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Potential Flow Of The Renormalisation Group In A Simple Two Component Model *

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ABSTRACT

The renormalisation group (RG) flow on the space of couplings of a simple model with two couplings is examined. The model considered is that of a single component scalar field with φ^4 self interaction coupled, via Yukawa coupling, to a fermion in flat four dimensional space. The RG flow on the two dimensional space of couplings, $\mathcal{G}$, is shown to be derivable from a potential to sixth order in the couplings, which requires two loop calculations of the β-functions. The identification of a potential requires the introduction of a metric on $\mathcal{G}$ and it is shown that the metric defined by Zamalodchikov, in terms of two point correlation functions of composite operators, gives potential flow to this order.

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§1 The c-theorem And Potential Flow Of The Renormalisation Group

The question of the nature of the renormalisation group (RG) flow on the space of coupling constants for a quantum field theory is a recurrent one in physics. It has been shown [1] in two dimensional Euclidean field theory, assuming certain positivity conditions on the Hilbert space of the theory, that there exists a function on the space of coupling constants which is non-increasing along the RG trajectories (the c-theorem). This has very important and far reaching implications for the theory because it puts constraints on the way that the RG flow can be realised, for example it can never come back to visit a point where it has already been, thus eliminating the possibility of limit cycles. The non-increasing function, c, can be interpreted as a measure of the number of degrees of freedom of the theory and its decreasing nature as the length scale, l, is increased as being due to "integrating out" the degrees of freedom on scales less than l. At fixed points (conformal field theories) it is the central charge of the theory.

A stronger condition than the c-theorem is that of potential flow. The possibility of potential flow was emphasised in [2] and [3] . In the latter reference the three loop β-functions for massless φ^4 theory, with several φ^4 couplings, were shown to be derivable from a potential, and it has been conjectured that this property should hold to all orders in perturbation theory [4] , [5] . An attempt at deriving general integrability conditions on the β-functions for potential flow is presented in [6] .

If the space of couplings is equipped with an invertible positive definite metric, G_{ab}, and the β functions are derivable from a potential, $V(g)$, in coupling constant space,

$$\beta^b G_{ba} := \beta_a = \frac{\partial}{\partial g^a} V(g), \tag{1}$$

then a c-theorem follows easily. (Here $a = 1, \ldots, n$ labels the dimensionless real couplings g^a which can be thought of as co-ordinates on the n-dimensional space of interactions, denoted by $\mathcal{G}$.) The metric is necessary since the β-functions are naturally defined as vectors,

$$\beta^a = \kappa \frac{dg^a}{d\kappa}, \tag{2}$$

and the gradient of a function is a co-vector. The c-theorem follows from (1) by differentiation of the potential,

$$\kappa \frac{dV}{d\kappa} = \beta^a \partial_a V = G_{ab} \beta^a \beta^b \geq 0, \tag{3}$$

where the last inequality holds when positive definiteness of the metric is assumed. Hence the potential, $V(g)$, is a function on $\mathcal{G}$ which is non-decreasing along the RG flow and so $c(g) = -V(g)$ is a function on $\mathcal{G}$ which is non-increasing along RG trajectories. Of course this analysis depends crucially on the choice of metric and various possibilities for a metric tensor on the space of interactions have been considered in the literature, [1] [2] [3] [7] [8] [9] [10] . In particular reference [3] examines a concrete model, however the technical aspects are necessarily rather complicated, since they involve the three loop calculations of [11] . At the two loop level there is no question about whether or not the

