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Nonlinear Problems in Accelerator Physics

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Abstract

In the following report we want to review nonlinear problems in accelerator physics. Theoretical tools and methods are introduced and discussed, and it is shown how these concepts can be applied to the study of various nonlinearities in storage rings. The first part treats Hamiltonian systems (proton accelerators) whereas the second part is concerned with explicitly stochastic systems (e.g. electron storage rings).

*invited lectures at the International Workshop on Nonlinear Problems in Accelerator Physics, Berlin 1992

Contents

1 Introduction

As synchrotron radiation sources and as colliders, storage rings have become an important tool in physical research. Colliders are devices which allow two beams of ultrarelativistic charged particles circulating in opposite directions to be accumulated, stored and collided. (see Figure 1)

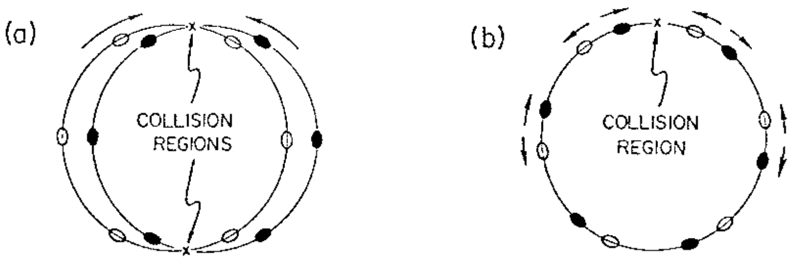

Figure 1: storage rings

The technical components of such an accelerator are magnets, a toroidal vacuum chamber and accelerating rf structures (cavities). Usually the stored beams consist of bunches, each of them containing 10^{10} to 10^{11} particles. The size of these bunches ranges from a tenth of a millimeter to a few centimeters. Some details of the electron-proton storage ring HERA at DESY in Hamburg are listed below:

	$electron-ring$	$proton-ring$
$circumference$	$6.336km$	$6.336km$
$energy$	$26/30GeV$	$820GeV$
$No.\ of\ bunches$	210	210
$particles/bunch$	3×10^{10}	10^{11}
$bunch-width\ at\ interaction\ point$	$0.27mm$	$0.29mm$
$bunch-height\ at\ interaction\ point$	$0,017mm$	$0,095mm$
$bunch-length\ at\ maximum\ energy$	$30mm$	$440mm$

An accelerator constitutes a complex many body system - namely an ensemble of 10^{10} to 10^{11} charged ultrarelativistic particles subject to external electromagnetic fields, radiation fields and various other influences such as restgas scattering, space charge effects and wakefields.

Although collective phenomena, as for example instabilities, are very important for the performance of an accelerator we restrict ourselves in this lecture to the classical single particle dynamics, i.e. we study the equations of motion of a single charged ultrarelativistic particle under the influence of external electromagnetic fields and radiation effects. In general these equations are nonlinear [1] [2]. The main nonlinearities are due to beam-beam interaction, due to nonlinear cavity fields or due to transverse multipole fields. These multipole fields are either introduced artificially, e.g. by sextupoles which compensate the natural chromaticity or they occur naturally as deviations from linear fields due to errors.

Because of these nonlinearities a storage ring acts as a nonlinear device schematically sketched in Figure 2. a_{in} is some initial amplitude (position, momentum given by the injection conditions)

Figure 2: storage ring as a nonlinear device

and a_{fin} is the amplitude after N (10^8 - 10^{10}) revolutions in the ring. In accelerator physics one often tries to define different zones according to a_{in}. For small amplitudes up to a certain boundary a_{lin} - the linear aperture - the storage ring behaves more or less like a linear element (at least for the time scales of interest, i.e. 10 - 20 hours storage time). For larger amplitudes the behaviour becomes more and more nonlinear and eventually at a_{dyn} - the dynamic aperture - the particle motion becomes unbounded. One problem of accelerator physics is to make quantitative predictions of these different zones, or stated in a different way, to calculate quantities such as the linear aperture a_{lin} or the dynamic aperture a_{dyn}. Furthermore one wants to know how these quantities depend on various machine parameters and the type of the nonlinearity. A better and - from a practical point of view - more relevant question is: what is the lifetime of the particle, or what is the probability for the particle to hit the vacuum chamber (first passage time) if it is injected into a certain volume in phase space. In order to solve these problems, various numerical and analytical tools have been developed, some of which will be described in the following.

This survey lecture is organized as follows: In the first part we will consider storage rings where radiation phenomena can be neglected, i.e. accelerators for protons or heavy ions. In HERA for example the radiation losses of a proton are a factor 10^{-7} less than the losses of an electron. Thus these storage rings can be modelled mathematically by nonlinear (in general nonintegrable) Hamiltonians. Nonintegrable means that the corresponding nonlinear equations of motion cannot be solved analytically. As we will see later the phase space dynamics of these systems shows a very rich and complicated structure. The questions we want to answer in the first part are:

- what does the Hamiltonian for the particle dynamics look like?
- what is in principle possible in these systems? (qualitative theory)
- which analytical (i.e. perturbative) tools are available for a quantitative study of these problems?

In the second part of this survey we will treat systems where radiation effects or noise effects are important. Because of the stochastic emission of the radiation, radiative systems can be modelled by explicit stochastic dynamical systems. A staightforward way to extend deterministic systems to include noise effects and explicit stochastic phenomena is to write down stochastic differential equations. In this lecture we will illustrate some of the subtleties related to stochastic differential equations including Gaussian white noise, and we will mention and illustrate some applications in accelerator physics.

This lecture cannot cover the whole subject exhaustively, we can only sketch the basic ideas and illustrate these ideas with simple (sometimes oversimplified) models. For many details we have to refer the reader to the references. Our main aim is to show that the single particle dynamics of storage rings represents an interesting field for nonlinear dynamics with a practical background.

2 Hamiltonian dynamics

2.1 Hamiltonian for coupled synchro-betatron motion

Starting point is the following relativistic Lagrangian for a charged particle under the influence of an electromagnetic field described by a vector potential $\vec{A}$ [3] :

$$\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{\dot{\vec{r}}^2}{c^2}} + \frac{e}{c}(\dot{\vec{r}}\vec{A}) \tag{1}$$

with

- e=elementary charge
- c=speed of light
- m_0=rest mass of the particle
- $\dot{\vec{r}}$=particle velocity

Usually one changes to a Hamiltonian description of motion and one introduces the curvilinear coordinate system depicted in Figure 3 [4].

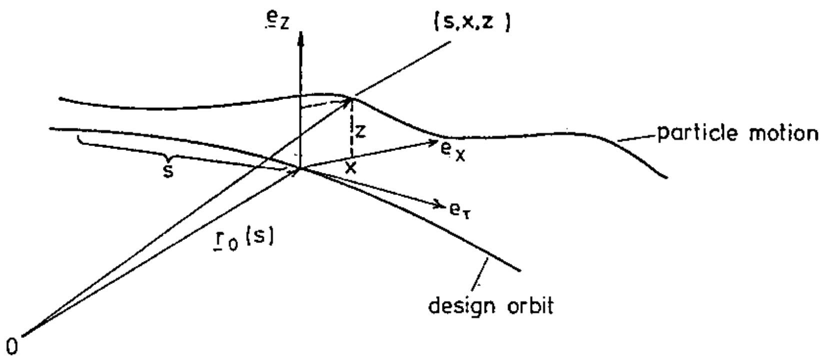

Figure 3: curvilinear coordinate system

It consists of three unit vectors $\vec{e}_\tau$,$\vec{e}_x$, $\vec{e}_z$ attached to the design orbit of the storage ring. s is the pathlength along this trajectory. For simplicity we have assumed a plane reference orbit with horizontal curvature κ only. Using s as an independent variable and introducing difference variables with respect to an equilibrium particle on the design orbit one obtains

$$\begin{aligned} &\mathcal{H}(x, z, \tau, p_x, p_z, p_\tau; s) = \\ &\quad -(1 + \kappa x)\{(1 + p_\tau)^2 - (p_x - \frac{e}{E_0}A_x)^2 - (p_z - \frac{e}{E_0}A_z)^2\}^{1/2} - \\ &\quad -(1 + \kappa x) \cdot \frac{e}{E_0} \cdot A_\tau + (1 + p_\tau) \end{aligned} \tag{2}$$

where we have used

- $v \approx c$ (ultrarelativistic particles)
- $\tau = s - ct$
- $p_\tau = \frac{\Delta E}{E_0}$
- E_0=energy of design particle.

The corresponding equations of motion are:

$$\begin{aligned}
\frac{d}{ds}\,x &= +\frac{\partial\mathcal{H}}{\partial p_x}; \quad \frac{d}{ds}\,p_x = -\frac{\partial\mathcal{H}}{\partial x} \\
\frac{d}{ds}\,z &= +\frac{\partial\mathcal{H}}{\partial p_z}; \quad \frac{d}{ds}\,p_z = -\frac{\partial\mathcal{H}}{\partial z} \\
\frac{d}{ds}\,\tau &= +\frac{\partial\mathcal{H}}{\partial p_\tau}; \quad \frac{d}{ds}\,p_\tau = -\frac{\partial\mathcal{H}}{\partial \tau}\,.
\end{aligned} \tag{3}$$

$\vec{A}(x,z,s) = (A_x(x,z,s), A_z(x,z,s), A_\tau(x,z,s))$ is the vector potential which determines the external electromagnetic fields. The transverse coordinates (x, z, p_x, p_z) describe the *betatron* motion and the longitudinal coordinates (τ, p_τ) describe the *synchrotron* motion. Some examples for the vector potential $\vec{A}(x,z,s)$ are shown below:

rf - cavity:

$$A_\tau = -\frac{L}{2\pi k}\cdot V_0\cdot\cos(k\frac{2\pi}{L}\tau)\cdot\delta(s-s_0) \tag{4}$$

with

- V_0=peak voltage of cavity
- L=circumference of storage ring
- k=harmonic number
- $\delta(s-s_0)$=delta function (localized cavity)

bending (dipole) magnet:

$$\frac{e}{E_0}A_\tau = -\frac{1}{2}(1+\kappa\cdot x) \tag{5}$$

with

- $\kappa = \frac{e}{E_0}B_z(x=z=0)$=horizontal curvature of design orbit. B_z=z-component of magnetic field

quadrupole:

$$\frac{e}{E_0}A_\tau = \frac{1}{2}g_0\cdot(z^2-x^2) \tag{6}$$

with

- $g_0 = \frac{e}{E_0}\cdot(\frac{\partial B_z}{\partial x})_{x=z=0}$= focusing strength of quadrupole

multipole (sextupole):

$$\frac{e}{E_0}\cdot A_\tau = -\frac{1}{6}\cdot\lambda_0\cdot(x^3-3xz^2) \tag{7}$$

with

- $\lambda_0 = \frac{e}{E_0}\cdot(\frac{\partial^2 B_z}{\partial x^2})_{x=z=0}$= strength of sextupole

multipole (octupole):

$$\frac{e}{E_0} \cdot A_\tau = \frac{1}{24} \cdot \mu_0 \cdot (z^4 - 6x^2z^2 + x^4) \tag{8}$$

with

- $\mu_0 = \frac{e}{E_0} \cdot (\frac{\partial^3 B_x}{\partial z^3})_{x=z=0}$=strength of octupole

Further examples for other types of electromagnetic fields can be found in [4].

Generally, by expanding the square root in equation (2) and the vector potential $\vec{A}(x,z,s)$ into a Taylor series around a reference orbit, various examples of nonlinear motion can be investigated.The linear part of the Hamiltonian is given by [5]:

$$\begin{aligned} &\mathcal{H}_0(x,z,\tau,p_x,p_z,p_\tau;s) = \\ &\quad \frac{1}{2}p_x^2 + \frac{1}{2}(\kappa^2(s) + g_0(s)) \cdot x^2 + \frac{1}{2}p_z^2 - \frac{1}{2}g_0(s) \cdot z^2 - \frac{1}{2}V(s) \cdot \tau^2 - \kappa(s) \cdot x \cdot p_\tau \end{aligned} \tag{9}$$

where $V(s) = V_0 \cdot \delta_p(s - s_0)$ with $\delta_p(s-s_0) = \sum_{n=-\infty}^{n=+\infty} \delta(s - (s_0 + n \cdot L))$ describes a localized cavity at position s_0 and where $g_0(s)$ characterizes the (periodic) focusing strength of the magnet system. $\mathcal{H}_0$ describes three coupled linear Floquet oscillators [5].

Two simple examples of nonlinear motion are given below:

Example 1:Nonlinear Cavity

$$\begin{aligned} &\mathcal{H}(x,z,\tau,p_x,p_z,p_\tau;s) = \\ &\quad \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 + \frac{1}{2}g_0(s) \cdot (x^2 - z^2) + \frac{1}{2}\kappa^2(s) \cdot x^2 - \kappa(s) \cdot x \cdot p_\tau + V(s) \cdot \cos(\tau) \end{aligned} \tag{10}$$

Introducing the dispersion function D defined by

$$D''(s) = -(\kappa^2(s) + g_0(s)) \cdot D(s) + \kappa(s) \tag{11}$$

with

$$(.)' = \frac{d}{ds}$$

via the canonical transformation [6],[7],[8] (depending on the *old* coordinates x, z, τ and the *new* momenta $\bar{p}_x, \bar{p}_z, \bar{p}_\tau$)

$$\begin{aligned} &F_2(x,z,\tau,\bar{p}_x,\bar{p}_z,\bar{p}_\tau;s) = \\ &\quad \bar{p}_x \cdot (x - \bar{p}_\tau \cdot D(s)) + \bar{p}_\tau \cdot D'(s) \cdot x + \bar{p}_\tau \cdot \tau + \bar{p}_z \cdot z - \frac{1}{2} \cdot D(s) \cdot D'(s) \cdot \bar{p}_\tau^2 \end{aligned} \tag{12}$$

and the corresponding transformation rules,

$$\left\{ \begin{aligned} x &= \bar{x} + \bar{p}_\tau \cdot D(s) \\ z &= \bar{z} \\ \tau &= \bar{\tau} + \bar{p}_x \cdot D(s) - \bar{x} \cdot D'(s) \end{aligned} \right. \tag{13}$$

$$\left\{ \begin{aligned} p_x &= \bar{p}_x + \bar{p}_\tau \cdot D'(s) \\ p_z &= \bar{p}_z \\ p_\tau &= \bar{p}_\tau \end{aligned} \right. \tag{14}$$

one obtains the Hamiltonian in the *new* variables $(\bar{x}, \bar{z}, \bar{\tau}, \bar{p}_x, \bar{p}_z, \bar{p}_\tau)$ as follows

$$\begin{aligned}
&\bar{\mathcal{H}}(\bar{x}, \bar{z}, \bar{\tau}, \bar{p}_x, \bar{p}_z, \bar{p}_\tau; s) = \\
&\quad \frac{1}{2}\bar{p}_x^2 + \frac{1}{2}(g_0(s) + \kappa^2(s)) \cdot \bar{x}^2 + \frac{1}{2}\bar{p}_z^2 - \frac{1}{2}g_0(s) \cdot \bar{z}^2 \\
&\quad -\frac{1}{2}\kappa(s) \cdot D(s) \cdot \bar{p}_\tau^2 + V(s) \cdot \cos(\bar{\tau} + D(s) \cdot \bar{p}_x - D'(s) \cdot \bar{x})
\end{aligned} \tag{15}$$

If there is no dispersion in the cavity region $(V \cdot D = 0)$, the synchrotron motion $(\bar{\tau}, \bar{p}_\tau)$ is completely decoupled from the betatron motion $(\bar{x}, \bar{z}, \bar{p}_x, \bar{p}_z)$ [9].

