\alpha\beta\gamma} &= (F \times (F' \times F''))_{\delta\epsilon\kappa}(\tau_{\delta\alpha} \otimes \tau'_{\epsilon\beta} \otimes \tau''_{\kappa\gamma})(\varphi) \\ (ii) \quad (F \times (F' \times F''))_{\alpha\beta\gamma} &= ((F \times F') \times F'')_{\delta\epsilon\kappa}(\tau_{\delta\alpha} \otimes \tau'_{\epsilon\beta} \otimes \tau''_{\kappa\gamma})(\varphi^{-1}) \ . \end{aligned}$$

4. *If $G \in \mathcal{A}$ is $\mathcal{G}^*$-invariant then*

$$(G \times F)_\alpha = GF_\alpha \ , \ (F \times G)_\alpha = F_\alpha G \ . \tag{1.6}$$

All items in this proposition follow from the properties of the re-associator φ. Proofs are spelled out in [59].

Now we turn to the special example of an algebra $\mathcal{B}$ which is generated by elements $\xi \in \mathcal{G}^*$ and one additional covariant tupel (Z_a) subject to *braid relations.* To define $\mathcal{B}$ we fix a representation $\tau = \tau^f$ of $\mathcal{G}^*$ and a complex number $c_1 \in \mathbf{C}$. The latter is supposed to coincide with one of the eigenvalues of $(\tau^f \otimes \tau^f)(R)$. Although all results in this section are independent of this choice, different values of c_1 would not lead to examples of quasi-associative differential geometry.

Definition 3 (Algebra $\mathcal{B}$) *The associative algebra $\mathcal{B}$ is generated by elements Z_a, $(a = 1...N)$, and the elements of $\mathcal{G}^*$ such that*

1. *the unit element e of $\mathcal{G}^*$ acts as a unit element of $\mathcal{B}$, i.e. $Z_a e = Z_a = eZ_a$,*
2. *the tupel (Z_a) transforms covariantly according to the representation τ^f,*

$$\xi Z_a = Z_b(\tau^f_{ba} \otimes id)(\Delta(\xi)) \quad \text{for all} \quad \xi \in \mathcal{G}^* ,$$

3. *elements Z_a satisfy braid relations*

$$(Z \times Z)_{ab} = c_1^{-1}(Z \times Z)_{cd}(\tau^f_{da} \otimes \tau^f_{cb})(R) \quad .$$

To write the braid relations in *3.* we used the $\times$-product. In this form their consistency with the transformation law in *2.* is apparent. The inversion formula (1.5) can be applied to substitute covariant products by ordinary multiplication in $\mathcal{B}$. Along these lines we find the expression

$$\tilde{\mathcal{R}}_{dc,ab} = (\tau^f_{da} \otimes \tau^f_{cb} \otimes id)(\varphi_{213}(R \otimes e)\varphi) \in \mathcal{G}^*$$

for the matrix $\tilde{\mathcal{R}}$ in (0.1). Here $\varphi_{213} = \sum \varphi^2_\sigma \otimes \varphi^1_\sigma \otimes \varphi^3_\sigma$ as usual. As we announced in the introduction, the matrix elements of $\tilde{\mathcal{R}}$ are non-numerical elements of $\mathcal{G}^*$ if $\varphi \neq e \otimes e \otimes e$.

Let us collect some useful results on the structure of $\mathcal{B}$. It follows from our remark after definition 1 that $\mathcal{B}$ is spanned by products

$$Z_{a_n} \dots Z_{a_1}\eta \quad \eta \in \mathcal{G}^* \ , \ n \geq 0 \quad .$$

One often prefers to use a covariant product of elements $Z_{a_n}, \dots, Z_{a_1}$. Therefore we abbreviate

$$Z^n_\alpha \equiv (Z \times (Z \times ... \times (Z \times Z)...))_{a_n...a_1}$$

with multiindex $\alpha = (a_n, \dots, a_1)$ of order n. As a consequence of the inversion formula (1.5) we obtain:

Proposition 4 (covariant products span $\mathcal{B}$) *Every element of $\mathcal{B}$ is a complex linear combination of elements of the form*

$$Z^n_\alpha \eta = (Z \times (Z \times ... \times (Z \times Z)...))_{a_n...a_1}\eta \tag{1.7}$$

with $n \geq 0$ and $\eta \in \mathcal{G}^*$.

