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Critical Exponents from the Effective Average Action

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Abstract

We compute the critical behaviour of three-dimensional scalar theories using a new exact non-perturbative evolution equation. Our values for the critical exponents agree well with previous precision estimates.

1. Introduction

A new exact non-perturbative evolution equation for the scale dependence of an effective action has been proposed recently [1]. The effective average action Γ_k obtains by integrating out only the quantum fluctuations with momenta $q^2 \geq k^2$. Then the usual effective action (the generating functional of the 1PI Green functions) is recovered in the limit $k \to 0$ where all quantum fluctuations are included. The effective average action can also be viewed as the action for averages of fields, similar in spirit to the block spin action [2, 3] in lattice theories. The dependence of the effective average action on the scale k is described by the exact evolution equation. This equation has the form of a one-loop expression, differentiated with respect to k. The non-perturbative content, which makes this equation exact, consists of using the exact inverse propagator, as given by the second functional derivative of the effective average action Γ_k, instead of the inverse propagator. Furthermore, an infrared cutoff suppressing the fluctuations with $q^2 \ll k^2$ is added to the inverse propagator. In consequence, one obtains an equation where the scale dependence of Γ_k is expressed in terms of its second functional derivative with respect to the fields ($t = \ln k$)

$$\frac{\partial}{\partial t}\Gamma_k = \frac{1}{2}\mathrm{Tr}\Big\{(\Gamma_k^{(2)} + R_k)^{-1}\frac{\partial}{\partial t}R_k\Big\}. \qquad (\mathbf{1}.1)$$

(Here R_k is the effective infrared cutoff - details will be given in the next section.) The momentum integration implied by the trace is both ultraviolet and infrared finite in arbitrary dimensions. This makes our non-perturbative equation suitable for dealing with theories which are plagued by infrared problems in perturbation theory. (Scalar theories in less than four dimensions or at non-vanishing temperature near a second order phase transition and non-abelian gauge theories figure among the examples.)

The most important new feature seems to us not the exactness of eq. (**1.**1), but its simple and intuitive form which makes it suitable for approximate solutions. Exact equations have been derived earlier, as for example the Schwinger-Dyson equations [4] or the "exact renormalization group equations" [3, 5]. (There exists actually a formal relation between the latter and eq. (**1.**1) [6] - [9]. There should also be a formal relation to the Schwinger-Dyson equations - after all, exact relations describing the same Green functions should all be equivalent.) The evolution equation (**1.**1) is a functional differential equation. It can be rewritten as an infinite system of ordinary coupled non-linear differential equations for the infinite number of couplings appearing in the most general form of Γ_k consistent with the symmetries [9]. (For example, the general potential of a scalar field theory contains terms $\sim \phi^6$, $\sim \phi^8$ etc.) An exact solution of this infinite system seems almost impossible for an interacting theory. The r.h.s. of eq. (**1.**1) involves the exact propagator which is as difficult to compute as Γ_k itself. The main point is that for many situations of interest the propagator is approximately known. Then one can employ the strategy to describe the propagator in terms of a finite number of parameters α_i which may depend on k. Typically, such parameters are masses, wave function renormalization constants etc. This approximation corresponds to an ansatz for the average action Γ_k in terms of a finite number of undetermined functions $\alpha_i(k)$. The evolution equation (**1.**1)

is now truncated and turned into a finite number of differential equations

$$\frac{d\alpha_i}{dt} = \beta_i(\alpha_j). \tag{1.2}$$

Such a finite system can be solved numerically or sometimes analytically. The sucess of our method ultimately depends on a good guess about the general form of the exact propagator, which in turn depends on the proper choice of degrees of freedom. (Sometimes it may be useful to include composite fields [10].) Fortunately, eq. (**1.**1) has a form very similar to the one-loop expression in standard perturbation theory. (It reproduces perturbation theory in its range of applicability. The loop expansion can be recovered from eq. (**1.**1) by a systematic expansion of the solution in terms of the small coupling.) This allows to build on the experience gained from the perturbative renormalization group analysis [4], [11] - [13] and provides useful checks for situations of overlap. Nevertheless, our equation is genuinely non-perturbative. Scalar theories at the critical temperature or in two and three dimensions are not accessible to perturbation theory and are described by our method with success [14, 15].

For a scalar theory the inverse propagator is characterized by a mass matrix and a generalized kinetic term $\sim Zq^2$. The mass matrix is directly related to derivatives of the scalar potential U_k which describes the non-derivative part of Γ_k. In general Z depends on fields and momenta and its detailed structure may be quite complicated. Nevertheless, this quantity plays essentially the role of a field and momentum dependent wave function renormalization. As a result its dependence on the various mass scales of the model is governed by the anomalous dimension. In particular, one expects a weak dependence on fields, momenta and the scale k if the anomalous dimension η is small. This is typically the case in three- and four-dimensional scalar theories. We will exploit this fact in the present paper.

The aim of this paper is a precision test of our method in a context which is not accessible to perturbation theory. We consider the three-dimensional $O(N)$-symmetric scalar theory in the phase with spontaneous symmetry breaking. The presence of the Goldstone bosons leads to severe infrared divergences in perturbation theory and excludes a perturbative treatment in three dimensions. On the other hand, the model exhibits a second order phase transition. The critical exponents at the phase transitions have been computed by various other methods [12, 13] with high precision. They serve as a valuable test. We shall see that even a truncation to only a few parameters α_i gives good quantitative results for the critical exponents, typically with an accuracy at the percent level. In view of the fact that the momentum integration in eq. (**1.**1) is the same as for a one-loop calculation, and that no small coupling characterizes the interactions at the phase transition, this result seems quite remarkable.

Since all our computations are done directly in three dimensions, we are not limited to the vicinity of the phase transition. In particular, we can employ our method in the case of the four-dimensional scalar theory at non-vanishing temperature and follow the transition from the effective four-dimensional behaviour for $k \gg T$ to an effective three-dimensional behaviour for $k \ll T$ [15]. The second order character of the high temperature

transition in $O(N)$-symmetric scalar theories has been established previously [15], together with a crude estimate of the critical exponents. The present work can be seen as a refinement of [15] near the critical temperature where the three-dimensional behaviour dominates. A more accurate quantitative description of the temperature dependence of masses, expectation values and couplings away from the critical temperature, or the study of the full temperature dependent scalar potential will require to redo the computations of [15], including the additional couplings discussed in the present paper.

This paper is organized as follows: We present the general formalism in section 2 and derive in section 3 an exact evolution equation for the scalar potential formulated in terms of dimensionless couplings. No explicit mass scale appears in this equation, which is therefore well suited for a description of the scale independent physics at the second order phase transition. In section 4 we approximate this equation with a uniform (field and momentum independent) wave function renormalization. This approximation becomes exact in the large N limit. In section 5 we solve the exact evolution equation in the large N limit and establish consistency with earlier results [16, 17]. In section 6 we discuss the general form of the effective scalar potential at the phase transition for arbitrary values of N by taking the limit $k \to 0$ for fixed values of the fields. In section 7 we exploit the knowledge that the anomalous dimension is small and sketch a truncation adapted to this property. The anomalous dimension is finally computed in section 8. Our results for the critical exponents are presented in section 9 and section 10 gives our conclusions.

2. Exact evolution equation for the effective average action

The scale dependent effective action Γ_k, for a theory described by a classical action S, results from the effective integration of degrees of freedom with characteristic momenta larger than a given infrared cutoff k. The dependence of the effective action on the scale k is described by an exact evolution equation. In this section we summarize the formalism for the implementation of the above ideas in the case of a $O(N)$-symmetric theory of real scalar fields. For a detaided discussion, proof of the exactness of the evolution equation and modification of the conventions for the discussion of complex fields, we refer the reader to ref. [1]. For our conventions see also the appendix A of ref. [7].

We consider a theory of N real scalar fields χ^a, in d dimensions, with an $O(N)$-symmetric action $S[\chi]$. We specify the action together with some ultraviolet cutoff Λ, so that the theory is properly regulated. We add to the kinetic term a piece

$$\Delta S = \frac{1}{2} \int d^d q R_k(q) \chi_a^*(q) \chi^a(q). \qquad (\mathbf{2}.1)$$

The function R_k is employed in order to prevent the propagation of modes with characteristic momenta $q^2 < k^2$. This can be achieved, for example, by the choice

$$R_k(q) = \frac{Z_k q^2 f_k^2(q)}{1 - f_k^2(q)}, \qquad (\mathbf{2}.2)$$

with

$$f_k^2(q) = \exp\left(-\frac{q^2}{k^2}\right). \qquad (\mathbf{2}.3)$$

As a result the inverse propagator for the action $S+\Delta S$ has a minimum $\sim k^2$. The modes with $q^2 \gg k^2$ are unaffected, while the low frequency modes with $q^2 \ll k^2$ are cut off:

$$\lim_{q^2\to 0} R_k = Z_k k^2. \qquad (\mathbf{2}.4)$$

The quantity Z_k in eq. (**2**.2) is an appropriate wave function renormalization whose precise definition will be given below. We emphasize at this point that the above form of R_k is not unique and many alternative choices are possible. We subsequently introduce sources and define the generating functional for the connected Green functions for the action $S+\Delta S$. Through a Legendre transformation we obtain the generating functional for the 1PI Green functions $\tilde{\Gamma}_k[\phi^a]$, where ϕ^a is the expectation value of the field χ^a in the presence of sources. The use of the modified propagator for the calculation of $\tilde{\Gamma}_k$ results in the effective integration of only the fluctuations with $q^2 \geq k^2$. Finally, the scale dependent effective action is obtained by removing the infrared cutoff

$$\Gamma_k[\phi^a] = \tilde{\Gamma}_k[\phi^a] - \frac{1}{2}\int d^dq R_k(q)\phi_a^*(q)\phi^a(q). \qquad (\mathbf{2}.5)$$

For k equal to the ultraviolet cutoff Λ, Γ_k becomes equal to the classical action S (no effective integration of modes takes place), while for $k \to 0$ it tends towards the effective action Γ corresponding to S (all the modes are included). The interpolation of Γ_k between the classical and the effective action makes it a very useful field theoretical tool. The means for practical calculations is provided by an exact evolution equation which describes the response of the scale dependent effective action to variations of the infrared cutoff ($t = \ln(k/\Lambda)$) [1]:

$$\frac{\partial}{\partial t}\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\Big\{(\Gamma_k^{(2)}[\phi] + R_k)^{-1}\frac{\partial}{\partial t}R_k\Big\}. \qquad (\mathbf{2}.6)$$

Here $\Gamma_k^{(2)}$ is the second functional derivative of the scale dependent effective action with respect to ϕ^a. For real fields in momentum space *, it reads

$$(\Gamma_k^{(2)})_b^a(q,q') = \frac{\delta^2\Gamma_k}{\delta\phi_a^*(q)\delta\phi^b(q')}, \qquad (\mathbf{2}.7)$$

with

$$\phi^a(-q) = \phi_a^*(q). \qquad (\mathbf{2}.8)$$

Note also that, although not explicitly indicated for simplicity, R_k in eq. (**2**.6) stands for

$$(R_k)_b^a(q,q') = R_k(q)\delta_b^a\delta(q-q'). \qquad (\mathbf{2}.9)$$

*In order to avoid excessive formalism, we shall use the same notation for functions or operators in position space and their Fourier transforms, defined with the convention $\phi(q) = (2\pi)^{-\frac{d}{2}}\int d^dx \exp(iq_\mu x^\mu)\phi(x)$.

In momentum space, for a system whose volume Ω is taken to infinity at the end, the trace reads

$$\mathrm{Tr} = \Omega \sum_a \int \frac{d^d q}{(2\pi)^d}, \tag{2.10}$$

and it is assumed to act on the diagonal part of the generalized matrix. The exact evolution equation gives the response of the scale dependent effective action Γ_k to variations of the scale k, through a one-loop expression involving the exact inverse propagator $\Gamma_k^{(2)}$ together with an infrared cutoff provided by R_k. It has a simple graphical representation (fig. 1). Our non-perturbative exact evolution equation can be viewed as a partial differential equation for the infinitely many variables t and $\phi^a(q)$. Its usefulness depends on the existence of appropriate truncations which permit its solution.

Before presenting the formalism which leads to approximate solutions of eq. (**2**.6), we briefly comment on the relation of the scale dependent effective action to the average action, which has been used for practical calculations in the past. The average action was introduced in ref. [7, 14] in order to describe the dynamics of averages of fields over volumes $\sim k^{-d}$. The implementation of an infrared cutoff k was naturally incorporated through the averaging procedure. The dependence of the average action on k was computed through an one-loop renormalization group equation which corresponds to a truncation of eq. (**2**.6). It was shown in ref. [8] that a modification of the averaging procedure leads to an improved average action which is identical to the scale dependent effective action up to an explicitely known ultraviolet regulator in the improved average action. For this reason we call Γ_k the effective average action. Within the approximations used so far there is no difference for low momentum quantities between the originally proposed average action [7, 14], the improved average action [8] and the effective average action [1]. As a result, all previous calculations can be viewed as approximate solutions of eq. (**2**.6) with some appropriate truncation. More specifically, these previous studies have described correctly the phase structure of the two- and three- dimensional scalar theories (including the Kosterlitz-Thouless phase transition) [14], the high temperature phase transition of the four-dimensional theory (with a reliable description of the effectively three-dimensional critical behaviour and a rough determination of critical exponents) [15, **?**], and the approach to convexity for the effective potential in the phase with spontaneous symmetry breaking [18].

As we have already mentioned, for the solution of eq. (**2**.6) one has to develop an efficient truncation scheme. We consider an effective action of the form

$$\Gamma_k = \int d^d x \Big\{ U_k(\rho) + \frac{1}{2}\partial^\mu \phi_a Z_k(\rho, -\partial^\nu \partial_\nu)\partial_\mu \phi^a + \frac{1}{4}\partial^\mu \rho Y_k(\rho, -\partial^\nu \partial_\nu)\partial_\mu \rho \Big\}, \tag{2.11}$$

where $\rho = \frac{1}{2}\phi_a \phi^a$ and appropriate normal ordering is assumed for the derivative operators [1]. In order to turn the evolution equation for the effective average action into equations for U_k, Z_k and Y_k, we have to evaluate the trace in eq. (**2**.6) for properly chosen background field configurations. For the evolution equation for U_k we have to expand around a constant field configuration. This calculation is carried out in section 3 and the study of the resulting equation occupies sections 4-7. The evolution of the wave function

renormalizations, which leads to the determination of the anomalous dimensions, requires an expansion around a background with a small momentum dependence. This calculation is presented in section 8 and appendix B. The evolution equations for U_k, Z_k and Y_k are partial differential equations for independent variables t and ρ. In most cases their study is possible when they are turned into an (infinite) set of coupled ordinary differential equations for independent variable t. This is achieved by Taylor expanding U_k, Z_k and Y_k around some value of ρ. Since in this work we are interested in the vacuum structure of the theory, we shall use an expansion around the k dependent minimum $\rho_0(k)$ of U_k. In the limit $k \to 0$ this minimum specifies the vacuum of the theory, and $U = U_0$, $Z = Z_0$ and $Y = Y_0$ and their ρ-derivatives at the minimum give the renormalized masses, couplings and wave function renormalizations.

In a first approximation we can neglect the q^2 dependence of $Z_k(\rho, q^2)$. We can now define the quantity Z_k appearing in eq. (**2**.2) as

$$Z_k = Z_k(\rho_0(k)). \tag{**2**.12}$$

For studies which concentrate on the minimum of U_k, the above definition permits the combination of the leading kinetic contribution to $\Gamma_k^{(2)}$ and R_k in eq. (**2**.6), into an effective inverse propagator (for massless fields)

$$P(q^2) = Z_k q^2 + R_k = \frac{Z_k q^2}{1 - f_k^2(q)}. \tag{**2**.13}$$

For $q^2 \gg k^2$ the inverse "average" propagator $P(q)$ approaches the standard inverse propagator $Z_k q^2$ exponentially fast, whereas for $q^2 \ll k^2$ the infrared cutoff guarantees $P(0) = Z_k k^2$. Throughout most of this paper, we shall work with the approximation of q^2 independent $Z_k(\rho)$ and use the definition of eq. (**2**.12). When the q^2 dependence of $Z_k(\rho, q^2)$ is considered, eq. (**2**.12) has to be replaced by an appropriate generalization, for example

$$Z_k = Z_k(\rho_0(k), q^2 = 0). \tag{**2**.14}$$

Having summarized the basic formalism, we obtain in the next section the exact evolution equation for the effective average potential U_k.

3. Exact evolution equation for the effective average potential and its scaling form

The evolution equation for $U_k(\rho)$ can be obtained by calculating the trace in eq. (**2**.6) for small field fluctuations around a constant background configuration $\phi_a(q = 0) = \phi\delta_{a1}$, $\rho = \frac{1}{2}\phi^2$. We define

$$\phi_a(q) = \phi\delta_{a1} + \chi_a(q) \tag{**3**.1}$$

and find [1]

$$(\Gamma_k^{(2)})_{ab}(q, q') = \left[\left(Z_k(\rho, q^2)\delta_{ab} + \rho Y_k(\rho, q^2)\delta_{a1}\delta_{b1}\right) q^2 + M_{ab}^2\right] \delta(q - q'), \tag{**3**.2}$$

where the mass matrix M^2_{ab} reads

$$M^2_{ab} = U'_k(\rho)\delta_{ab} + 2\rho U''_k(\rho)\delta_{a1}\delta_{b1}. \tag{3.3}$$

Here primes denote derivatives with respect to ρ. Eq. (**2.**6) now reads

$$\frac{\partial}{\partial t}U_k(\rho) = \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\frac{\partial}{\partial t}R_k(q)\left(\frac{N-1}{M_0} + \frac{1}{M_1}\right), \tag{3.4}$$

with $R_k(q)$ given by eqs. (**2.**2) - (**2.**3),

$$\begin{aligned} M_0(\rho, q^2) &= Z_k(\rho, q^2)q^2 + R_k(q) + U'_k(\rho) \\ M_1(\rho, q^2) &= \tilde{Z}_k(\rho, q^2)q^2 + R_k(q) + U'_k(\rho) + 2\rho U''_k(\rho) \end{aligned} \tag{3.5}$$

and

$$\tilde{Z}_k(\rho, q^2) = Z_k(\rho, q^2) + \rho Y_k(\rho, q^2). \tag{3.6}$$

For ρ different from zero it is easy to recognize the first term in eq. (**3.**4) as the contribution from the $N-1$ Goldstone bosons (U' vanishes at the minimum). The second term is then related to the radial mode. The effective average potential U_k has a k dependent minimum $\rho_0(k)$. We introduce the wave function renormalizations for the Goldstone and radial modes (in agreement with eq. (**2.**14))

$$\begin{aligned} Z_k &= Z_k(\rho_0(k), q^2 = 0) \\ Y_k &= Y_k(\rho_0(k), q^2 = 0) \\ \tilde{Z}_k &= \tilde{Z}_k(\rho_0(k), q^2 = 0) = Z_k + \rho_0(k)Y_k \end{aligned} \tag{3.7}$$

When the q^2 dependence of $Z_k(\rho, q^2)$, $\tilde{Z}_k(\rho, q^2)$, $Y_k(\rho, q^2)$ is neglected the wave function renormalizations are defined according to eq. (**2.**12) and read

$$\begin{aligned} Z_k &= Z_k(\rho_0(k)) \\ Y_k &= Y_k(\rho_0(k)) \\ \tilde{Z}_k &= \tilde{Z}_k(\rho_0(k)) = Z_k + \rho_0(k)Y_k. \end{aligned} \tag{3.8}$$