Example 2:multipole

As a second example of nonlinear motion we consider the influence of transverse multipole fields with the following Hamiltonian:

$$\mathcal{H}(x, z, p_x, p_z, s) = \frac{1}{2}p_x^2 + \frac{1}{2}p_z^2 - \frac{e}{E_0} \cdot A_\tau(x, z, s) \tag{16}$$

The equations of motion are given by

$$\left\{ \begin{aligned}
\frac{d}{ds}\, x &= p_x \\
\frac{d}{ds}\, z &= p_z \\
\frac{d}{ds}\, p_x &= \frac{e}{E_0} \cdot \frac{\partial A_\tau}{\partial x} = -\frac{e}{E_0} \cdot B_z(x, z, s) \\
\frac{d}{ds}\, p_z &= \frac{e}{E_0} \cdot \frac{\partial A_\tau}{\partial z} = \frac{e}{E_0} \cdot B_x(x, z, s)
\end{aligned} \right. \tag{17}$$

The magnetic field components B_x and B_z are usually expressed in terms of the skew and normal multipole expansion coefficients a and b according to :

$$(B_z + iB_x) = B_0 \cdot \sum_{n=2}^{\infty}(b_n + ia_n) \cdot (x + iz)^{n-1} \tag{18}$$

It is an easy exercise to verify, that these simple examples (15) and (16) contain the standard map [10] [11]:

$$\left\{ \begin{aligned}
\bar{\tau}(n) &= \bar{\tau}(n-1) + \bar{p}_\tau(n) \\
\bar{p}_\tau(n) &= \bar{p}_\tau(n-1) + V \cdot \sin(\bar{\tau}(n-1))
\end{aligned} \right. \tag{19}$$

and the quadratic map of Henon [12]:

$$\begin{pmatrix} x(n+1) \\ p_x(n+1) \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} x(n) \\ p_x(n) \end{pmatrix} + \begin{pmatrix} 0 \\ x^2(n+1) \end{pmatrix} \tag{20}$$

as limiting cases.These maps are extensively studied in nonlinear dynamics and show a very complex behaviour.Regular and chaotic motion is intricately mixed in phase space.For the quadratic map of Henon this is illustrated in Figure 4.

Thus one can expect, that the original system as described by (2) also shows highly nontrivial behaviour.

In order to get a better understanding of this complex dynamical phase space pattern, we will briefly repeat some facts from the *qualitative* theory of nonintegrable Hamiltonian systems.

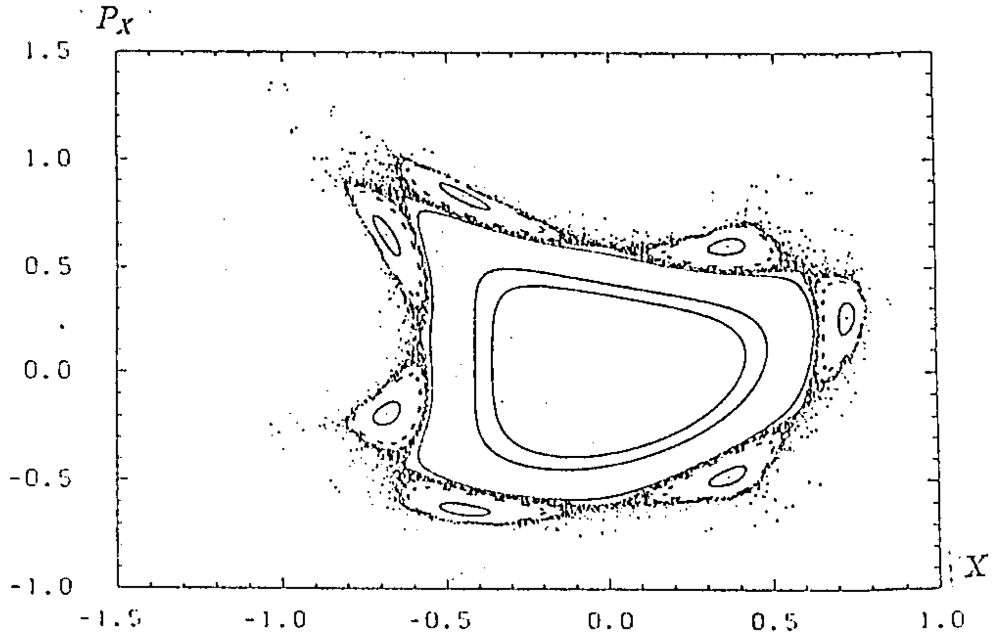

Figure 4: Hénon map (20) for an angle $\phi = 0.7071$

2.2 Qualitative theory of nonlinear Hamiltonian systems

Excellent and detailed reviews can be found in [10], [11],[13],[14],[15],[16].

The easiest way to investigate weakly perturbed nonintegrable Hamiltonian systems is via a map. The reduction of a Hamiltonian system to a nonlinear mapping has been a well-known procedure since Poincaré (1890). Consider for example a two-dimensional Hamiltonian system without explicit time (or s-) dependence $\mathcal{H}(q_1, q_2, p_1, p_2)$. The corresponding phase space is four-dimensional, and since $\mathcal{H}$ itself is a constant of the motion, the physically accessible phase space is three-dimensional. Consider a surface Σ in this three-dimensional -not necessarily Euclidean- space as depicted for example in Figure 5. The bounded particle motion induced by the Hamiltonian $\mathcal{H}$ will generally intersect this surface in different points $(P_0, P_1, ...P_n...)$. If one is not interested in the fine details of the orbit but only in the behaviour over longer time scales it is sufficient to consider the consecutive points $(P_0, P_1....)$ of intersection. These contain complete information on the Hamiltonian system. In this sense one has reduced the Hamiltonian dynamics to a mapping of Σ to itself, which is in general nonlinear (Poincaré surface of section technique). The Hamiltonian character is reflected in the symplectic structure of the map. Symplectic means that the Jacobian $\underline{J}$ of the map is a symplectic matrix with

$$\underline{J}^T \cdot \underline{S} \cdot \underline{J} = \underline{S} \tag{21}$$

where $\underline{J}^T$ is the transpose of $\underline{J}$ and where $\underline{S}$ is the symplectic unity

$$\begin{pmatrix} \underline{0} & \underline{1} \\ -\underline{1} & \underline{0} \end{pmatrix} \tag{22}$$

($\underline{1}$ designates the unit matrix). Similar mappings can also be derived for Hamiltonian systems with explicit periodic time (s-) dependence (this is normally the case in storage rings).

Another important fact and, after the work of Chirikov [11] one of the few beacons among an otherwise still dense mist of diverse phenomena, is the KAM-theorem (Kolmogorov,Arnold,Moser see for example [10]). We will only illustrate this theorem in the two-dimensional case and instead of concentrating on mathematical rigour we will discuss its physical implications.

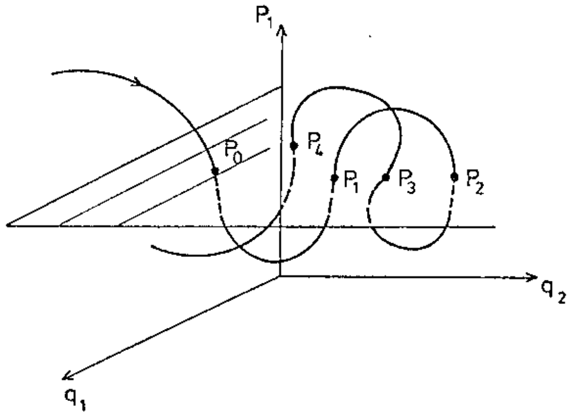

Figure 5: Poincaré surface of section method

Consider first the bounded motion of a two-dimensional autonomous (not explicitly time (s-) dependent) Hamiltonian system which is integrable. Roughly speaking, an n-dimensional system $\mathcal{H}(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ is integrable if there exists a canonical transformation to action-angle variables $(I_1, I_2, ..., I_n, \Theta_1, \Theta_2, ..., \Theta_n)$ such that the transformed Hamiltonian depends only on the n (constant) action variables $(I_1, I_2, ..., I_n)$ alone. For the two-dimensional case under consideration this implies, that $\mathcal{H}(q_1, q_2, p_1, p_2)$ is transformed into $\mathcal{H}(I_1, I_2)$ with the corresponding equations of motion:

$$\begin{cases} \frac{d}{ds} I_1 = 0 \\ \frac{d}{ds} I_2 = 0 \\ \frac{d}{ds} \Theta_1 = \frac{\partial \mathcal{H}}{\partial I_1} = \omega_1(I_1, I_2) = \text{const} \\ \frac{d}{ds} \Theta_2 = \frac{\partial \mathcal{H}}{\partial I_2} = \omega_2(I_1, I_2) = \text{const} \end{cases} \tag{23}$$

The motion is restricted to a two-torus, parametrized by the two angle variables Θ_1 and Θ_2, as depicted in Figure 6.

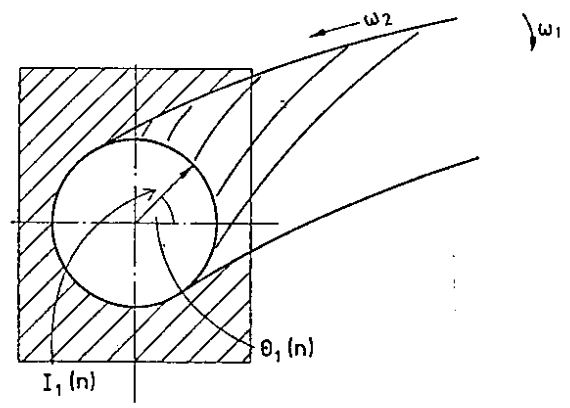

Figure 6: surface of section method for a two-dimensional integrable system

As surface of section one can choose the $(I_1 - \Theta_1)$ plane for Θ_2=constant. In this surface of section, which may be chosen to be just the plane of the page, the motion of the integrable two-dimensional system looks very simple. During the motion around the torus from one crossing of the plane to the next the radius of the torus (action variable) does not change (see (23))

$$I_1(n) = I_1(n-1) \tag{24}$$

and the angle Θ_1 changes according to (see (23))

$$\Theta_1(n) = \Theta_1(n-1) + \omega_1 \cdot T \tag{25}$$

where T is just the revolution time in Θ_2-direction from one intersection of the plane to the next, namely

$$T = \frac{2\pi}{\omega_2} \, . \tag{26}$$

Thus, for an integrable system one obtains the so-called twist-mapping:

$$\left\{ \begin{array}{c} I_1(n) = I_1(n-1) \\ \Theta_1(n) = \Theta_1(n-1) + 2\pi \cdot \alpha(I_1(n)) \end{array} \right. \tag{27}$$

The term $\alpha = \frac{\omega_1}{\omega_2}$ is called the winding number and is the ratio of the two frequencies of the system. In general α will depend on the actions. If α is irrational, the $\Theta_1(n)$ form a dense circle while if α is rational the $\Theta_1(n)$ close after a finite sequence of revolutions (periodic orbit or resonance).Thus, there are invariant curves (circles) under the mapping which belong to rational and irrational winding numbers. What happens now if a perturbation is switched on, i.e. if

$$\left\{ \begin{array}{c} I_1(n) = I_1(n-1) + \varepsilon \cdot f(I_1(n), \Theta_1(n-1)) \\ \Theta_1(n) = \Theta_1(n-1) + 2\pi \cdot \alpha(I_1(n)) + \varepsilon \cdot g(I_1(n), \Theta_1(n-1)) \ ? \end{array} \right. \tag{28}$$

In particular, can one still find invariant curves? The KAM-theorem says that this is indeed the case if the following conditions are fulfilled

- perturbation must be weak
- $\alpha = \frac{\omega_1}{\omega_2}$ must be sufficiently irrational, i.e. $|\alpha - \frac{p}{q}| \geq \frac{k(\varepsilon)}{q^{2+\delta}}$ with p, q integers, $\delta > 0$ and $k(\varepsilon) \to 0$ for $\varepsilon \to 0$

together with some requirements of differentiability and periodicity for f and g. For further details see for example [10], [16]. Under these assumptions most of the unperturbed tori survive the perturbation although in slightly distorted form.

The rational and some nearby tori, however, are destroyed, only a finite number of fixed points of the rational tori survive - half of them are stable (elliptic orbits around this fixed point), half of them are unstable (hyperbolic orbits).This is a consequence of the Poincaré-Birkhoff fixed point theorem [13].

The hyperbolic fixed points with their stable and unstable branches, which generally intersect in the homoclinic points, see Figure 7, are the source of chaotic motion in phase space, i.e. motion which is extremely sensitive to the variation of initial conditions. From a historical point of view it is interesting to note, that these facts were already known by Poincaré 100 years ago [13].

The motion around the elliptic fixed points can be considered as motion around a torus with smaller radius and the arguments used till now can be repeated on this smaller scale giving rise to the - now well known - schematic picture shown below (see Figure 8 ,"chaos-scenario" of weakly perturbed two-dimensional twist maps).

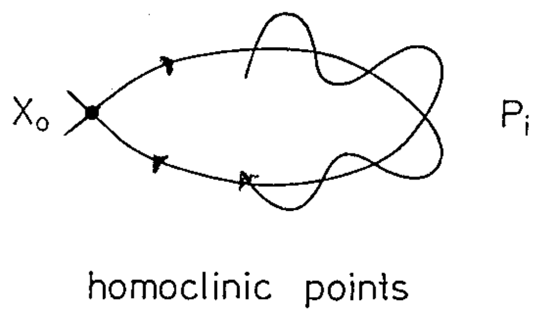

Figure 7: homoclinic intersections of stable and unstable branch of fixed point $\vec{X}_0$

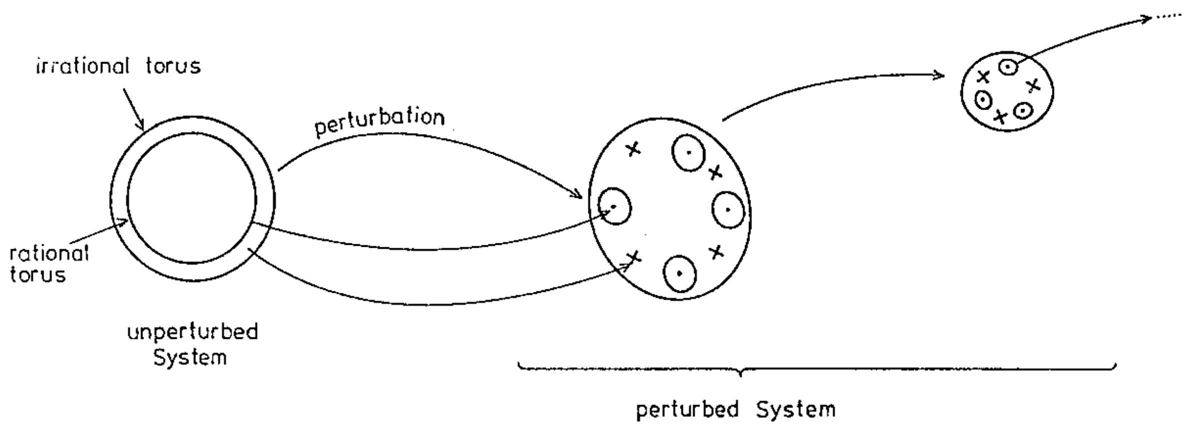

Figure 8: chaos scenario

Thus, the phase space pattern of a weakly perturbed integrable two-dimensional system looks extremely complicated. There are regular orbits confined to tori with chaotic trajectories delicately distributed among them. As the strength of the perturbation increases more and more KAM tori break up giving rise to larger and larger chaotic regions. This onset of large scale chaos has been the subject of many studies [10],[17]. For the standard map (see equation (19)) the situation is depicted in Figure 9.

One comment is pertinent at this point - two-dimensional systems are special in that the existence of KAM circles implies exact stability for orbits starting inside such an invariant curve. Since these trajectories cannot escape without intersecting the KAM tori, they are forever trapped inside.

The situation is much more complex and less well-understood for higher-dimensional systems like our storage ring (six-dimensional, explicitly s- dependent Hamiltonian system). The KAM theorem predicts three-tori in six-dimensional phase space, four-tori in eight-dimensional phase space etc. In this case chaotic trajectories can in principle always escape and explore all the accessible phase space although the motion can be obstructed strongly by existing tori. Chaotic regions can form a connected web along which the particle can diffuse, as has been demonstrated by Arnold (Arnold diffusion see for example [10], [17]).We will come back to this point later.