The last result in this subsection concerns braid relations for products of covariant tupels. Since similar formulae have been discussed in some detail in part I, we can state the following proposition without further remarks.

Proposition 5 (braid relations for composite operators) *Suppose that $F = (F_\alpha)$, $F' = (F'_\beta)$ and $F'' = (F''_\gamma)$ transform covariantly according to representations τ, τ' and τ'' of $\mathcal{G}^*$.*
(i) Suppose that the braid relations

$$\begin{aligned} (F \times F')_{\alpha\beta} &= c'(F' \times F)_{\mu\nu}(\tau_{\nu\alpha} \otimes \tau'_{\mu\beta})(R) \\ (F \times F'')_{\alpha\gamma} &= c''(F'' \times F)_{\mu\rho}(\tau_{\rho\alpha} \otimes \tau''_{\mu\beta})(R) \end{aligned}$$

hold true. Then F and $F' \times F''$ satisfy braid relations

$$\begin{aligned} (F \times (F' \times F''))_{\alpha\beta\gamma} &= c'c''((F' \times F'') \times F)_{\mu\nu\rho}(\tau_{\rho\alpha} \otimes (\tau' \boxtimes \tau'')_{\mu\nu,\beta\gamma})(R) \\ ((F' \times F'') \times F)_{\beta\gamma\alpha} &= c'c''(F \times (F' \times F''))_{\rho\mu\nu}((\tau' \boxtimes \tau'')_{\mu\nu,\beta\gamma} \otimes \tau_{\rho\alpha})(R). \end{aligned}$$

(ii) If F_α, F'_β are complex linear combinations of $\times$-products of Z_a's (with brackets in arbitrary positions), and are homogeneous in the Z's of degree n and m, respectively, then

$$(F \times F')_{\alpha\beta} = (F' \times F)_{\delta\gamma} c_1^{-nm}(\tau \otimes \tau')(R) \ . \tag{1.8}$$

In particular, if $F' \in \mathcal{B}$ is $\mathcal{G}^*$–invariant then

$$F_\alpha F' = F' F_\alpha c_1^{-nm} \ . \tag{1.9}$$

1.2 The quasi-associative algebra $\mathcal{F}$

We will construct a $\mathcal{B}$–module $\mathcal{F}$ – i.e. a space which is a representation space for $\mathcal{G}^*$ and on which Z_a can act. It is obtained as a coset space $\mathcal{F} = \mathcal{B}/\mathcal{G}^*$ so that elements are equivalence classes of elements of $\mathcal{B}$. Since $\mathcal{G}^*$ is an algebra and not a group, we need a homomorphism

$$\epsilon : \mathcal{G}^* \to \mathbf{C}$$

to define the equivalence relation. This homomorphism is given by the co-unit. Recalling proposition 4 and the notations therein we define the equivalence relation by

$$Z^n_\alpha \xi \sim Z^m_\beta \eta \quad \text{if} \quad m = n \ , \ \alpha = \beta \ , \ \epsilon(\xi) = \epsilon(\eta) \ .$$

We denote the maps into cosets by

$$\varepsilon : \mathcal{B} \to \mathcal{F} = \mathcal{B}/\mathcal{G}^* \ . \tag{1.10}$$

$\mathcal{F}$ becomes a $\mathcal{B}$–module by setting

$$F\varepsilon(G) = \varepsilon(FG) \quad \text{for} \quad F, G \in \mathcal{B} \ . \tag{1.11}$$

We introduce a special notation for the image of the unit element e of $\mathcal{B}$.

$$\Omega = \varepsilon(e) \ . \tag{1.12}$$

Let us summarize some results on the $\mathcal{B}$-module $\mathcal{F}$ in the following proposition.

Proposition 6 (properties of the module $\mathcal{F}$) *Elements $\xi \in \mathcal{G}^* \subset \mathcal{B}$ act on $\mathcal{F}$ such that*

1. *the $\mathcal{G}^*$-module $\mathcal{F}$ decomposes into a direct sum $\mathcal{F} = \bigoplus \mathcal{F}^{(n)}$ where $\mathcal{F}^{(n)}$ is spanned by elements*

$$z^n_\alpha \equiv Z^n_\alpha \Omega \quad (n \geq 1) \tag{1.13}$$

and $\mathcal{F}^{(0)}$ is a one–dimensional subspace which contains Ω. Note that the relation $(id \otimes id \otimes \epsilon)(\varphi) = \Delta(e)$ implies $z^n_\alpha = Z_{a_n} \dots Z_{a_1}\Omega$.