The k dependence of these functions is given by the anomalous dimensions

$$\begin{aligned} \eta(k) &= -\frac{d}{dt}(\ln Z_k) \\ \tilde{\eta}(k) &= -\frac{d}{dt}(\ln \tilde{Z}_k). \end{aligned} \tag{3.9}$$

At this stage the r.h.s. of eq. (**3.**4) depends explicitely on the scale k once the momentum integration is performed. The aim of this paper is a study of the critical behaviour at and near the second order phase transition. At the phase transition one expects a scaling behaviour of the effective average action Γ_k and, in particular, of the effective average

potential U_k. This holds provided approximate renormalized dimensionless couplings are used. By a proper choice of variables the evolution equation (**3.**4) should therefore be cast into a form where the scale k no longer appears explicitely. In this formulation it will become easy to study the fixed point behaviour. We perform the transformation of eq. (**3.**4) into its scaling form in several steps. First we introduce the variable $x = q^2$ and define the dimensionless functions

$$\begin{aligned} r_k(x) &= \frac{R_k(x)}{Z_k x} \\ s_k(x) &= \frac{1}{Z_k}\frac{\partial}{\partial t}\left(\frac{R_k(x)}{x}\right) = \frac{\partial}{\partial t} r_k(x) + \frac{\partial}{\partial t}(\ln Z_k) r_k(x) = -2x\frac{\partial}{\partial x} r_k(x) - \eta r_k(x). \end{aligned} \tag{3.10}$$

Here $R_k(x)/Z_k x$ is a dimensionless function of the ratio x/k^2 (a property of R_k independent of the specific choice (**2.**2)). Eq. (**3.**4) can then be written in the form

$$\frac{\partial}{\partial t} U_k(\rho) = v_d \int_0^\infty dx x^{\frac{d}{2}} s_k \left(\frac{N-1}{M_0/Z_k} + \frac{1}{M_1/Z_k}\right), \tag{3.11}$$

with

$$v_d^{-1} = 2^{d+1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right). \tag{3.12}$$

The ratios M_0/Z_k and M_1/Z_k read

$$\begin{aligned} M_0(\rho, x)/Z_k &= z_k(\rho, x)x + r_k(x)x + Z_k^{-1} U_k'(\rho) \\ M_1(\rho, x)/Z_k &= \hat{z}\tilde{z}_k(\rho, x)x + r_k(x)x + Z_k^{-1} U_k'(\rho) + 2Z_k^{-1}\rho U_k''(\rho), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} z_k(\rho, x) &= Z_k(\rho, x)/Z_k \\ \tilde{z}_k(\rho, x) &= \tilde{Z}_k(\rho, x)/\tilde{Z}_k \end{aligned} \tag{3.14}$$

and

$$\hat{z}(k) = \tilde{Z}_k/Z_k. \tag{3.15}$$

This suggests the use of a dimensionless renormalized variable

$$\tilde{\rho} = Z_k k^{2-d}\rho. \tag{3.16}$$

In terms of $\tilde{\rho}$ one obtains

$$\left.\frac{\partial U_k(\tilde{\rho})}{\partial t}\right|_{\tilde{\rho}} = (d - 2 + \eta)\tilde{\rho}\frac{\partial U_k(\tilde{\rho})}{\partial \tilde{\rho}} + v_d k^{-2}\int_0^\infty dx x^{\frac{d}{2}} s_k \left(\frac{N-1}{m_0} + \frac{1}{m_1}\right), \tag{3.17}$$

where we have used the identity

$$\begin{aligned} \left.\frac{\partial U_k}{\partial t}\right|_{\tilde{\rho}} &= \left.\frac{\partial U_k}{\partial t}\right|_{\rho} + \left.\frac{\partial U_k}{\partial \rho}\right|_t \left.\frac{\partial \rho}{\partial t}\right|_{\tilde{\rho}} = \left.\frac{\partial U_k}{\partial t}\right|_{\rho} + \tilde{\rho}\left.\frac{\partial U_k}{\partial \tilde{\rho}}\right|_t \left.\frac{\partial \ln\tilde{\rho}}{\partial \ln\rho}\right|_t \left.\frac{\partial \ln\rho}{\partial t}\right|_{\tilde{\rho}} \\ &= \left.\frac{\partial U_k}{\partial t}\right|_{\rho} + (d - 2 + \eta)\tilde{\rho}\left.\frac{\partial U_k}{\partial \tilde{\rho}}\right|_t. \end{aligned} \tag{3.18}$$

The dimensionless ratios m_0 and m_1 are given by

$$m_0(\tilde{\rho},x) = \frac{M_0}{Z_k k^2} = \frac{x}{k^2}\left[z_k(\tilde{\rho},x) + r_k(x)\right] + k^{-d}\frac{\partial U_k(\tilde{\rho})}{\partial\tilde{\rho}}$$
$$m_1(\tilde{\rho},x) = \frac{M_1}{Z_k k^2} = \frac{x}{k^2}\left[\hat{z}\tilde{z}_k(\tilde{\rho},x) + r_k(x)\right] + k^{-d}\left[\frac{\partial U_k(\tilde{\rho})}{\partial\tilde{\rho}} + 2\tilde{\rho}\frac{\partial^2 U_k(\tilde{\rho})}{\partial\tilde{\rho}^2}\right]. \quad (\mathbf{3}.19)$$

We finally switch to a dimensionless potential

$$u_k(\tilde{\rho}) = k^{-d}U_k(\tilde{\rho}) \quad (\mathbf{3}.20)$$

and use a dimensionless momentum variable

$$y = x/k^2. \quad (\mathbf{3}.21)$$

We adopt the convention that the prime on $u_k(\tilde{\rho})$ denotes a derivative with respect to $\tilde{\rho}$ at fixed t, while the prime on $U_k(\rho)$ denotes a derivative with respect to ρ at fixed t. At this point it is useful to fix the notation which we shall use in the next sections. We denote the minimum of $U_k(\rho)$ with $\rho_0(k)$ and the minimum of $u_k(\tilde{\rho})$ with $\kappa(k) = \tilde{\rho}_0(k)$. For $\rho_0(k) \neq 0$ (spontaneously broken regime) we have $U_k'(\rho_0) = 0$, $u_k'(\kappa) = 0$. We define $U_k''(\rho_0) = \bar{\lambda}(k)$, $U_k^{(n)}(\rho_0) = U_n(k)$ and $u_k''(\kappa) = \lambda(k)$, $u_k^{(n)}(\kappa) = u_n(k)$. For $\rho_0(k) = 0$ (symmetric regime) we define $U_k'(0) = \bar{m}^2(k)$, $u_k'(0) = m^2(k)/k^2 = \tilde{m}^2(k)$ and as above for the higher derivatives. We point out the relations

$$\begin{aligned}
\kappa &= Z_k k^{2-d}\rho_0 \\
\tilde{m}^2 &= Z_k^{-1}k^{-2}\bar{m}^2 = k^{-2}m^2 \\
\lambda &= Z_k^{-2}k^{d-4}\bar{\lambda} \\
u_n &= Z_k^{-n}k^{(n-1)d-2n}U_n.
\end{aligned} \quad (\mathbf{3}.22)$$

We can now write the evolution equation in a scale independent form

$$\frac{\partial}{\partial t}u_k(\tilde{\rho}) = -du_k(\tilde{\rho}) + (d-2+\eta)\tilde{\rho}u_k'(\tilde{\rho}) + v_d\int_0^\infty dy\, y^{\frac{d}{2}}s(y)\left(\frac{N-1}{m_0} + \frac{1}{m_1}\right), \quad (\mathbf{3}.23)$$

with

$$\begin{aligned}
m_0(\tilde{\rho},y) &= y\left[z_k(\tilde{\rho},y) + r(y)\right] + u_k'(\tilde{\rho}) \\
m_1(\tilde{\rho},y) &= y\left[\hat{z}\tilde{z}_k(\tilde{\rho},y) + r(y)\right] + u_k'(\tilde{\rho}) + 2\tilde{\rho}u_k''(\tilde{\rho}).
\end{aligned} \quad (\mathbf{3}.24)$$

Indeed, $r_k = r(y)$ only depends on y, whereas $s_k = s(y,\eta) = -2y\frac{\partial}{\partial y}r(y) - \eta r(y)$ depends on y and η. We have, therefore, dropped the label k for these functions.

Eq. (**3**.23) is the scaling form of eq. (**3**.4) we were looking for. The behaviour exactly at the second order phase transition should be given by a k-independent solution of this equation. The scaling solution of (**3**.23) obtains therefore for $\partial u_k/\partial t = 0$. Since no explicit

k dependence is present on the r.h.s. of (**3.**23) we conclude that all the k dependent quantities in (**3.**23) obtain constant (fixed point) values. In particular $\eta(k)$, $\tilde{\eta}(k)$ and $\hat{z}(k)$ take values η_*, $\tilde{\eta}_*$ and $\hat{z}_*$ for this solution. The definitions (**3.**9), (**3.**15) then imply $\eta_* = \tilde{\eta}_*$. Similarly $u_*(\tilde{\rho})$ and $z_*(\tilde{\rho}, y)$, $\tilde{z}_*(\tilde{\rho}, y)$ become functions which are independent of k. The nature of the fixed point becomes apparent through the observation that the minimum of the potential appears for a constant value

$$\kappa(k) = \kappa_*. \tag{3.25}$$

As a result, for the fixed point solution, the minimum of the effective potential is given by

$$\rho_0 = \lim_{k\to 0} \rho_0(k) \sim \lim_{k\to 0} k^{d-2+\eta_*}\kappa_* = 0. \tag{3.26}$$

We conclude that the fixed point of the evolution equation indeed corresponds to the phase transition between the spontaneously broken and the symmetric regime. The scaling form of the potential $u_*(\tilde{\rho})$ is determined by the equation

$$du_*(\tilde{\rho}) - (d-2+\eta_*)\tilde{\rho}u_*'(\tilde{\rho}) + 2v_d \int_0^\infty dy y^{\frac{d}{2}} \left(y\frac{\partial r(y)}{\partial y} + \frac{\eta_*}{2} r(y)\right)\left(\frac{N-1}{m_{0*}} + \frac{1}{m_{1*}}\right) = 0, \tag{3.27}$$

with

$$\begin{aligned} m_{0*}(\tilde{\rho}, y) &= y\,[z_*(\tilde{\rho}, y) + r(y)] + u_*'(\tilde{\rho}) \\ m_{1*}(\tilde{\rho}, y) &= y\,[\hat{z}_*\tilde{z}_*(\tilde{\rho}, y) + r(y)] + u_*'(\tilde{\rho}) + 2\tilde{\rho}u_*''(\tilde{\rho}). \end{aligned} \tag{3.28}$$

If the functions $z_*(\tilde{\rho}, y)$, $\tilde{z}_*(\tilde{\rho}, y)$ and the constants η_*, $\hat{z}_*$ are known and the y-integration is performed, eqs. (**3.**27), (**3.**28) are reduced to a non-linear second order differential equation for $u_*(\tilde{\rho})$.

We point out that eqs. (**3.**23), (**3.**27) are exact, since eq. (**2.**11) contains the most general terms which contribute to $\Gamma_k^{(2)}$ when evaluated at a constant background field. We have, therefore, obtained an exact non-perturbative evolution equation which does not involve any explicit mass scale.

4. Approximation with uniform wave function renormalization

The solution $u_*(\tilde{\rho})$ of the scaling equation (**3.**27) together with the evolution equation (**3.**23) contains all the important information about the behaviour at and near the phase transition. Indeed, eq. (**3.**23) gives the functional dependence of the various β-functions on the couplings at the phase transition. (In fact, $u_*(\tilde{\rho})$ describes infinitely many couplings and eq. (**3.**23) specifies infinitely many β-functions.) This allows to compute derivatives of the β-functions with respect to the couplings. From there one may compute the matrix of anomalous dimensions and infer the critical exponents at the phase transition. (For

example, the critical exponent ν is related to the anomalous mass dimension.) Even though eqs. (**3**.23) and (**3**.27) are exact the r.h.s. of eq. (**3**.27) still involves the unknown constants η_*, $\hat{z}_*$ and functions $z_*(\tilde{\rho}, y)$, $\tilde{z}_*(\tilde{\rho}, y)$. If these are given one may attempt to compute the y-integral in eq. (**3**.27) numerically. The result is a function depending on $\tilde{\rho}$, $u'_*(\tilde{\rho})$ and $u''_*(\tilde{\rho})$. The remaining non-linear second order differential equation can also be treated with numerical methods. This program requires some effort, but the result - exact values of the critical exponents - would be rewarding.

Unfortunately, exact information on η_*, $\hat{z}_*$, $z_*(\tilde{\rho}, y)$ and $\tilde{z}_*(\tilde{\rho}, y)$ is not available and the problem seems only postponed. However, the situation is not so hopeless, at least for the three-dimensional models for which η_* is typically a small quantity and $z_*(\tilde{\rho}, y)$, $\tilde{z}_*(\tilde{\rho}, y)$ are expected to depend only weakly on $\tilde{\rho}$ and y. Then, a relatively crude approximation for η_*, $\hat{z}_*$, $z_*(\tilde{\rho}, y)$, $\tilde{z}_*(\tilde{\rho}, y)$, which may be obtained by truncating the exact evolution equation for these quantities or by alternative methods, should already give excellent results. We shall demonstrate the power of our method by making the very simple approximation of uniform wave function renormalization, i.e.

$$z_k(\tilde{\rho}, y) = \tilde{z}_k(\tilde{\rho}, y) = 1. \tag{**4**.1}$$

This corresponds to neglecting the field and momentum dependence of the wave function renormalization. We observe that the dominant scale dependence of the wave function renormalization - the anomalous dimension - is always included by the k-dependence of Z_k and $\tilde{Z}_k$ (the definition (**3**.8) applies). For non-zero values of the infrared cutoff k the more general wave function renormalization function $Z_k(\rho, q^2)$ should be analytic at $q^2 = 0$ even at the phase transition. This is a consequence of the presence of the infrared cutoff. One expects a very mild momentum dependence of $Z_k(\rho_0, q^2)$ for q^2 between zero and k^2 whereas for $q^2 \gg k^2$ the anomalous dimension should determine the behaviour near the phase transition ($Z_k(\rho_0, q^2) \sim (q^2)^{-\frac{\eta}{2}}$). We note, however, that all momentum integrals are dominated by $q^2 \simeq k^2$. Similarly, an expansion of $Z_k(\rho, q^2)$ in the field ρ should be analytic at $\rho_0(k)$ as long as k is strictly positive. (See section 6 for more details.) Our approximation (**4**.1) should, therefore, correctly describe all qualitative features at the phase transition and be a good quantitative approximation. Of course, if a high precision calculation is attempted one will have to go beyond the truncation (**4**.1). In the approximation (**4**.1) it is useful to define the integrals

$$\begin{aligned}
\tilde{L}_0^d(\tilde{w}, \eta, \hat{z}) = \; & 2\int_0^\infty dy y^{\frac{d}{2}+1} \frac{\partial r(y)}{\partial y} \left[y(\hat{z} + r(y)) + \tilde{w}\right]^{-1} \\
& + \eta \int_0^\infty dy y^{\frac{d}{2}} r(y) \left[y(\hat{z} + r(y)) + \tilde{w}\right]^{-1} \\
\tilde{L}_n^d(\tilde{w}, \eta, \hat{z}) = \; & 2n\int_0^\infty dy y^{\frac{d}{2}+1} \frac{\partial r(y)}{\partial y} \left[y(\hat{z} + r(y)) + \tilde{w}\right]^{-(n+1)} \\
& + n\eta \int_0^\infty dy y^{\frac{d}{2}} r(y) \left[y(\hat{z} + r(y)) + \tilde{w}\right]^{-(n+1)} \qquad \text{for } n \geq 1,
\end{aligned} \tag{**4**.2}$$

which obey

$$\frac{\partial}{\partial\tilde{w}}\tilde{L}_0^d(\tilde{w},\eta,\hat{z}) = -\tilde{L}_1^d(\tilde{w},\eta,\hat{z})$$
$$\frac{\partial}{\partial\tilde{w}}\tilde{L}_n^d(\tilde{w},\eta,\hat{z}) = -n\tilde{L}_{n+1}^d(\tilde{w},\eta,\hat{z}) \qquad \text{for } n\geq 1. \quad (\mathbf{4}.3)$$

Similarly we introduce

$$L_n^d(\tilde{w},\eta) = \tilde{L}_n^d(\tilde{w},\eta,1). \quad (\mathbf{4}.4)$$

In the following, for notational simplicity, we shall not write down explicitly the dependence of L_n^d and $\tilde{L}_n^d$ on η and $\hat{z}$. Also, for the practical calculations in section 9, we shall omit the contribution to L_n^d, $\tilde{L}_n^d$ proportional to η (we shall study the critical three-dimensional theory, for which $\eta_* \simeq 0-0.04$) and we shall set $\hat{z}=1$ (which is again a good approximation near the fixed point). With these simplifications we need only the detailed form of $L_n^d(\tilde{w}) = L_n^d(\tilde{w},0)$. For the choice (**2**.2) - (**2**.3) for $R_k(q)$ these integrals are discussed in appendix A. We emphasize, however, that all our formulae remain valid for $\eta\neq 0$ and $\hat{z}\neq 1$, even though the dependence on these parameters is suppressed in the notation.

In terms of L_n^d, $\tilde{L}_n^d$ eq. (**3**.23) reads

$$\frac{\partial u}{\partial t} = -du + (d-2+\eta)\tilde{\rho}u' - v_d(N-1)L_0^d(u') - v_d\tilde{L}_0^d(u'+2\tilde{\rho}u''), \quad (\mathbf{4}.5)$$

where, for simplicity, we drop the subscript k of u_k from now on. The evolution equations for the derivatives of u are easily obtained by differentiating (**4**.5) with respect to $\tilde{\rho}$:

$$\begin{aligned}\frac{\partial u'}{\partial t} &= (-2+\eta)u' + (d-2+\eta)\tilde{\rho}u'' \\ &\quad + v_d(N-1)u''L_1^d(u') + v_d(3u''+2\tilde{\rho}u''')\tilde{L}_1^d(u'+2\tilde{\rho}u'') \quad (\mathbf{4}.6)\\ \frac{\partial u''}{\partial t} &= (d-4+2\eta)u'' + (d-2+\eta)\tilde{\rho}u''' \\ &\quad - v_d(N-1)\Big\{[u'']^2L_2^d(u') - u'''L_1^d(u')\Big\} \\ &\quad - v_d\Big\{(3u''+2\tilde{\rho}u''')^2\tilde{L}_2^d(u'+2\tilde{\rho}u'') - (5u'''+2\tilde{\rho}u^{(4)})\tilde{L}_1^d(u'+2\tilde{\rho}u'')\Big\}. \quad (\mathbf{4}.7)\end{aligned}$$

In the spontaneously broken regime, at the minimum $\kappa(k)$ we have $u'(\kappa)=0$, $\lambda = u''(\kappa)$ and $u_n = u^{(n)}(\kappa)$. The running of the minimum is obtained by taking a total t-derivative of the condition $u'(\kappa)=0$

$$\begin{aligned}\frac{d\kappa}{dt} &= \beta_\kappa = -[u''(\kappa)]^{-1}\left.\frac{\partial u'}{\partial t}\right|_{\tilde{\rho}=\kappa} \\ &= -(d-2+\eta)\kappa - v_d(N-1)L_1^d(0) - v_d\left(3+\frac{2\kappa u_3}{\lambda}\right)\tilde{L}_1^d(2\lambda\kappa). \quad (\mathbf{4}.8)\end{aligned}$$

Eq. (**4**.5) remains a complicated partial differential equation for two variables t and $\tilde{\rho}$, even if L_0^d, $\tilde{L}_0^d$ take a simple functional form. If we consider u in the vicinity of some fixed

value of $\tilde{\rho}$, e.g. $\tilde{\rho} = \kappa$, we can parametrize u by its derivatives at this fixed value, e.g. λ, u_n. The evolution equation is then expressed as an infinite system of coupled differential equations for the couplings λ, u_n. In particular, the β-function for λ reads

$$\begin{aligned}\frac{d\lambda}{dt} = \beta_\lambda &= \left.\frac{\partial u''}{\partial t}\right|_{\tilde{\rho}=\kappa} + u_3 \frac{d\kappa}{dt} \\ &= (d-4+2\eta)\lambda - v_d(N-1)\lambda^2 L_2^d(0) \\ &\quad - v_d(3\lambda + 2\kappa u_3)^2 \tilde{L}_2^d(2\lambda\kappa) + v_d\left(2u_3 + 2\kappa u_4 - \frac{2\kappa u_3^2}{\lambda}\right)\tilde{L}_1^d(2\lambda\kappa) \qquad (\mathbf{4}.9)\end{aligned}$$

and it involves the higher derivatives u_3 and u_4. In turn, the β-function for u_3 contains u_4 and u_5 and so on (see section 7 for the details). It should be also pointed out that the functions L_n^d, $\tilde{L}_n^d$ automatically introduce a "threshold" behaviour in the β-functions, which leads to the vanishing of the contributions from the massive radial mode at scales k much smaller than its running mass.