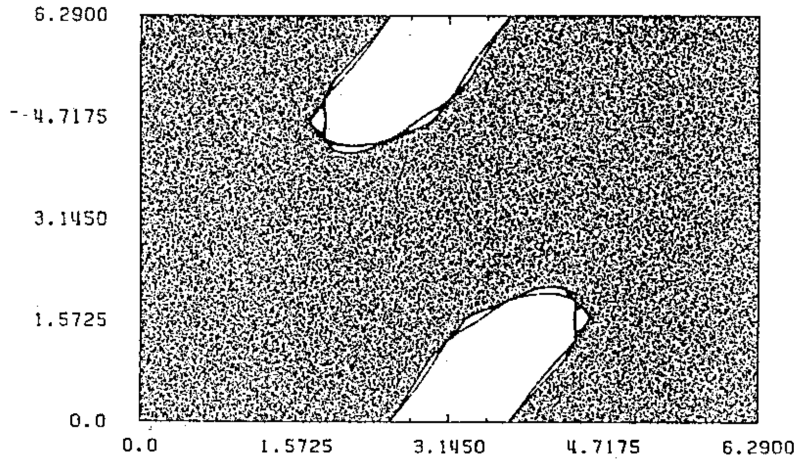

Figure 9: $(\bar{p}_\tau - \bar{\tau})$-phase space plot of the standard map (19) showing global chaos for $V = 3.3$

Figure 10 shows examples of regular and chaotic trajectories in a realistic model of a storage ring [18],[19]. We have used the characteristic Lyapunov exponent λ to distinguish between regular and chaotic motion [10], [20],[21]:

$$\lambda = \lim_{t\to\infty, d(0)\to 0} \frac{1}{t} \cdot \ln\left(\frac{|d(t)|}{|d(0)|}\right) \tag{29}$$

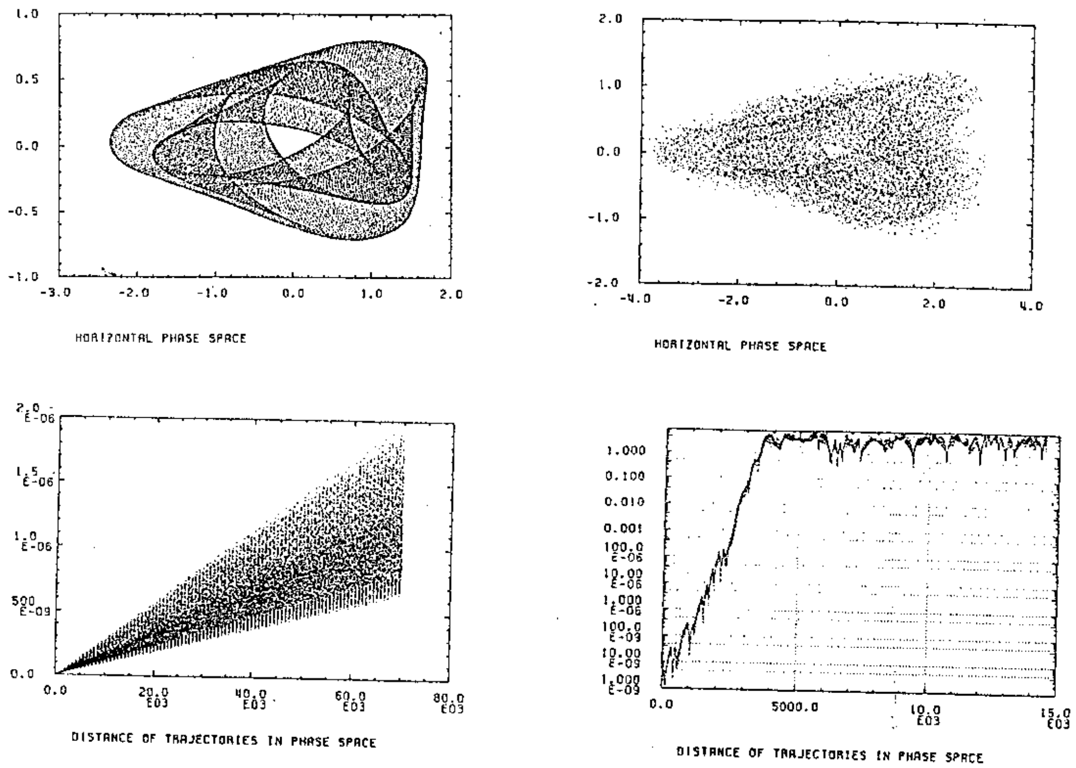

Figure 10: regular and chaotic trajectories in a realistic model of a storage ring and the evolution of the distance of two adjacent initial conditions

$d(t)$ describes how the (Euclidean) distance between two adjacent phase space points evolves with time and $d(0)$ is the initial distance. In a chaotic region of phase space this distance will grow exponentially fast and a non-zero Lyapunov-exponent λ is a quantitative measure for this separation. For the details of an explicit numerical calculation of the characteristic Lyapunov-exponents (for continuous and discrete dynamical systems) the reader is referred to [10], [20],[21].

Another method derived from the homoclinic structure of nearly integrable symplectic mappings is due to Melnikov, and this method belongs to one of the few analytical tools for investigating chaotic behaviour. It is applicable to dynamical systems of the form:

$$\left\{ \begin{array}{c} \frac{d}{ds}\,\vec{x}(s) = \vec{F}(\vec{x}) + \varepsilon \cdot \vec{G}(\vec{x}, s) \\ \vec{x} = (x_1, x_2)^T \in \mathbf{R^2} \\ s \in \mathbf{R} \\ \vec{F} = (F_1, F_2)^T \\ \vec{G} = (G_1, G_2)^T \end{array} \right. \tag{30}$$

where $\vec{F}$ usually describes a Hamiltonian system. The perturbation $\varepsilon\vec{G}$, which may also be weakly dissipative, is periodic in s, and the unperturbed system

$$\frac{d}{ds}\,\vec{x}(s) = \vec{F}(\vec{x}) \tag{31}$$

has a homoclinic orbit belonging to a saddle point $\vec{x}_0$. Homoclinic orbit means smooth joining of the stable and unstable branch of the saddle or hyperbolic fixed point (see Figure 11). The Melnikov method enables a kind of directed distance between the stable and unstable branch of the perturbed saddle $\vec{x}_0^\varepsilon$ to be calculated and thus allows the existence of homoclinic points to be predicted, a prerequisite of chaotic dynamics. A derivation of the Melnikov function and further details and applications can be found in [22].

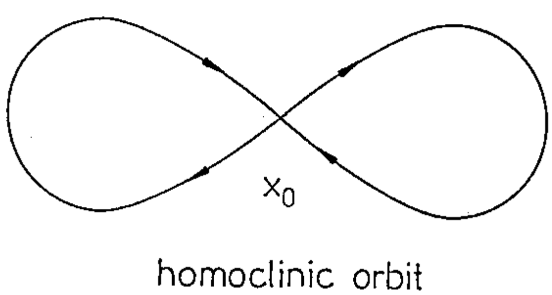

Figure 11: homoclinic orbit belonging to saddle $\vec{x}_0$

Remark:

High resolution 3D-colour graphics can be a very helpful tool for visualizing the dynamics of nonlinear four-dimensional mappings [19] [23]. Toy models like

$$\vec{x}(n+1) = \underline{R} \cdot \vec{x}(n) + \vec{f}(\vec{x}(n)) \tag{32}$$

with

$$\vec{x}(n) = \begin{pmatrix} x(n) \\ p_x(n) \\ z(n) \\ p_z(n) \end{pmatrix} \tag{33}$$

and

$$\underline{R}(\phi, \theta) = \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 & 0 \\ -\sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{34}$$

and

$$\left\{ \begin{array}{c} \vec{f}(\vec{x}(n)) = \begin{pmatrix} 0 \\ \frac{\partial f}{\partial x}(x(n+1), z(n+1)) \\ 0 \\ \frac{\partial f}{\partial z}(x(n+1), z(n+1)) \end{pmatrix} \\ (f = f(x, z)) \end{array} \right. \tag{35}$$

can help to get a better understanding of the break-up mechanism of invariant tori and the role periodic orbits play in this process [23],[24]. ($(n+1)$ periodic orbits are defined by: $\vec{x}(n+1) = \underline{T}(\vec{x}(n)) = \vec{x}(0)$ where $\underline{T}$ is some nonlinear (symplectic) map).

In the last chapter we have seen that the single particle dynamics of a proton in a storage ring can be modelled by nonintegrable Hamiltonians. The qualitative theory we have briefly sketched predicts a very rich and complicated phase space structure - regular and chaotic regions are intricately mixed in phase space.

When applying these concepts to accelerators one is immediately faced with questions such as: *What is the relevance of chaos for the practical performance of a storage ring? How do KAM tori break up as the strength of the nonlinearity increases? Can we somehow estimate the size of the chaotic regions in phase space? What is the character of the particle motion in this region? Can it be described by diffusion-like models? Is it possible to calculate escape rates of the particle if it is in such a chaotic region of phase space?*

A quantitative analysis of these and other questions makes extensive use of perturbation theory and numerical simulations of the system.

2.3 Perturbation theory

Perturbation theory for weakly perturbed integrable Hamiltonian systems is a vast field and an active area of research, which we cannot treat exhaustively in this survey lecture. We will only illustrate some of the basic results and ideas. Before we enter into detail we will briefly repeat some facts from the linear theory of particle motion in storage rings (linear theory of synchro-betatron oscillations [5])

In simple cases, as for example pure x- or z-motion without any coupling, the system is described by Floquet oscillators of the form:

$$\mathcal{H}(q, p, s) = \frac{1}{2} \cdot p^2 + \frac{1}{2} \cdot g_0(s) \cdot q^2 \tag{36}$$

with $p = p_x$, p_z, $q = x$, z and $g_0(s) = g_0(s + L)$ periodic function of circumference L.

It is well known that these Floquet type systems can be solved exactly. Using the optical functions $\alpha(s)$,$\beta(s)$ and $\gamma(s)$ defined by the following set of differential equations:

$$\frac{d}{ds}\,\alpha(s) = -\gamma(s) + \beta(s)\cdot g_0(s) \tag{37}$$

$$\frac{d}{ds}\,\beta(s) = -2\cdot\alpha(s) \tag{38}$$

$$\frac{d}{ds}\,\gamma(s) = 2\cdot\alpha(s)\cdot g_0(s) \tag{39}$$

one can find a canonical transformation to action angle variables I and Θ such that the Hamiltonian in equation (36) is transformed into [25]:

$$\bar{\mathcal{H}}(\Theta, I) = \frac{2\pi\cdot Q}{L}\cdot I \tag{40}$$

with

$$Q = \frac{1}{2\pi}\cdot\int_0^L \frac{ds'}{\beta(s')} \tag{41}$$

(Q is the so-called tune of the machine) and

$$I = \frac{q^2}{2\beta(s)}\cdot\{1 + (\frac{\beta(s)\cdot p}{q} + \alpha(s))^2\} \tag{42}$$

(I is called Courant-Snyder invariant see [26]).

In realistic cases there is always some coupling between the different degrees of freedom and the situation is more complicated. In these cases machine physicists rely on the one-turn matrix $\underline{M}$ relating some initial state phase space vector $\vec{y}(s_{in})$ to the final state vector $\vec{y}(s_{fin})$ after one complete revolution around the ring:

$$\vec{y}(s_{fin} = s_{in} + L) = \underline{M}(s_{in} + L, s_{in})\cdot\vec{y}(s_{in}) \tag{43}$$

In general $\vec{y}$ is six-dimensional and consists of the phase space coordinates $x, z, \tau, p_x, p_z, p_\tau$. The linear one-turn map $\underline{M}(s_{in} + L, s_{in})$ contains all the information about the system. For example the stability of the particle motion depends on the eigenvalue spectrum of the (symplectic) matrix $\underline{M}$ [5] - stability is only guaranteed if the eigenvalues lie on the complex unit circle (see also Figure 12).

What happens now if we perturb such a linear system with some nonlinear terms? How can we extend the linear analysis to the nonlinear case?

In simple models we can start with a perturbed Hamiltonian

$$\mathcal{H}(q,p,s) = \mathcal{H}_0(q,p,s) + \varepsilon\cdot\mathcal{H}_1(q,p,s) \tag{44}$$

Using the action angle variables of the unperturbed system one can apply conventional Hamiltonian perturbation theory [25],[27], which we will sketch in a moment. The advantages of such an approach are that one easily gets simple analytical expressions for interesting machine parameters of the perturbed system in terms of the unperturbed quantities. The price one has to pay, however, is an over-simplification of the problem. Realistic machines with all their nonlinearities and perturbations are extremely complex and cannot be handled efficiently in such a way. In such a case one should try to extend the concept of the one-turn map to the nonlinear case. This "contemporary" approach of Hamiltonian-free perturbation theory for particle dynamics in storage rings has been strongly advanced by E.Forest [28] and a recent description can be found in [29].

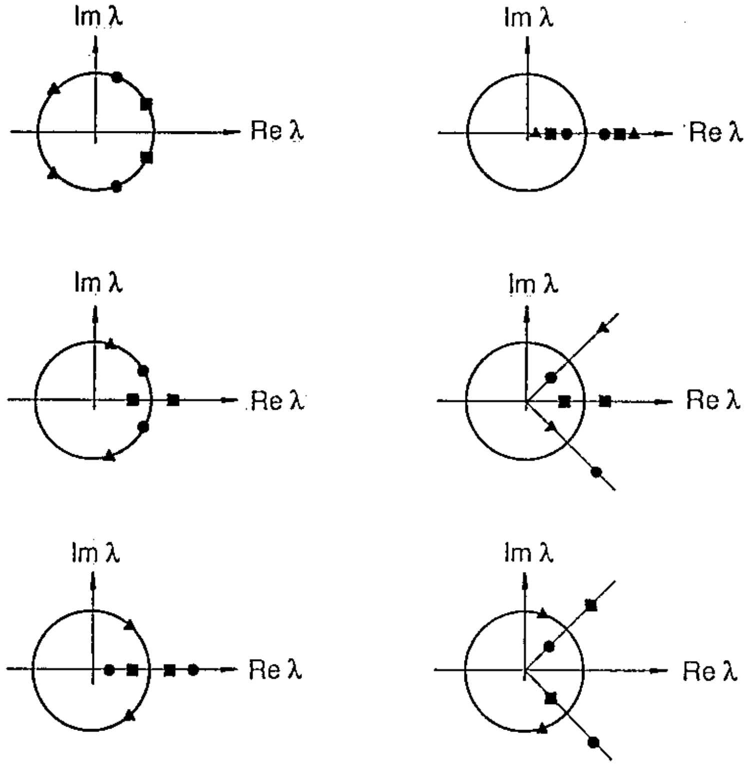

Figure 12: eigenvalue spectrum of a six-dimensional symplectic matrix

The basic idea of perturbation theory (common to both the direct Hamiltonian approach and the contemporary Hamiltonian-free approach) is to find - in a way to be specified - new variables, such that the system becomes solvable or at least easier to handle in this new set.

Let us first consider the Hamiltonian formalism.Various realizations exist for this kind of perturbation theory:Poincaré-von Zeipel [25],[27],[30], Lie methods [31], [32], and normal form algorithms [33],[34].

Here we will illustrate the Poincaré-von Zeipel method.

Assume our Hamiltonian is of the form

$$\mathcal{H}(\vec{q},\vec{p}) = \mathcal{H}_0(\vec{q},\vec{p}) + \varepsilon \cdot \mathcal{H}_1(\vec{q},\vec{p}) \tag{45}$$

where the vectors for the coordinates and momenta $\vec{q}$ and $\vec{p}$ may have arbitrary dimension (3 in the storage ring case)

$$\vec{q} = \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_n \end{pmatrix}, \qquad \vec{p} = \begin{pmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{pmatrix}. \tag{46}$$

Introducing the action angle variable vectors $\vec{I}$ and $\vec{\Theta}$ of the unperturbed system $\mathcal{H}_0$

$$\vec{I} = \begin{pmatrix} I_1 \\ \cdot \\ \cdot \\ \cdot \\ I_n \end{pmatrix}, \qquad \vec{\Theta} = \begin{pmatrix} \Theta_1 \\ \cdot \\ \cdot \\ \cdot \\ \Theta_n \end{pmatrix} \tag{47}$$

the Hamiltonian (45) can be rewritten in the form:

$$\mathcal{H}(\vec{I},\vec{\Theta}) = \mathcal{H}_0(\vec{I}) + \varepsilon \cdot \mathcal{H}_1(\vec{I},\vec{\Theta}) \tag{48}$$

The problem would be trivial, if we could find a transformation to new variables $\vec{J} = (J_1...J_n)^T$ and $\vec{\psi} = (\psi_1...\psi_n)^T$ such that the transformed Hamiltonian depends only on the new action variables $J_1...J_n$ alone. Since most Hamiltonian systems are nonintegrable [13],[14] this cannot be done exactly. The best one can achieve is to push the nonlinear perturbation to higher and higher orders in ε i.e. after a sequence of N canonical transformations

$$\mathcal{H}(\vec{I},\vec{\Theta}) = \mathcal{H}_0(\vec{I}) + \varepsilon \cdot \mathcal{H}_1(\vec{I},\vec{\Theta})$$

is transformed into a form given by :

$$\begin{cases} \bar{\mathcal{H}}(\vec{J}^{(N)},\vec{\psi}^{(N)}) = \bar{\mathcal{H}}_0(\vec{J}^{(N)}) + \varepsilon^{N+1} \cdot R_N(\vec{\psi}^{(N)},\vec{J}^{(N)}) \\ \bar{\mathcal{H}}_0(\vec{J}^{(N)}) = \sum_{i=0}^{N} \varepsilon^i \cdot \mathcal{H}_0^{(i)}(\vec{J}^{(N)}) \\ \bar{\mathcal{H}}_0^{(0)}(\vec{J}^{(N)}) = \mathcal{H}_0(\vec{J}^{(N)}) \end{cases} \tag{49}$$

where

$$\vec{J}^{(N)} = \begin{pmatrix} J_1^{(N)} \\ \cdot \\ \cdot \\ \cdot \\ J_n^{(N)} \end{pmatrix}, \qquad \vec{\psi}^{(N)} = \begin{pmatrix} \psi_1^{(N)} \\ \cdot \\ \cdot \\ \cdot \\ \psi_n^{(N)} \end{pmatrix} \tag{50}$$

are the new variables after N transformations.Neglecting the remainder $\varepsilon^{N+1} \cdot R_N$ (which is of order ε^{N+1} i.e. one order higher than the first part in equation (49)) the system is then trivially solvable.

