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Subfactors and Coset Models*

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Abstract: Some facts about von Neumann algebras and finite index inclusions of factors are viewed in the context of local quantum field theory. The possibility of local fields intertwining superselection sectors with braid group statistics is explored. Conformal embeddings and coset models serve as examples. The associated symmetry concept is pointed out.

1 Introduction

The present article is a pedestrian's view of some basic mathematical facts about von Neumann algebras and the structure theory of subfactors relevant for the physical problem of implementability of superselection sectors by charged fields. Its intention is to facilitate the access to abstract results by formulating them in terms of algebraic relations of rather direct physical significance, and to illustrate them by a large class of models (coset models of conformal quantum field theory) well known to a broad audience. The present work is a precursory study related to a joint program in progress with K. Fredenhagen, R. Longo, and J.E. Roberts about local extensions of local quantum field theories.

The general analysis we shall present below was incited by a recent result of a model study of chiral current algebras [1], which at first sight quite puzzled the author. The authors of [1] were searching for *local* extensions of chiral $SU(2)$ current algebras which are identified by their local conformal block functions satisfying the $SU(2)$ Ward identities. They found a solution at level 10 for the isospin 3 sector, clearly different from the standard non-local vertex operator (Coulomb gas) solution.

The "fusion rules" which are read off the local solution are

$$[3][3] = [0] + [3] \tag{1.1}$$

in contrast to the standard fusion rules

$$[3][3] = [0] + [1] + [2] + [3] + [4]. \tag{1.2}$$

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This seems to contradict the message from the general theory of superselection sectors [2] that the fusion rules are *intrinsic* to a given local quantum field theory. Moreover, the isospin 3 sector is known to have non-trivial braid group statistics, so there is the surprising fact that one can associate it with local correlation functions. In fact, the local solution found in [1] has been identified as the conformal embedding of the $SU(2)$ current algebra at level 10 into the $SO(5)$ (or $Sp(4)$) current algebra at level 1. It is well known that the vacuum representation of the latter splits into the representations [0] and [3] of the former, where the primary field for the representation [3] is the isospin 3 multiplet of $SO(5)$ currents orthogonal to the embedded $SU(2)$ currents. The conformal blocks and the fusion rules (1.1) are just the correlation functions and the operator product expansions of these $SO(5)$ currents.

The second solution found in [1] is much less surprising. It is the conformal embedding [3] of $SU(2)$ at level 4 into $SU(3)$ at level 1, generating the isospin 2 sector. This sector has permutation group statistics, and the fusion rule $[2][2] = [0]$ coincides with the standard one. In fact, the detailed analysis shows that the $SU(2)_4$ theory is exactly the subtheory of $\mathbb{Z}_2$-invariants of the $SU(3)_1$ theory, where the global symmetry group $\mathbb{Z}_2$ is the involutive Lie group automorphism of $SU(3)$ associated with the symmetric space $SU(3)/SU(2)$, lifted to the current algebra.

The previous solution is an example of a model, where a sector of a local quantum field theory, which has genuine braid group statistics, is generated by local fields of some larger local quantum field theory. In fact, conformal coset models generically exhibit this feature, and the $SU(2)_4 \subset SU(3)_1$ case is rather the exception. The aim of the present article is a general analysis of this scheme, as an alternative to the widespread conviction that charged fields generating sectors with braid group statistics should exhibit braid-like commutation relations at space-like distance.

The apparent conflict between locality and braid group statistics dissolves due to some intrinsic "defect". As we shall see, one can only find *intertwining* fields

$$\psi a = \varrho(a)\psi \tag{1.3}$$

for the endomorphism corresponding to the sector in question [2], but not *implement* the endomorphism itself:

$$\varrho(a) \stackrel{!}{=} \sum_i \psi^i \, a \, \psi^{i*} \tag{1.4}$$

as it is the case with permutation group statistics [4]. The defect is responsible for a mismatch between the abstract fusion rules of the theory of superselection sectors [2] and the multiplicative structure (operator product expansions) of the generating fields. This then solves the previously mentioned puzzle. Namely, the fusion rules (1.1) are not the intrinsic ones but refer to the embedding into a larger theory.