As a starting point for the understanding of the solutions of (**4**.5) we need first the behaviour of the scaling solution for $\partial u/\partial t = 0$. The scaling equation

$$-du_* + (d-2+\eta_*)\tilde{\rho}u_*' - v_d(N-1)L_0^d(u_*') - v_d\tilde{L}_0^d(u_*' + 2\tilde{\rho}u_*'') = 0 \qquad (\mathbf{4}.10)$$

is a non-linear differential equation for $\tilde{\rho}$, which remains difficult to solve analytically. Numerical work for its solution is in progress [19]. For the present paper we rather concentrate on analytical aspects and present, in the next two sections, results for limiting cases.

5. Solution of the evolution equation for the effective average potential for $N \to \infty$

The main results of this work will be obtained through use of the evolution equation (**4**.5) for the study of the three-dimensional theory. This will require a certain amount of numerical work. It is useful, therefore, to have analytical results in some limiting case, in order to obtain conceptual understanding and verification of the numerical solution. This is the purpose of this section in which we concentrate on the three-dimensional theory in the large N limit [20]. For $N \to \infty$ the evolution equation (**4**.5) simplifies considerably. The first simplification obtains from the dominance of the Goldstone contributions in the evolution equation. The second one results from the vanishing of the anomalous dimension η. (This is a known result [12] which will be reproduced in section 9). This means that we can set $Z_k = 1$ and omit all terms involving η in our expressions. It is more convenient to study the equation (**4**.6) for u'. The potential can be obtained at the end through simple integration. In the large N limit eq. (**4**.6) takes the form ($d = 3$)

$$\frac{\partial u'}{\partial t} = -2u' + \tilde{\rho}u'' + v_3 N u'' L_1^3(u'). \qquad (\mathbf{5}.1)$$

For the choice of R_k of eqs. (**2**.2), (**2**.3) the function $L_1^3(u')$ does not have a simple analytical form. We use the notation of eq. (**A**.5) and and make, in this section, the approximation (with l_1^3 a constant of order unity)

$$\begin{aligned} L_1^3(u') &= -2l_1^3 s_1^3(u') \\ s_1^3(u') &= (1+u')^{-2}. \end{aligned} \tag{**5**.2}$$

Notice that the above choice preserves the correct behaviour of $L_1^3(u')$ as discussed in appendix A: monotonicity, existence of a pole at $u' = -1$, and asymptotic dependence on u' for $u' \to \infty$. We emphasize that the choice of R_k is arbitrary within some general conditions [1]. It may be possible to find an R_k such that eqs. (**5**.2) become exact, with a suitably chosen value of l_1^3. (See also footnote on page 17.)

Eq. (**5**.1) can now be written as

$$\frac{\partial u'}{\partial t} - \tilde{\rho}\frac{\partial u'}{\partial \tilde{\rho}} + \frac{NC}{(1+u')^2}\frac{\partial u'}{\partial \tilde{\rho}} + 2u' = 0, \tag{**5**.3}$$

where $C = 2v_3 l_1^3 = l_1^3/4\pi^2$. We have also kept explicitly the factor of N for consistency checks through the N dependence of the couplings. The most general solution of the partial differential equation (**5**.3) is given by the relations

$$\frac{\tilde{\rho}}{\sqrt{u'}} - \frac{NC}{\sqrt{u'}} - \frac{NC}{2}\frac{\sqrt{u'}}{1+u'} + \frac{3}{2}NC \arctan\left(\frac{1}{\sqrt{u'}}\right) = F\left(u'e^{2t}\right) \qquad \text{for } u' > 0 \tag{**5**.4}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'}} - \frac{NC}{\sqrt{-u'}} + \frac{NC}{2}\frac{\sqrt{-u'}}{1+u'} - \frac{3}{4}NC \ln\left(\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}\right) = F\left(u'e^{2t}\right) \quad \text{for } u' < 0. \tag{**5**.5}$$

The function F is undetermined until boundary conditions are specified. For $t = 0$ ($k = \Lambda$), U_k coincides with the classical potential which we choose

$$U_\Lambda(\rho) = V(\rho) = -\bar{\mu}_\Lambda^2 \rho + \frac{\bar{\lambda}_\Lambda}{2}\rho^2. \tag{**5**.6}$$

The boundary condition, therefore, reads

$$u'(\tilde{\rho}, t=0) = -\mu_\Lambda^2 + \lambda_\Lambda \tilde{\rho}, \tag{**5**.7}$$

where

$$\mu_\Lambda^2 = \frac{\bar{\mu}_\Lambda^2}{\Lambda^2}, \qquad \lambda_\Lambda = \frac{\bar{\lambda}_\Lambda}{\Lambda}. \tag{**5**.8}$$

This uniquely specifies F and we obtain

$$\begin{aligned} &\frac{\tilde{\rho}}{\sqrt{u'}} - \frac{NC}{\sqrt{u'}} - \frac{NC}{2}\frac{\sqrt{u'}}{1+u'} + \frac{3}{2}NC \arctan\left(\frac{1}{\sqrt{u'}}\right) = \\ &\qquad \frac{\sqrt{u'}e^t}{\lambda_\Lambda} + \left(\frac{\mu_\Lambda^2}{\lambda_\Lambda} - NC\right)\frac{1}{\sqrt{u'}e^t} - \frac{NC}{2}\frac{\sqrt{u'}e^t}{1+u'e^{2t}} + \frac{3}{2}NC \arctan\left(\frac{1}{\sqrt{u'}e^t}\right) \qquad \text{for } u' > 0 \end{aligned} \tag{**5**.9}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'}}-\frac{NC}{\sqrt{-u'}}+\frac{NC}{2}\frac{\sqrt{-u'}}{1+u'}-\frac{3}{4}NC\ln\left(\frac{1-\sqrt{-u'}}{1+\sqrt{-u'}}\right)=$$
$$-\frac{\sqrt{-u'}e^t}{\lambda_\Lambda}+\left(\frac{\mu_\Lambda^2}{\lambda_\Lambda}-NC\right)\frac{1}{\sqrt{-u'}e^t}+\frac{NC}{2}\frac{\sqrt{-u'}e^t}{1+u'e^{2t}}-\frac{3}{4}NC\ln\left(\frac{1-\sqrt{-u'}e^t}{1+\sqrt{-u'}e^t}\right)\quad\text{for}\quad u'<0. \tag{5.10}$$

Even though the approximation of eq. (**5**.2) does not permit quantitative accuracy in all respects, eqs. (**5**.9), (**5**.10) contain all the qualitative information for the non-trivial behaviour of the three-dimensional theory. There is a critical value for the minimum of the classical potential

$$\kappa(\Lambda)=\frac{\mu_\Lambda^2}{\lambda_\Lambda}=\kappa_{cr}=NC, \tag{5.11}$$

for which a scale invariant (fixed point) solution is approached in the limit $t\to-\infty$. This solution is given by the relations

$$\frac{\tilde{\rho}}{\sqrt{u'_*}}-\frac{NC}{\sqrt{u'_*}}-\frac{NC}{2}\frac{\sqrt{u'_*}}{1+u'_*}+\frac{3}{2}NC\arctan\left(\frac{1}{\sqrt{u'_*}}\right)=\frac{3\pi}{4}NC\qquad\text{for}\quad u'_*>0 \tag{5.12}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'_*}}-\frac{NC}{\sqrt{-u'_*}}+\frac{NC}{2}\frac{\sqrt{-u'_*}}{1+u'_*}-\frac{3}{4}NC\ln\left(\frac{1-\sqrt{-u'_*}}{1+\sqrt{-u'_*}}\right)=0\qquad\text{for}\quad u'_*<0. \tag{5.13}$$

On the other hand, the general solution of the scaling equation, which results from eq. (**5**.3) by setting $\partial u'/\partial t=0$, is given by

$$\frac{\tilde{\rho}}{\sqrt{u'_*}}-\frac{NC}{\sqrt{u'_*}}-\frac{NC}{2}\frac{\sqrt{u'_*}}{1+u'_*}+\frac{3}{2}NC\arctan\left(\frac{1}{\sqrt{u'_*}}\right)=c_+\qquad\text{for}\quad u'_*>0 \tag{5.14}$$

$$\frac{\tilde{\rho}}{\sqrt{-u'_*}}-\frac{NC}{\sqrt{-u'_*}}+\frac{NC}{2}\frac{\sqrt{-u'_*}}{1+u'_*}-\frac{3}{4}NC\ln\left(\frac{1-\sqrt{-u'_*}}{1+\sqrt{-u'_*}}\right)=c_-\qquad\text{for}\quad u'_*<0, \tag{5.15}$$

with c_+, c_- arbitrary constants. It is remarkable that there is only one solution of the scaling equation which is continuous and finite for all finite $\tilde{\rho}$. This is exactly given by eqs. (**5**.12), (**5**.13). This criterion specifies more generally the selection of the two free integration constants which are necessarily present for the general solution of eq. (**4**.10).

Eqs. (**5**.12), (**5**.13) describe a potential u which has a minimum at a constant value

$$\kappa(k)=\kappa_*=NC, \tag{5.16}$$

and, according to (**3**.26), corresponds to the phase transition between the spontaneously broken and the symmetric regime. (The values for κ_{cr} and κ_* coincide, but this is accidental.) For the second $\tilde{\rho}$-derivative of u at the minimum $\lambda=u''(\kappa)$ we find

$$\lambda(k)=\lambda_*=\frac{1}{2NC}, \tag{5.17}$$

and similar fixed point values for the higher derivarives of u. For $1 \ll \tilde{\rho}/NC \ll (3\pi/4)e^{-t}$ the rescaled potential u has the form

$$u'_*(\tilde{\rho}) = \left(\frac{4}{3\pi NC}\right)^2 \tilde{\rho}^2. \tag{5.18}$$

Notice that the region of validity of eq. (**5.**18) extends to infinite $\tilde{\rho}$ for $t \to -\infty$. From eq. (**5.**18) with $t \to -\infty (k \to 0)$ we obtain for the effective potential at the phase transition

$$U_*(\rho) = \frac{1}{3}\left(\frac{4}{3\pi NC}\right)^2 \rho^3 \tag{5.19}$$

and a critical exponent $\delta = 5$, in agreement with standard large N results [12]. (For the definition of δ and detailed discussion for non-zero η see section 6.)

Through eqs. (**5.**9), (**5.**10) we can also study solutions which deviate slightly from the scale invariant one. For this purpose we define a classical potential with a minimum

$$\kappa(\Lambda) = \kappa_{cr} + N\delta\kappa_\Lambda, \tag{5.20}$$

with $|\delta\kappa_\Lambda| \ll 1$. We find for the minimum of the potential

$$\kappa(k) = \kappa_* + N\delta\kappa_\Lambda e^{-t}, \tag{5.21}$$

and for λ

$$\lambda(k) = \frac{\lambda_*}{1 + \left(\frac{\lambda_*}{\lambda_\Lambda} - 1\right) e^t}. \tag{5.22}$$

Eq. (**5.**21) indicates that the minimum of the potential stays close to the fixed point value κ_* given by (**5.**16), for a very long "time" $|t| < -\ln|\delta\kappa_\Lambda|$. For $|t| > -\ln|\delta\kappa_\Lambda|$ it deviates from the fixed point, either towards the phase with spontaneous symmetry breaking (for $\delta\kappa_\Lambda > 0$), or the symmetric one (for $\delta\kappa_\Lambda < 0$). Eq. (**5.**22) implies an attractive fixed point for λ, with a value given by eq. (**5.**17). Similarly the higher derivatives are attracted to their fixed point values. The full phase diagram corresponds to a second order phase transition. For $\delta\kappa_\Lambda > 0$ the system ends up in the phase with spontaneous symmetry breaking, with an order parameter given by

$$\rho_0 = \lim_{k\to 0} \rho_0(k) = \lim_{k\to 0} k\kappa(k) = N\delta\kappa_\Lambda \Lambda. \tag{5.23}$$

This leads to a critical exponent

$$\beta = \lim_{\delta\kappa_\Lambda \to 0} \frac{d\left[\ln\sqrt{\rho_0}\right]}{d\left[\ln \delta\kappa_\Lambda\right]} = 0.5, \tag{5.24}$$

again in agreement with standard large N results [12]. (For a detailed discussion of the exponents β, ν, ζ see section 9.) In this phase the renormalized quartic coupling approaches zero linearly with k

$$\lambda_R = \lim_{k\to 0} k\lambda(k) = \lim_{k\to 0} k\lambda_* = 0. \tag{5.25}$$

The fluctuations of the Goldstone bosons lead to an infrared free theory in the phase with spontaneous symmetry breaking. For $\delta\kappa_\Lambda < 0$, $\kappa(k)$ becomes zero at a scale

$$t_s = -\ln\left(\frac{\kappa_*}{N|\delta\kappa_\Lambda|}\right) \tag{5.26}$$

and the system ends up in the symmetric regime ($\rho_0 = 0$). From (**5**.9), in the limit $t \to -\infty$, with $u', u'', u''' \to \infty$, so that $u'e^{2t} \sim |\delta\kappa_\Lambda|^2$, $u''e^t \sim |\delta\kappa_\Lambda|/N$, $u''' \sim 1/N^2$, we find

$$U(\rho) = U_0(\rho) = \left(\frac{4}{3\pi C}\right)^2 \left[[\delta\kappa_\Lambda]^2\Lambda^2\rho - \frac{1}{N}\delta\kappa_\Lambda\Lambda\rho^2 + \frac{1}{3N^2}\rho^3\right], \tag{5.27}$$

in qualitative agreement with previous studies [12, **?**]. In particular, we emphasize that the critical behaviour of the three-dimensional theory coincides with the behaviour of the four-dimensional theory at non-vanishing temperature near the critical temperature T_{cr}. Eq. (**5**.27) yields for both the symmetric phase and the phase with spontaneous symmetry breaking ($\rho \geq \rho_{min}$)

$$U'(\rho_4) = \left(\frac{2}{3l_1^3}\right)^2 \left(\frac{8\pi\rho_4}{NT} + \frac{\pi}{3}\frac{T^2 - T_{cr}^2}{T}\right)^2, \tag{5.28}$$

provided we identify

$$\delta\kappa_\Lambda\ \Lambda = \frac{1}{24}\frac{T_{cr}^2 - T^2}{T} \tag{5.29}$$

and use the four-dimensional normalization for the field $\rho_4 = \rho T$. Up to a possible difference [†] in the overall normalization of U' this is exactly the high temperature result of ref. [17]. One obtains the following well known values for the critical exponents ν, ζ in the symmetric phase

$$\nu = \lim_{\delta\kappa_\Lambda \to 0} \frac{d\left[\ln m_R\right]}{d\left[\ln|\delta\kappa_\Lambda|\right]} = \lim_{\delta\kappa_\Lambda \to 0} \frac{d\left[\ln\sqrt{U'(0)}\right]}{d\left[\ln|\delta\kappa_\Lambda|\right]} = 1 \tag{5.30}$$

$$\zeta = \lim_{\delta\kappa_\Lambda \to 0} \frac{d\left[\ln\lambda_R\right]}{d\left[\ln|\delta\kappa_\Lambda|\right]} = \lim_{\delta\kappa_\Lambda \to 0} \frac{d\left[\ln U''(0)\right]}{d\left[\ln|\delta\kappa_\Lambda|\right]} = 1. \tag{5.31}$$

Finally we find

$$\lim_{\delta\kappa_\Lambda \to 0} \frac{\lambda_R}{m_R} = \frac{8}{3\pi NC}, \tag{5.32}$$

a result connected to the resolution of the problem of infrared divergences of perturbation theory [13, **?**, **?**].

Another approach for the study of eq. (**5**.3) would make use of a parametrization of u in terms of its minimum κ and the derivatives at the minimum λ, u_n, as discussed at

[†] Eq. (3.15) in ref. [17] obtains for $l_1^3 = 2/3$. It differs from eq. (**5**.28) in the present paper by an overall factor if we insert the value (**A**.4), since $l_1^3 = \sqrt{\pi}/2$. This difference is due to the approximation of eq. (**5**.2). Note that it may be possible to alter the definition of R_k in eqs. (**2**.2), (**2**.3) so that eq. (**5**.2) becomes exact. Then l_1^3 should come out to be 2/3 with this special choice.

the end of the previous section. For the approximation of eq. (**5**.2), and in the large N limit, eqs. (**4**.8), (**4**.9) read

$$\frac{d\kappa}{dt} = -\kappa + NC \tag{**5**.33}$$

$$\frac{d\lambda}{dt} = -\lambda + 2NC\lambda^2. \tag{**5**.34}$$

The solution of the above equations, for a classical potential of the form of eq. (**5**.6), is given by eqs. (**5**.21), (**5**.22). Notice, however, that the disappearance of the higher derivatives of the potential in eqs. (**5**.33), (**5**.34) is a very convenient simplification appearing only for $N \to \infty$. In the next sections, where we shall study the full eqs. (**4**.8), (**4**.9) for small values of N, more elaborate approximation schemes will be necessary.

As a final comment we point out that, for a theory with spontaneous symmetry breaking , we can use eq. (**5**.10) in order to study the "inner" part of the potential. In particular, for $\tilde{\rho} = 0$ and $t \to -\infty$ eq. (**5**.10) predicts a potential u which asymptotically behaves as

$$\lim_{t \to -\infty} u'(0) = -1. \tag{**5**.35}$$

This leads to an effective average potential U_k which becomes convex with

$$\lim_{k \to 0} U_k'(0) = -k^2, \tag{**5**.36}$$

in agreement with the detailed study of ref. [18].