For example in first order of perturbation theory this is achieved by a canonical transformation (depending on the *old* coordinates $\vec{\Theta}$ and the *new* momenta $\vec{J}$)

$$F_2(\vec{\Theta},\vec{J}) = \vec{\Theta} \cdot \vec{J} + \varepsilon \cdot S_1(\vec{\Theta},\vec{J}) \tag{51}$$

where $S_1(\vec{\Theta},\vec{J})$ is given by:

$$S_1(\vec{\Theta},\vec{J}) = -\frac{1}{i} \cdot \sum_{\vec{n}\neq 0} \frac{\mathcal{H}_{1,\vec{n}}(\vec{J})}{\vec{\omega}\cdot\vec{n}} \cdot \exp(i\vec{n}\cdot\vec{\Theta}) \tag{52}$$

$\vec{\omega}$ designates the frequency vector

$$\vec{\omega}(\vec{J}) = \frac{\partial \mathcal{H}_0(\vec{J})}{\partial \vec{J}} \tag{53}$$

and $\mathcal{H}_{1,\vec{n}}(\vec{J})$ is defined by the Fourier expansion of $\mathcal{H}_1(\vec{\Theta},\vec{J})$

$$\mathcal{H}_1(\vec{\Theta},\vec{J}) = \sum_{\vec{n}} \mathcal{H}_{1,\vec{n}}(\vec{J}) \cdot \exp(i\vec{n}\cdot\vec{\Theta}). \tag{54}$$

However, there is a serious problem concerning the convergence of our perturbative approach: even if we exclude the *nonlinear resonances* $\vec{n} \cdot \vec{\omega} = 0$ in equation (52) the infinite sum always contains n_i's such that the denominator in equation (52) can become arbitrarily small (small divisor problem) making this whole enterprise very doubtful. Generally these expansions diverge. Nevertheless the hope is that these expansions can be useful as asymptotic series and one hopes that the new invariants $(J_1, ...J_n)$ calculated in this way approximate in some sense our original system. However, for finite perturbations, there is no proof for the accuracy, if any, of such an approximation. So some care is always needed when one applies perturbation theories of this kind. A careful analysis of the convergence properties of (52) leads immediately to the heart of the KAM theory and requires sophisticated mathematical tools, which are far beyond the scope of this survey lecture.

Let us now briefly illustrate the perturbative techniques for mappings [28],[35],[36],[37],[38].

As mentioned already in the introduction, an accelerator acts as a nonlinear device and an initial state phase space vector $\vec{y}(s_{in})$ is nonlinearly related to the final state $\vec{y}(s_{fin})$ by a symplectic map

$$\vec{y}(s_{fin}) = \mathcal{M}(\vec{y}(s_{in})) \quad . \tag{55}$$

Let us assume this map can be Taylor expanded up to some order N with respect to $\vec{y}(s_{in})$

$$y_i(s_{fin}) = \sum_j A_{ij} \cdot y_j(s_{in}) + \sum_{jk} B_{ijk} \cdot y_j(s_{in}) \cdot y_k(s_{in}) + ... \tag{56}$$

with the transfer matrix or aberration coefficients A_{ij}, B_{ijk} etc.

Dragt and Finn [39],[40] have shown that Lie algebraic techniques can be very efficient for handling maps like (56). The factorization theorem [37] for example states that $\mathcal{M}$ can be expressed as a product of *Lie transforms*

$$\mathcal{M} = e^{:f_1:} \; \ldots \; e^{:f_k:} \; .. \tag{57}$$

where $: f_i :$ denotes a *Lie operator* related to a homogeneous polynomial of degree i in the variables $y_i(s_{in})$ and $: f :$ acts on the space of phase space functions g via the Poisson bracket operation of classical mechanics [27]

$$: f : g \equiv \{f, g\} \tag{58}$$

Example

The map $e^{:\frac{a}{3}x^3:}$ gives the known expression for a sextupole in thin lens (kick) approximation [37], [38]:

$$\begin{pmatrix} x(s_{fin}) \\ p_x(s_{fin}) \end{pmatrix} = \begin{pmatrix} x(s_{in}) \\ p_x(s_{in}) + a \cdot x^2(s_{in}) \end{pmatrix} \tag{59}$$

because

$$\left\{ \begin{array}{c} e^{:\frac{a}{3}x^3(s_{in}):} \; x(s_{in}) = x(s_{in}) \\ e^{:\frac{a}{3}x^3(s_{in}):} \; p_x(s_{in}) = p_x(s_{in}) + a \cdot x^2(s_{in}) \end{array} \right. \tag{60}$$

In principle one could now try to construct the one-turn map for a nonlinear accelerator using these Lie algebraic tools. However, beyond an order $N = 3$ in the Taylor expansion, this becomes incredibly tedious and complicated. So we will discuss a more efficient way of obtaining Taylor expanded maps for one turn later.

The advantage of using maps in the form (57) is formal and lies in the Lie algebraic tools that are available for treating these systems. There is an elegant extension of the normal form theory to such cases [41]. The problem is -roughly stated- that given a map $\mathcal{M}$ one has to determine a map $\mathcal{A}$ such that

$$\mathcal{N} = \mathcal{A} \cdot \mathcal{M} \cdot \mathcal{A}^{-1} \tag{61}$$

is as simple as possible. Simple means that the action of the map is simple. We can easily illustrate this fact with the following map which describes the action of a single multipole in kick approximation [35],[42]

$$\begin{pmatrix} x(n+1) \\ p_x(n+1) \end{pmatrix} = \begin{pmatrix} \cos(2\pi \cdot Q) & \sin(2\pi \cdot Q) \\ -\sin(2\pi \cdot Q) & \cos(2\pi \cdot Q) \end{pmatrix} \cdot \begin{pmatrix} x(n) \\ p_x(n) + \varepsilon \cdot x^p(n) \end{pmatrix} \tag{62}$$

In complex notation $z = x + ip_x$ equation (62) can be rewritten as:

$$z(n+1) = \exp(-i \cdot 2\pi \cdot Q) \cdot \{z(n) + \frac{i\varepsilon}{2^p} \cdot (z(n) + z^*(n))^p\} \tag{63}$$

where z^* designates the complex conjugate of z.

Finding a map $\mathcal{A}$ implies that one transforms to a new set of variables

$$z \longrightarrow \xi \tag{64}$$

such that equation (63) takes the following form in the new variables:

$$\xi(n+1) = \exp\{-i\Omega(\xi(n) \cdot \xi^*(n))\}\, \xi(n) + o(\varepsilon^2) + ... \tag{65}$$

Now, up to order ε, the action of the map is very simple - it is just a rotation in the ξ-plane with a frequency (winding number) Ω which depends on the distance from the origin (see Figure 13). This kind of perturbative analysis has been developed in detail in [28], [35] and has led to a powerful strategy for investigating the nonlinear motion of particles in storage rings. We will come back to this point after the description of numerical simulations in the next section.

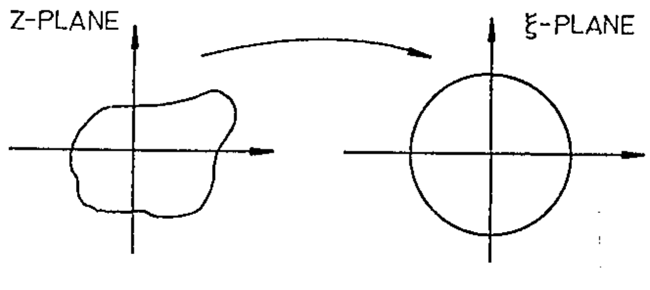

Figure 13: normal form theory of maps

2.4 Numerical simulations and particle tracking

The main idea of numerical simulations is to track particles over many revolutions in realistic models of the storage ring and to observe the amplitude of the particle at a special point s_0 [43], [44]. Given the initial amplitude $\vec{y}(s_0) = (x(s_0), z(s_0), \tau(s_0), p_x(s_0), p_z(s_0), p_\tau(s_0))$ one needs to know $\vec{y}(s_0 + n \cdot L)$ for n of the order of 10^9 (corresponding to a storage time of a particle of about 10 hours in HERA). Different methods and codes have been developed to evaluate $\vec{y}(s_0 + n \cdot L)$. Among others there are COSY INFINITY [45],TEAPOT [46],MARYLIE [47],TRANSPORT [48], RACETRACK [49]. We will not go into the details of these codes - much more will be said about these things in other contributions to this workshop - we restrict ourselves to some general remarks and facts instead.

An ideal code should be fast and accurate. It should allow for six-dimensional phase space calculations, thus allowing all kinds of coupling between the synchrotron and betatron oscillations. Error simulations of the storage ring should be possible as well as the calculation of interesting physical quantities such as tunes, (perturbed) invariants, nonlinear resonance widths etc.

One way of achieving this is by naive element to element tracking.

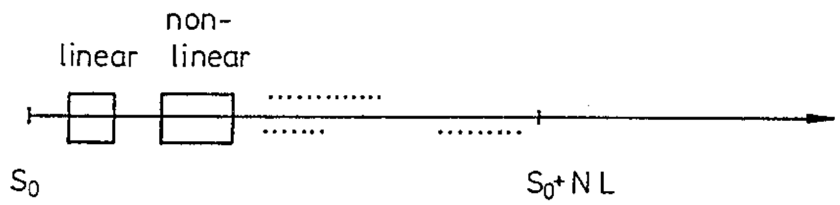

Figure 14: beam line

One has to solve the corresponding equations of motion for each element (linear or nonlinear). In each case the symplectic structure of the underlying Hamiltonian system has to be preserved by using suitable symplectic integration schemes [50], [51]. Such a code would be accurate but also extremely slow especially for large colliders like HERA and the SSC. A modification of these element-to-element tracking codes is the so-called kick approximation.Nonlinear elements described for example by terms

$$\mathcal{H}_1 = \sum_{n,m} a_{n,m}(s) \cdot x^n \cdot z^m$$

in the Hamiltonian are replaced by

$$\mathcal{H}_1 = \sum_{n,m} \bar{a}_{n,m} \cdot x^n \cdot z^m \cdot \delta_p(s - s_i)$$

where s_i denotes the localization of the nonlinear kick. These codes can speed up the calculations considerably and they also preserve the symplectic structure of the underlying equations automatically. However, one has to check carefully the accuracy of this kind of approximation.

Recent developments in particle tracking and numerical simulations use differential algebra tools as developed by M. Berz and described in other contributions to this workshop.

A typical numerical investigation of particle motion in nonlinear storage rings then comprises the following steps:

1. specification of the storage ring model by a Hamiltonian $\mathcal{H}$
2. numerical integration of the corresponding equations of motion for one complete revolution using symplectic integrators

3. extraction of the Taylor expanded form of the one-turn map (see equation (56)) from this calculation

4. use of this map for long-time tracking and for a perturbative analysis (to get physical quantities of interest such as invariants, perturbed frequencies and tunes of the synchro-betatron oscillations, nonlinear resonance widths etc.).

Step 3 is elegantly solved by using the powerful differential algebra package developed by M. Berz [52], [53] [54], [55], [56]. Figure 15 shows a flow chart for this approach.

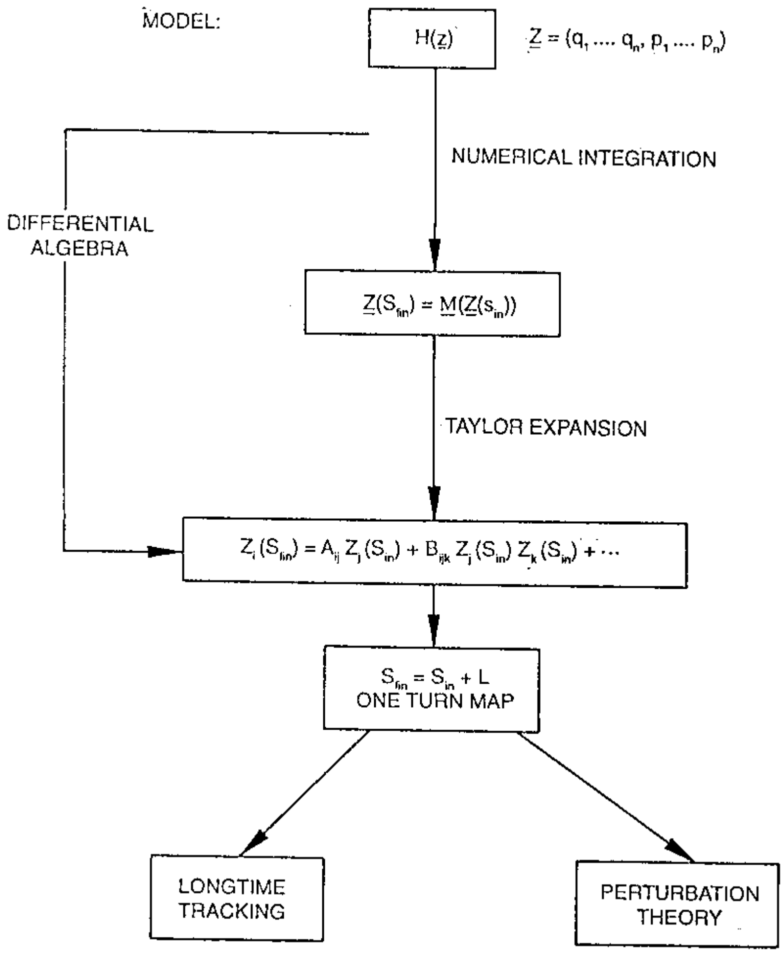

Figure 15: particle tracking

Let us conclude this section by mentioning some problems related to tracking namely the unavoidable rounding errors of the computers and the limited CPU time. The rounding errors depend on the number system used by the compiler and they can destroy the symplectic structure of the nonlinear mappings. Thus these rounding errors can simulate non-physical damping (anti-damping) effects [57]. In order to estimate the order of magnitude of these effects, one can switch to a higher precision structure in the computer hardware or software and observe the differences. Another way is to compare the differences between forward tracking of the particles and backward tracking [58].

The limited CPU time could be improved by developing special tracking processors [59]. Special processors have been successfully used in celestial mechanics for studies of the long-time stability of the solar system [60].

Besides these technical problems there are also some physical problems related to the evaluation and interpretation of the tracking data. For example fast instabilities with an exponential increase of amplitudes beyond a certain boundary (dynamic aperture) can easily be detected, whereas slow, diffusion-like processes which are very important for an understanding of the long-time dynamics are much more difficult to detect.

Nevertheless, tracking is the only way to obtain realistic estimates for the dynamic aperture up to $10^5 - 10^6$ revolutions, but it is very difficult and sometimes dangerous to extrapolate these data to longer times (10^9 revolutions or more). Furthermore, tracking is always very important for checking perturbative calculations because of the divergence problems in perturbation theories as mentioned above. We conclude this chapter on Hamiltonian systems with some final remarks.

2.5 Remarks

As mentioned already in the introduction, in accelerator physics one often tries to define different zones or regions corresponding to the importance of the nonlinearities. For small nonlinearities the accelerator behaves more or less like a linear element. A quantitative measure for this quasilinear behaviour is the so-called smear, a concept developed during the design studies for the SSC [61]. This quantity indicates how much the invariants of the linear machine are changed due to the nonlinear perturbations (see Figure 16). Another measure could be the amplitude dependence of the tunes. In the weakly nonlinear region one would expect that perturbation theory is the adequate theoretical tool.