RG flow in this model is potential, [2], since a quick look at the possible Feynman diagrams contributing to the β-functions easily shows, without any calculation whatsoever, that it is always possible to find a potential simply by adjusting the co-efficients of the appropriate vaccuum diagrams which can contribute to a potential. This is because every diagram that contributes to the four point function, and therefore to the RG flow, at the two loop level can be obtained by removing a vertex (differentiating with respect to a coupling) from a vacuum diagram and each possible vacuum diagram contributes only one diagram to the RG flow. Hence whatever the numerical contribution of any given diagram to the two loop four point function, this can always be obtained from a potential involving vacuum diagrams just by adjusting the co-efficient of the relevant vacuum diagram appropriately, i.e. there is a one to one correspondence between the diagrams contributing to the four point function at the two loop level and the vacuum diagrams from which these can be obtained by differentiation with respect to the couplings. At the three loop level this is no longer true, one finds more than one diagram contributing to the three loop four point functions coming from a single vacuum diagram and there is no guarantee that the coefficients of the four point diagrams are compatible with the possibility of being obtained by differentiation of a single vacuum diagram. The observation of Wallace and Zia was that it is possible to *choose* a metric which makes the co-efficients match consistently, at the order in which they were working.

The purpose of this paper is to present a simpler model where, even at two loops, looking at the Feynman diagrams is not sufficient and their contributions to the β-functions must be evaluated to determine whether or not the flow is potential. The model studied is that of a scalar field with φ^4 interaction coupled with Yukawa couplings to a single fermion in four dimensions. Thus the space of couplings is two dimensional. This model was briefely mentioned at the end of reference [3], but no calculations were presented and so no conclusions drawn. We shall see that the technical aspects are simpler than the three loop calculations of the model studied in [3]. In particular it allows for a physical interpretation of the metric which is exactly that of Zamolodchikov's metric [1], [9]. The approach presented here will be different to that of [3], in that we shall not assume potential flow to derive a metric but rather assume Zamolodchikov's metric and prove that this leads to potential flow of the renormalisation group equations, at least to fifth order in the couplings appearing in the β-functions (sixth order in the potential).

§2 The Model And The Metric

Consider a single massless scalar field coupled, via Yukawa couplings, to a massless four component fermion in four dimensional Euclidean space. The action is

$$S = \int d^4x H(x) = \int d^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\lambda}{4!}\varphi^4 + i\overline{\psi}\gamma^\mu\partial_\mu\psi + g\overline{\psi}\psi\varphi \right), \tag{4}$$

where λ and g are real couplings. We shall take $\lambda \sim g^2$ and calculate to order g^5. Following

Zamolodchikov [1], [9] we shall define a metric on the two dimensional space of couplings

$$G_{ab} = |x|^8 < \hat{\Phi}_a(x)\hat{\Phi}_b(0) > \Big|_{|x| = l}, \tag{5}$$

where $\hat{\Phi}_a(x) = \partial_a H(x) - < \partial_a H(x) >$ and $l = \kappa^{-1}$ is a renormalisation length. The metric is defined in terms of renormalised composite operators and the differentiations are with respect to renormalised couplings which are dimensionless. The factor of $|x|^8$ is included to make the metric dimensionless.

To lowest order in the couplings the metric can be evaluated by taking $\hat{\Phi}_a$ to be normal ordered and using Wick's theorem. Thus

$$\begin{aligned} G_{\lambda\lambda} &= \frac{|x|^8}{(4!)^2} <: \varphi^4(x) :: \varphi^4(0) :>= \frac{1}{4!}\frac{1}{(4\pi^2)^4} \ + \ o(\lambda) \\ G_{gg} &= |x|^8 <: \overline{\psi}\psi\varphi(x) :: \overline{\psi}\psi\varphi(0) :>= \frac{1}{4\pi^6} \ + \ o(g^2) \\ G_{g\lambda} &= \frac{|x|^8}{4!} <: \overline{\psi}\psi\varphi(x) :: \varphi^4(0) :>= 0 \ + \ o(\lambda g, g^3), \end{aligned} \tag{6}$$

independent of the value of $|x|$. The propagators are defined with the standard conventions,

$$< \varphi(x)\varphi(0) >= \frac{1}{4\pi^2|x|^2} \qquad \text{and} \qquad < \psi(x)\overline{\psi}(0) >= \frac{i}{2\pi^2}\frac{\gamma^\mu x_\mu}{|x|^4}. \tag{7}$$

The interactions contribute terms of order g^2 but we will not need to calculate them. Thus

$$G_{ab} = \frac{1}{(16\pi^2)\pi^4}\begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4!(16\pi^2)} \end{pmatrix} + o(g^2) \tag{8}$$

so this metric is flat and, one might have thought, trivial but we shall see that the factors in (8) are crucial for the property of potential flow. This metric could also have been obtained using momentum space renormalisation conditions in (5) rather then x-space conditions. For massless theories the result is the same, up to overall factors of $4\pi^2$. The lowest order form of the metric (8) also co-incides with that appearing in [10], but this equality will not hold at higher orders.