2. *elements $F_\alpha\Omega \in \mathcal{F}$ generated from Ω by components of a covariant tupel (F_α) transform according to*

$$\xi F_\alpha \Omega = F_\beta \Omega \tau_{\beta\alpha}(\xi) \tag{1.14}$$

if (F_α) transforms covariantly according to the representation τ of $\mathcal{G}^*$. In particular Ω is $\mathcal{G}^*$-invariant in the sense that $\xi\Omega = \Omega\epsilon(\xi)$ and z^n_α transform according to the representation $\tau^{(n)}$, i.e. $\xi z^n_\alpha = z^n_\beta \tau^{(n)}_{\beta\alpha}(\xi)$. Here $\tau^{(n)}$ is defined to be

$$\tau^{(n)} \equiv (\tau^f \boxtimes (\tau^f \boxtimes \ldots \boxtimes (\tau^f \boxtimes \tau^f)\ldots)) \quad (n \text{ factors}) . \tag{1.15}$$

One might suspect that the multiplicative structure of B gets lost when we divide by $\mathcal{G}^*$ to construct $\mathcal{F}$. This is indeed the case, if the weak quasi quantum group $\mathcal{G}^*$ has a non-trivial re-associator $\varphi \neq e \otimes e \otimes e$. However the $\times$ product survives and yields a quasi-associative product $\cdot$ in $\mathcal{F}$. Let $Z^0 = e$ and Z^n_α be defined as above so that $z^n_\alpha = Z^n_\alpha \Omega$. Define

$$(z^n_\alpha \cdot z^m_\beta) = (Z^n \times Z^m)_{\alpha\beta}\Omega \tag{1.16}$$

for $n, m \geq 0$. In particular, $\Omega = z^0$ is the unit element of the algebra $\mathcal{F}$. A thorough discussion reveals that the product $\cdot$ is well defined [59].

Theorem 7 (properties of the algebra $\mathcal{F}$) *Rel.(1.16) defines a quasi-associative and braid-commutative product on $\mathcal{F}$ which is compatible with the action of B. In detail this means the following.*

1. *Generators $Z_a \in B$ act on $\mathcal{F}$ by multiplication with z_a*

$$Z_a f = (z_a \cdot f) \ .$$

2. *$\xi \in \mathcal{G}^*$ acts on the algebra $\mathcal{F}$ as a generalized derivation*

$$\begin{aligned} \xi z^n_\alpha &= z^n_\beta \tau^{(n)}_{\beta\alpha}(\xi) , & (1.17)\\ \xi(f^1 \cdot f^2) &= m(\Delta(\xi)[f^1 \otimes f^2]) & (1.18)\end{aligned}$$

where $\tau^{(n)}$ is the n-fold tensor product $\tau^f \boxtimes (\tau^f \boxtimes \cdots \boxtimes (\tau^f \boxtimes \tau^f)\cdots))$ of fundamental representations τ^f of $\mathcal{G}^*$ and $m : \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}$ is multiplication in $\mathcal{F}$.

3. *The algebra $\mathcal{F}$ is quasi-associative in the sense that the product $z_{a_n} \cdots z_{a_1}$ with arbitrary specification of the position of brackets can be written as a complex linear combination of products $z_{b_n} \cdots z_{b_1}$ with any other specification of brackets. Re-association is performed with the help of the formulae*

$$\begin{aligned} ((f_\alpha \cdot f'_\beta) \cdot f''_\gamma) &= (f_\delta \cdot (f'_\epsilon \cdot f''_\kappa))(\tau_{\alpha\delta} \otimes \tau'_{\beta\epsilon} \otimes \tau''_{\gamma\kappa})(\varphi) , & (1.19)\\ (f_\delta \cdot (f'_\epsilon \cdot f''_\kappa)) &= ((f_\alpha \cdot f'_\beta) \cdot f''_\gamma)(\tau_{\delta\alpha} \otimes \tau'_{\epsilon\beta} \otimes \tau''_{\kappa\gamma})(\varphi^{-1}) . & (1.20)\end{aligned}$$

They are valid if $f, f', f'' \in \mathcal{F}$ transform according to representations τ, τ', τ'' of $\mathcal{G}^*$.