6. Scaling solution for $k \to 0$ at fixed ρ

We now turn back to the finite values of N. The scaling equation (**4**.10) takes a particularly simple form for $\tilde{\rho} \to \infty$ which we will study in this section. We observe that the integrals $L_0^d(\tilde{w})$, $\tilde{L}_0^d(\tilde{w})$ rapidly decrease for $\tilde{w} \gg 1$ (see appendix A). For large values of $\tilde{\rho}$ we, therefore, obtain the asymptotic equation

$$-du_* + (d - 2 + \eta_*)\tilde{\rho}u_*' = 0. \tag{**6**.1}$$

This has the simple general solution

$$u_*(\tilde{\rho}) = c\tilde{\rho}^\tau \tag{**6**.2}$$

with

$$\tau = \frac{d}{d - 2 + \eta_*} \tag{**6**.3}$$

and c an integration constant.

The limit $k \to 0$ for fixed ρ corresponds to $\tilde{\rho} \to \infty$ for $d - 2 + \eta_* > 0$ (see eq. (**3**.16)). In this limit U_k becomes the effective potential $U_0 = U$. As a result eq. (**6**.2) gives the scaling form of $U(\rho)$, namely

$$U_*(\rho) = cZ_c^\tau \rho^\tau. \tag{**6**.4}$$

Here the constant Z_c appears in the scaling form of the wave function renormalization

$$Z_k = Z_c k^{-\eta_*}. \tag{6.5}$$

The form of the potential (**6.**4) is directly related to the critical exponent δ characterizing the relation between the magnetization ϕ and the magnetic field $B = \frac{\partial U}{\partial \phi} = \phi \frac{\partial U}{\partial \rho}$

$$B \sim \phi^{2\tau - 1} \sim \phi^{\delta}. \tag{6.6}$$

One obtains the well known scaling law

$$\delta = \frac{d + 2 - \eta_*}{d - 2 + \eta_*}. \tag{6.7}$$

At this point the reader may be somewhat surprised that the simple approximation (**6.**1) automatically produces the correct relation between δ and η_*. One would expect that for k^{d-2} much smaller than $Z_k \rho$ the wave function renormalization $Z_k(\rho)$ becomes independent of k, since no low mass modes are present. The identification of $Z_k(\rho)$ with Z_k as defined in eq. (**3.**8) becomes then problematic. Indeed, the presence of two different infrared cutoffs: k^2 and

$$m^2 = Z_k^{-1}(\rho) \frac{\partial U}{\partial \rho} \tag{6.8}$$

suggests a qualitative behaviour

$$Z_k(\rho) \sim (k^2 + C m^2)^{-\frac{\eta_*}{2}}. \tag{6.9}$$

This implies that $Z_k(\rho)$ and $Z_k = Z_k(\rho_0(k))$ differ substantially for $\tilde{\rho} \gg 1$. Nevertheless, the asymptotic form of the scaling equation, given by eq. (**6.**1), remains valid. This can be seen by introducing a different scaling variable

$$\bar{\rho} = Z_k(\rho) k^{2-d} \rho \tag{6.10}$$

which absorbs the dominant effects of the wave function renormalization for large $\tilde{\rho}$ as well. Using the scaling behaviour given by eqs. (**6.**4), (**6.**9) for large $\bar{\rho}$ we find

$$Z_k(\rho) \sim m^{-\eta_*} \sim \rho^{-\frac{\eta_*(\tau - 1)}{2 - \eta_*}} \sim \rho^{-\frac{\eta_*}{d - 2 + \eta_*}} \tag{6.11}$$

$$\bar{\rho} \sim k^{2-d} \rho^{\frac{d-2}{d-2+\eta_*}}. \tag{6.12}$$

We replace eq. (**3.**18) by

$$\begin{aligned} \left. \frac{\partial U_k}{\partial t} \right|_{\bar{\rho}} &= \left. \frac{\partial U_k}{\partial t} \right|_{\rho} + \left. \frac{\partial U_k}{\partial \rho} \right|_t \left. \frac{\partial \rho}{\partial t} \right|_{\bar{\rho}} = \left. \frac{\partial U_k}{\partial t} \right|_{\rho} + \bar{\rho} \left. \frac{\partial U_k}{\partial \bar{\rho}} \right|_t \left. \frac{\partial \ln \bar{\rho}}{\partial \ln \rho} \right|_t \left. \frac{\partial \ln \rho}{\partial t} \right|_{\bar{\rho}} \\ &= \left. \frac{\partial U_k}{\partial t} \right|_{\rho} + (d-2) \bar{\rho} \left. \frac{\partial U_k}{\partial \bar{\rho}} \right|_t . \end{aligned} \tag{6.13}$$

The asymptotic solution obtains in a similar way as before

$$u_*(\bar{\rho}) \sim \bar{\rho}^{\frac{d}{d-2}} \tag{6.14}$$

and coincides with eq. (**6.**4). We conclude that, in a leading approximation, the scaling equation (**6.**1) remains valid for large $\tilde{\rho}$ despite the difference between $Z_k(\rho)$ and Z_k.

We finally observe that corrections to the asymptotic solution can be obtained from eq. (**4.**10) by factoring out the asymptotic behaviour

$$\hat{u}(\tilde{\rho}) = u(\tilde{\rho})\tilde{\rho}^{-\tau}. \tag{6.15}$$

This leads to

$$\begin{aligned}(d-2+\eta_*)\tilde{\rho}\hat{u}_*' - v_d(N-1)\tilde{\rho}^{-\tau}L_0^d\left(\hat{u}_*'\tilde{\rho}^{\tau} + \tau\hat{u}_*\tilde{\rho}^{\tau-1}\right) \\ - v_d\tilde{\rho}^{-\tau}\tilde{L}_0^d\left(2\hat{u}_*''\tilde{\rho}^{\tau+1} + (1+4\tau)\hat{u}_*'\tilde{\rho}^{\tau} + \tau(2\tau-1)\hat{u}_*\tilde{\rho}^{\tau-1}\right) = 0.\end{aligned} \tag{6.16}$$

The integrals L_0^d, $\tilde{L}_0^d$ can be expanded for large values of the argument $\tilde{w}$. (For the asymptotic behaviour of L_0^d for $\eta = 0$ see eq. (**A.**6) in appendix A.) In this way one can compute the leading corrections to eq. (**6.**2) for large $\tilde{\rho}$. We postpone this discussion for a future publication [19].

7. The η expansion

In three dimensions, where

$$\tau = \frac{3}{1+\eta_*} \tag{7.1}$$

we observe that, according to eq. (**6.**2), the third derivative of $u_*(\tilde{\rho})$ goes to zero for $\tilde{\rho} \to \infty$ since η_* is positive

$$u_*'''(\tilde{\rho}) = c\tau(\tau-1)(\tau-2)\tilde{\rho}^{-\frac{3\eta_*}{1+\eta_*}}. \tag{7.2}$$

This also holds for the higher derivatives of u_*. On the other hand we learn from eq. (**6.**4) that $U_*'''(\rho \to 0)$ diverges at the phase transition. This shows that the ϕ^6 coupling plays an important role at and near the phase transition. Since the fourth and higher derivatives of U_* also diverge as $\rho \to 0$ one may suspect that all these higher derivatives have to be included for a quantitative estimate of the properties near the phase transition. This would require a solution of eq. (**4.**5) without the possibility of truncating the corresponding system of infinitely many differential equations for $u^{(n)}$ (see the discussion at the end of section 4) by setting $u^{(n)} = 0$ for n higher than a certain integer.

Fortunately, the situation is much better due to the smallness of the anomalous dimension η_*. (For the three-dimensional theory $\eta_* \simeq 0 - 0.04$, see section 9.) We note that for $\eta_* = 0$ the fourth and higer derivatives of $u_*(\tilde{\rho})$ vanish identically for $\tilde{\rho} \to \infty$. Numerical studies of eq. (**4.**10) indicate that this is approximately true for small values of $\tilde{\rho}$ as well

[19]. Moreover, infinitesimally close to the phase transition the system spends arbitrarily long "time" $|t| = -\ln(k/\Lambda)$ near the fixed point, before deviating towards the phase with spontaneous symmetry breaking or the symmetric one. As a result the value of η which is relevant for the dynamics is the fixed point value η_*. There should, therefore, exist an expansion in the value of η – the η expansion – where the derivatives $u^{(n)}$ for $n \geq 4$ give only small corrections $\sim \eta$. For $\tilde{\rho} \to \infty$ we observe the relations

$$\tilde{\rho} u^{(4)}(\tilde{\rho}) = -\frac{3\eta}{1+\eta} u'''(\tilde{\rho}) = -\epsilon u'''(\tilde{\rho}) \tag{7.3}$$

$$\tilde{\rho}^2 u^{(5)}(\tilde{\rho}) = \epsilon(\epsilon+1) u'''(\tilde{\rho}) \tag{7.4}$$

and similar for higher derivatives. All derivatives $u^{(n)}$ with $n \geq 4$ are of order $\epsilon \sim \eta$ and combinations of the type $\tilde{\rho}^2 u^{(5)} + \tilde{\rho} u^{(4)}$ are even suppressed by higher powers of ϵ according to

$$\left(\tilde{\rho}\frac{\partial}{\partial \tilde{\rho}}\right)^n u'''(\tilde{\rho}) = (-\epsilon)^n u'''(\tilde{\rho}). \tag{7.5}$$

A truncation of (**4.**5) should give reliable results at least for small values of η_*. Let us now try to exploit these properties also near the minimum of u and device a systematic truncation of eq. (**4.**5). The evolution of u', u'' is given by equations (**4.**6), (**4.**7) respectively. For the next two derivatives one finds

$$\begin{aligned}
\frac{\partial u'''}{\partial t} = {} & (2d - 6 + 3\eta) u''' + (d - 2 + \eta)\tilde{\rho} u^{(4)} \\
& + v_d(N-1)\Big\{2[u'']^3 L_3^d(u') - 3u'' u''' L_2^d(u') + u^{(4)} L_1^d(u')\Big\} \\
& + v_d\Big\{2(3u'' + 2\tilde{\rho} u''')^3 \tilde{L}_3^d(u' + 2\tilde{\rho} u'') - 3(3u'' + 2\tilde{\rho} u''')(5u''' + 2\tilde{\rho} u^{(4)}) \tilde{L}_2^d(u' + 2\tilde{\rho} u'') \\
& + (7u^{(4)} + 2\tilde{\rho} u^{(5)}) \tilde{L}_1^d(u' + 2\tilde{\rho} u'')\Big\}
\end{aligned} \tag{7.6}$$

$$\begin{aligned}
\frac{\partial u^{(4)}}{\partial t} = {} & (3d - 8 + 4\eta) u^{(4)} + (d - 2 + \eta)\tilde{\rho} u^{(5)} \\
& - v_d(N-1)\Big\{6[u'']^4 L_4^d(u') - 12[u'']^2 u''' L_3^d(u') + \Big(4u'' u^{(4)} + 3[u''']^2\Big) L_2^d(u') - u^{(5)} L_1^d(u')\Big\} \\
& - v_d\Big\{6(3u'' + 2\tilde{\rho} u''')^4 \tilde{L}_4^d(u' + 2\tilde{\rho} u'') - 12(3u'' + 2\tilde{\rho} u''')^2 (5u''' + 2\tilde{\rho} u^{(4)}) \tilde{L}_3^d(u' + 2\tilde{\rho} u'') \\
& + \Big[4(3u'' + 2\tilde{\rho} u''')(7u^{(4)} + 2\tilde{\rho} u^{(5)}) + 3(5u''' + 2\tilde{\rho} u^{(4)})^2\Big] \tilde{L}_2^d(u' + 2\tilde{\rho} u'') \\
& - (9u^{(5)} + 2\tilde{\rho} u^{(6)}) \tilde{L}_1^d(u' + 2\tilde{\rho} u'')\Big\}.
\end{aligned} \tag{7.7}$$

As discussed earlier we can use these equations for a Taylor expansion of u around some fixed value. We choose $\rho = \rho_0(k)$ if the minimum of u is away from the origin and $\rho = 0$ otherwise.

In the spontaneously broken regime ($\rho_0(k) \neq 0$), the evolution of the minimum κ of $u(\tilde{\rho})$ is given by eq. (**4.**8). We consider the couplings $\lambda = u''(\kappa)$, $u_n = u^{(n)}(\kappa)$ which obey

the evolution equations

$$\frac{du_n}{dt} = \beta_n = \left.\frac{\partial u^{(n)}}{\partial t}\right|_{\tilde{\rho}=\kappa} + u_{n+1}\frac{d\kappa}{dt}. \tag{7.8}$$

The running of λ is given by eq. (**4**.9) and the next two couplings evolve according to

$$\begin{aligned}\frac{du_3}{dt} = \beta_3 &= (2d-6+3\eta)u_3 + v_d(N-1)\Big\{2\lambda^3 L_3^d(0) - 3\lambda u_3 L_2^d(0)\Big\} \\ &+ v_d\Big\{2(3\lambda+2\kappa u_3)^3\tilde{L}_3^d(2\lambda\kappa) - 3(3\lambda+2\kappa u_3)(5u_3+2\kappa u_4)\tilde{L}_2^d(2\lambda\kappa) \\ &+ \left(4u_4 + 2\kappa u_5 - \frac{2\kappa u_3 u_4}{\lambda}\right)\tilde{L}_1^d(2\lambda\kappa)\Big\} \end{aligned} \tag{7.9}$$

$$\begin{aligned}\frac{du_4}{dt} = \beta_4 &= (3d-8+4\eta)u_4 - v_d(N-1)\Big\{6\lambda^4 L_4^d(0) - 12\lambda^2 u_3 L_3^d(0) + (4\lambda u_4 + 3u_3^2)L_2^d(0)\Big\} \\ &- v_d\Big\{6(3\lambda+2\kappa u_3)^4\tilde{L}_4^d(2\lambda\kappa) - 12(3\lambda+2\kappa u_3)^2(5u_3+2\kappa u_4)\tilde{L}_3^d(2\lambda\kappa) \\ &+ \Big[4(3\lambda+2\kappa u_3)(7u_4+2\kappa u_5) + 3(5u_3+2\kappa u_4)^2\Big]\tilde{L}_2^d(2\lambda\kappa) \\ &- \left(6u_5 + 2\kappa u_6 - \frac{2\kappa u_3 u_5}{\lambda}\right)\tilde{L}_1^d(2\lambda\kappa)\Big\}. \end{aligned} \tag{7.10}$$

We observe that eqs. (**4**.8), (**4**.9), (**7**.9), (**7**.10) in leading order in N, i.e. keeping the contribution $(n(d-2+\eta)-d)u_n$ and the term $\sim (N-1)$, can be solved level by level. In fact, β_κ, β_λ involve only κ and λ, β_3 depends only on κ, λ, u_3 and β_4 on κ, λ, u_3, u_4. In this limit no truncation is necessary for the determination of the infrared fixed point κ_*, λ_*, u_{3*}, u_{4*}. This behaviour generalizes to arbitrary n. The problem arises from the last bracket in the expressions for β_3, β_4 which involves the couplings u_5, u_6 which are undetermined at this stage. The same problem shifts to higher couplings if higher β_n are computed. The function β_n always involves the couplings u_{n+1}, u_{n+2} in addition to u_m with $m \leq n$. At this place some truncation of the infinite system of coupled differential equations is required. We need some information on u_5 and u_6. A first possibility is to put all u_n with $n > 4$ to zero (truncation I). A second choice (truncation II) is suggested by eq. (**7**.5). We approximate

$$\begin{aligned} &u_5 = -\kappa^{-1}u_4, \quad u_6 = 2\kappa^{-2}u_4 && \text{for } \ 0.01 < 2\lambda\kappa < 100 \\ &u_5 = u_6 = 0 && \text{for } \ 2\lambda\kappa < 0.01 \ \text{ or } \ 2\lambda\kappa > 100. \end{aligned} \tag{7.11}$$

As we shall see in section 9, at the fixed point $2\kappa_*\lambda_* \simeq 1$, independent of N. Through the truncation II we approximate u_5, u_6 by expressions inspired by the analysis of the scaling solution, for the part of the evolution during which the system is near the fixed point. For the rest of the evolution we neglect them. Numerical values for the fixed point for the two different truncations are given in section 9, where all the numerical results are presented.

In the symmetric regime ($\rho_0(k) = 0$) we use the couplings $\tilde{m}^2 = u'(0)$, $\lambda = u''(0)$, $u_n = u^{(n)}(0)$. Their evolution equations read (in the symmetric regime $\hat{z} = 1$, see eq. (**3.**15), and the L_n^d, $\tilde{L}_n^d$ integrals coincide)

$$\frac{d\tilde{m}^2}{dt} = \beta_m = (-2+\eta)\tilde{m}^2 + v_d(N+2)\lambda L_1^d(\tilde{m}^2) \tag{**7.**12}$$

$$\frac{d\lambda}{dt} = \beta_\lambda = (d-4+2\eta)\lambda - v_d(N+8)\lambda^2 L_2^d(\tilde{m}^2) + v_d(N+4)u_3 L_1^d(\tilde{m}^2) \tag{**7.**13}$$

$$\begin{aligned}\frac{du_3}{dt} = \beta_3 &= (2d-6+3\eta)u_3 + 2v_d(N+26)\lambda^3 L_3^d(\tilde{m}^2) - 3v_d(N+14)\lambda u_3 L_2^d(\tilde{m}^2) \\ &\quad + v_d(N+6)u_4 L_1^d(\tilde{m}^2)\end{aligned} \tag{**7.**14}$$

$$\begin{aligned}\frac{du_4}{dt} = \beta_4 &= (3d-8+4\eta)u_4 - 6v_d(N+80)\lambda^4 L_4^d(\tilde{m}^2) + 12v_d(N+44)\lambda^2 u_3 L_3^d(\tilde{m}^2) \\ &\quad - 4v_d(N+20)\lambda u_4 L_2^d(\tilde{m}^2) - 3v_d(N+24)u_3^2 L_2^d(\tilde{m}^2) + v_d(N+8)u_5 L_1^d(\tilde{m}^2).\end{aligned} \tag{**7.**15}$$

For the numerical work we shall neglect u_5 in the above equations, an approximation consistent with both truncations I and II.

The last necessary ingredients for the numerical study of the evolution equations are the anomalous dimensions η, $\tilde{\eta}$, which we compute in the following section.