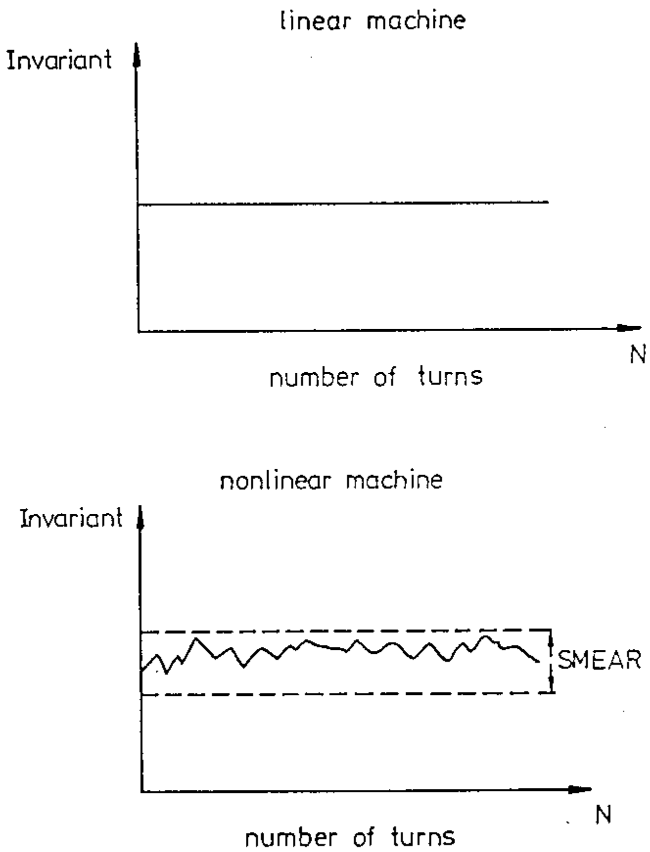

Figure 16: concept of smear

For stronger nonlinearities the dynamics becomes more and more nonlinear and chaotic i.e. sensitively dependent on the initial conditions. A quantitative measure for the onset of large scale chaos can be derived from Chirikov's resonance overlap criterion.One estimates the resonance widths and resonance distances and the criterion roughly states that no KAM tori survive in the region where resonance overlap occurs which leads to a completely chaotic particle motion in this area of phase space. The formal steps for applying this criterion are carefully described in [11] and [62]. Direct application to the standard map (19) for example yields a critical nonlinearity parameter of $V \approx 2.47$, which is in qualitative agreement with numerical simulations (see Figure 9).

The problem of beam-beam interaction in storage rings is an example where this method has been applied extensively by Tennyson et. al. [63], [64], [65].

An interesting and important question is: how does the particle motion look in this extended chaotic region of phase space? Can it be described by a diffusion-like process and can probabilistic concepts be used successfully in this context [66], [67]?

In order to illustrate some of the ideas and techniques used in this case we choose the following simple model [10], [11], [17]:

$$\begin{aligned} &\mathcal{H}(\psi_1, \psi_2, J_1, J_2, t) = \\ &\quad \frac{1}{2} \cdot (J_1^2 + J_2^2) + \varepsilon \cdot (\cos\psi_1 - 1) \cdot (1 + \mu \cdot \sin\psi_2 + \mu \cdot \cos t) \end{aligned} \tag{66}$$

which in extended phase space $(J_1, J_2, p, \psi_1, \psi_2, x = t)$ can be written as :

$$\begin{aligned} &\mathcal{K}(J_1, J_2, p, \psi_1, \psi_2, x) = \\ &\quad \frac{1}{2} \cdot (J_1^2 + J_2^2) + p + \varepsilon \cdot (\cos\psi_1 - 1) - \mu \cdot \varepsilon \cdot \sin\psi_2 - \mu \cdot \varepsilon \cos x + \\ &\quad \frac{\mu \cdot \varepsilon}{2}(\sin(\psi_2 - \psi_1) + \sin(\psi_2 + \psi_1) + \cos(\psi_1 - x) + \cos(\psi_1 + x)) \end{aligned} \tag{67}$$

$\mathcal{K}$ represents now an autonomous system in six-dimensional phase space. The primary resonances of the system (67) and the corresponding resonance widths are given by:

$$\left\{ \begin{array}{c} \frac{d}{dt}\,\psi_1 \approx J_1 = 0;\ \text{width} \sim \sqrt{\varepsilon} \\ \frac{d}{dt}\,\psi_2 \approx J_2 = 0;\ \text{width} \sim \sqrt{\varepsilon \cdot \mu} \\ \frac{d}{dt}\,(\psi_1 \pm \psi_2) \approx J_1 \pm J_2 = 0;\ \text{width} \sim \sqrt{\varepsilon \cdot \mu} \\ \frac{d}{dt}\,(\psi_1 \pm x) \approx J_1 \pm 1 = 0;\ \text{width} \sim \sqrt{\varepsilon \cdot \mu}. \end{array} \right. \tag{68}$$

For small ε, μ the energy surface is approximated by

$$\mathcal{K}_0(J_1, J_2, p) \approx \frac{1}{2}(J_1^2 + J_2^2) + p \tag{69}$$

and the resonance zones are given approximately by the intersection of the resonance surfaces (68) with the unperturbed energy surface (69), see Figures 17,18.

If $\varepsilon \gg \varepsilon \cdot \mu$, $J_1 = 0$ is the dominant resonance (guiding resonance). The motion (transport, diffusion) along the guiding resonance and the resonances which intersect it is called Arnold diffusion, see Figures 17,18.

For $\mu = 0$ the Hamiltonian in equation (67) is integrable (nonlinear pendulum) and a constant of the motion. For $\mu \neq 0$ the Hamiltonian is nonintegrable and the separatrix of the $J_1 = 0$ resonance will be replaced by a chaotic layer. Furthermore, we expect some diffusive variation of the energy in this case. One can calculate this variation approximately [10], [11], [17].

Using

$$\Delta\mathcal{H}(J_1, J_2, \psi_1, \psi_2, t) =$$
$$\int_{-\infty}^{\infty} dt \frac{d\mathcal{H}}{dt} = \int_{-\infty}^{\infty} \varepsilon \cdot \mu \sin t \cdot (1 - \cos \psi_1) dt \tag{70}$$

(see equation (66)) and replacing $\psi_1(t)$ by the unperturbed separatrix expression $\psi_{1sx}(t)$ the evaluation of the resulting Melnikov-Arnold integral [11] gives:

$$< (\Delta\mathcal{H})^2 > \approx 8\pi^2 \cdot \mu^2 \cdot \exp\{-\frac{\pi}{\sqrt{\varepsilon}}\} \tag{71}$$

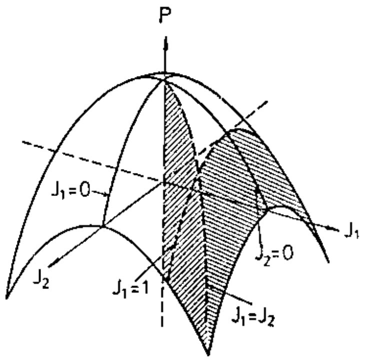

Figure 17: energy surface of unperturbed system (69)

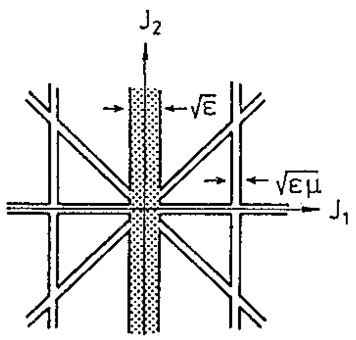

Figure 18: projection of resonance curves on $(J_1 - J_2)$ plane

Equation (71) is an estimate for the short time variation of the energy of the system for trajectories deep inside the chaotic layer of the $J_1 = 0$ resonance.Further details can be found in [10], [11], [17]. This kind of analysis has been applied in accelerator physics by Brüning see [68].

As mentioned already, the outstanding problem of accelerator physics is the long time stability of particle motion under the influence of various nonlinearities such as magnetic multipoles, rf fields and beam-beam forces.

In perturbation theory one usually approximates a nonintegrable system by an integrable (solvable) system. Whether this approximation really reflects the "reality" of the nonintegrable case has to be checked very carefully especially because integrable systems have no chaotic regions in phase space and because the dominant instability mechanisms are related to chaotic diffusion or transport [69]. For two-dimensional systems chaotic transport is in general only possible by breaking KAM tori (but see also [70]).For higher-dimensional systems the chaotic layers can form a connected web along which diffusion like motion is always possible.

To extract information about the long time stability of particle motion from numerical simulations is also a difficult task as mentioned above. So-called survival plots (see Figure 19) [71] can be helpful in getting some insight into the problem.

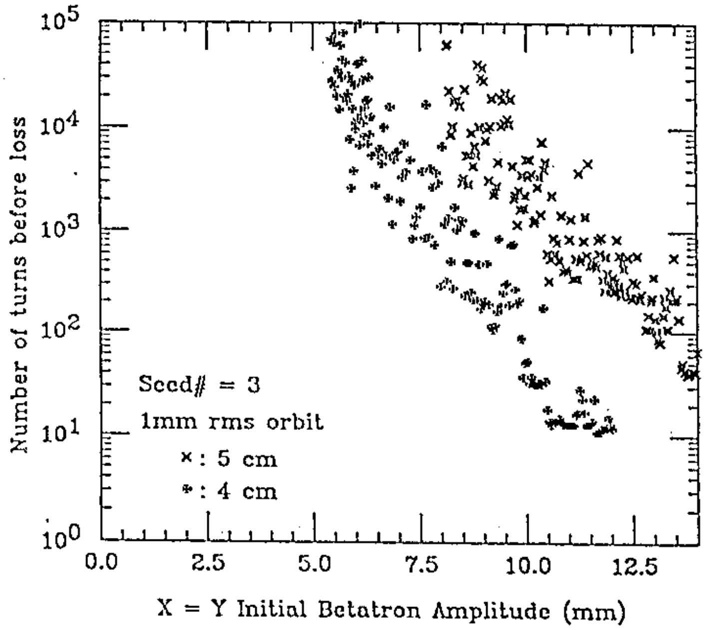

Figure 19: survival plot for the SSC, i.e. number of turns of the particle before loss versus initial betatron amplitude of the particle

An ideal and realistic - but mathematically very complicated - approach would be to consider the perturbed system

$$\mathcal{H}(\vec{I},\vec{\Theta}) = \mathcal{H}_0(\vec{I}) + \varepsilon \cdot \mathcal{H}_1(\vec{I},\vec{\Theta})$$

and to find some rigorous estimates for the time variation of the actions $\vec{I}$ - for example to predict a time T for which the variation of $\vec{I}$ is less than some fixed upper limit. This is the spirit of Nekhoroshev's theory [72]. First promising attempts to apply these ideas to accelerator physics problems have been made by Warnock, Ruth and Turchetti see [73],[74], [75] for more details.

Since there are no exact solutions available for the complicated nonlinear dynamics in storage rings and in order to check and test the theoretical concepts and tools described above, existing accelerators at FERMILAB, CERN and at the University of Indiana have been used for experimental investigations of the particle motion. Summaries of these results and further details can be found in [76], [77], [78].

As an example of an experimentally observed phase space plot of a nonlinear machine we show Figure 20 [79].

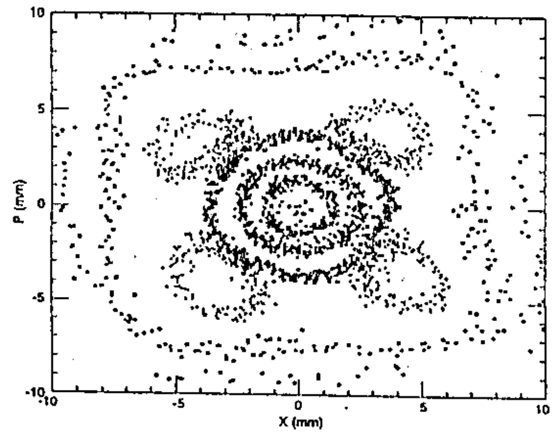

Figure 20: measured transverse phase space plot near a $Q_x = 15/4$ nonlinear resonance in the IUCF

Figure 21 summarizes the status of the art of the nonlinear particle motion in storage rings.

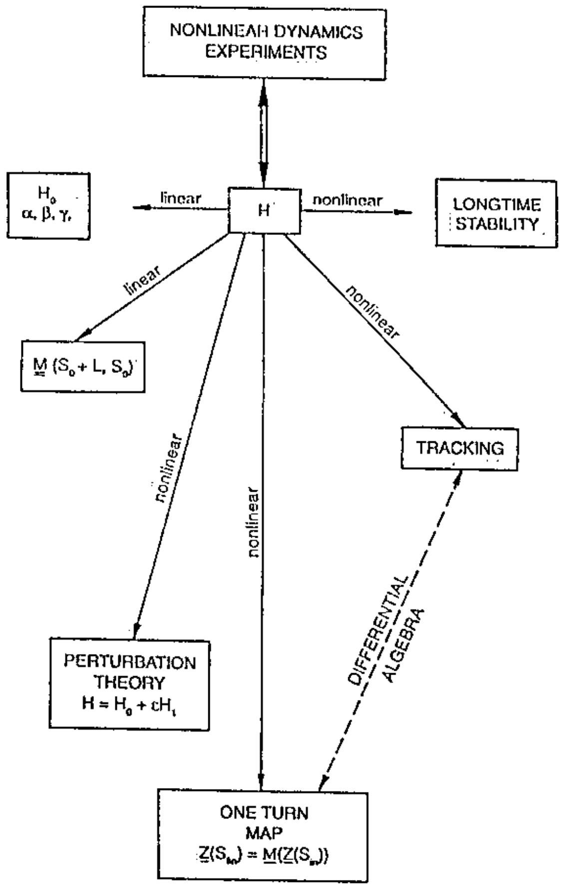

Figure 21: nonlinear dynamics of storage rings

In the next chapter we will investigate explicitly stochastic systems i.e. systems subject to noise or stochastic forces.

3 Stochastic dynamics in storage rings

In the first part of this lecture we have shown how the single particle dynamics of proton storage rings can be described by nonintegrable Hamiltonians. These systems show a very complex dynamics - regular and chaotic motion is intricately mixed in phase space. We have mentioned the concept of Arnold diffusion and chaotic transport and we have asked whether probabilistic methods can be applied successfully in this context.

In the second part of our survey we want to investigate systems where probabilistic tools are necessary, because we want to study the influence of stochastic forces and noise. In this case the equations of motion, which describe the dynamics, take the form :

$$\frac{d}{dt}\,\vec{x}(t) = \vec{f}(\vec{x}, t; \vec{\xi}(t)) \tag{72}$$

or in the discrete time (mapping) case

$$\vec{x}(n+1) = \vec{f}(\vec{x}(n), \vec{\xi}(n)) \tag{73}$$

where $\vec{\xi}(t)$ or $\vec{\xi}(n)$ designates some explicit stochastic vector process with known statistical properties. Our aim will be to study the temporal evolution of $\vec{x}(t)$ or $\vec{x}(n)$ under the influence of these explicit stochastic forces. We will call this kind of (probabilistic) dynamics *stochastic dynamics* in contrast to the (deterministic) *chaotic dynamics* investigated in the first part of this review. Questions we want to answer in the following are:

Given the statistical properties of the random forces, what are the statistical properties of $\vec{x}(t)$ or $\vec{x}(n)$? How can we treat these systems mathematically? And how can we calculate, for example, average values $< x_i(t) >$ or correlations $< x_i(t)\, x_j(t') >$?

This part of the review is organized as follows. At first we will summarize some basic results of probability theory and the theory of stochastic processes. Then we will concentrate on stochastic differential equations and their use in accelerator physics problems. In the case that the fluctuating random forces are modelled by Gaussian white noise processes (which is quite often a very good approximation) we will illustrate the mathematical subtleties related to these processes.

Examples of stochastic differential equations are

1. Langevin equation approach to Brownian motion

$$\frac{d}{dt}\,v = -\eta \cdot v + \xi(t) \tag{74}$$

   with v particle velocity, η friction coefficient and $\xi(t)$ fluctuating random force

2. stochastically driven harmonic oscillator

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix} \tag{75}$$

3. spin diffusion or Brownian motion on the unit sphere [83] (see Figure 22)

$$\frac{d}{dt}\,\vec{S}(t) = \vec{H}(t) \times \vec{S}(t) \tag{76}$$

   where $\vec{H}(t)$ denotes a fluctuating field.

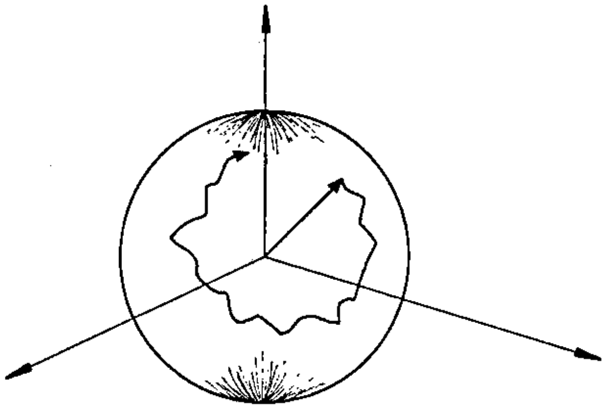

Figure 22: Brownian motion on the unit sphere

As we will see later, single particle dynamics in accelerators is a rich source for stochastic differential equations. Before we start with a systematic study of these systems we have to repeat some basic facts.