4. *The product $\cdot$ is braid-commutative in the sense that*

$$(z^n_\alpha \cdot z^m_\beta) = (z^m_\gamma \cdot z^n_\delta) c_1^{-nm} (\tau^{(n)}_{\delta\alpha} \otimes \tau^{(m)}_{\gamma\beta})(R) . \tag{1.21}$$

Before we conclude this section we would like to stress the meaning of the last theorem. It shows that we still have an *algebra* $\mathcal{F}$ of "functions on the quasi-quantum plane". But the possibility of quasi-associative products takes us beyond the framework of non-commutative geometry. The same will turn out to hold for the differential calculus.

2 Quasi-associative differential calculus on the quasi-quantum plane

2.1 Differential forms and exterior derivatives

Our next task is to define a differential calculus of $\mathcal{F}$. Therefore we extend the algebra $\mathcal{F}$ to a quasi-associative algebra

$$\Lambda\mathcal{F} = \bigoplus_{n \geq 0} \Lambda^n \mathcal{F} \tag{2.1}$$

by adjoining θ_b to $\mathcal{F}$, $\Lambda^0\mathcal{F} = \mathcal{F}$. The space $\Lambda^n\mathcal{F}$ will be spanned by elements of the form

$$x^n_{\alpha\beta} = (z_{a_m} \cdot (z_{a_{m-1}} \cdot \ldots (z_{a_1} \cdot (\theta_{b_n} \cdot (\theta_{b_{n-1}} \cdot \ldots (\theta_{b_2}\theta_{b_1}) \ldots))) \ldots)) . \tag{2.2}$$

with $\alpha = (a_m \ldots a_1), \beta = (b_n \ldots b_1), m = 0, 1, \ldots$. The quasi-associative product in $\Lambda\mathcal{F}$ is written as $\cdot$ in what follows.

Under restrictive assumptions on the representation τ^f an exterior derivative d will be defined which acts on $\Lambda\mathcal{F}$ and enjoys the standard properties

$$\begin{aligned} d : \Lambda^n\mathcal{F} &\mapsto \Lambda^{n+1}\mathcal{F} , & (2.3)\\ dz_a &= \theta^a , & (2.4)\\ d^2 &= 0 , & (2.5)\\ d(x \cdot y) &= (dx \cdot y) + (-1)^n (x \cdot dy) \quad \text{if } x \in \Lambda^n\mathcal{F} , \ y \in \Lambda\mathcal{F} \ (\textit{Leibniz rule}). & (2.6)\end{aligned}$$

The action of $\mathcal{G}^*$ on $\mathcal{F}$ extends to an action on $\Lambda\mathcal{F}$ by generalized derivations.

We will obtain the quasi-associative algebra $\Lambda\mathcal{F}$ as a quotient from an associative algebra

$$\Lambda B = \bigoplus_{n \geq 0} \Lambda^n B \tag{2.7}$$

which is generated by $\mathcal{G}^*$ and Z_a, Θ_a subject to the relations (e= unit element of $\mathcal{G}^*$)

$$\begin{aligned} Z_a e = eZ_a &= Z_a , & (2.8)\\ \Theta_a e = e\Theta_a &= \Theta_a , & (2.9)\\ \xi Z_a &= Z_b(\tau^f_{ba} \otimes id)(\Delta(\xi)) \quad \text{and} & (2.10)\\ \xi \Theta_a &= \Theta_b(\tau^f_{ba} \otimes id)(\Delta(\xi)) \quad \text{for } \xi \in \mathcal{G}^*, & (2.11)\\ (Z \times Z)_{ab} &= (Z \times Z)_{dc} c_1^{-1} (\tau^f_{ca} \otimes \tau^f_{db})(R) , & (2.12)\\ (\Theta \times \Theta)_{ab} &= (\Theta \times \Theta)_{dc} c_2^{-1} (\tau^f_{ca} \otimes \tau^f_{db})(R) , & (2.13)\\ (\Theta \times Z)_{ab} &= -(Z \times \Theta)_{dc} c_2^{-1} (\tau^f_{ca} \otimes \tau^f_{db})(R) . & (2.14)\end{aligned}$$

All our above constructions and results on B and $\mathcal{F}$ can be extended to a situation of more than one tupel of generators (Z_a) and thus to ΛB and $\Lambda\mathcal{F}$. In particular, the co-unit ϵ induces a map $\epsilon : \Lambda B \mapsto \Lambda\mathcal{F} = \Lambda B/\mathcal{G}^*$. If we set $z_a \equiv \epsilon(Z_a)$ and $\theta_a \equiv \epsilon(\Theta_a)$, $\Lambda^n\mathcal{F}$ is indeed spanned by elements of the form (2.2). The product $\cdot$ on ΛF is defined as above.