8. Anomalous dimensions

In this section we compute the anomalous dimensions η and $\tilde{\eta}$ (defined in eqs. (**3.**9)), as well as the evolution equation for the ratio $\hat{z} = \tilde{Z}_k/Z_k$ (defined in eq. (**3.**15)). The starting point is again the exact evolution equation (**2.**6), with Γ_k parametrized according to eq. (**2.**11). In order to obtain the evolution equation for Z_k (for $N \geq 2$), we consider a background field configuration with a small momentum dependence given in momentum space by

$$\phi_1(0) = \phi, \qquad \delta\phi_2(Q) = \delta\phi, \tag{**8.**1}$$

with $\delta\phi \ll \phi$, and $\phi_a = 0$ for $a > 2$. (We remind the reader that we are concentrating on real fields for which $\phi(-Q) = \phi^*(Q)$.) Inserting the above configuration into (**2.**11) we obtain

$$Z_k(\rho) = Z_k(\rho, Q^2 = 0) = \lim_{Q^2\to 0} \frac{\partial}{\partial Q^2} \frac{\delta\Gamma_k}{\delta(\delta\phi^*\delta\phi)}(\phi; \delta\phi = 0). \tag{**8.**2}$$

Similarly the evolution equation for $\tilde{Z}_k$ is obtained from eq. (**2.**6) by considering a configuration

$$\phi_1(0) = \phi, \qquad \delta\phi_1(Q) = \delta\phi, \tag{**8.**3}$$

with $\delta\phi \ll \phi$, and $\phi_a = 0$ for $a \geq 2$. Eq. (**2.**11) then gives

$$\tilde{Z}_k(\rho) = \tilde{Z}_k(\rho, Q^2 = 0) = \lim_{Q^2\to 0} \frac{\partial}{\partial Q^2} \frac{\delta\Gamma_k}{\delta(\delta\phi^*\delta\phi)}(\phi; \delta\phi = 0). \tag{**8.**4}$$

For the evaluation of the r.h.s. of eq. (**2**.6) we need an expansion of the effective average action around the configurations (**8**.1), (**8**.3). This calculation has been done in ref. [14]. The results are summarized in appendix B, where the general expressions for η and $\tilde{\eta}$ are also derived. In this section we limit our discussion to the approximation of uniform wave function renormalization introduced in section 4 (see eq. (**4**.1)). This approximation corresponds to considering only the value of $Z_k(\rho)$, $\tilde{Z}_k(\rho)$ at the minimum $\rho_0(k)$, while neglecting their derivatives. The relevant expressions are obtained by inserting the truncations

$$\begin{aligned} Z_k'(\rho_0) &= Z_k''(\rho_0) = 0 \\ \tilde{Z}_k'(\rho_0) &= Z_k'(\rho_0) + Y_k(\rho_0) + Y_k'(\rho_0)\rho_0 = 0 \\ \tilde{Z}_k''(\rho_0) &= Z_k''(\rho_0) + 2Y_k'(\rho_0) + Y_k''(\rho_0)\rho_0 = 0 \end{aligned} \tag{**8**.5}$$

into eqs. (**B**.22) and (**B**.39) (where the primes denote ρ-derivatives). It is convenient to introduce the dimensionless quantity

$$y_0 = Z_k^{-2} k^{d-2} Y_k \tag{**8**.6}$$

such that

$$\hat{z} = \frac{\tilde{Z}_k}{Z_k} = 1 + \kappa y_0. \tag{**8**.7}$$

It should be noted that for $N = 1$ there is only one kinetic invariant. The wave function renormalization should be identified with $\tilde{Z}_k$, while Y_k can be set to zero.

In the spontaneously broken regime one finds for the anomalous dimension η

$$\eta = -\frac{2v_d}{d}\kappa^{-1}\hat{M}_2^d(2\lambda\kappa) - v_d y_0 \left[\tilde{L}_1^d(2\lambda\kappa) - \frac{2}{d} y_0 \kappa \tilde{L}_2^{d+2}(2\lambda\kappa)\right] \tag{**8**.8}$$

and similarly for $\tilde{\eta}$

$$\begin{aligned} \tilde{\eta} = {} & v_d(N-1)y_0\hat{z}^{-1}\left[L_1^d(0) + 2\lambda\kappa L_2^d(0)\right] \\ & - \frac{4v_d}{d}\hat{z}^{-1}\kappa\left[(N-1)\lambda^2 Z_k^2 M_{4,0}^d(0) + (3\lambda + 2u_3\kappa)^2 Z_k^2 \tilde{M}_{4,0}^d(2\lambda\kappa)\right]. \end{aligned} \tag{**8**.9}$$

The functions $M_{4,0}^d$, $\tilde{M}_{4,0}^d$ and $\hat{M}_2^d$ are defined in eqs. (**B**.16), (**B**.35) and (**B**.27) respectively. Notice that in eqs. (**8**.8), (**8**.9) we are using rescaled arguments with the conventions of eq. (**3**.22). We shall need the function $\hat{M}_2^d$ for the numerical calculations of the next section. Within the approximation of uniform wave function renormalization and with the notation introduced in section 3, it can be written as

$$\begin{aligned} \hat{M}_2^d(\tilde{w},\eta,\hat{z}) = {} & -2\int_0^\infty dy\, y^{\frac{d}{2}}\left(1 + r + y\frac{\partial r}{\partial y}\right)\left[\frac{1}{(1+r)y} - \frac{1}{(\hat{z}+r)y + \tilde{w}}\right]^2 \\ & \left\{2y\frac{\partial r}{\partial y} + 2\left(y\frac{\partial}{\partial y}\right)^2 r + \eta r + \eta y\frac{\partial r}{\partial y}\right. \\ & \left. - y\left(1 + r + y\frac{\partial r}{\partial y}\right)\left(\eta r + 2y\frac{\partial r}{\partial y}\right)\left[\frac{1}{(1+r)y} + \frac{1}{(\hat{z}+r)y + \tilde{w}}\right]\right\}. \end{aligned} \tag{**8**.10}$$

(The dependence of $\hat{M}_2^d$ (and the other functions) on η and $\hat{z}$ has been suppressed in the notation of eq. (**8**.8) for simplicity.) This integral simplifies considerably if the contributions proportional to η are neglected and we set $\hat{z} = 1$. With these simplifications it depends only on $\tilde{w}$ and is discussed in the appendix C. The evolution equation for $\hat{z}$ can be expressed in terms of η and $\tilde{\eta}$

$$\frac{d\hat{z}}{dt} = \frac{d(\kappa y_0)}{dt} = \hat{z}(\eta - \tilde{\eta}). \tag{**8**.11}$$

For the scaling solution $\hat{z}$ is at its infrared stable fixed point $\hat{z}_*$ (see discussion at the end of section 3). From eq. (**8**.11) we conclude that the fixed point values η_* and $\tilde{\eta}_*$ coincide. In other words, $\hat{z}$ adjusts itself so that the two anomalous dimensions become identical. This behaviour is expected to be a general feature, independent of the particular truncation used in this paper.

In the symmetric regime, where $\rho_0(k) = 0$, we obtain from eq. (**B**.40)

$$\eta = \tilde{\eta} = -v_d y_0 L_1^d(\tilde{m}^2). \tag{**8**.12}$$

At the transition between the spontaneously broken and symmetric regime (for $\kappa = 0$, $\tilde{m}^2 = 0$) this expression agrees with eq. (**8**.8) but not with eq. (**8**.9). The reason is that for small κ the truncation $\tilde{Z}_k'(\rho_0) = 0$ ceases to be consistent with a nonvanishing $Y_k(\rho_0)$ (and $Z_k'(\rho_0) = 0$). A reliable computation of $\tilde{\eta}$ for very small nonvanishing κ should improve over the truncation of eq. (**8**.5). For similar reasons the present truncation does not guarantee $\hat{z} \to 1$ for $\kappa \to 0$ in the spontaneously broken regime. Such a behaviour is required by continuity in a treatment without truncations, since $\hat{z}$ exactly equals one in the symmetric regime. We note that beyond the approximation $Z_k'(\rho_0) = \tilde{Z}_k'(\rho_0) = 0$ the values of η or $\tilde{\eta}$ do not necessarily coincide at the transition between the two regimes, since the definition of the anomalous dimensions involves additional terms from the k dependence of $\rho_0(k)$ in the spontaneously broken regime. For very small κ we may approximate $\tilde{\eta}$ in the spontaneously broken regime by neglecting the contribution from $\tilde{Z}^{(b)}$ of eq. (**B**.32) and setting $Z_k'(\rho_0) = 0$, $Y_k'(\rho_0)\rho_0 = 0$, $U'''(\rho_0)\rho_0 = 0$

$$\tilde{\eta} = v_d \hat{z}^{-1} y_0 \Big\{ 2\tilde{L}_1^d(2\lambda\kappa) + (N-1)L_1^d(0) + \frac{y_0}{\lambda}\tilde{L}_1^{d+2}(2\lambda\kappa) \Big\}. \tag{**8**.13}$$

Finally we need the evolution equation for y_0 in the symmetric regime

$$\frac{dy_0}{dt} = (d - 2 + 2\eta)y_0 + Z_k^{-2} k^{d-2} \frac{dY_k}{dt} \tag{**8**.14}$$

$$\frac{dY_k}{dt} = \lim_{\rho \to 0} \frac{1}{\rho} \frac{\partial}{\partial t} \left[\tilde{Z}_k(\rho) - Z_k(\rho) \right]. \tag{**8**.15}$$

Here $\partial Z_k(\rho)/\partial t$ and $\partial \tilde{Z}_k(\rho)/\partial t$ can be extracted from eqs. (**B**.7), (**B**.8) and (**B**.31), (**B**.32) respectively, with $w_1 = w_2 = U_k'(\rho)$ and $Z_k(\rho) = \tilde{Z}_k(\rho)$. We observe that eq. (**8**.15) is well defined since

$$\lim_{\rho \to 0} \frac{\partial}{\partial t} \tilde{Z}_k(\rho) = \lim_{\rho \to 0} \frac{\partial}{\partial t} Z_k(\rho) = v_d k^{d-2} Z_k^{-1} \left[N Z_k'(0) + Y_k(0) \right] L_1^d(\tilde{m}^2). \tag{**8**.16}$$

Due to the presence of massive modes, Y_k quickly stops running in the symmetric regime and y_0 approaches zero for $d > 2$. Since in the present paper the running in the symmetric regime plays only a secondary role, we shall use in this regime a somewhat stronger truncation by setting $Y_k(0) = 0$. This implies for the symmetric regime

$$\eta = \tilde{\eta} = 0. \tag{8.17}$$

9. Critical exponents of the three-dimensional theory

In this section we use the formalism developed in the previous sections in order to study the phase transition of the three-dimensional N-component ϕ^4 theory. As we have already discussed in section 2, the effective average action interpolates between the classical action (for $k = \Lambda$) and the effective action (for $k = 0$). Therefore, our strategy is to define the classical theory at the ultraviolet cutoff scale and solve the evolution equation for the effective average action (in some appropriate truncation scheme), in order to determine the renormalized theory in the limit $k \to 0$. Subsequently, we can adjust the classical (bare) parameters of the theory in order to approach the fixed point in the critical region and obtain all the quantitative information for the phase transition.

The evolution equations for the parameters of the effective average potential are given by expressions (**4.**8), (**4.**9), (**7.**9), (**7.**10) for the spontaneously broken regime, and (**7.**12) - (**7.**15) for the symmetric regime. The derivatives of the potential higher than the fourth are either neglected (truncation I and symmetric regime) or estimated according to eq. (**7.**11) (truncation II). For the numerical calculation we consider only the anomalous dimension for the Goldstone modes η. We have shown in section 8 that, for the scaling (fixed point) solution, the anomalous dimensions for the radial and Goldstone modes coincide: $\tilde{\eta}_* = \eta_*$. Since in this work we are interested only in the critical region, it is a good approximation to drop the distinction between the two anomalous dimensions. For practical purposes we use eq. (**8.**8), keeping only the first term and neglecting the ones proportional to y_0. It should be noted that this approximation is not valid for $N = 1$, where the only anomalous dimension is $\tilde{\eta}$. It is expected, however, from continuity of our expressions in the parameter N, that eq. (**8.**8) should give a good approximation to $\tilde{\eta}_*$ even for $N = 1$. In the symmetric phase we set $\eta = 0$ according to eq. (**8.**17). For the numerical calculation we also drop the distinction between the integrals L_n^d and $\tilde{L}_n^d$ as explained in section 4 after eq. (**4.**4). In this approximation these integrals are discussed in appendix A. Since they cannot be evaluated in closed form for a general argument, we perform numerical fits of $L_n^3(\tilde{w})$, which connect the expressions (**A.**4) for $\tilde{w} = 0$ with the asymptotic expressions (**A.**6). Another numerical fit is used for the integral $\hat{M}_2^3(\tilde{w})$ which is discussed in appendix C.

A typical example of the numerical integration of the evolution equations is displayed in fig. 2, for $N = 3$. The truncation II is used, even though the parameters u_4, u_5, u_6 are not plotted. The evolution starts at $k = \Lambda$ with the classical action, so that $Z_\Lambda = 1$, $u_3(\Lambda) = u_4(\Lambda) = u_5(\Lambda) = u_6(\Lambda) = 0$. The quartic coupling is arbitrarily chosen $\lambda(\Lambda) = \bar{\lambda}(\Lambda)/\Lambda = 0.1$. Two values of $\kappa(\Lambda)$ are selected so that the system is very close

to the two sides of the critical line. The two values are indistinguishable in the graph and are given by $\kappa(\Lambda) = \kappa_{cr} + \delta\kappa_\Lambda$, with $\kappa_{cr} = [\rho_0(\Lambda)]_{cr}/\Lambda \simeq 0.107$, $|\delta\kappa_\Lambda| \ll \kappa_{cr}$ and $\delta\kappa_\Lambda$ positive or negative. With these initial conditions the system evolves towards the fixed point in the critical region and remains close to it for a long "time" $|t| = -\ln(k/\Lambda)$. The fixed point values of the different parameters are listed in the last row of table 1. By staying near the fixed point for several orders of magnitude in t, the system loses memory of the initial conditions at the cutoff. As a result the critical behaviour is indepenent of the detailed short distance physics and is determined by the fixed point, as long as the evolution starts sufficiently close to the critical line. In the final part of the evolution, as $k \to 0$, the theory settles down either in the phase with spontaneous symmetry breaking or the symmetric one, for $\delta\kappa_\Lambda$ positive or negative respectively.
a) For positive $\delta\kappa_\Lambda$, the physical parameters of the theory approach their renormalized values in the limit $k \to 0$. Due to the presence of the mass for the radial mode, the anomalous dimension η becomes zero for $k \to 0$, and the wave function renormalization approaches a finite value $Z_0 = Z$. We define

$$\begin{aligned} \rho_{0R} &= \lim_{k\to 0} k\kappa(k) = Z^{-1}\rho_0 \\ \lambda_R &= \lim_{k\to 0} k\lambda(k) = Z^{-2}\bar{\lambda}_0 \\ U_{nR} &= \lim_{k\to 0} k^{3-n}u_n(k) = Z^{-n}U_0^{(n)}(\rho_0). \end{aligned} \tag{9.1}$$

For $N \neq 1$ and $k \to 0$, the quantity $\lambda(k)$ approaches a finite value. As a result, due to the fluctuations of the Goldstone modes, the renormalized quartic coupling λ_R is zero in the phase with spontaneous breaking and the theory is infrared free. This is not the case for $N = 1$, where λ diverges in the infrared so that λ_R remains non-zero.
b) For negative $\delta\kappa_\Lambda$, the quantity $\kappa(k)$ becomes zero at some non-zero k_s. The evolution is continued with the equations for the symmetric regime, with $\tilde{m}^2(k_s) = 0$ and $u_n(k_s)$ assuming their values at the end of the running in the spontaneously broken regime. This part of the evolution is not displayed in fig. 2 for simplicity. Notice that the anomalous dimension approaches zero at k_s and remains zero in the symmetric regime. For $k \to 0$ the physical parameters of the theory approach their renormalized values

$$\begin{aligned} m_R^2 &= \lim_{k\to 0} k^2\tilde{m}^2(k) \\ \lambda_R &= \lim_{k\to 0} k\lambda(k) \\ U_{nR} &= \lim_{k\to 0} k^{3-n}u_n(k). \end{aligned} \tag{9.2}$$

Very close to the phase transition, m_R and λ_R are very small compared to Λ, with a fixed value for the ratio m_R/λ_R. We shall return to this point later in this section.

In table 1 the fixed point values of various parameters are listed, for $N = 3$ and various truncations of the evolution equations. For the first row only two evolution equations are considered for κ and λ, and the higher derivatives of the potential as well as the anomalous dimension are set to zero. Then, the number of evolution equations is enlarged to three

with the inclusion of u_3. Subsequently η and u_4 are added (truncation I), and finally u_5 and u_6 are approximated according to eq. (**7.**11) and are included in the system of five coupled evolution equations (truncation II). It becomes apparent from table 1, that the influence of the higher derivatives of the potential on the fixed point, and therefore on the critical dynamics, is small. In fact, for an efficient truncation it would suffice to consider the "relevant" first three ρ-derivatives of the potential and the anomalous dimension. The inclusion of higher derivatives gives small improvements in agreement with the analysis of section 7. In table 2 we list the fixed point values of the various parameters for several values of N. It is apparent that the anomalous dimension at the fixed point is always a small quantity, and that $2\lambda_*\kappa_* \simeq 1$ for all N. This justifies eq. (**7.**11) for the truncation II.

The critical exponents parametrize the singular behaviour of the theory as the phase transition is approached. A measure of the distance from the phase transition is the small difference

$$\delta\kappa_\Lambda = \kappa(\Lambda) - \kappa_{cr} \tag{**9.**3}$$

for any given value of $\lambda(\Lambda)$. If $\kappa(\Lambda)$ is interpreted as a function of temperature in a realistic three-dimensional model, the deviation $\delta\kappa_\Lambda$ is proportional to the deviation from the critical temperature, $\delta\kappa_\Lambda = A_\kappa(T_{cr} - T)$. The critical exponent β parametrizes the behaviour of the "unrenormalized" order parameter

$$\rho_0 = \lim_{k\to 0} \rho_0(k) = Z^{-1}\rho_{0R} \tag{**9.**4}$$

according to

$$\rho_0 \sim (\delta\kappa_\Lambda)^{2\beta}, \tag{**9.**5}$$

for $\delta\kappa_\Lambda \to 0^+$. Similarly, the critical exponent ν describes the divergence near the phase transition of the correlation length, which is equal to the inverse of the renormalized mass m_R. As we have already mentioned, the theory is infrared free in the phase with spontaneous breaking for $N \neq 1$. This means that λ_R vanishes and, since ρ_{0R} is fixed, the renormalized mass m_R vanishes [‡] As a result, for $\delta\kappa_\Lambda > 0$ the critical exponent ν can be defined only for $N = 1$, according to

$$m_R = \sqrt{2\lambda_R\,\rho_{0R}} \sim (\delta\kappa_\Lambda)^{\nu-}. \tag{**9.**6}$$

For $\delta\kappa_\Lambda < 0$ the theory is in the symmetric phase and, for any N, we can define

$$m_R \sim |\delta\kappa_\Lambda|^{\nu+}. \tag{**9.**7}$$

(Here m_R is given by $m(k = 0)$.) The critical exponent γ is related to the response of the expectation value ϕ^a to an external magnetic field or source B^a. The magnetic susceptibility

$$\chi = \frac{\partial\phi^a}{\partial B^a}, \tag{**9.**8}$$

[‡] We could avoid this problem by defining λ_R and m_R by renormalized four and two point functions with non-vanishing external momenta.

evaluated for a vanishing magnetic field $B^a = 0$, is non-analytic for $\delta\kappa_\Lambda \to 0$

$$\chi \sim |\delta\kappa_\Lambda|^{-\gamma}. \tag{9.9}$$

The relation between ϕ^a and B^a is encoded in the effective potential and reads for small values of ϕ^a and $\delta\kappa_\Lambda \neq 0$

$$B^a = \frac{\partial U(\phi)}{\partial \phi_a} = U'\phi^a = \bar{m}^2\phi^a$$
$$\chi = \bar{m}^{-2}. \tag{9.10}$$

Here the "unrenormalized" mass term $\bar{m}^2$ is given by

$$\bar{m}^2 = \lim_{k\to 0} \bar{m}^2(k) = Zm_R^2. \tag{9.11}$$

For $N = 1$ and $\delta\kappa_\Lambda > 0$ we obtain

$$\bar{m}^2 = 2\bar{\lambda}_0\rho_0 = 2Z\lambda_R\ \rho_{0R} \sim (\delta\kappa_\Lambda)^{\gamma_-}. \tag{9.12}$$