3.1 Summary of mathematical facts

Fundamental concepts of probability theory are the *random experiment* (e.g. throwing dice) and the *probability space.* This space consists of the sample space Ω of outcomes ω of the random experiment, a sigma algebra of events $\mathcal{A}$ i.e. a family of sets defined over Ω such that

1. $\Omega \in \mathcal{A}$
2. for every $A_j \in \mathcal{A}, \quad \bar{A}_j \in \mathcal{A}$
3. for $A_j, j = 1, 2, ...$ with $A_j \in \mathcal{A}$

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$$

(where $\bar{A}$ denotes the complement of A with respect to Ω) and a probability measure Pr defined over $\mathcal{A}$:

$$Pr : \mathcal{A} \longrightarrow [0, 1].$$

Pr is a measure for the frequency of the occurance of an event in $\mathcal{A}$ and it satisfies the following axioms:

1. $Pr(\phi) = 0$
2. $Pr(\Omega) = 1$
3. $Pr(A_i \cup A_j) = Pr(A_i) + Pr(A_j)$ for $A_i \cap A_j = \phi$

(ϕ is the empty set and designates the impossible event whereas Ω is the certain event).

The aim of probability theory is not the calculation of the probability measure of the underlying sample space Ω, but it is concerned with the calculation of new probabilities from given ones [80]. A rigorous treatment needs sophisticated measure theoretic concepts and is beyond the scope of this review. We will restrict ourselves to some basic facts and results which will be needed later. Detailed presentations of probability theory can be found in the references [80], [81], [82], [84], [85], [86], [87], [88]. In the following summary we will closely follow the book of Horsthemke and Lefever [80].

3.1.1 Random variables (r.v.)

The first notion we need is that of a random variable (r.v.) $\mathcal{X}$. A random variable is a function from the sample space Ω to $\mathbf{R}$ i.e. $\mathcal{X} : \Omega \longrightarrow \mathbf{R}$ with the property that

$$A = \{\omega | \mathcal{X}(\omega) \leq x\} \in \mathcal{A} \tag{77}$$

for all $x \in \mathbf{R}$. This means that A is an event which belongs to $\mathcal{A}$ for all x.

Remark: One should always distinguish carefully between the random variable $\mathcal{X}$ (calligrahic letter) and the realization x of the r.v., i.e. the value $\mathcal{X}$ takes on $\mathbf{R}$.

In the following we will only consider continuous r.v. which can be characterized by probability density functions $p_{\mathcal{X}}(x)dx$ which - roughly stated - give the probability of finding $\mathcal{X}$ between x and $x + dx$ i.e.

$$p_{\mathcal{X}}(x)dx = Pr(\{\omega | x \leq \mathcal{X}(\omega) \leq x + dx\}) \tag{78}$$

or in shorthand notation

$$p_{\mathcal{X}}(x)dx = Pr(x \leq \mathcal{X} \leq x + dx)$$

Given this probability density one can define expectation value, moments and mean square deviation or variance of a r.v.:

1. expectation value of a random variable $\mathcal{X}$

$$E\{\mathcal{X}\} \equiv < \mathcal{X} > \equiv m_{\mathcal{X}} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x p_{\mathcal{X}}(x)dx. \tag{79}$$

2. moment of order r

$$E\{\mathcal{X}^r\} \equiv < \mathcal{X}^r > \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} x^r p_{\mathcal{X}}(x)dx. \tag{80}$$

3. mean square deviation or variance

$$E\{(\delta\mathcal{X})^2\} \equiv E\{(\mathcal{X} - < \mathcal{X} >)^2\} \equiv \sigma^2 \stackrel{\text{def}}{=} \\ \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} (x - m_{\mathcal{X}})^2 p_{\mathcal{X}}(x)dx \tag{81}$$

As an example we consider the Gaussian distribution, which because of the central limit theorem (see the references), plays an important role in statistics. This distribution is defined by (see also Figure 23):

$$p_{\mathcal{X}}(x) = [(2\pi)^{1/2}\sigma]^{-1} \cdot \exp\{-\frac{(x-m)^2}{2\sigma^2}\} \tag{82}$$

In this case equations (79), (80) and (81) yield

$$E\{\mathcal{X}\} = m \tag{83}$$

$$E\{(\delta\mathcal{X})^2\} = \sigma^2 \tag{84}$$

$$E\{(\delta\mathcal{X})^r\} = \begin{cases} 0, \text{ for } r \geq 1 \text{ odd} \\ (r-1)!! \cdot \sigma^r, \; r \text{ even} \end{cases} \tag{85}$$

where we have used the following definition $(r-1)!! = 1 \cdot 3 \cdot 5 \cdot .... \cdot (r-1)$.

A Gaussian variable is thus completely specified by its first two moments.

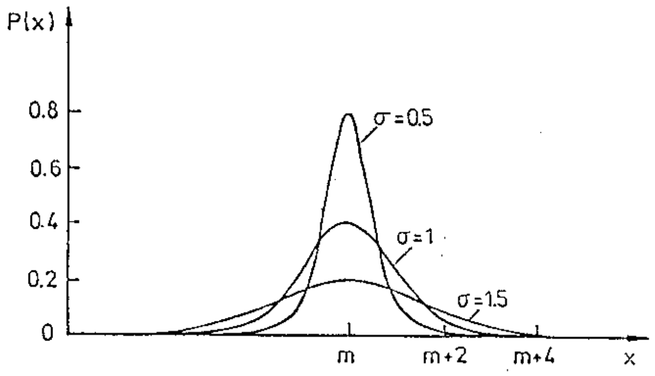

Figure 23: Gaussian distribution

Extending these considerations to the multivariable or random vector case $\mathcal{X}_1, ...\mathcal{X}_n$ requires the notion of joint probability densities i.e.

$$\begin{aligned} & p_{\mathcal{X}_1...\mathcal{X}_n}(x_1, ...x_n)dx_1...dx_n = \\ & \quad Pr(x_1 \leq \mathcal{X}_1 \leq x_1 + dx_1, \; ..., \; x_n \leq \mathcal{X}_n \leq x_n + dx_n). \end{aligned} \tag{86}$$

Moments, cross correlations, covariance matrix etc can then be defined. For example in the two-dimensional case $\vec{\mathcal{X}} = (\mathcal{X}, \mathcal{Y})^T$ mixed moments are defined by:

$$E\{\mathcal{X}^r \cdot \mathcal{Y}^p\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^r \cdot y^p p_{\mathcal{X},\mathcal{Y}}(x, y)dxdy \tag{87}$$

Another important definition we will need is the conditional density function $p_{\mathcal{X},\mathcal{Y}}(x|y)$. Conditional probability $p(A|B)$ means the probability that event A will take place, knowing with certainty that another event B has occured. For random variables the corresponding density is given by:

$$p_{\mathcal{X},\mathcal{Y}}(x|y) = \frac{p_{\mathcal{X},\mathcal{Y}}(x, y)}{p_{\mathcal{Y}}(y)} \tag{88}$$

or similarly

$$p_{\mathcal{X},\mathcal{Y}}(x, y) = p_{\mathcal{X},\mathcal{Y}}(x|y) \cdot p_{\mathcal{Y}}(y). \tag{89}$$

In the multivariable case one has accordingly:

$$\begin{aligned} & p_{\mathcal{X}_1...\mathcal{X}_n}(x_1...x_n) = \\ & \quad p_{\mathcal{X}_1...\mathcal{X}_n}(x_1|x_2...x_n) \cdot p_{\mathcal{X}_2...\mathcal{X}_n}(x_2|x_3...x_n) \cdot \\ & \quad ...p_{\mathcal{X}_{n-1},\mathcal{X}_n}(x_{n-1}|x_n) \cdot p_{\mathcal{X}_n}(x_n). \end{aligned} \tag{90}$$

3.1.2 stochastic processes (s.p.)

Next we introduce stochastic processes (s.p.) with the following definition: a family of random variables indexed by a parameter t, $\mathcal{X}_t$, is called a random or stochastic process (t may be continuous or discrete). A stochastic process thus depends on two arguments (t, ω). For fixed t, $\mathcal{X}_t$ is a random variable whereas for fixed ω and continuous t , $\mathcal{X}_{(.)}(\omega)$ is a real valued function of t which is called realization or sample path of the s.p. (realizations are designated by $x(t)$).

Generally, stochastic processes $\mathcal{X}_t$ are defined by an infinite hierarchy of joint distribution density functions

$$\begin{cases} p(x_1, t_1)dx_1 \\ p(x_1, t_1; x_2, t_2)dx_1dx_2 \\ \cdot \\ \cdot \\ \cdot \\ p(x_1, t_1; ...x_n, t_n)dx_1...dx_n \\ \cdot \\ \cdot \end{cases} \tag{91}$$

and are a complicated mathematical object. A proper treatment requires the so-called stochastic calculus see for example [82].

In a similar way as for random variables we can define moments and correlation functions. For example

$$\begin{aligned} E\{\mathcal{X}_{t_1}\mathcal{X}_{t_2}\} &=< x(t_1)x(t_2) >= \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2p(x_1, t_1; x_2, t_2)dx_1dx_2 \end{aligned} \tag{92}$$

is called the two-time correlation function of the stochastic process $\mathcal{X}_t$. Higher order correlations are obtained analogously

$$\begin{aligned} E\{\mathcal{X}_{t_1}...\mathcal{X}_{t_n}\} &=< x(t_1)...x(t_n) >= \\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} x_1...x_n \cdot p(x_1, t_1; ...; x_n, t_n)dx_1...dx_n \end{aligned} \tag{93}$$

One way of characterizing a stochastic process is by looking at its history or memory. The completely independent process is defined by

$$p(x_1, t_1; ...; x_n, t_n) = \prod_{i=1}^{n} p(x_i, t_i) \tag{94}$$

i.e. only the one-time distribution density is needed to classify and determine this process.

The next simplest case is the so-called *Markov process*. It is defined by

$$p(x_n, t_n|x_{n-1}, t_{n-1}; ..; x_1, t_1) = p(x_n, t_n|x_{n-1}, t_{n-1}) \tag{95}$$

with

$$t_1 \leq t_2 \leq ... \leq t_n \tag{96}$$

Equation (95) implies that if the present state is known, any additional information on the past history is totally irrelevant for predicting the future evolution.

Markov processes are completely specified by the transition probability density $p(x_n, t_n|x_{n-1}, t_{n-1})$ and the one-time probability density $p(x, t)$. Because of (95) and (90) we have:

$$\begin{aligned} &p(x_n, t_n; ...; x_1, t_1) = \\ &= p(x_n, t_n|x_{n-1}, t_{n-1}) \cdot p(x_{n-1}, t_{n-1}|x_{n-2}, t_{n-2}) \cdot ... \cdot p(x_2, t_2|x_1, t_1) \cdot p(x_1, t_1). \end{aligned} \tag{97}$$

The transition probability densities fulfill the following nonlinear functional relation (Chapman-Kolmogorov equation):

$$p(x_3, t_3|x_1, t_1) = \\ = \int_{-\infty}^{\infty} p(x_3, t_3|x_2, t_2) \cdot p(x_2, t_2|x_1, t_1) dx_2 \tag{98}$$

Examples of stochastic processes are:

1. Gaussian stochastic processes $\mathcal{X}_t$
   $\mathcal{X}_t$ is specified by

$$\begin{cases} p(x,t) \\ p(x_1, t_1; x_2, t_2) \\ \cdot \\ \cdot \\ \cdot \\ p(x_1, t_1; ...; x_n, t_n) \\ \cdot \\ \cdot \end{cases}$$

   If all the $m-$th order distributions are Gaussian i.e.

$$p(x_1, t_1; ...; x_m, t_m) = \\ (2\pi)^{-m/2} \cdot (\det \underline{\Lambda})^{-1/2} \cdot \exp\{-\frac{1}{2}(\vec{x} - \vec{m})^T \cdot \underline{\Lambda}^{-1} \cdot (\vec{x} - \vec{m})\} \tag{99}$$

   with $\vec{m}^T = (m_{\mathcal{X}}(t_1), ..., m_{\mathcal{X}}(t_m))$ and $\underline{\Lambda} = \Lambda_{ij} = E\{(\mathcal{X}_{t_i} - m_{\mathcal{X}}(t_i))(\mathcal{X}_{t_j} - m_{\mathcal{X}}(t_j))\}$, $\mathcal{X}_t$ is called a Gaussian stochastic process.

2. The Wiener process $\mathcal{W}_t$ which plays an important role in probability theory and which is defined by:

$$p(w_n, t_n; ...; w_0, t_0) = \prod_{i=0}^{n-1} p(w_{i+1}, t_{i+1}|w_i, t_i) \cdot p(w_0, t_0) = \\ = \prod_{i=0}^{n-1} [2\pi(t_{i+1} - t_i)]^{-1/2} \cdot \exp\{-\frac{(w_{i+1} - w_i)^2}{2(t_{i+1} - t_i)}\} \cdot p(w_0, t_0). \tag{100}$$

   The Wiener process is an example for an independent increment process. This can be seen as follows: defining the increments

$$\Delta \mathcal{W}_i = \mathcal{W}_i - \mathcal{W}_{i-1} \tag{101}$$

   and

$$\Delta t_i = t_i - t_{i-1} \tag{102}$$

   we obtain

$$p(\Delta w_n, t_n; ...; \Delta w_1, t_1; w_0, t_0) = \\ = \prod_{i=1}^{n} (2\pi \Delta t_i)^{-1/2} \cdot \exp\{-\frac{(\Delta w_i)^2}{2\Delta t_i}\} \cdot p(w_0, t_0) \tag{103}$$

i.e. the random variables $\Delta\mathcal{W}_{t_i}$ are statistically independent. Furthermore one calculates in this case

$$E\{\mathcal{W}_t\} = 0 \tag{104}$$

$$E\{\mathcal{W}_t\mathcal{W}_s\} = \min(t, s) \tag{105}$$

$$E\{\mathcal{W}_t^2\} = t. \tag{106}$$

A typical path of a Wiener process is shown in Figure 24. These paths are continuous but nowhere differentiable.

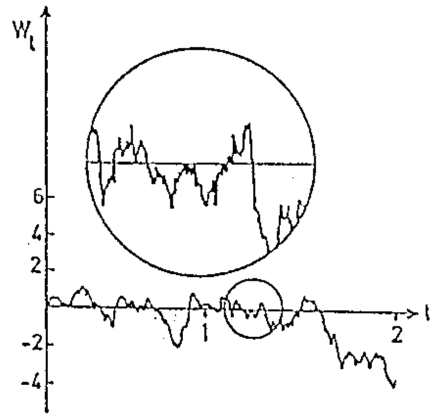

Figure 24: typical path of a Wiener process

3. Gaussian white noise process $\mathcal{Z}_t$. It is a completely random process with

$$p(z_1, t_1; ...; z_n, t_n) = \prod_{i=1}^{n} p(z_i) \tag{107}$$

i.e. independent values at every instant of time. It has

$$E\{\mathcal{Z}_t\} = 0 \tag{108}$$

and the two-time correlation function is given by

$$E\{\mathcal{Z}_t\mathcal{Z}_s\} = \delta(t-s) \ . \tag{109}$$

Since $\mathcal{Z}_t$ is Gaussian all odd correlations vanish automatically (see equation (85)) and the even correlations are given by [84]

$$E\{\mathcal{Z}_{t_1} \cdot \ \dots \ \cdot \mathcal{Z}_{t_{2n}}\} = \sum_{P_i} \delta(t_{i_1} - t_{i_2}) \cdot \ \dots \ \cdot \delta(t_{i_{2n-1}} - t_{i_{2n}}) \tag{110}$$

where the sum is taken over $(2n)!/(2^n\ n!)$ permutations. Gaussian white noise is a mathematical idealization and does not occur in nature. It plays a similar role in the theory of stochastic processes as the Dirac δ-function in functional analysis. One can show, that in a generalized sense [80] Gaussian white noise is the derivative of the Wiener process.

The last concept we need is that of a Markovian diffusion process. This is a Markov process with continuous sample paths. Diffusion processes play an important role in physics and in the context of stochastic differential equations with Gaussian white noise, as we will see in the next section. The temporal evolution of these diffusion processes is described by the so-called Fokker-Planck equation. This is a linear partial differential equation for the transition density $p(x, t|x_0, t_0)$ or the one-time density $p(x, t)$.

3.1.3 stochastic differential equations (s.d.e.)