So far the complex constant c_2 and the representation τ^f remained arbitrary. This changes if we try to adjoin an exterior derivative d to ΛB. It is required to be $\mathcal{G}^*$–invariant,

$$d\xi = \xi d \quad \text{for } \xi \in \mathcal{G}^* , \tag{2.15}$$

and subject to the relations

$$\begin{aligned} d^2 &= 0 , \\ dZ_a &= \Theta_a + Z_a d \\ d\Theta_a &= -\Theta_a d . \end{aligned} \tag{2.16}$$

These relations are actually not independent. The last relation follows from the first two relations, as is seen by multiplying the second one with d from left or right. A simple consistency check of (2.16) with braid relations (2.12,2.13,2.14) already reveals the most important restrictions on c_i and τ^f. To see this, let us multiply (2.12) by d from the left. After shifting d to the right by application of (2.16) we obtain

$$(Z \times \Theta)_{ab} + (\Theta \times Z)_{ab} = ((Z \times \Theta)_{dc} + (\Theta \times Z)_{dc}) c_1^{-1} (\tau^f_{ca} \otimes \tau^f_{db})(R) .$$

In this expression we already subtracted terms of the type $(Z \times Z)d$. Now we use (2.14) to deduce

$$(Z \times \Theta)_{ab} (\tau^f_{be} \otimes \tau^f_{af})(e \otimes e - c_2^{-1} R)(\tau^f_{fc} \otimes \tau^f_{ed})(e \otimes e - c_1^{-1} R) = 0$$

When applied to an element in $\Lambda^r \mathcal{F}$ this amounts to a new linear relation in $\Lambda^{r+1}\mathcal{F}$, if the coefficients of the $Z \times \Theta$-terms are non-zero. To obtain a map $d : \Lambda^r \mathcal{F} \mapsto \Lambda^{r+1}\mathcal{F}$, these coefficients have to vanish,

$$(\tau^f_{be} \otimes \tau^f_{af})(e \otimes e c_2 - R)(\tau^f_{fc} \otimes \tau^f_{ed})(e \otimes e c_1 - R) = 0 . \tag{2.17}$$

This means that the matrix $(\tau^f \otimes \tau^f)(R)$ is allowed to have at most two distinct eigenvalues. For a differential calculus to exist it is necessary that the complex constants c_1, c_2 in (2.12,2.13) agree with these eigenvalues. Since we already made a corresponding choice of c_1 in the last section, c_2 is now fixed to be the other eigenvalue of $(\tau^f \otimes \tau^f)(R)$.

2.2 Partial derivatives

Let us try to construct the exterior derivative d explicitly. The strategy is to extend the algebra $\Lambda\mathcal{B}$ by partial derivatives $\partial_{\dot a}$ and to find an element d with the properties (2.16) in this larger algebra. We need some preparation before we give the details. Remember that the contragredient representation $\bar\tau$ of a representation τ of $\mathcal{G}^*$ is defined with the help of the antipode S (cf. Appendix A in part I).

$$\bar\tau(\xi) = {}^t\tau(S^{-1}(\xi)) .$$

Tensor products $\bar\tau^f \boxtimes \tau^f$ and $\tau^f \boxtimes \bar\tau^f$ always contain the trivial representation ϵ as a subrepresentation. The corresponding intertwiners yield two matrices with complex valued entries $g_{\dot a b}$ and $g^{a\dot b}$.

$$\begin{aligned} g_{\dot c d}(\bar\tau^f \boxtimes \tau^f)_{\dot c d \dot a b}(\xi) &= g_{\dot a b}\epsilon(\xi) , &(2.18)\\ (\tau^f \boxtimes \bar\tau^f)_{c\dot d a \dot b}(\xi) g^{a\dot b} &= g^{c\dot d}\epsilon(\xi) . &(2.19) \end{aligned}$$

In addition we impose the following normalization condition

$$g_{\dot f g}(\tau^f_{ha} \otimes \bar\tau^f_{\dot j \dot b} \otimes \tau^f_{gc})(\varphi) g^{a\dot b} = \delta_{hc} \tag{2.20}$$

As a consequence of Schurs' lemma, left and right hand side of this equation are proportional, if τ^f is irreducible. Therefore the normalization condition can be ensured, unless the right hand side of eq.(2.20) is 0.