In the symmetric phase ($\delta\kappa_\Lambda < 0$) we have for arbitrary N

$$\bar{m}^2 = Zm_R^2 \sim |\delta\kappa_\Lambda|^{\gamma_+}. \tag{9.13}$$

For the numerical determination of the critical exponents we integrate the evolution equations with the truncation II, for several very small values of $\delta\kappa_\Lambda$. The critical exponents are then obtained as the slope of the logarithm of the relevant quantity at $k = 0$, when plotted as a function of $\ln|\delta\kappa_\Lambda|$. More specifically, for the determination of β we compute the slope of $\ln\rho_0$ as a function of $\ln\delta\kappa_\Lambda$, and similarly for the other critical exponents. The behaviour of the exponents for decreasing $|\delta\kappa_\Lambda|$ is depicted in fig. 3 for $N = 4$. As $\delta\kappa_\Lambda \to 0$ they approach universal values. The deviation from the universal behaviour for larger values of $\delta\kappa_\Lambda$ depends on the details of the bare theory. Fig. 3 corresponds to a bare theory (defined at $k = \Lambda$) with $N = 4$, $\lambda(\Lambda) = 0.5$, $\kappa_{cr} \simeq 0.120$. We have verified that the asymptotic values of the critical exponents are universal by determining them for different $\lambda(\Lambda)$ and κ_{cr}. For the anomalous dimension the relevant value is the fixed point value η_*. The results of our calculation are summarized in table 3 for various N. For $N = 1$ the critical exponents ν and γ satisfy $\nu_+ = \nu_-$ and $\gamma_+ = \gamma_-$ with an accuracy of 0.1%. For $N \neq 1$ we list the values for ν_+ and γ_+. The critical exponents satisfy the scaling laws

$$\beta = (1 + \eta_*)\frac{\nu}{2} \tag{9.14}$$

$$\gamma = (2 - \eta_*)\nu \tag{9.15}$$

with an accuracy of 0.1%. We mention also the scaling law of eq. (**6.**7), which gives the critical exponent δ and was obtained in section 6. In table 3 we have included values for the critical exponents obtained through other methods (summed perturbation series in

three dimensions, ϵ-expansion, lattice calculations and $1/N$-expansion), for comparison. We observe agreement at the 1-5 % level for the exponents β, ν, and γ. For the anomalous dimension η_* we observe satisfactory agreement of our results with the quoted values, even though η_* is a small quantity and most severely affected by the various approximations in our method. In fact, significant improvement for all the results is expected if the anomalous dimension for the radial mode $\tilde{\eta}$ is included. Preliminary results indicate that this requires a truncation that includes at least the first ρ-derivative of the wave function renormalizations. We postpone this discussion for a future publication [19]. An interesting point, which is connected to the resolution of the problem of infrared divergences of perturbation theory [13, **?**, **?**], concerns the vanishing of the renormalized quartic coupling λ_R at the phase transition. More specifically, the ratio λ_R/m_R takes an N dependent value in the critical region. As a result, the behaviour of λ_R in the critical region can be characerized by a critical exponent ζ defined according to

$$\lambda_R \sim |\delta\kappa_\Lambda|^\zeta \tag{**9**.16}$$

and

$$\zeta = \nu. \tag{**9**.17}$$

In table 4 we list our results for the ratio λ_R/m_R, as well as results obtained with other methods for comparison. For $N \neq 1$, due to the vanishing of λ_R in the phase with spontaneous symmetry breaking, the ratio was calculated in the symmetric phase. For $N = 1$, the values obtained in the phase with spontaneous symmetry breaking and the symmetric one agree at the 0.1 % level.

10. Conclusions

The effective average action Γ_k results from the effective integration of fluctuations with characteristic momenta larger than a given infrared cutoff k. It is the appropriate quantity for the study of the physics at the scale k. It interpolates between the classical action S for $k = \Lambda$ (Λ being the ultraviolet cutoff of the theory, much larger than any other physical scale) and the effective action Γ for $k = 0$. The dependence of Γ_k on k is given by the exact evolution equation (**2**.6) [1]. No ultraviolet or infrared divergences appear in this evolution equation, and the formalism leads to an efficient treatment of the infrared problems which plague theories with massless modes in less than four dimensions. In this work we have studied the exact non-perturbative evolution equation for the effective potential. It is well suited for the study of critical phenomena since it can be cast into a scale independent form. In particular, a second order phase transition corresponds to fixed points for the various couplings appearing in the potential. They can be computed as the solution of a non-linear second order differential equation for the field dependence of the potential. We have given the explicit solution of this differential equation in the large N limit of the $O(N)$-symmetric scalar theory in three dimensions. It describes the fixed points for infinitely many dimensionless couplings. We also have solved the more general evolution equation away from the phase transition.

For a finite number of components a truncation to a finite number of couplings becomes necessary and the evolution equation has to be solved numerically. We have concentrated on the phase transition of the three-dimensional theory. Our results for the critical exponents β, ν, γ, δ agree at the 1-5 % level with the results of the most sophisticated calculations through other methods (summed perturbation series in three dimensions, ϵ-expansion, lattice calculations, $1/N$-expansion). The anomalous dimension η is also well determined, even though significant improvement is expected with a more efficient treatment. We also emphasize that, apart from the precise determination of the critical dynamics, the method of the effective average action permits the discussion of the non-universal behaviour of the theory, such as the behaviour away from the critical temperature for the finite temperature four-dimensional theory [15]. In this way permits the determination of the region of attraction of the possible infrared fixed points. The effective average action is not only useful for a study of fixed points. More generally it describes the relevant physics at a given lenght scale k^{-1}. In practice, this can be used through the identification of the infrared cutoff scale k with the appropriate physical infrared cutoff scale of the problem (such as the critical bubble scale for first order phase transitions or the Hubble parameter for inflationary cosmology).

We conclude that the formalism of the effective average action provides intuitive understanding, as well as computational power, for a large number of physical problems. Work in progress focuses on the equation of state for the $O(N)$-symmetric theory [19], the further improvement of the precision for the calculation of the critical exponents, and the study of scalar theories with different symmetries and possible first order phase transitions [22]. The concept of the effective integration of degrees of freedom through averaging has been developed for fermionic and gauge fields [23], and an exact evolution equation has been formulated for gauge theories [24]. Work in this direction will lead to the study of more complicated systems, with exciting possibilities for new physical behaviour.

Appendix A: The integrals L_n^d

In this appendix we disuss the integrals $L_n^d(\tilde{w}) = L_n^d(\tilde{w}, 0)$, where $L_n^d(\tilde{w}, \eta)$ are defined in eq. (**4.**4) with $\tilde{L}_n^d(\tilde{w}, \eta, \hat{z})$ given by eqs. (**4.**2). With the choice (**2.**2) - (**2.**3) for $R_k(q)$, they read

$$L_0^d(\tilde{w}) = -2\int_0^\infty dy y^{\frac{d}{2}+1} \frac{\exp(-y)}{[1-\exp(-y)]^2} \left[\frac{y}{1-\exp(-y)} + \tilde{w}\right]^{-1}$$

$$L_n^d(\tilde{w}) = -2n\int_0^\infty dy y^{\frac{d}{2}+1} \frac{\exp(-y)}{[1-\exp(-y)]^2} \left[\frac{y}{1-\exp(-y)} + \tilde{w}\right]^{-(n+1)} \quad \text{for} \quad n \geq 1 \quad (\mathbf{A}.1)$$

They obey the relations

$$\frac{\partial}{\partial \tilde{w}} L_0^d(\tilde{w}) = -L_1^d(\tilde{w})$$

$$\frac{\partial}{\partial \tilde{w}} L_n^d(\tilde{w}) = -nL_{n+1}^d(\tilde{w}) \qquad \text{for} \quad n \geq 1. \qquad (\mathbf{A}.2)$$

For $\tilde{w} > -1$, the integrals $|L_n^d(\tilde{w})|$ are finite, monotonically decreasing functions of $\tilde{w}$ [§]. We define

$$L_0^d(0) = -2l_n^d \qquad (\mathbf{A}.3)$$

and give the values of l_n^d for the first five values of n

$$l_0^d = \zeta\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}+1\right)$$

$$l_1^d = \Gamma\left(\frac{d}{2}\right)$$

$$l_2^d = 2\left(1-2^{1-\frac{d}{2}}\right)\Gamma\left(\frac{d}{2}-1\right)$$

$$l_3^d = 3\left(1-2^{3-\frac{d}{2}}+3^{2-\frac{d}{2}}\right)\Gamma\left(\frac{d}{2}-2\right)$$

$$l_4^d = 4\left(1-3\times 2^{3-\frac{d}{2}}+3^{4-\frac{d}{2}}-4^{3-\frac{d}{2}}\right)\Gamma\left(\frac{d}{2}-3\right). \qquad (\mathbf{A}.4)$$

We define the threshold functions $s_n^d(\tilde{w})$ through

$$L_n^d(\tilde{w}) = -2l_n^d s_n^d(\tilde{w}). \qquad (\mathbf{A}.5)$$

They are decreasing functions of $\tilde{w}$, and $1 \geq s_n^d(\tilde{w}) > 0$ for $0 \leq \tilde{w} < \infty$. The first two terms of the asymptotic expansion of $L_n^d(\tilde{w})$ for large arguments $\tilde{w} \to \infty$ read

$$L_0^d(\tilde{w}) = -2\,\zeta\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}+2\right)\tilde{w}^{-1} + \left[\zeta\left(\frac{d}{2}+1\right)+\zeta\left(\frac{d}{2}+2\right)\right]\Gamma\left(\frac{d}{2}+3\right)\tilde{w}^{-2}...$$

[§] The pole structure of $L_n^d(\tilde{w})$ is not relevant for this work. For a discussion see ref. [18].

$$L_n^d(\tilde{w}) = -2n\,\zeta\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}+2\right)\tilde{w}^{-(n+1)}$$
$$+\,n(n+1)\left[\zeta\left(\frac{d}{2}+1\right)+\zeta\left(\frac{d}{2}+2\right)\right]\Gamma\left(\frac{d}{2}+3\right)\tilde{w}^{-(n+2)}\ldots \quad \text{for} \quad n\geq 1. \quad (\mathbf{A}.6)$$

Appendix B: Wave function renormalization

In this appendix we derive general expressions for the anomalous dimensions η, $\tilde{\eta}$. We consider an effective action of the form of eq. (**2**.11). This is no longer the most general ansatz contributing to the wave function renormalization and our equations are therefore not exact, in contrast to the ones for the effective average potential. In addition, we neglect the q^2 dependence of the couplings $Z_k(\rho, q^2)$. This truncation corresponds to the most general action containing no more than two derivatives. For notational simplicity, we denote in this appendix $U_k(\rho)$, $Z_k(\rho)$, $\tilde{Z}_k(\rho)$, $Y_k(\rho)$ by U, Z, $\tilde{Z}$, Y respectively. Primes on U, Z, $\tilde{Z}$, Y denote derivatives with respect to ρ. The wave function renormalizations Z_k, $\tilde{Z}_k$, Y_k are defined at the minimum of the potential $\rho_0(k)$ according to eq. (**3**.8) and the anomalous dimensions according to eqs. (**3**.9). Also primes on Z_k, $\tilde{Z}_k$, Y_k denote ρ-derivatives of $Z_k(\rho)$, $\tilde{Z}_k(\rho)$, $Y_k(\rho)$ at $\rho=\rho_0$. We point out that for $N=1$ one should set $Y=0$ and $\tilde{Z}$ is the relevant wave function renormalization.

For a calculation of the evolution of Z (which is connected to the anomalous dimension η) we need an expansion of the effective average action Γ_k around the configuration given by eq. (**8**.1). This calculation can be found in section 5 of ref. [14]. Here we present the result for the second functional derivative $\Gamma_k^{(2)}$. It can be written as

$$(\Gamma_k^{(2)})_{ab}(q,q') = (\Gamma_0)_{ab}(q,q') + (\Gamma_1)_{ab}(q,q') + (\Gamma_2)_{ab}(q,q'), \qquad (\mathbf{B}.1)$$

where we have separated the terms containing zero, one and two powers of $\delta\phi$. The first term reads

$$(\Gamma_0)_{ab}(q,q') = \left[(Z\delta_{ab}+\rho Y\delta_{a1}\delta_{b1})\,q^2 + M_{ab}^2\right]\delta(q-q'), \qquad (\mathbf{B}.2)$$

where

$$M_{ab}^2 = U'\delta_{ab} + 2\rho U''\delta_{a1}\delta_{b1}. \qquad (\mathbf{B}.3)$$

The second has only an off diagonal contribution

$$(\Gamma_1)_{11}(q,q') = (\Gamma_1)_{22}(q,q') = 0, \qquad (\Gamma_1)_{ab}(q,q') = 0 \quad \text{for} \quad a,b>2$$
$$(\Gamma_1)_{12}(q,q') = \hat{\Gamma}_1(q,Q)\delta(q-q'-Q) + \hat{\Gamma}_1^*(q,-Q)\delta(q-q'+Q)$$
$$(\Gamma_1)_{21}(q,q') = (\Gamma_1^*)_{12}(q',q)$$
$$\text{with} \qquad \hat{\Gamma}_1(q,Q) = \phi\delta\phi\Big\{U'' + \frac{1}{2}Yq^2 - Z'(q-Q)Q\Big\}. \qquad (\mathbf{B}.4)$$

The diagonal elements of the third term read

$$(\Gamma_2)_{aa}(q,q) = \delta\phi^*\delta\phi\Big\{U'' + 2U''\delta_{a2} + 2U'''\rho\delta_{a1}$$
$$+\,q^2(Z' + Y\delta_{a2} + Y'\rho\delta_{a1}) + Q^2(Z' + Y\delta_{a2} + 2Z''\rho\delta_{a1})\Big\}. \qquad (\mathbf{B}.5)$$

Subsequently one has to evaluate the trace on the r.h.s. of eq. (**2.**6) and extract the piece proportional to $Q^2(\delta\phi^*\delta\phi)$, which gives the evolution of Z according to eq. (**8.**2). The details for the calculation of the trace are given in ref. [14]. For the scale dependence of Z $(= Z_k(\rho))$ at fixed ρ one obtains

$$\left.\frac{\partial Z}{\partial t}\right|_\rho = \left.\frac{\partial Z^{(a)}}{\partial t}\right|_\rho + \left.\frac{\partial Z^{(b)}}{\partial t}\right|_\rho , \tag{**B.**6}$$

with

$$\left.\frac{\partial Z^{(a)}}{\partial t}\right|_\rho = v_d[(N-1)Z' + Y]Z_k^{-1}k^{d-2}L_1^d(w_1) + v_d(Z' + 2Z''\rho)Z_k^{-1}k^{d-2}\tilde{L}_1^d(w_2) \tag{**B.**7}$$

$$\begin{aligned}\left.\frac{\partial Z^{(b)}}{\partial t}\right|_\rho &= 4v_d[U'']^2\rho k^{d-6}Q_{2,1}^{d,0}(w_1,w_2) + 4v_dYU''\rho k^{d-4}Q_{2,1}^{d,1}(w_1,w_2) \\ &\quad + v_dY^2\rho k^{d-2}Q_{2,1}^{d,2}(w_1,w_2) - 8v_dZ'U''\rho k^{d-4}L_{1,1}^d(w_1,w_2) \\ &\quad - \frac{4v_d}{d}[Z']^2\rho k^{d-2}L_{1,1}^{d+2}(w_1,w_2) - 4v_dZ'Y\rho k^{d-2}L_{1,1}^{d+2}(w_1,w_2) \\ &\quad + \frac{16v_d}{d}Z'U''\rho k^{d-4}N_{2,1}^d(w_1,w_2) + \frac{8v_d}{d}Z'Y\rho k^{d-2}N_{2,1}^{d+2}(w_1,w_2).\end{aligned} \tag{**B.**8}$$

Here we have used the variables

$$\begin{aligned}w_1 &= U' \\ w_2 &= U' + 2\rho U''.\end{aligned} \tag{**B.**9}$$

It is also convenient to define the shorthands

$$\begin{aligned}P &= Zx + R_k(x) \\ \tilde{P} &= \tilde{Z}x + R_k(x)\end{aligned} \tag{**B.**10}$$

with $R_k(q)$ given by eq. (**2.**2) and $x = q^2$. The integrals L_n^d, $\tilde{L}_n^d$ are given by

$$\begin{aligned}L_n^d(w) &= k^{2n-d}Z_k^n\int_0^\infty dx x^{\frac{d}{2}-1}\frac{\partial}{\partial t}(P+w)^{-n} \\ \tilde{L}_n^d(w) &= k^{2n-d}Z_k^n\int_0^\infty dx x^{\frac{d}{2}-1}\frac{\partial}{\partial t}(\tilde{P}+w)^{-n},\end{aligned} \tag{**B.**11}$$

where we emphasize that $\partial/\partial t$ acts on the r.h.s. only on $R_k(x)$. In the approximation of uniform wave function renormalization (when the ρ dependence of Z, $\tilde{Z}$ is neglected) these integrals are discussed in the beginning of section 4 and in appendix A. Notice the connection between the arguments w and $\tilde{w}$: $\tilde{w} = w/Z_k k^2$. The dimensionless integrals

$$L_{n_1,n_2}^d(w_1,w_2) = k^{2(n_1+n_2)-d}\int_0^\infty dx x^{\frac{d}{2}-1}\frac{\partial}{\partial t}\Big\{(P+w_1)^{-n_1}(\tilde{P}+w_2)^{-n_2}\Big\} \tag{**B.**12}$$

are generalizations of the integrals L_n^d, $\tilde{L}_n^d$ with

$$\begin{aligned} L_{n,0}^d(w_1) &= Z_k^{-n} L_n^d(w_1) \\ L_{0,n}^d(w_2) &= Z_k^{-n} \tilde{L}_n^d(w_2). \end{aligned} \tag{B.13}$$

Similarly, the integrals $N_{n_1,n_2}^d(w_1,w_2)$ are defined by

$$N_{n_1,n_2}^d(w_1,w_2) = k^{2(n_1+n_2-1)-d} \int_0^\infty dx x^{\frac{d}{2}} \frac{\partial}{\partial t} \Big\{ \frac{\partial P}{\partial x} (P+w_1)^{-n_1} (\tilde{P}+w_2)^{-n_2} \Big\}. \tag{B.14}$$

Finally, the integrals

$$\begin{aligned} Q_{n_1,n_2}^{d,\alpha}(w_1,w_2) &= k^{2(n_1+n_2-\alpha)-d} \int_0^\infty dx x^{\frac{d}{2}-1+\alpha} \\ &\frac{\partial}{\partial t} \Bigg\{ \left[\frac{\partial P}{\partial x} + \frac{2x}{d} \frac{\partial^2 P}{\partial x^2} - \frac{4x}{d} \left(\frac{\partial P}{\partial x} \right)^2 (P+w_1)^{-1} \right] (P+w_1)^{-n_1} (\tilde{P}+w_2)^{-n_2} \Bigg\} \end{aligned} \tag{B.15}$$

are related by partial integration to other integrals

$$M_{n_1,n_2}^d(w_1,w_2) = k^{2(n_1+n_2-1)-d} \int_0^\infty dx x^{\frac{d}{2}} \frac{\partial}{\partial t} \left\{ \left(\frac{\partial P}{\partial x} \right)^2 (P+w_1)^{-n_1} (\tilde{P}+w_2)^{-n_2} \right\} \tag{B.16}$$

through

$$\begin{aligned} Q_{n_1,n_2}^{d,\alpha}(w_1,w_2) = & \frac{2n_1-4}{d} M_{n_1+1,n_2}^{d+2\alpha}(w_1,w_2) + \frac{2n_2}{d} M_{n_1,n_2+1}^{d+2\alpha}(w_1,w_2) \\ & + \frac{2n_2}{d} \rho Y N_{n_1,n_2+1}^{d+2\alpha}(w_1,w_2) - \frac{2\alpha}{d} N_{n_1,n_2}^{d+2\alpha-2}(w_1,w_2). \end{aligned} \tag{B.17}$$

The evolution equation for Z depends, in this approximation, on U', U'', Z, Z', Z'' and Y. There is also an η dependence resulting from the t-derivative acting on Z_k in $R_k(q)$.