As mentioned already stochastic differential equations are the natural extension of deterministic systems, if one wants to include noise effects

$$\frac{d}{dt}\,\vec{x}(t) = \vec{f}(\vec{x},t;\vec{\xi}(t)) \ .$$

Often $\vec{\xi}(t)$ is modelled by a Gaussian white noise process. This approximation is well justified if the fluctuating forces show only short-time correlations compared to other typical time-scales of the system. Introducing Gaussian white noise in dynamical systems is related to some mathematical problems which we want to illustrate now. In order to keep the notation as simple as possible we will restrict ourselves for the moment to scalar equations of motion (multiplicative stochastic processes [1]) of the form

$$\frac{d}{dt}\,\mathcal{X}_t = f(\mathcal{X}_t) + g(\mathcal{X}_t)\cdot\mathcal{Z}_t \tag{111}$$

where $\mathcal{Z}_t$ designates Gaussian white noise and where we have switched to the notation introduced above in order to make clear that we are treating stochastic processes.

Before we start our investigation let us repeat what Horsthemke and Lefever have written in this context:

The transition to Gaussian white noise sounds rather harmless but it is actually at this point that dangerous territory is entered, which contains hidden pitfalls and traps to ensnare the unwary theoretician ... if one succeeds in avoiding the various traps, either by luck, intuition or whatever, one captures a treasure which might be bane or boon: the white noise

The mathematical problems are related to the irregular behaviour of white noise and one has to ask which sense can be given to equations like (111). Remember, the sample paths of a Wiener process are continuous but nowhere differentiable, so what is the meaning of $\frac{d}{dt}$ in (111)?

We will not go through all the mathematical details, we only want to illustrate the subtleties, so that readers who are confronted with this problem are reminded of being careful when using stochastic differential equations with white noise. Excellent presentations of this problem can be found in [89],[90].

After these remarks we try to give a sense to the stochastic differential equation by rewriting it as an integral equation

$$\mathcal{X}_t = \mathcal{X}_0 + \int_0^t f(\mathcal{X}_s)ds + \int_0^t g(\mathcal{X}_s)\mathcal{Z}_s ds \tag{112}$$

which is eqivalent to

$$\mathcal{X}_t = \mathcal{X}_0 + \int_0^t f(\mathcal{X}_s)ds + \int_0^t g(\mathcal{X}_s)d\mathcal{W}_s. \tag{113}$$

As before $\mathcal{W}_s$ denotes the Wiener process. The second integral on the right hand side of equation (113) - a kind of a stochastic Stieltjes integral - is the main reason for the mathematical problems. Let us quote again Horsthemke and Lefever :

The problem is though a sense can be given to this integral and thus to the stochastic differential equation in spite of the extremely irregular nature of the white noise, there is no unique way to define it, precisely because white noise is so irregular. This has nothing to do with the different definitions of ordinary integrals by Riemann and Lebesgue. After all for the class of functions for which the Riemann integral as well as the Lebesgue integral can be defined, both integrals yield the same answer. The difference between the definitions for the above stochastic integral, connected with the names of Ito and Stratonovich , is much deeper; they give different results.

[1] the external noise is coupled in a multiplicative manner to $\mathcal{X}_t$ the statistical properties of which are sought

This difficulty can be illustrated as follows:
Consider a stochastic integral of the form

$$\mathcal{S}_t = \int_{t_0}^{t} \mathcal{W}_s d\mathcal{W}_s \tag{114}$$

If (114) would be Riemann integrable the result would be

$$\mathcal{S}_t = \frac{1}{2}(\mathcal{W}_t^2 - \mathcal{W}_{t_0}^2) \tag{115}$$

As in the Riemann case we try to evaluate (114) by a limit of approximating sums of the form

$$\mathcal{S}_n = \sum_{i=1}^{n} \mathcal{W}_{\tau_i^{(n)}} \cdot (\mathcal{W}_{t_i^{(n)}} - \mathcal{W}_{t_{i-1}^{(n)}}) \tag{116}$$

with a partition of the interval $[t_0, t]$

$$t_0 \leq t_1^{(n)} \leq t_2^{(n)} \leq \ldots \leq t_{n-1}^{(n)} \leq t$$

and

$$\tau_i^{(n)} \in [t_{i-1}^{(n)}, t_i^{(n)}]$$

or

$$\tau_i^{(n)} = (1-\alpha)t_{i-1}^{(n)} + \alpha t_i^{(n)}$$

with $0 \leq \alpha \leq 1$.

Using the stochastic calculus (the proper calculus to treat stochastic processes as mentioned above) one can show that the limit of $\mathcal{S}_n$ for $n \to \infty$ depends on the evaluation points $\tau_i^{(n)}$ or α [80], [81]

$$\lim_{n\to\infty} \mathcal{S}_n = \frac{1}{2}(\mathcal{W}_t^2 - \mathcal{W}_{t_0}^2) + (\alpha - \frac{1}{2})(t - t_0) \ . \tag{117}$$

Thus, the stochastic integral is no ordinary Riemann integral. However, an unambiguous definition of the integral can be given - and thus a consistent calculus is possible - if $\tau_i^{(n)}$ is fixed once and forever. Two choices are convenient

- $\alpha = 0$, Ito definition
- $\alpha = \frac{1}{2}$, Stratonovich definition.

Thus, a stochastic differential equation has always to be supplemented by a kind of interpretation rule for the stochastic integral. In both cases mentioned above one can show, that the solutions of the corresponding equations are Markovian diffusion processes. In the Ito case

$$(I) \ \ d\mathcal{X}_t = f(\mathcal{X}_t, t)dt + g(\mathcal{X}_t, t)d\mathcal{W}_t \tag{118}$$

the corresponding Fokker-Planck equation for the transition probability density $p(x, t|x_0, t_0)$ reads

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) =$$
$$= -\frac{\partial}{\partial x}[f(x,t)\cdot p(x,t|x_0,t_0)] + \frac{1}{2}\cdot\frac{\partial^2}{\partial x^2}[g^2(x,t)\cdot p(x,t|x_0,t_0)] \tag{119}$$

whereas in the Stratonovich case

$$(S)\ d\mathcal{X}_t = f(\mathcal{X}_t,t)dt + g(\mathcal{X}_t,t)d\mathcal{W}_t \tag{120}$$

the Fokker-Planck equation is given by

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) =$$
$$= -\frac{\partial}{\partial x}[f(x,t)\cdot p(x,t|x_0,t_0)] + \frac{1}{2}\cdot\frac{\partial}{\partial x}[g(x,t)\cdot\frac{\partial}{\partial x}(g(x,t)p(x,t|x_0,t_0))] \tag{121}$$

or equivalently by

$$\frac{\partial}{\partial t}p(x,t|x_0,t_0) =$$
$$= -\frac{\partial}{\partial x}([f(x,t) + \frac{1}{2}\frac{\partial g(x,t)}{\partial x}\cdot g(x,t)]\cdot p(x,t|x_0,t_0)) +$$
$$+\frac{1}{2}\cdot\frac{\partial^2}{\partial x^2}[g^2(x,t)\cdot p(x,t|x_0,t_0)]\ . \tag{122}$$

Equations (119) and (121) have to be supplemented with the initial conditions

$$p(x,t|x_0,t_0)_{t=t_0} = \delta(x-x_0)$$

and suitable boundary conditions for x.

The Ito calculus is mathematically more general but leads to unusual rules such as

$$\int_{t_0}^{t}\mathcal{W}_s d\mathcal{W}_s = \frac{1}{2}(\mathcal{W}_t^2 - \mathcal{W}_{t_0}^2) - \frac{1}{2}(t-t_0)$$

and some care is needed when one transforms from one process $\mathcal{X}_t$ to $\mathcal{R}_t = h(\mathcal{X}_t)$. For further details and a discussion of the relationship between Ito and Stratonovich approach (which preserves the "normal" rules of calculus) the reader is referred to the references.

Remarks:

1. The one-time probability density $p(x,t)$ of a Markovian diffusion process $\mathcal{X}_t$ also satisfies the Fokker-Planck equation (119) or (121).
2. In the case of purely additive noise where g does not depend on $\mathcal{X}_t$ there is no difference between the Ito and Stratonovich approach, so both stochastic differential equations define the same Markovian diffusion process.

3. Since Gaussian white noise is a mathematical idealization and can only approximately model real stochastic processes in nature, there is always the question how to interpret equation (111) in practical problems. In most physical cases one will rely on the Stratonovich interpretation as is suggested by a theorem due to Wong and Zakai [80] which roughly states :

   if we start with a phenomenological equation containing realistic noise $\mathcal{W}_t^{(n)}$ of the form

$$\frac{d}{dt}\,\mathcal{X}_t = f(\mathcal{X}_t) + g(\mathcal{X}_t)\cdot\frac{d}{dt}\,\mathcal{W}_t^{(n)} \tag{123}$$

   where all the integrals can be interpreted in the usual (e.g. Riemann) sense and if we pass to the white noise limit

$$\mathcal{W}_t^{(n)} \longrightarrow \mathcal{W}_t \tag{124}$$

   so that a stochastic differential equation of the form

$$\frac{d}{dt}\,\mathcal{X}_t = f(\mathcal{X}_t) + g(\mathcal{X}_t)\cdot\frac{d}{dt}\,\mathcal{W}_t \tag{125}$$

   is obtained (remember that Gaussian white noise is the derivative of the Wiener process $\mathcal{Z}_t = \frac{d}{dt}\,\mathcal{W}_t$) the latter has to be interpreted as a Stratonovich equation.

4. The above considerations can be extended to the multivariable case where $\mathcal{X}_t$, $f(\mathcal{X}_t)$ and $\mathcal{W}_t$ have to be replaced by vector quantities and where $g(\mathcal{X}_t)$ has to be replaced by a matrix. Now,the stochastic differential equation takes the form

$$d\vec{\mathcal{X}}_t = \vec{f}(\vec{\mathcal{X}}_t, t)dt + \underline{g}(\vec{\mathcal{X}}_t, t)d\vec{\mathcal{W}}_t \ . \tag{126}$$

   The Ito interpretation leads to a Fokker-Planck equation for the transition density $p(\vec{x}, t|\vec{x}_0, t_0)$ of the form

$$\begin{aligned}
&\frac{\partial}{\partial t}p(\vec{x}, t|\vec{x}_0, t_0) = \\
&\quad = -\sum_i \frac{\partial}{\partial x_i}[f_i(\vec{x}, t)\cdot p(\vec{x}, t|\vec{x}_0, t_0)] + \\
&\quad +\frac{1}{2}\cdot\sum_{i,j}\frac{\partial}{\partial x_i}\cdot\frac{\partial}{\partial x_j}[\{\underline{g}(\vec{x}, t)\underline{g}^T(\vec{x}, t)\}_{ij}\cdot p(\vec{x}, t|\vec{x}_0, t_0)]
\end{aligned} \tag{127}$$

   whereas the Stratonovich interpretation gives

$$\begin{aligned}
&\frac{\partial}{\partial t}p(\vec{x}, t|\vec{x}_0, t_0) = \\
&\quad = -\sum_i \frac{\partial}{\partial x_i}[f_i(\vec{x}, t)\cdot p(\vec{x}, t|\vec{x}_0, t_0)] + \\
&\quad +\frac{1}{2}\cdot\sum_{ijk}\frac{\partial}{\partial x_i}\{g_{ik}(\vec{x}, t)\frac{\partial}{\partial x_j}[g_{jk}(\vec{x}, t)\cdot p(\vec{x}, t|\vec{x}_0, t_0)]\} \ .
\end{aligned} \tag{128}$$

Examples

1. Wiener process

$$d\mathcal{X}_t = d\mathcal{W}_t \tag{129}$$

with the corresponding Fokker-Planck (diffusion) equation ($f = 0, g = 1$)

$$\frac{\partial}{\partial t} p(x,t|x_0,t_0) = \frac{1}{2} \cdot \frac{\partial^2}{\partial x^2} \cdot p(x,t|x_0,t_0) \tag{130}$$

2. The Ornstein Uhlenbeck process (see also equation (74))

$$d\mathcal{V}_t = -\eta \mathcal{V}_t dt + \sigma \cdot d\mathcal{W}_t \tag{131}$$

leads to the following Fokker-Planck equation ($f = -\eta, g = \sigma$)

$$\begin{aligned} &\frac{\partial}{\partial t} p(v,t|v_0,t_0) = \\ &= \frac{\partial}{\partial v}[\eta \cdot v \cdot p(v,t|v_0,t_0)] + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial v^2} \cdot p(v,t|v_0,t_0) \end{aligned} \tag{132}$$

3. Stochastically driven harmonic oscillator as an example of a multivariable system:

$$\begin{pmatrix} d\mathcal{X}_{1_t} \\ d\mathcal{X}_{2_t} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathcal{X}_{1_t}\, dt \\ \mathcal{X}_{2_t}\, dt \end{pmatrix} + \begin{pmatrix} 0 \\ d\mathcal{W}_t \end{pmatrix} \tag{133}$$

In this case one obtains as Fokker-Planck equation

$$\begin{aligned} &\frac{\partial}{\partial t} p(x_1,x_2,t|x_{1_0},x_{2_0},t_0) = \\ &= -\frac{\partial}{\partial x_1}[x_2 \cdot p(x_1,x_2,t|x_{1_0},x_{2_0},t_0)] + \frac{\partial}{\partial x_2}[x_1 \cdot p(x_1,x_2,t|x_{1_0},x_{2_0},t_0)] + \\ &+ \frac{1}{2} \cdot \frac{\partial^2}{\partial {x_2}^2} p(x_1,x_2,t|x_{1_0},x_{2_0},t_0) \ . \end{aligned} \tag{134}$$

Summarizing, we can say that stochastic differential equations are the natural extension of deterministic systems if one wants to study the influence of noise. Often these noise processes are approximated by Gaussian white noise, a mathematical idealization, which has to be treated with great care. However, the mathematical subtleties related to stochastic differential equations with white noise are outweighed by the results which are available for these Markovian diffusion processes namely the Fokker-Planck equation [86].

3.2 Stochastic dynamics problems in accelerator physics

After this - admittedly - very sketchy summary of mathematical results we will now investigate where stochastic differential equations arise in the single particle dynamics of accelerators [91].

At first we will study the particle motion in electron storage rings where radiation effects play an important role. Classically, radiation is taken into account by the following modified Lorentz equation [92], [93] :

$$\frac{d}{dt}\,(\frac{E}{c^2}\cdot\dot{\vec{r}}) = \frac{e}{c}\cdot\dot{\vec{r}}\times\vec{B}(\vec{r},t) + e\cdot\vec{\epsilon}(\vec{r},t) + \vec{R}^{rad}(\vec{r},t) \tag{135}$$

with

- $\vec{B}(\vec{r},t)$ magnetic field
- $\vec{\epsilon}(\vec{r},t)$ electric field
- $\vec{R}^{rad}(\vec{r},t)$ radiation reaction force
- E energy of particle

Because of the stochastic emission of the radiation, $\vec{R}^{rad}(\vec{r},t)$ is modelled by a stochastic force, which we divide into its average part and its fluctuating part according to

$$\vec{R}^{rad}(\vec{r},t) = <\vec{R}^{rad}(\vec{r},t)> + \delta\vec{R}^{rad}(\vec{r},t) \quad . \tag{136}$$

The average part is identified with the classical Lorentz-Dirac radiation reaction force, see for example [94]

$$<\vec{R}^{rad}(\vec{r},t)> = -\frac{2}{3}\cdot\frac{e^2}{c^5}\cdot\gamma^4\cdot\dot{\vec{r}}\cdot\{(\ddot{\vec{r}})^2 + \frac{\gamma^2}{c^2}\cdot(\dot{\vec{r}}\cdot\ddot{\vec{r}})^2\} \tag{137}$$

with $\gamma = \frac{E}{m_0 c^2}$.

$<\vec{R}^{rad}(\vec{r},t)>$ leads in general to the so-called radiation damping. The fluctuating part in equation (136),which mainly effects the energy variation in equation (135) is usually approximated by Gaussian white noise [5],[93],[95].

Thus, in the curvilinear coordinate system of the accelerator one generally obtains a system of six coupled nonlinear stochastic differential equations of the form [5]:

$$\frac{d}{ds}\,\vec{y}(s) = \vec{f}(\vec{y},s) + \underline{T}(\vec{y},s)\cdot\delta\vec{c}(s) \tag{138}$$

with $\vec{y}^T = (x, z, \tau, p_x, p_z, p_\tau)$. $\delta\vec{c}(s)$ designates a Gaussian white noise vector process.

Interesting physical questions one wants to answer are :

What are the average fluctuations of the particle around the closed orbit (beam emittances)? What is the particle distribution $p(\vec{y},s)$ i.e. what is the probability for finding the particle between $\vec{y}$ and $\vec{y}+d\vec{y}$ at location s ? Is there a stationary solution of this density i.e. what is $\lim_{s\to\infty}\, p(\vec{y},s)$? What is the particle lifetime in the finite vacuum chambers of the accelerator (time to hit the border) ?