It will be convenient in the next section to get rid of some of the factors of $16\pi^2$ in metric. To this end define

$$\tilde{\lambda} = \frac{\lambda}{16\pi^2} \qquad \tilde{g} = \frac{g}{\sqrt{16\pi^2}}\ . \tag{9}$$

In these co-ordinates the metric (8) becomes

$$G_{\tilde{a}\tilde{b}} = \frac{1}{\pi^4}\begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4!} \end{pmatrix} + \ o(\tilde{g}^2). \tag{10}$$

§3 The β-functions To Order g^5 And Potential Flow

At one loop the diagrams contributing to the β-functions are given in Figures 1 and 2 for g and λ respectively. β^g is of order g^3 and β^λ is already of order g^4 (remember $\lambda \sim g^2$). The contributions to the β-functions are easily calculated using standard techniques (dimensional regularisation and minimal subtraction were used for the results presented here),

$$\begin{aligned} \beta^g &= \frac{5g^3}{16\pi^2} \ + \ o(g^5) \\ \beta^\lambda &= \frac{1}{16\pi^2}\big(3\lambda^2 + 8\lambda g^2 - 48g^4\big) \ + \ o(g^6). \end{aligned} \tag{11}$$

To have a consistent expansion in powers of the coupling we must calculate the next highest contribution to β^g, which involves the two loop diagrams in Figure 3 as well as some other diagrams contributing a term in g^5 which we do not need to calculate. Note that a two loop contribution to β^λ would be of order g^6, thus a loop expansion is not the same as an expansion in powers of the coupling. Diagram (i) in Figure 3 has an overlapping divergence, but the calculation is standard. Diagram (ii) is easier, it does not have any divergent subdiagrams. Including these two diagrams in β^g gives the following β-functions, consistent to order g^5,

$$\begin{aligned} \beta^g &= \frac{1}{16\pi^2}\left[5g^3 + \frac{1}{16\pi^2}\Big(\frac{\lambda^2 g}{12} - 2\lambda g^3 + ag^5\Big)\right] \ + \ o(g^7) \\ \beta^\lambda &= \frac{1}{16\pi^2}\big(3\lambda^2 + 8\lambda g^2 - 48g^4\big) \ + \ o(g^6), \end{aligned} \tag{12}$$

or, in terms of the co-ordinates (9),

$$\begin{aligned} \beta^{\tilde g} &= 5\tilde g^3 + \frac{1}{12}\tilde\lambda^2\tilde g - 2\tilde\lambda\tilde g^3 + a\tilde g^5 \ + \ o(\tilde g^7) \\ \beta^{\tilde\lambda} &= 3\tilde\lambda^2 + 8\tilde\lambda\tilde g^2 - 48\tilde g^4 \ + \ o(\tilde g^6). \end{aligned} \tag{13}$$

The value of the constant a does not affect the ensuing analysis.

The vacuum diagrams from which figures 1-3 can be obtained, by differentiating with respect to the couplings, are shown in figure 4. Diagrams (i) of figure 2 and diagram (i) of figure 3 both come from the same diagram (v) of figure 4 and there is no reason why their numerical contributions to the β-functions should be compatible with them both coming from a single term in a potential. This is reflected in equations (13) where $\beta^{\tilde\lambda}$ would require a term $4\tilde\lambda^2\tilde g^2$ in a potential whereas $\beta^{\tilde g}$ would require a term $\frac{1}{24}\tilde\lambda^2\tilde g^2$ - with different coefficients. Similar coments apply to diagrams (iii) of figure 2 and diagram (ii) of figure 3, both of which come from diagram (vi) of figure: $\beta^{\tilde\lambda}$ would require a term $-48\tilde\lambda\tilde g^4$ in a potential whereas $\beta^{\tilde g}$ would require $-\frac{1}{2}\tilde\lambda\tilde g^4$ - again with different coefficients.