We define the anomalous dimension η by the variation of $\ln Z_k = \ln Z_k(\rho_0(k))$ (see eqs. (**3**.8), (**3**.9)). In the spontaneously broken regime, where $\rho_0(k) \neq 0$, this implies

$$\eta = -Z_k^{-1} \left(\left. \frac{\partial Z^{(a)}}{\partial t} \right|_{\rho_0} + \left. \frac{\partial Z^{(b)}}{\partial t} \right|_{\rho_0} \right) - Z_k^{-1} Z_k' \bar{\delta}. \tag{B.18}$$

Here $\bar{\delta} = d\rho_0(k)/dt$ describes the k dependence of the minimum of the potential

$$\begin{aligned} \bar{\delta} = & -v_d Z_k^{-1} k^{d-2} \Big\{ (N-1) L_1^d(0) + \left(3 + \frac{2U_3\rho_0}{\bar{\lambda}} \right) \tilde{L}_1^d(2\bar{\lambda}\rho_0) \Big\} \\ & - v_d Z_k^{-1} k^d \Big\{ (N-1) \frac{Z_k'}{\bar{\lambda}} L_1^{d+2}(0) + \frac{\tilde{Z}_k'}{\bar{\lambda}} \tilde{L}_1^{d+2}(2\bar{\lambda}\rho_0) \Big\}, \end{aligned} \tag{B.19}$$

with

$$\begin{aligned} &\bar{\lambda} = U_k''(\rho_0), \qquad U_n = U_k^{(n)}(\rho_0) \quad \text{for} \quad n \geq 3 \\ &w_1(\rho_0) = 0, \qquad w_2(\rho_0) = 2\bar{\lambda}\rho_0. \end{aligned} \tag{B.20}$$

Combining eqs. (**B.**7), (**B.**8), (**B.**18), (**B.**19) we obtain

$$\begin{aligned}
\eta = & -v_d k^{d-2} Z_k^{-2}\Big\{[(N-1)Z_k' + Y_k]\, L_1^d(0) + (Z_k' + 2\rho_0 Z_k'')\tilde{L}_1^d(2\bar{\lambda}\rho_0)\Big\} \\
& - \frac{8v_d}{d} Z_k^{-1}\bar{\lambda}^2\rho_0 k^{d-6}\left\{M_{2,2}^d(0,2\bar{\lambda}\rho_0) + \frac{Y_k k^2}{\bar{\lambda}} M_{2,2}^{d+2}(0,2\bar{\lambda}\rho_0) + \frac{1}{4}\left(\frac{Y_k k^2}{\bar{\lambda}}\right)^2 M_{2,2}^{d+4}(0,2\bar{\lambda}\rho_0)\right\} \\
& - \frac{8v_d}{d} Z_k^{-1} Y_k \bar{\lambda}^2\rho_0^2 k^{d-6}\left\{N_{2,2}^d(0,2\bar{\lambda}\rho_0) + \frac{Y_k k^2}{\bar{\lambda}} N_{2,2}^{d+2}(0,2\bar{\lambda}\rho_0) + \frac{1}{4}\left(\frac{Y_k k^2}{\bar{\lambda}}\right)^2 N_{2,2}^{d+4}(0,2\bar{\lambda}\rho_0)\right\} \\
& - \frac{8v_d}{d} Z_k^{-1}(2Z_k' - Y_k)\bar{\lambda}\rho_0 k^{d-4}\left\{N_{2,1}^d(0,2\bar{\lambda}\rho_0) + \frac{1}{2}\frac{Y_k k^2}{\bar{\lambda}} N_{2,1}^{d+2}(0,2\bar{\lambda}\rho_0)\right\} \\
& + 8v_d Z_k^{-1} Z_k'\bar{\lambda}\rho_0 k^{d-4} L_{1,1}^d(0,2\bar{\lambda}\rho_0) + \frac{4v_d}{d} Z_k^{-1} Z_k'(Z_k' + dY_k)\rho_0 k^{d-2} L_{1,1}^{d+2}(0,2\bar{\lambda}\rho_0) \\
& - Z_k^{-1} Z_k'\bar{\delta}. \qquad (\mathbf{B}.21)
\end{aligned}$$

It is convenient to bring η into the form

$$\begin{aligned}
\eta = & -\frac{2v_d}{d} Z_k^{-1}\rho_0^{-1} k^{d-2}\Big\{M_{2,0}^d(0) - 2M_{1,1}^d(0,2\bar{\lambda}\rho_0) + M_{0,2}^d(2\bar{\lambda}\rho_0)\Big\} \\
& - v_d Z_k^{-2} Y_k k^{d-2}\Big\{\tilde{L}_1^d(2\bar{\lambda}\rho_0) - \frac{2}{d} Z_k^{-1} Y_k\rho_0 \tilde{L}_2^{d+2}(2\bar{\lambda}\rho_0)\Big\} \\
& - v_d Z_k^{-1} Z_k' k^{d-2}\left\{(N-1)Z_k^{-1} L_1^d(0) - \frac{8}{d} N_{1,1}^d(0,2\bar{\lambda}\rho_0)\right. \\
& \left. + Z_k^{-1}\left(5 + \frac{2Z_k''\rho_0}{Z_k'}\right)\tilde{L}_1^d(2\bar{\lambda}\rho_0) - \frac{4}{d} Z_k'\rho_0 L_{1,1}^{d+2}(0,2\bar{\lambda}\rho_0)\right\} \\
& - Z_k^{-1} Z_k'\bar{\delta}. \qquad (\mathbf{B}.22)
\end{aligned}$$

For this purpose we have employed the following identities

$$\begin{aligned}
& N_{n_1,n_2}^d(w_1,w_2) = \\
& \qquad \frac{k^2}{w_2 - w_1}\Big\{N_{n_1,n_2-1}^d(w_1,w_2) - N_{n_1-1,n_2}^d(w_1,w_2) - Y\rho N_{n_1,n_2}^{d+2}(w_1,w_2)\Big\} \qquad (\mathbf{B}.23) \\
& N_{n,0}^d(0) = \frac{d}{2(n-1)} Z_k^{-(n-1)} L_{n-1}^d(0) \qquad \text{for } n > 1 \qquad (\mathbf{B}.24) \\
& N_{0,n}^d(w) = \frac{d}{2(n-1)} Z_k^{-(n-1)}\tilde{L}_{n-1}^d(w) - Z_k^{-n} Y\rho\tilde{L}_n^{d+2}(w) \quad \text{for } n > 1 \qquad (\mathbf{B}.25) \\
& M_{n_1,n_2}^d(w_1,w_2) = \\
& \qquad \frac{k^2}{w_2 - w_1}\Big\{M_{n_1,n_2-1}^d(w_1,w_2) - M_{n_1-1,n_2}^d(w_1,w_2) - Y\rho M_{n_1,n_2}^{d+2}(w_1,w_2)\Big\}. \qquad (\mathbf{B}.26)
\end{aligned}$$

We also define the function

$$\hat{M}_2^d(w) = M_{2,0}^d(0) - 2M_{1,1}^d(0,w) + M_{0,2}^d(w)$$

$$= k^{2-d}\int_0^\infty dx x^{\frac{d}{2}}\frac{\partial}{\partial t}\left\{\frac{\partial P}{\partial x}\left(\frac{1}{P}-\frac{1}{\tilde{P}+w}\right)\right\}^2. \qquad (\mathbf{B}.27)$$

For the calculation of the evolution of $\tilde{Z}$ (which is connected to $\tilde{\eta}$) we need an expansion of Γ_k around the configuration given by eq. (**8**.3). The second functional derivative $\Gamma_k^{(2)}$ is again given by eq. (**B**.1) with Γ_0 given by eqs. (**B**.2), (**B**.3). The other two terms now read

$$(\Gamma_1)_{ab}(q,q') = (\hat{\Gamma}_1)_{ab}(q,Q)\delta(q-q'-Q) + (\hat{\Gamma}_1^*)_{ab}(q,-Q)\delta(q-q'+Q)$$
$$(\Gamma_1)_{ba}(q,q') = (\Gamma_1^*)_{ab}(q',q)$$
$$\text{with}\quad (\hat{\Gamma}_1)_{ab}(q,Q) = \phi\delta\phi\Big\{\Big[U''+(q^2-qQ)Z'+\frac{1}{2}Q^2Y\Big]\delta_{ab}$$
$$+\Big[2U''+2U'''\rho+(q^2-qQ)(Y+Y'\rho)+Q^2(Z'+\frac{1}{2}Y+Y'\rho)\Big]\delta_{a1}\delta_{b1}\Big\} \qquad (\mathbf{B}.28)$$

and

$$(\Gamma_2)_{ab}(q,q) = \delta\phi^*\delta\phi\Big\{(U''+2U'''\rho)\delta_{ab}+(2U''+10U'''\rho+4U^{(4)}\rho^2)\delta_{a1}\delta_{b1}$$
$$+(Z'+Y'\rho)Q^2\delta_{ab}+(Y+2Z''\rho+4Y'\rho+2Y''\rho^2)Q^2\delta_{a1}\delta_{b1}$$
$$+(Z'+2Z''\rho)q^2\delta_{ab}+(Y+5Y'\rho+2Y''\rho^2)q^2\delta_{a1}\delta_{b1}\Big\}. \qquad (\mathbf{B}.29)$$

For the calculation of the trace we refer the reader again to ref. [14] and we give the result for

$$\left.\frac{\partial\tilde{Z}}{\partial t}\right|_\rho = \left.\frac{\partial\tilde{Z}^{(a)}}{\partial t}\right|_\rho + \left.\frac{\partial\tilde{Z}^{(b)}}{\partial t}\right|_\rho. \qquad (\mathbf{B}.30)$$

One finds (with w_1, w_2 given by eqs. (**B**.9))

$$\left.\frac{\partial\tilde{Z}^{(a)}}{\partial t}\right|_\rho = v_d(N-1)(Z'+Y'\rho)Z_k^{-1}k^{d-2}L_1^d(w_1)$$
$$+v_d(Z'+2Z''\rho+Y+5Y'\rho+2Y''\rho^2)Z_k^{-1}k^{d-2}\tilde{L}_1^d(w_2) \qquad (\mathbf{B}.31)$$
$$\left.\frac{\partial\tilde{Z}^{(b)}}{\partial t}\right|_\rho = 2v_d(N-1)[U'']^2\rho k^{d-6}Q_{3,0}^{d,0}(w_1)+4v_d(N-1)U''Z'\rho k^{d-4}Q_{3,0}^{d,1}(w_1)$$
$$+2v_d(N-1)[Z']^2\rho k^{d-2}Q_{3,0}^{d,2}(w_1)+2v_d(3U''+2U'''\rho)^2\rho k^{d-6}\tilde{Q}_{3,0}^{d,0}(w_2)$$
$$+4v_d(Z'+Y+Y'\rho)(3U''+2U'''\rho)\rho k^{d-4}\tilde{Q}_{3,0}^{d,1}(w_2)+2v_d(Z'+Y+Y'\rho)^2\rho k^{d-2}\tilde{Q}_{3,0}^{d,2}(w_2)$$
$$+\frac{8v_d}{d}(N-1)Z'U''\rho k^{d-4}N_{3,0}^d(w_1)+\frac{8v_d}{d}(N-1)[Z']^2\rho k^{d-2}N_{3,0}^{d+2}(w_1)$$
$$+\frac{8v_d}{d}(Z'+Y+Y'\rho)(3U''+2U'''\rho)\rho k^{d-4}\tilde{N}_{3,0}^d(w_2)+\frac{8v_d}{d}(Z'+Y+Y'\rho)^2\rho k^{d-2}\tilde{N}_{3,0}^{d+2}(w_2)$$
$$-2v_d(N-1)YU''Z_k^{-2}\rho k^{d-4}L_2^d(w_1)-2v_d(N-1)\left(Z'Y+\frac{1}{d}[Z']^2\right)Z_k^{-2}\rho k^{d-2}L_2^{d+2}(w_1)$$

$$- 4v_d(Z' + Y + Y'\rho)(3U'' + 2U'''\rho)Z_k^{-2}\rho k^{d-4}\tilde{L}_2^d(w_2)$$
$$- 2\left(2 + \frac{1}{d}\right)v_d(Z' + Y + Y'\rho)^2 Z_k^{-2}\rho k^{d-2}\tilde{L}_2^{d+2}(w_2). \qquad (\mathbf{B}.32)$$

Here we use the definitions

$$\tilde{N}^d_{n_1,n_2}(w_1,w_2) = N^d_{n_1,n_2}(w_1,w_2;P\leftrightarrow\tilde{P}) = N^d_{n_2,n_1}(w_2,w_1) + Y\rho L^{d+2}_{n_2,n_1}(w_2,w_1) \qquad (\mathbf{B}.33)$$

and

$$\tilde{Q}^{d,\alpha}_{n_1,n_2}(w_1,w_2) = Q^{d,\alpha}_{n_1,n_2}(w_1,w_2;P\leftrightarrow\tilde{P}). \qquad (\mathbf{B}.34)$$

The integrals $\tilde{Q}^{d,\alpha}_{n_1,n_2}$ are related to $\tilde{M}^d_{n_1,n_2}$ by an equation analogous to eq. (**B**.17). The relation between $\tilde{M}^d_{n_1,n_2}$ and $M^d_{n_1,n_2}$ reads

$$\tilde{M}^d_{n_1,n_2}(w_1,w_2) = M^d_{n_1,n_2}(w_1,w_2;P\leftrightarrow\tilde{P}) =$$
$$M^d_{n_2,n_1}(w_2,w_1) + 2Y\rho N^d_{n_2,n_1}(w_2,w_1) + Y^2\rho^2 L^{d+2}_{n_2,n_1}(w_2,w_1). \qquad (\mathbf{B}.35)$$

We also observe a general relation for the integrals $N^d_{n_1,n_2}(w_1,w_2)$

$$(n_1-1)N^d_{n_1,n_2}(w_1,w_2) = \frac{d}{2}L^d_{n_1-1,n_2}(w_1,w_2) - n_2\tilde{N}^d_{n_2+1,n_1-1}(w_2,w_1) =$$
$$\frac{d}{2}L^d_{n_1-1,n_2}(w_1,w_2) - n_2 Y\rho L^{d+2}_{n_1-1,n_2+1}(w_1,w_2) - n_2 N^d_{n_1-1,n_2+1}(w_1,w_2). \qquad (\mathbf{B}.36)$$

From this, or directly from partial integration of eq. (**B**.14), follow the identities

$$N^d_{3,0}(w) = \frac{d}{4}L^d_{2,0}(w)$$
$$\tilde{N}^d_{3,0}(w) = \frac{d}{4}L^d_{0,2}(w)$$
$$N^d_{0,3}(w) = \frac{d}{4}L^d_{0,2}(w) - Y\rho L^{d+2}_{0,3}(w)$$
$$\tilde{N}^d_{0,3}(w) = \frac{d}{4}L^d_{2,0}(w) + Y\rho L^{d+2}_{3,0}(w). \qquad (\mathbf{B}.37)$$

Using the various relations among the integrals we can express $\partial\tilde{Z}/\partial t$ in terms of $M^d_{4,0}$, $\tilde{M}^d_{4,0}$, L^d_n and $\tilde{L}^d_n$ only. We conclude that the evolution equation for $\tilde{Z}$ $(=\tilde{Z}_k(\rho))$ depends on U', U'', U''', Z, Z', Z'', Y, Y' and Y'' in our approximation. There is also an η dependence resulting from the t-derivative acting on Z_k in $R_k(q)$.

The anomalous dimension for the radial mode is defined in the spontaneously broken regime by

$$\tilde{\eta} = -\tilde{Z}_k^{-1}\left(\left.\frac{\partial\tilde{Z}^{(a)}}{\partial t}\right|_{\rho_0} + \left.\frac{\partial\tilde{Z}^{(b)}}{\partial t}\right|_{\rho_0}\right) - \tilde{Z}_k^{-1}\tilde{Z}_k'\bar{\delta}. \qquad (\mathbf{B}.38)$$

We find

$$\begin{aligned}\tilde{\eta} = &- v_d Z_k^{-1}\tilde{Z}_k^{-1}k^{d-2}\Big\{(N-1)(Z_k'+Y_k'\rho_0)L_1^d(0)+(Z_k'+2Z_k''\rho_0+Y_k+5Y_k'\rho_0+2Y_k''\rho_0^2)\tilde{L}_1^d(2\bar{\lambda}\rho_0)\Big\}\\
&-\frac{4v_d}{d}\tilde{Z}_k^{-1}\rho_0 k^{d-6}\Big\{(N-1)\bar{\lambda}^2M_{4,0}^d(0)+2(N-1)Z_k'\bar{\lambda}k^2M_{4,0}^{d+2}(0)+(N-1)[Z_k']^2k^4M_{4,0}^{d+4}(0)\\
&+(3\bar{\lambda}+2U_3\rho_0)^2\tilde{M}_{4,0}^d(2\bar{\lambda}\rho_0)+2(Z_k'+Y_k+Y_k'\rho_0)(3\bar{\lambda}+2U_3\rho_0)k^2\tilde{M}_{4,0}^{d+2}(2\bar{\lambda}\rho_0)\\
&+(Z_k'+Y_k+Y_k'\rho_0)^2k^4\tilde{M}_{4,0}^{d+4}(2\bar{\lambda}\rho_0)\Big\}\\
&+2v_dZ_k^{-2}\tilde{Z}_k^{-1}\rho_0k^{d-4}\Big\{(N-1)Y_k\bar{\lambda}L_2^d(0)+(N-1)\left(Z_k'Y_k+\frac{1}{d}[Z_k']^2\right)k^2L_2^{d+2}(0)\\
&+2(Z_k'+Y_k+Y_k'\rho_0)(3\bar{\lambda}+2U_3\rho_0)\tilde{L}_2^d(2\bar{\lambda}\rho_0)+\left(2+\frac{1}{d}\right)(Z_k'+Y_k+Y_k'\rho_0)^2k^2\tilde{L}_2^{d+2}(2\bar{\lambda}\rho_0)\Big\}\\
&-\tilde{Z}_k^{-1}(Z_k'+Y_k+Y_k'\rho_0)\bar{\delta}. \qquad (\mathbf{B}.39)\end{aligned}$$

Finally, we need the anomalous dimensions in the symmetric regime where $\rho_0(k)=0$. In this regime Z_k and $\tilde{Z}_k$ coincide. The contribution from $\partial Z^{(b)}/\partial t$ or $\partial\tilde{Z}^{(b)}/\partial t$ vanishes identically for $\rho_0=0$, since there is no cubic vertex. One finds from eq. (**B**.7) or eq. (**B**.31)

$$\eta=\tilde{\eta}=-\frac{d\ln Z_k(0)}{dt}=-\frac{d\ln\tilde{Z}_k(0)}{dt}=-v_dZ_k^{-2}(NZ_k'(0)+Y_k(0))k^{d-2}L_1^d(\bar{m}^2) \qquad (\mathbf{B}.40)$$

with $\bar{m}^2=U_k'(0)$.