These questions have been extensively studied in the linear case [92], [93], [95],[96], [97]

$$\frac{d}{ds}\,\vec{y}(s) = (\underline{A}(s) + \delta\underline{A}(s))\cdot\vec{y}(s) + \vec{c}(s) + \delta\vec{c}(s) \tag{139}$$

$\underline{A}$ designates the Hamiltonian part of the motion (six-dimensional coupled synchro-betatron oscillations) [5]; $\delta\underline{A}(s)$ describes the radiation damping due to $< \vec{R}^{rad} >$ (see equation (137)); $\delta\vec{c}$ is the fluctuating part of the radiative force and $\vec{c}$ denotes some additional field errors of the system. In this case one obtains compact and easily programmable expressions for the important beam parameters (beam emittances) [95], [96]. Furthermore, the corresponding Fokker-Planck equation for the probability distribution can be solved exactly [98], [99], [100].

An investigation of the nonlinear system is much more complicated and is an active area of research. Nonlinear systems such as an octupole-dipole wiggler or the beam-beam interaction in electron storage rings have been analyzed by various authors [101], [102], [103], [104], [105], [106]. Let us consider the latter case in more detail. The main problem is to understand the motion of a test particle under the influence of the nonlinear electromagnetic fields of the counter rotating beam [107] (see Figure 25).

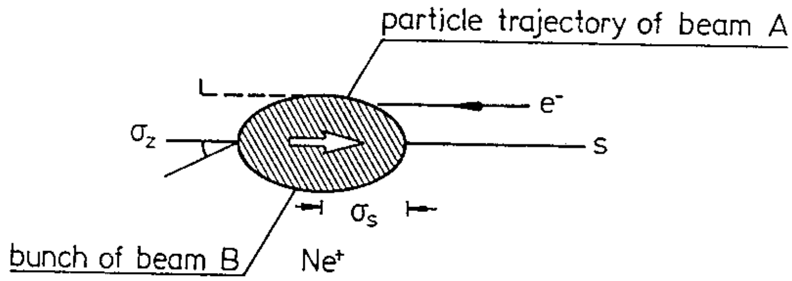

Figure 25: deflection of a test particle in beam A in the field of the counter rotating beam B

This so-called weak-strong model of the beam-beam interaction is mathematically described by the following set of equations [107] (perturbed Hamiltonian system) :

$$\begin{cases} \frac{d}{ds}\, x(s) = \frac{\partial}{\partial p_x}\, \mathcal{H}(x,z,p_x,p_z,s) \\ \frac{d}{ds}\, p_x(s) = -\frac{\partial}{\partial x}\, \mathcal{H}(x,z,p_x,p_z,s) - \gamma_x \cdot p_x + d_x \cdot \Gamma \\ \frac{d}{ds}\, z(s) = \frac{\partial}{\partial p_z}\, \mathcal{H}(x,z,p_x,p_z,s) \\ \frac{d}{ds}\, p_z(s) = -\frac{\partial}{\partial z}\, \mathcal{H}(x,z,p_x,p_z,s) - \gamma_z \cdot p_z + d_z \cdot \Gamma \end{cases} \tag{140}$$

The Hamiltonian $\mathcal{H}(x,z,p_x,p_z,s)$ consists of the linear part and the nonlinear potential due to the beam-beam interaction

$$\begin{aligned} &\mathcal{H}(x,z,p_x,p_z,s) = \\ &= \frac{p_x^2}{2} + K_x(s) \cdot \frac{x^2}{2} + \frac{p_z^2}{2} + K_z(s) \cdot \frac{z^2}{2} + U(x,z) \cdot \delta_p(s-s_0). \end{aligned} \tag{141}$$

This nonlinear term is given by [107]

$$U(x,z) = \frac{N_b \cdot r_e}{\gamma} \cdot \int_0^\infty \frac{1 - \exp\{-\frac{x^2}{2\sigma_x^2+q} - \frac{z^2}{2\sigma_z^2+q}\}}{(2\sigma_x^2+q)^{1/2} \cdot (2\sigma_z^2+q)^{1/2}} \cdot dq. \tag{142}$$

In equations (140), (141) and (142) we have used the following definitions : $\delta_p(s-s_0)$ periodic delta function, r_e classical electron radius, N_b number of particles in the counter rotating bunch, σ_x, σ_z

rms beam sizes of the strong bunch. Radiation damping is described by the two damping constants γ_x, γ_z and the strength of the stochastic excitation Γ is denoted by d_x, d_z.

These equations have been extensively used in numerical simulations. These simulations are very helpful for understanding the complicated interplay of nonlinearity, damping and stochastic excitation in lepton colliders. Figure 26 [108] shows such a calculation. The combined effect of quantum fluctuations and nonlinearity can move a particle starting near the origin in phase space to a (nonlinear) resonance island before it is damped again and eventually pushed to another resonance nearby.

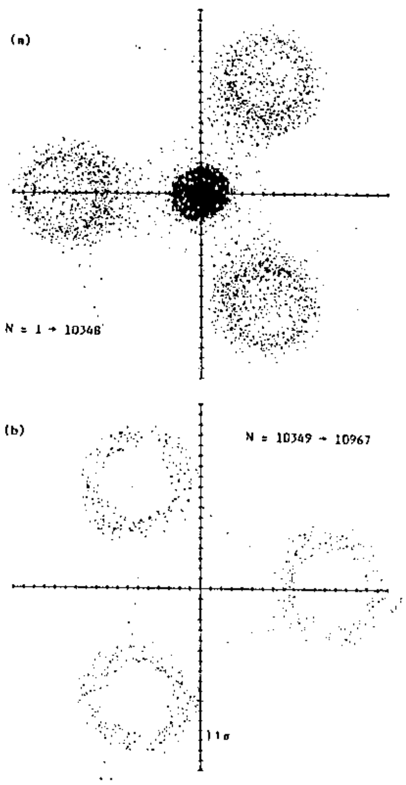

Figure 26: $(p_z - z)$ phase diagram of a simulation

An interesting and important question in this context is :

How does the distribution function $p(x, z, p_x, p_z, s)$ of a stochastic system like (140) evolve with s or time?

The corresponding Fokker-Planck equation would be a five-dimensional partial differential equation, and because of the highly singular behaviour of its coefficients (delta functions), this equation would be extremely complicated to solve.In this case stochastic mappings are more suitable. In the following we will sketch an algorithm for calculating the temporal evolution of the density function for such a stochastic map. This algorithm is based on an idea of Gerasimov and gives much faster results than direct numerical simulations (see also [109]).

We will illustrate this approach with a simple two-dimensional model of the beam-beam interaction. The details are described in [110] and in a PhD thesis of Pauluhn [111].

The considered model is given by (see also [65]):

$$\begin{pmatrix} x(n+1) \\ p_x(n+1) \end{pmatrix} = \begin{pmatrix} \cos(2\pi Q) & \beta \cdot \sin(2\pi Q) \\ -\frac{1}{\beta} \cdot \sin(2\pi Q) & \cos(2\pi Q) \end{pmatrix} \cdot \begin{pmatrix} x(n) \\ p_x(n) - \gamma_x p_x(n) + u(x(n)) + d_x \Gamma \end{pmatrix} \quad (143)$$

Γ is now a random variable, Q is the tune of the storage ring, β is the beta-function at the interaction point and $u(x(n))$ is given by

$$u(x(n)) = -\frac{4\pi\xi}{\beta} \cdot x(n) \cdot \frac{1 - \exp(-\frac{x^2(n)}{2\sigma^2})}{\frac{x^2(n)}{2\sigma^2}} \tag{144}$$

with ξ beam-beam strength parameter [107].
The main steps of this algorithm are:

1. discretization of the two-dimensional phase space
2. use of the microscopic dynamics (see equation (143)) to calculate the transition rates A_{ij} between the discretized bins of the phase space
3. use of this (stochastic) transition matrix A_{ij} as macroscopic propagator for the time evolution of an initial particle distribution

Figure 27 shows how an initially constant and homogeneous distribution evolves with time (after 1000, 3000, 15000, 99000 turns respectively). These results are in excellent agreement with direct numerical simulations [111].

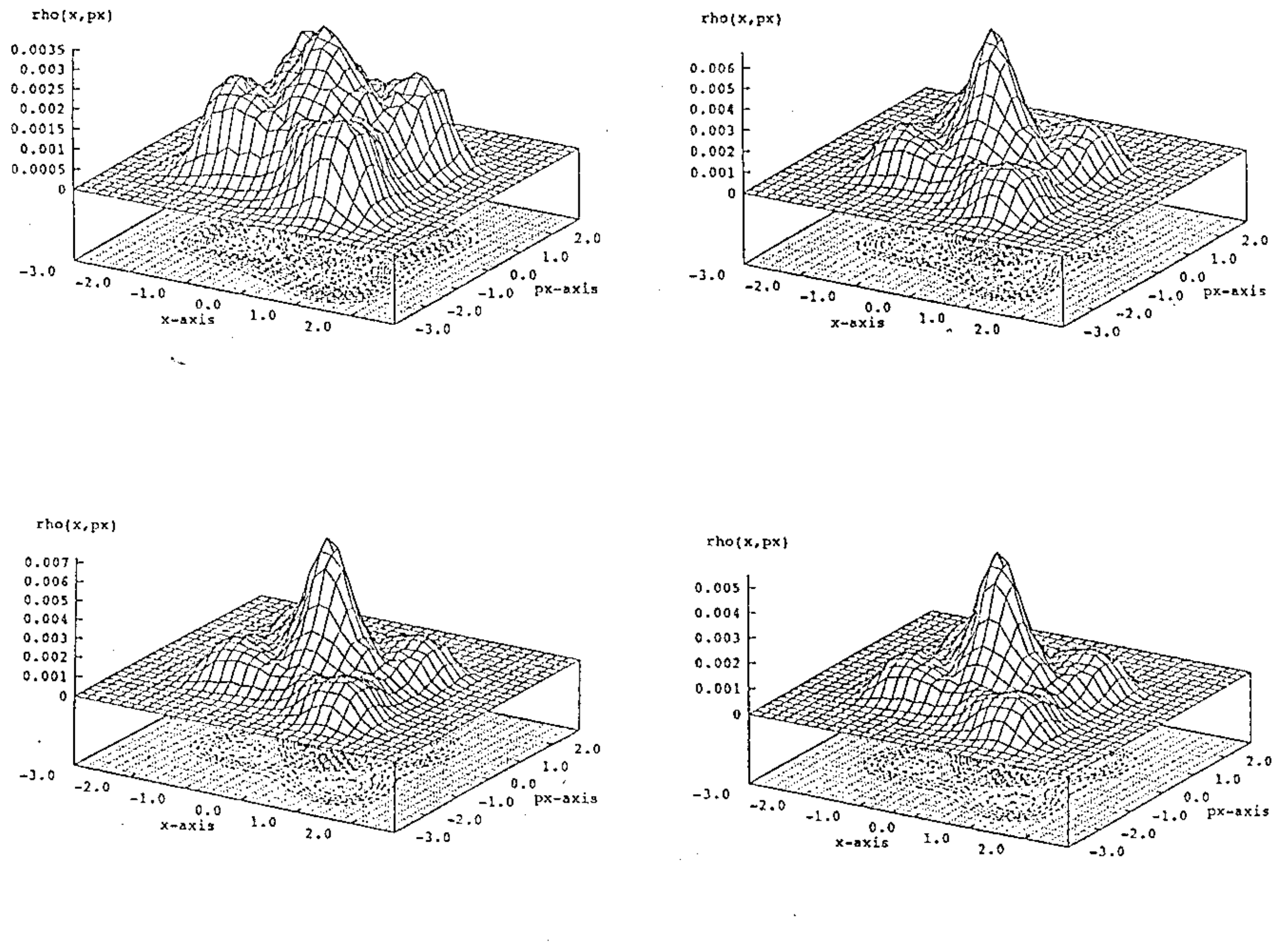

Figure 27: evolution of the density function for the stochastic map (143) (here denoted by $\rho(x, p_x, t)$) as a function of time

Spin dynamics in electron storage rings constitutes another interesting application of stochastic differential equations in beam dynamics. In this case the orbital equations of motion (138)

$$\frac{d}{ds}\,\vec{y}(s) = \vec{f}(\vec{y},s) + \underline{T}(\vec{y},s)\cdot\delta\vec{c}(s)$$

have to be supplemented by the spin equation of motion, the so-called BMT equation (Bargman, Michel, Telegdi see for example [3])

$$\frac{d}{ds}\,\vec{S} = \vec{\Omega}(\vec{y})\times\vec{S} \tag{145}$$

where the field $\vec{\Omega}$ depends on the orbital degrees of freedom. Since (138) is a stochastic differential equation, we end up with a kind of spin diffusion or random motion on the unit sphere [83].

The linearized spin -orbit motion in electron storage rings has been studied in [112], [113] and the reader is referred to these references for further details.

Until now we have seen, that radiation is a natural source for noise in electron accelerators. However, there are also other sources for stochastic forces such as rf noise, power supply noise, random ground motion or restgas scattering. These influences can also be present in proton (Hamiltonian) systems and some of these sources could for example be included in the vector potential $\vec{A}$ of equation (2).

The influence of rf noise on the beam dynamics in storage rings has been investigated extensively by several authors [114],[115], [116], [117], [118].In the smooth approximation (oscillator model) [7], where one averages $\kappa(s)D(s)$ and $V(s)$ over the circumference L of the storage ring one obtains the following Hamiltonian (see equation (15))

$$\begin{aligned}\bar{\mathcal{H}}(\bar{\tau},\bar{p}_\tau) &= \\ &= -\frac{1}{2}\mu\cdot\bar{p}_\tau^2 + \frac{L}{2\pi k}\cdot\frac{e\cdot\bar{V}_0}{E_0}\cdot\cos(\frac{2\pi k}{L}\cdot\bar{\tau}) + \\ &+\frac{L}{2\pi k}\cdot\frac{e\cdot\delta\bar{V}_0}{E_0}\cdot\cos(\frac{2\pi k}{L}\cdot\bar{\tau}) - \delta\bar{\tau}\cdot\frac{e\cdot\bar{V}_0}{E_0}\cdot\sin(\frac{2\pi k}{L}\cdot\bar{\tau})\end{aligned} \tag{146}$$

with

$$\mu = \frac{1}{L}\cdot\int_0^L \kappa(s)\cdot D(s)\cdot ds \tag{147}$$

and

$$\bar{V}_0 = \frac{1}{L}\cdot\int_0^L V(s)\cdot ds. \tag{148}$$

$\delta\bar{V}_0$ and $\delta\bar{\tau}$ denote the amplitude and phase noise (for example Gaussian white noise) of a cavity. The corresponding stochastic differential equations now read

$$\left\{\begin{array}{c}\frac{d}{ds}\,\bar{\tau} = -\mu\cdot\bar{p}_\tau \\ \frac{d}{ds}\,\bar{p}_\tau = \frac{e}{E_0}\cdot(\bar{V}_0+\delta\bar{V}_0)\cdot\sin(\frac{2\pi k}{L}\cdot\bar{\tau}) + \frac{e\cdot\bar{V}_0}{E_0}\cdot\frac{2\pi k}{L}\cdot\delta\bar{\tau}\cdot\cos(\frac{2\pi k}{L}\cdot\bar{\tau})\ .\end{array}\right. \tag{149}$$

Using perturbation methods the corresponding Fokker-Planck equation has been solved in [114], [115], [117]. An alternative to this perturbative approach would be a numerical integration of the exact Fokker-Planck equation or a direct numerical treatment of the stochastic differential equations (149). Numerical methods to solve stochastic differential equations are described in detail in a recently published book [119]. These methods have been applied in [111], [120].

If one treats the cavity as as a strongly localized element, one should investigate the stochastic map, in this case the standard map with explicit stochasticity and an interesting problem, one is then faced with, is:

How is the interplay between deterministic chaos and explicit stochasticity?

Only few results are available for this case [121], [122], and these stochastic nonintegrable Hamiltonian systems remain a challenging and interesting problem not only in accelerator physics.

4 Summary and conclusions

The main aim of this review was to illustrate the problems of nonlinear beam dynamics in storage rings and to introduce some of the concepts and tools to study these systems. Although a lot of facts are known and powerful techniques - especially in the Hamiltonian case - have been developed, designing new accelerators (like the SSC) remains a challenging problem of nonlinear dynamics. The design of such a machine requires a lot of numerical simulations, the knowledge of the basic facts of the qualitative theory of dynamical systems and perturbative investigations. Besides these theoretical issues, future accelerator developments have to rely on the experience with existing machines and on special nonlinear dynamics experiments performed with these machines.

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