In order to place the above example in a general context, and to elucidate the point where the standard reasoning fails, we resort to the theory of von Neumann algebras and subfactors. By the comparison of the general case of a subfactor $A \subset B$ with the

case when the subfactor is given by the invariants under the action of a symmetry group, we shall very explicitly see the role of the gauge group (of the first kind) to exclude the occurrence of the defect, thus leading to the standard duality theory [4].

The general case is described by a faithful normal conditional expectation

$$\mu : B \to A \tag{1.5}$$

i.e. a positive unit preserving map of B onto A satisfying

$$\mu(abc) = a\mu(b)c \qquad (a, c \in A,\ b \in B). \tag{1.6}$$

A conditional expectation generalizes the abstract properties of a group average

$$\mu(b) := |G|^{-1} \sum_{g \in G} \alpha_g(b), \tag{1.7}$$

which special case we shall always compare to the general situation.

Associated with the conditional expectation is a "canonical" endomorphism $\gamma : B \to A$, the structure theory of which is treated in Sect. 2. It is of direct physical significance: the sector decomposition of γ as an endomorphism of B allows to recover the symmetry group in the case (1.7), and naturally leads to Ocneanu's generalized symmetry concept ("paragroups" [5]) in the general case, see Sect. 4. The sector decomposition of γ as an endomorphism of A corresponds to the decomposition of a representation of B upon restriction to A, provided the former is obtained from an invariant state $\omega_0 = \omega_0 \circ \mu$. In the quantum field theoretical context (i.e. A and B are a pair of local von Neumann algebras from local quantum field theories $\mathcal{A}$ and $\mathcal{B}$ and ω_0 is the vacuum state in which the symmetry is unbroken), this is the "branching rule" for the vacuum sector of the larger theory. The sectors of the subtheory thus obtained are interpolated by fields from the larger theory.

The canonical endomorphism uniquely characterizes the underlying inclusion $A \subset B$. We show how to recover detailed structure information about the latter in terms of intertwining operators and their algebraic properties. In Haag's spirit that "fields are just coordinates on local algebras", we are putting coordinates on an inclusion.

Our conclusion in the quantum field theoretical context is that indeed some sectors with braid group statistics can be generated by a local field algebra. The exact relation to the statistics is discussed in Sect. 3. In fact, this possibility is realized in (almost) all coset models of conformal quantum field theory.

The models we have in mind include the previously mentioned conformal embedding $SU(2)_{10} \subset SO(5)_1$, but also coset models [6] of the form

$$Diag(G \times G)_{k+l} \otimes W \subset G_k \otimes G_l \tag{1.8}$$

where G_k stands for the level k chiral current algebra of the compact simple Lie group G, or still more generally

$$H_l \otimes W \subset G_k \tag{1.9}$$

where $H \subset G$ is a pair of compact semi-simple Lie groups (not necessarily simple such that the levels may have several components for the simple components of H and G). In all cases, W stands for a local quantum field theory decoupled from the subgroup currents. It is trivial for conformal embeddings and otherwise contains the coset energy-momentum tensor given by

$$T^{(G)} = T^{(H)} + T^{\text{coset}} \tag{1.10}$$

in terms of the Sugawara expressions for $T^{(G)}$, $T^{(H)}$. If the central charge of T^{coset} is $c > 1$, then W may contain further primary fields of higher scaling dimensions.

We consider a coset model as an extension of the local theory of observables $\mathcal{A}$ into a local theory of "charged" fields $\mathcal{B}$. The detailed knowledge of these models precisely confirms and exemplifies our general findings. In Sect. 4, we briefly sketch the generalized symmetry associated with such an extension.

According to Wassermann [7], the local von Neumann algebras of current algebra models are defined – in purely group theoretical terms – as $\pi_0(L_I G)''$ where LG is the group of smooth maps (loops) : $S^1 \to G$, the local subgroup $L_I G$ consists of those maps with support in the interval I of the circle, and π_0 is the projective vacuum representation of level k. The term "vacuum" refers to the implementation of the group of diffeomorphisms of the circle which act as automorphisms of the loop group, and in particular to the spectrum of the "rigid" rotations, while the level refers to the cocycle associated with a projective representation. Similar definitions can be given for the algebra of the (coset) energy-momentum tensor in terms of the diffeomorphism group of S^1 and its local subgroups.