Theorem 8 (partial derivatives) *Suppose that the matrices $(\tau^f \otimes \tau^f)(R)$ and $(\bar\tau^f \otimes \bar\tau^f)(R)$ have two distinct eigenvalues each and that c_1, c_2 are the eigenvalues of $(\tau^f \otimes \tau^f)(R)$. Denote the eigenvalues of $(\bar\tau^f \otimes \bar\tau^f)(R)$ by $\bar c_1, \bar c_2$. If $c_2 = \bar c_2$ the algebra $\Lambda\mathcal{B}$ can be extended to an associative algebra $\mathcal{D}$ which is generated by elements $\xi \in \mathcal{G}^*$, Z_a, Θ_a and $\partial_{\dot a}$ subject to the following relations:*
(i) The unit element $e \in \mathcal{G}^*$ is also unit element of $\mathcal{D}$.
(ii) The tupels $(Z_a), (\Theta_b)$ transform as in (2.10,2.11) and $(\partial_{\dot c})$ transforms covariantly according to the contragredient representation $\bar\tau^f$.

$$\xi\partial_{\dot a} = \partial_{\dot b}(\bar\tau^f_{\dot b \dot a} \otimes id)(\xi) \quad \text{for all } \xi \in \mathcal{G}^* . \tag{2.21}$$

(iii) Braid relations (2.12,2.13,2.14) hold together with

$$\begin{aligned} (\partial \times \partial)_{\dot a \dot b} &= (\partial \times \partial)_{\dot d \dot c} \bar c_1^{-1} (\bar\tau^f_{\dot c \dot a} \otimes \bar\tau^f_{\dot d \dot b})(R) , &(2.22)\\ (\Theta \times \partial)_{a \dot b} &= -(\partial \times \Theta)_{\dot d c} c_2 (\tau^f_{ca} \otimes \bar\tau^f_{\dot d \dot b})(R) , &(2.23)\\ (\partial \times Z)_{\dot a b} &= g_{\dot a b} e - (Z \times \partial)_{d \dot c} c_2 (\bar\tau^f_{\dot c \dot a} \otimes \tau^f_{db})(R) . &(2.24) \end{aligned}$$

Inverse braid relations involving R^{-1} can be derived from those stated here. The braid relations are written in $\mathcal{G}^*$–covariant form, but they can be transformed into relations involving ordinary products. One verifies that all braid relations, including (2.24), are $\mathcal{G}^*$–covariant.

It follows from the definitions that elements of $\mathcal{D}$ can be written as a linear combination of terms

$$Z_{a_1} \dots Z_{a_n} \Theta_{b_1} \dots \Theta_{b_m} \partial_{\dot c_1} \dots \partial_{\dot c_p} \xi \tag{2.25}$$

with $\xi \in \mathcal{G}^*$. Alternatively, one might use covariant products as in proposition 4.

Next we consider the space $\Lambda\mathcal{F}$ of forms. We reconstruct this linear space as a factor space of $\mathcal{D}$. In this way it becomes a $\mathcal{D}$-module. The elements of $\mathcal{D}$ can act on it as multiplication and differentiation operators. $\Lambda\mathcal{F}$ is constructed as a coset space $\mathcal{D}/\mathcal{J}$ with the help of a homomorphism $\epsilon : \mathcal{J} \to \mathbf{C}$. The coset space consists of equivalence classes, $X\xi \sim X\epsilon(\xi)$ if $\xi \in \mathcal{J}$.