Appendix C: The integral $\hat{M}_2^d$

In this appendix we disuss the integral $\hat{M}_2^d(\tilde{w})=\hat{M}_2^d(\tilde{w},0,1)$, where $\hat{M}_2^d(\tilde{w},\eta,\hat{z})$ is defined in eq. (**8**.10). Since $\hat{z}=1$, it can be written as

$$\hat{M}_2^d(\tilde{w})=\tilde{w}^2Z_k^2M_{2,2}^d(0,\tilde{w}), \qquad (\mathbf{C}.1)$$

where $M_{n_1,n_2}^d(w_1,w_2)$ is defined in eq. (**B**.16). With the above approximations (uniform wave function renormalization, $\eta=0$, $\hat{z}=1$) and the choice (**2**.2) - (**2**.3) for $R_k(q)$, this integral reads

$$\begin{aligned}M_{2,2}^d(0,\tilde{w})=&-2Z_k^{-2}\int_0^\infty dy\,y^{\frac{d}{2}-2}\frac{1+r+y\frac{\partial r}{\partial y}}{(1+r)^2\left[(1+r)y+\tilde{w}\right]^2}\\
&\left\{2y\frac{\partial r}{\partial y}+2\left(y\frac{\partial}{\partial y}\right)^2r-2y^2\left(1+r+y\frac{\partial r}{\partial y}\right)\frac{\partial r}{\partial y}\left[\frac{1}{(1+r)y}+\frac{1}{(1+r)y+\tilde{w}}\right]\right\},\end{aligned} \qquad (\mathbf{C}.2)$$

with

$$r(y) = \frac{\exp(-y)}{1-\exp(-y)}. \tag{C.3}$$

The function $M^d_{2,2}(0,\tilde{w})$ is monotonically decreasing for positive $\tilde{w}$. We define

$$M^d_{2,2}(0,0) = -2Z_k^{-2} m_4^d \tag{C.4}$$

and find

$$m_4^d = 2\Gamma\left(\frac{d}{2}-1\right)\left[1-2^{-\frac{d}{2}}\left(1+\frac{d}{2}\right)\right]. \tag{C.5}$$

We can define the function $t^d_{2,2}(\tilde{w})$ through

$$M^d_{2,2}(0,\tilde{w}) = -2Z_k^{-2} m_4^d t^d_{2,2}(\tilde{w}). \tag{C.6}$$

It is a decreasing function of $\tilde{w}$, and $1 \geq t^d_{2,2}(\tilde{w}) > 0$ for $0 \leq \tilde{w} < \infty$. The first term of the asymptotic expansion of $M^d_{2,2}(0,\tilde{w})$ for $\tilde{w} \to \infty$ reads

$$M^d_{2,2}(0,\tilde{w}) = 4Z_k^{-2}\Gamma\left(\frac{d}{2}\right)\left\{\left[1-d+\frac{d(d+2)}{8}\right]\zeta\left(\frac{d}{2}\right)+\frac{(2-d)d}{8}\zeta\left(\frac{d}{2}+1\right)\right\}\tilde{w}^{-2}... \tag{C.7}$$

For $d=2$ the leading term takes the particularly simple form

$$M^2_{2,2}(0,\tilde{w}) = -2Z_k^{-2}\tilde{w}^{-2}... \tag{C.8}$$

and

$$\lim_{\tilde{w}\to\infty} \hat{M}^2_2(\tilde{w}) = -2. \tag{C.9}$$

References

[1] C. Wetterich, Phys. Lett. B **301**, 90 (1993).

[2] L.P. Kadanoff, Physics **2**, 263 (1966).

[3] K.G. Wilson, Phys. Rev. B **4**, 3174;3184 (1971); K.G. Wilson and I.G. Kogut, Phys. Rep. **12**, 75 (1974); F.J. Wegner, in: Phase Transitions and Critical Phenomena, vol. 6, eds. C. Domb and M.S. Greene, Academic Press (1976).

[4] F.J. Dyson, Phys. Rev. **75**, 1736 (1949); J. Schwinger, Proc. Nat. Acad. Sc. **37**, 452;455 (1951).

[5] J. Polchinski, Nucl. Phys. B **231**, 269 (1984);

[6] U. Ellwanger, private communication; M. Bonini, M. D'Attanasio and G. Marchesini, Parma preprint UPRF-92-360.

[7] C. Wetterich, Nucl. Phys. B **352**, 529 (1991).

[8] C. Wetterich, Heidelberg preprint HD-THEP-92-64, in press Z. Phys. C.

[9] C. Wetterich, Heidelberg preprint HD-THEP-93-17.

[10] U. Ellwanger, Heidelberg preprint HD-THEP-92-33.

[11] C.G. Callan, Phys. Rev. D **2**, 1541 (1970); K. Symanzik, Commun. Math. Phys. **18**, 227 (1970).

[12] J. Zinn-Justin, Quantum field theory and critical phenomena, Oxford Science Publications (1989);

[13] G. Parisi, J. Stat. Phys. **23**, 49 (1980); Statistical field theory, Addison-Wesley (1988).

[14] C. Wetterich, Z. Phys. C **57**, 451 (1993).

[15] N. Tetradis and C. Wetterich, Nucl. Phys. B. **398**, 659 (1993).

[16] V. Jain, Nucl. Phys. B **394**, 707 (1993); H. Meyer-Ortmanns and A. Patkos, Phys. Lett. B **297**, 331 (1992).

[17] M. Reuter, N. Tetradis and C. Wetterich, preprint DESY-93-004, in press Nucl. Phys. B.

[18] A. Ringwald and C. Wetterich, Nucl. Phys. B **334**, 506 (1990); N. Tetradis and C. Wetterich, Nucl. Phys. B **383**, 197 (1992).

[19] N. Tetradis and C. Wetterich, in preparation.

[20] H.J. Schnitzer, Phys. Rev. D **10**, 1800 (1974); S. Coleman, R. Jackiw and H.D. Politzer, Phys. Rev. D **10**, 2491 (1974); L.F. Abbott, J.S. Kang and H.J. Schnitzer, Phys. Rev. D **13**, 2212 (1976); W.A. Bardeen and M. Moshe, Phys. Rev. D **28**, 1372 (1983).

[21] I. Kondor and T. Temesvari, J. Phys. Lett. (Paris) **39**, L99 (1978).

[22] S. Bornholdt, N. Tetradis and C. Wetterich, in preparation.

[23] C. Wetterich, Z. Phys. C **48**, 693 (1990); M. Reuter and C. Wetterich, Nucl. Phys. B **391**, 147 (1993); Heidelberg preprint HD-THEP-92-62; S. Bornholdt and C. Wetterich, Phys. Lett. B **282**, 399 (1992).

[24] M. Reuter and C. Wetterich, in preparation.

Tables

κ_*	λ_*	u_{3*}	η_*	u_{4*}	u_{5*}	u_{6*}
6.57×10^{-2}	11.5					
8.01×10^{-2}	7.27	52.8				
7.86×10^{-2}	6.64	42.0	3.58×10^{-2}			
7.75×10^{-2}	6.94	43.5	3.77×10^{-2}	95.7		
7.71×10^{-2}	7.03	43.4	3.83×10^{-2}	111	-1.43×10^{3}	3.72×10^{4}

Table 1: Fixed point values for the minimum of u, its derivatives and the anomalous dimension, for various truncations of the infinite system of evolution equations. $N = 3$.

N	κ_*	λ_*	u_{3*}	u_{4*}	η_*	$2\kappa_*\lambda_*$
1	4.05×10^{-2}	9.25	87.4	598	4.49×10^{-2}	0.749
2	5.81×10^{-2}	8.08	61.5	249	4.23×10^{-2}	0.939
3	7.71×10^{-2}	7.03	43.4	111	3.83×10^{-2}	1.08
4	9.71×10^{-2}	6.15	31.3	56.6	3.42×10^{-2}	1.20
10	2.26×10^{-1}	3.26	7.60	6.01	1.87×10^{-2}	1.48
20	4.49×10^{-1}	1.77	2.14	0.944	1.02×10^{-2}	1.59
100	2.24	0.375	9.32×10^{-2}	9.05×10^{-3}	2.17×10^{-3}	1.68

Table 2: Fixed point values for the minimum of u, its derivatives and the anomalous dimension, for various values of N, with the truncation II (eq. (**7.**11)).

N	β		ν		γ		η_*	
1	0.333	$0.3250(15)^a$ $0.3270(15)^b$ $0.312(5)^c$	0.638	$0.6300(15)^a$ $0.6310(15)^b$ $0.6305(15)^c$	1.247	$1.2405(15)^a$ $1.2390(25)^b$ $1.2385(25)^c$	0.045	$0.032(3)^a$ $0.0375(25)^b$
2	0.365	$0.3455(20)^a$ $0.3485(35)^b$	0.700	$0.6695(20)^a$ $0.671(5)^b$ $0.672(7)^c$	1.371	$1.316(25)^a$ $1.315(7)^b$ $1.33(2)^c$	0.042	$0.033(4)^a$ $0.040(3)^b$
3	0.390	$0.3645(25)^a$ $0.368(4)^b$ $0.38(3)^c$	0.752	$0.705(3)^a$ $0.710(7)^b$ $0.715(20)^c$	1.474	$1.386(4)^a$ $1.39(1)^b$ $1.38(2)^c$	0.038	$0.033(4)^a$ $0.040(3)^b$
4	0.409		0.791		1.556		0.034	
10	0.461	0.449^d	0.906	0.877^d	1.795	1.732^d	0.019	0.025^d
20	0.481	0.477^d	0.952	0.942^d	1.895	1.872^d	0.010	0.013^d
100	0.497	0.496^d	0.992	0.989^d	1.981	1.975^d	0.002	0.003^d

Table 3: Critical exponents of the three-dimensional theory for various values of N. For comparison we list results obtained with other methods as summarized in [12] and [21]:
a) From summed perturbation series in fixed dimension 3 at six-loop order.
b) From the ϵ-expansion at order ϵ^5.
c) From lattice calculations.
d) From the $1/N$-expansion at order $1/N^2$.

N	1		2		3		4		10		20		100	
$\frac{\lambda_R}{m_R}$	9.6	7.9^a	8.4	7.1^a	7.4	6.4^a	6.6		3.8	5.0^b	2.1	2.5^b	0.49	0.50^b

Table 4: The ratio λ_R/m_R for the three-dimensional theory in the critical region, for various values of N. For comparison we list results obtained with other methods:
a) From summed perturbation series in fixed dimension 3 [13, 12].
b) From large N calculations [17, 12].

Figures

Fig. 1 Graphical representation of the exact evolution equation for the effective average action Γ_k.

Fig. 2 Solution of the evolution equations for $\kappa(k)$, $\lambda(k)$, $u_3(k)$ and $\eta(k)$ in the critical region, for initial values slightly above and below the critical line. The fixed point for the second order phase transition is displayed, as well as the final running towards the phase with spontaneous symmetry breaking or the symmetric one. $N = 3$.

Fig. 3 The critical exponents as the phase transition is appoached. $N = 4$.

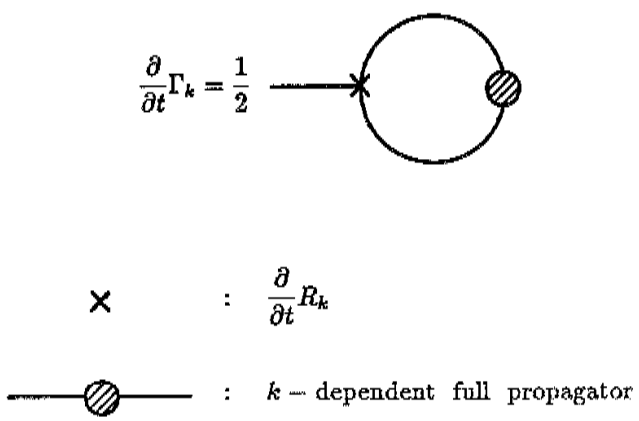
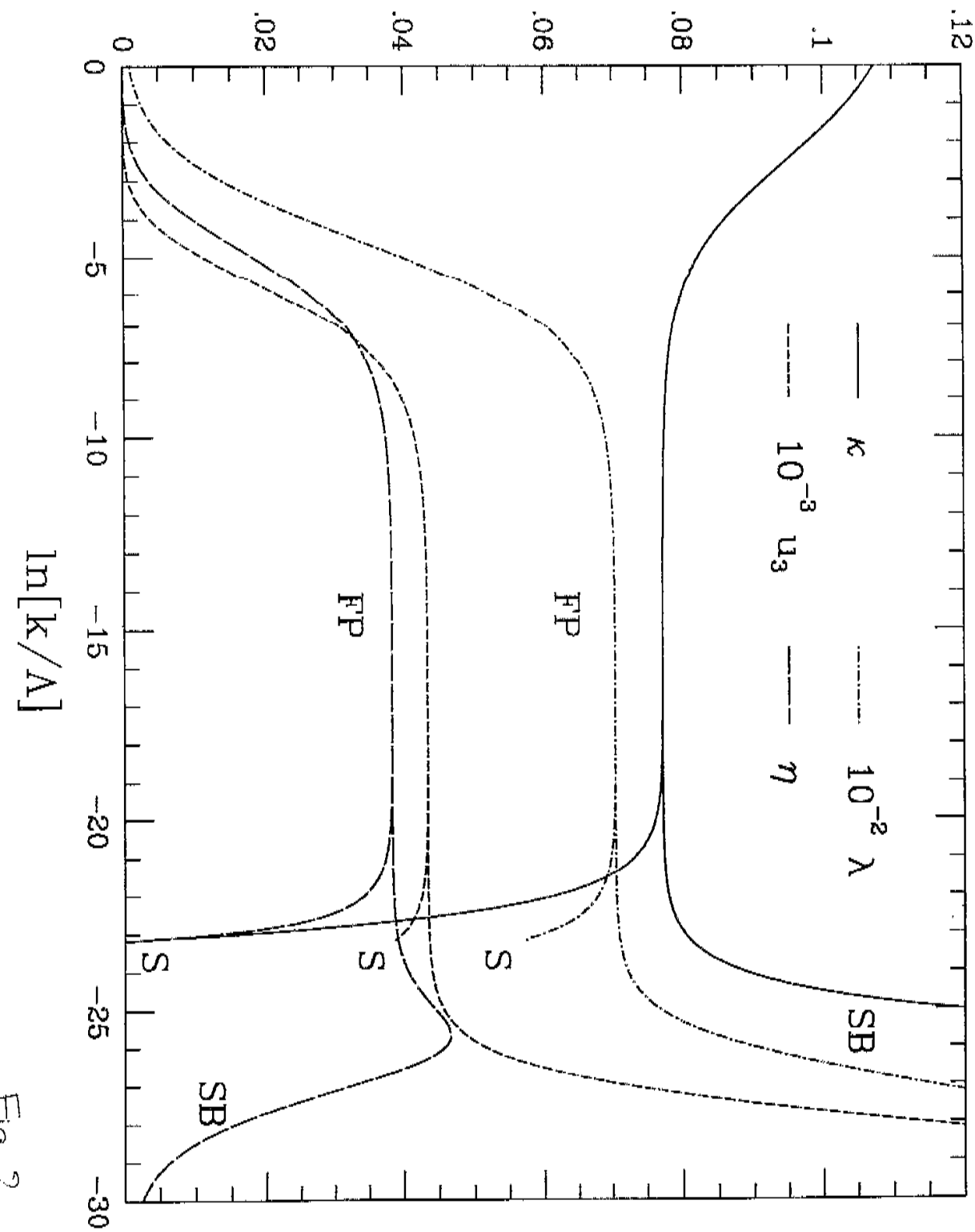

$\frac{\partial}{\partial t}\Gamma_k = \frac{1}{2}$
$\times$: $\frac{\partial}{\partial t}R_k$
: $k-$ dependent full propagator

Fig. 1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.1
.08
.06
.04
.02
0
0
-5
-10
-15
-20
-25
-30
ln[k/Λ]
κ
10^-3 u3
10^-2 λ
η
FP
FP
S
S
S
SB
SB

Fig 2

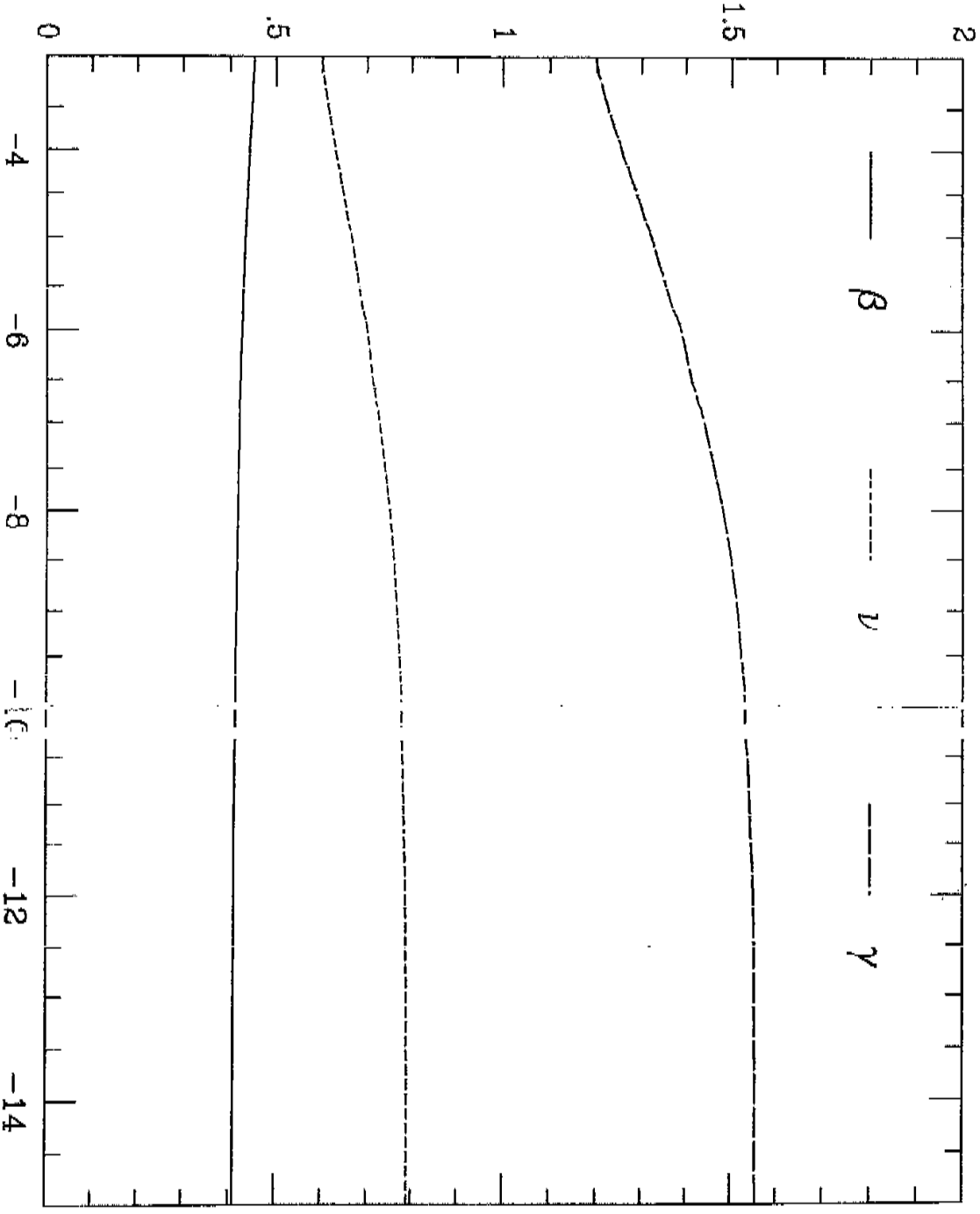

Fig. 2