However (13) are vectors, not co-vectors, and it is only co-vectors that can be obtained from differentiation of a potential. So we use the metric (10) to convert (13) into co-vectors, giving

$$\begin{pmatrix} \beta_{\tilde g} \\ \beta_{\tilde\lambda} \end{pmatrix} = \frac{1}{\pi^4}\begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{4!} \end{pmatrix}\begin{pmatrix} \beta^{\tilde g} \\ \beta^{\tilde\lambda} \end{pmatrix} = \frac{1}{\pi^4}\begin{pmatrix} 20\tilde g^3 + \frac{1}{3}\tilde\lambda^2\tilde g - 8\tilde\lambda\tilde g^3 + a'\tilde g^5 \\ \frac{1}{8}\tilde\lambda^2 + \frac{1}{3}\tilde\lambda\tilde g^2 - 2\tilde g^4 \end{pmatrix}, \tag{14}$$

where a' is a constant which is the sum of the co-efficient of $\tilde{g}^2$ in $G_{\tilde{g}\tilde{g}}$ and $4a$. It is immediately obvious that now

$$\beta_{\tilde{a}} = \partial_{\tilde{a}} V(g) \tag{15}$$

with the potential given by

$$V(g) = C + \frac{1}{\pi^4}\left(5\tilde{g}^4 + \frac{1}{4!}\tilde{\lambda}^3 + \frac{1}{6}\tilde{\lambda}^2\tilde{g}^2 - 2\tilde{\lambda}\tilde{g}^4 + \frac{a'}{6}\tilde{g}^6\right) + \ o(\tilde{g}^8) \tag{16}$$

(there are no terms of $o(\tilde{g}^7)$, the potential can only contain even powers of $\tilde{g}$).
Here C is an undetermined integration constant, independent of the couplings. When $\tilde{\lambda} = \tilde{g} = 0$, C might be interpreted as the central charge for a free massless scalar and a free massless fermion (with both left and right components) in four dimensions, but obviously it cannot be calculated by the methods used here.

It is remarkable that such a simple metric, with such a strong physical motivation, [9], does the trick. It would be very interesting to see whether or not this property holds to next order in the couplings, but this would require three loop calculations of the β-functions as well as the calculation of the higher order terms in the metric, up to $o(\tilde{g}^4)$, and this will not be attempted here.

§4 **Conclusions**

It has been shown that, to sixth order in $\tilde{g}$, the β-functions of the model presented in §2 can be obtained from the potential (16), with the constant C undetermined (a' can be calculated straightforwardly, but we have not done so because it is not necessary). It is tempting to speculate that this property may continue to higher, indeed all, orders in perturbation theory. If this were true it would be very interesting to determine what class of field theories has this property. If it is true, it is unlikely to be a fluke of one or two models and it is more probable that it has a deeper significance. The set of such models would clearly constitute a very important class of field theories.

An obvious question is, would it hold for massive theories? It is argued in [3] that the very existence of mass independent regularisation schemes (such as dimensional regularisation) shows that the introduction of masses should not affect the conclusions, but the details would become more complicated and it is not clear (at least to the author) that the simple interpretation of the metric as the two point function presented in (5) would continue to hold in the massive case. Perhaps a detailed analysis would show that it does still hold, or perhaps this form of the metric would have to be changed in the massive case and a different metric, e.g. that of O'Connor and Stephens [8], would be required for potential flow.

The ideas presented here are not new, but the model is a little simpler and, I believe, the physics a little clearer than in the three loop $\lambda\varphi^4$ model of Wallace and Zia and a connection has been made with Zamolodchikov's metric.

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