Proposition 9 ($\mathcal{D}$–module $\Lambda\mathcal{F}$) *Let $\mathcal{J} \subset \mathcal{D}$ be generated by elements $\xi \in \mathcal{G}^*$ and by $\partial_{\dot a}$ ($\dot a = 1 \dots N$). Extend the co-unit of $\mathcal{G}^*$ to a homomorphism $\epsilon : \mathcal{J} \to \mathbf{C}$ by setting $\epsilon(\partial_{\dot a}) = 0$. Define*

$$\Lambda\mathcal{F} = \mathcal{D}/\mathcal{J} . \tag{2.26}$$

where cosets are formed with the help of ϵ as explained above. Writing $\varepsilon : \mathcal{D} \to \Lambda\mathcal{F}$ for the map to cosets, $E \in \mathcal{D}$ acts on $\Lambda\mathcal{F}$ in the obvious way

$$E\varepsilon(X) = \varepsilon(EX) . \tag{2.27}$$

The special element $\Omega = \varepsilon(e)$ is annihilated by differential operators

$$\partial_{\dot a}\Omega = 0 .$$

Now we are prepared to construct the exterior derivative d under the assumptions stated in theorem 8. The next proposition asserts that d can be identified with a certain element in $\mathcal{D}$.

Proposition 10 (exterior derivative) *Define $d = g^{a\dot b}(\Theta \times \partial)_{a\dot b}$. Then d is $\mathcal{G}^*$–invariant, and*

$$\begin{aligned} d^2 &= 0 , &(2.28)\\ dZ_a &= \Theta_a + Z_a d . &(2.29) \end{aligned}$$

Detailed proofs of theorem 8 and proposition 10 can be found in [59]. In conclusion we demonstrated that the assumptions in proposition 8 suffice to obtain a quasi-associative differential calculus $(\Lambda\mathcal{F}, d)$ on the quasi quantum plane $\mathcal{F}$. Even though existence of a suitable representation τ^f provides a strong restriction on the weak quasi quantum group $\mathcal{G}^*$, non-trivial examples can be found.

3 Example: Truncated quantum planes

Examples of quasi-quantum planes are associated with the truncated quantum group algebras $U_q^T(sl_2), q^p = 1$, which were introduced in part I. Let q be a primitive p-th root of unity. For simplicity we restrict attention to $p \geq 5$. For the fundamental representation τ^f we select the two-dimensional representation $\tau^{1/2}$. The matrix $(\tau^f \otimes \tau^f)(R)$ is the same as in the non-truncated case and thus the fundamental braid relations differ from the ordinary quantum plane only through the appearance of the $\times$-product.

$$(Z \times Z)_{12} = q^{-1/2}(Z \times Z)_{21} \quad . \tag{3.1}$$

One obtains the corresponding algebras $\mathcal{F}$ as in the general case. We call them "truncated quantum planes" and use the symbol $\mathcal{F}_q^T$ instead of $\mathcal{F}$. Before we come to the differential calculus we mention some special features of the truncated quantum planes $\mathcal{F}_q^T$.

Proposition 11 (Structure of $\mathcal{F}_q^{T(n)}$) [59] *Let $U_q^T(sl_2)$ be the truncated quantum group algebra associated with $U_q(sl_2)$, q a primitive p-th root of unity, $p \geq 4$, and let τ^f be its fundamental two-dimensional representation. The $U_q^T(sl_2)$-module $\mathcal{F}_q^T$ decomposes into a direct sum of subspaces $\mathcal{F}_q^{T(n)}$ as in proposition 6. For the subspaces $\mathcal{F}_q^{T(n)}$ one finds*

(i) $\mathcal{F}^{T(n)} = 0$ for all $n \geq p-1$,

(ii) $\mathcal{F}^{T(n)}$ carries the $n+1$-dimensional irreducible representation of $\mathcal{G}^*$ if $n \leq p-2$.

In tensor products of up to $p-2$ fundamental two-dimensional representations τ^f of $U_q^T(sl_2)$, no truncation appears. As a consequence, the product of less than $p-1$ factors z_a is associative. Due to the truncation (i) in this proposition, higher order products vanish so that the algebras $\mathcal{F}_q^T$ turn out to be associative.

The representation $\bar\tau^f = \bar\tau^{1/2}$ is equivalent to $\tau^{1/2}$. For $p \geq 5$, truncation is absent in threefold tensor products of the two-dimensional representation so that $(\tau^f \otimes \bar\tau^f \otimes \tau^f)(\varphi) = 1$ and consequently, normalization (2.20) of the tensors $g_{\alpha b}, g^{\alpha\beta}$ can be ensured. The matrices $(\tau^f \otimes \tau^f)(R)$ and $(\bar\tau^f \otimes \bar\tau^f)(R)$ are the same as in the non-truncated case and satisfy the assumptions stated in proposition 8. As a result, differential calculus and partial derivatives exist and braid relations are the same as in the ordinary quantum plane, except that covariant products substitute for ordinary ones. Thus the $\partial \times \partial$ and $\Theta \times \Theta$ braid relations reduce to

$$\begin{aligned} (\partial \times \partial)_{12} &= q^{1/2}(\partial \times \partial)_{21} \ , \\ (\Theta \times \Theta)_{12} &= -q^{1/2}(\Theta \times \Theta)_{21} \ , \\ (\Theta \times \Theta)_{11} = & \ 0 \ = (\Theta \times \Theta)_{22} \quad . \end{aligned}$$

The $(\partial \times Z)$ braid relations read

$$\begin{aligned} (\partial \times Z)_{12} &= q^{-1/2}(Z \times \partial)_{21} \\ (\partial \times Z)_{21} &= q^{-1/2}(Z \times \partial)_{12} \\ (\partial \times Z)_{11} &= e + q^{-1}(Z \times \partial)_{11} + (q^{-1} - 1)(Z \times \partial)_{22} \\ (\partial \times Z)_{22} &= e + q^{-1}(Z \times \partial)_{22} \quad . \end{aligned} \tag{3.2}$$

Braid relations for ∂, Θ are of the same form, except that the inhomogeneous terms are absent. The exterior derivative is given by

$$d = (\Theta \times \partial)_{11} + (\Theta \times \partial)_{22} \tag{3.3}$$

In [74] we have shown that $\mathcal{F}_q^T$ admits a $U_q^T(sl_2)$- invariant scalar product such that the adjoint of operators Z_a on $\mathcal{F}_q^T$ is of the "Bargmann type".

$$Z_a^* = \partial_b(\bar\tau_{ba}^f \otimes id)(R^{-1})c \tag{3.4}$$

where evaluation on $\mathcal{F}_q^T$ is understood and c is a special element in the center of $U_q^T(sl_2)$.

4 Discussion and Outlook

The spaces $\Lambda^n\mathcal{F}^T \supset \mathcal{F}^T$ are $\mathcal{F}^T$-bimodules. In the algebraic setting, such modules substitute for spaces of sections in vectorbundles. Elements of $\Lambda^n\mathcal{F}^T$ generalize n-th rank antisymmetric tensor fields on the plane. As a consequence of the constructions and results presented here, covariant exterior derivatives

$$D : \Lambda^n\mathcal{F}^T \mapsto \Lambda^{n+1}\mathcal{F}^T \ . \tag{4.1}$$

can be introduced as

$$D = d + A \ , \ A \in \Lambda^1\mathcal{B} \ . \tag{4.2}$$

They can act on elements of $\Lambda\mathcal{F}^T$; $D\omega = d\omega + A\omega$, where the second term involves the action of elements of $\Lambda\mathcal{B}$ on $\Lambda\mathcal{F}^T = \Lambda\mathcal{B}/\mathcal{G}^*$.

If Ξ is a gauge transformation, i.e. $\Xi \in \mathcal{B}$ is an invertible element in $\mathcal{B}$, then

$$D\Xi\omega = \Xi D'\omega \quad \text{for} \quad \omega \in \Lambda\mathcal{F}^T \ , \tag{4.3}$$

$$D' = d + A' \ , \tag{4.4}$$

$$A' = \Xi^{-1}A\Xi + \Xi^{-1}[d, \Xi] \ . \tag{4.5}$$

The field strength tensor $F \in \Lambda^2$, is defined by

$$F = D^2 = \{d, A\} + AA \ . \tag{4.6}$$

$A \in \Lambda^1\mathcal{B}$ contains a single factor Θ. It follows from this and from the standard properties of d that indeed $\{d, A\} \in \Lambda^2\mathcal{F}^T$.

Further examples of quasi-associative (differential-) geometry should be associated with the dual $\mathcal{G}$ of weak quasi quantum groups $\mathcal{G}^*$. It is well known that the co-product Δ in $\mathcal{G}^*$ yields a product in the dual $\mathcal{G}$. If Δ is only quasi-co-associative, the multiplication in $\mathcal{G}$ is quasi-associative in the sense that products with different positions of brackets are linear combinations of each other. One can hope to work out examples of quasi-associative differential calculi on $\mathcal{G}$ as it is done in the case of quantum groups (e.g. [83, 84, 48, 78]).

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