Following [7], we shall argue in Sect. 5 that the index of the inclusion of the corresponding local von Neumann algebras can be finite only when the vacuum representation of $\mathcal{B}$ decomposes finitely into representations of $\mathcal{A}$, and give a formula to compute the index. The branching rules for coset models being known, this formula is very explicit.

The contents of this article are not really new, although we sometimes adopt an unconventional point of view. By viewing a class of physical models in the light of the abstract structure theory for the canonical endomorphism, some model findings are systematically related to the general algebraic structures which underly local extensions of local quantum field theories. Moreover, the latter are formulated in physically evident terms. Some novel insight concerns the properties of intertwining fields and the relation between their local commutativity and braid group statistics, the precise identification of the points where more general subtheories depart from the rather special gauge symmetry inclusions, as well as some algebraic control over the defect going along with non-integer statistical dimensions.

2 Inclusions of type III factors

We collect some well known results about finite index inclusions of type *III* factors [8, 9, 10, 11, 12], which we shall need for the subsequent discussion, and qualify the abstract statements in the special case when the inclusion is given by the action of a symmetry group. The generalization to Hopf algebra actions is also known [13].

We consider an irreducible inclusion $A \subset B$ of type *III* factors. With this pair, we have in mind a local von Neumann algebra $\mathcal{B}(\mathcal{O})$ from some quantum field theory and a local von Neumann algebra $\mathcal{A}(\mathcal{O})$ from some subtheory. However, the purely mathematical statements in this section do not refer to the physical context. The local aspects will be treated in Sect. 3.

Proposition 1: Let $A \subset B$ be an irreducible inclusion of type *III* factors with finite index λ. There are a unique normal conditional expectation $\mu : B \to A$ which satisfies the operator estimate

$$\mu(bb^*) \geq \lambda^{-1} \cdot bb^* \qquad (b \in B), \tag{2.1}$$

and an isometry $V \in B$ such that

$$\begin{array}{ll} (i) & \mu(bV^*)V = V^*\mu(Vb) = \lambda^{-1}b \qquad (b \in B) \\ (ii) & \mu(VV^*) = \lambda^{-1}. \end{array} \tag{2.2}$$

The map $\gamma : B \to A \subset B$

$$\gamma(b) := \lambda \cdot \mu(VbV^*) \tag{2.3}$$

is an endomorphism, called the canonical endomorphism, and V is an intertwiner : $id \to \gamma$, i.e.

$$V\, b = \gamma(b)\, V. \tag{2.4}$$

The canonical endomorphism was originally introduced in [14] in terms of modular conjugations. It is very useful in representation theory, since it allows the restriction of an endomorphism σ of B to an endomorphism $\gamma \circ \sigma|_A$ of A, and the induction of an endomorphism σ of A to an endomorphism $\sigma \circ \gamma$ of B. Considering endomorphisms as representations of spatial von Neumann algebras, these prescriptions correspond to the restriction and (Mackey) induction of representations.

We specify the statement of Prop. 1 in the case when A are the "gauge invariants" of B with respect to some symmetry group.

Proposition 1′: Let in Prop. 1 $A = B^\alpha$ be the fixpoint subalgebra under an outer action $\alpha : G \to Aut(B)$ of a symmetry group G. Then $\lambda = |G|$ (in particular, G is finite), and μ is the group average:

$$\mu = |G|^{-1} \sum_{g \in G} \alpha_g \,. \tag{2.5}$$

The operators

$$V_g := \alpha_g(V) \tag{2.6}$$

form a complete system of orthonormal isometries:

$$V_g^* V_h = \delta_{gh} \qquad \text{and} \qquad \sum_{g\in G} V_g V_g^* = 1, \tag{2.7}$$

carrying the left regular representation of G:

$$\alpha_g(V_h) = V_{gh}. \tag{2.8}$$

The canonical endomorphism is

$$\gamma(b) = \sum_{g\in G} V_g \alpha_g(b) V_g^*. \tag{2.9}$$

The completeness of V_g ist just $(2.2(ii))$: $|G|\cdot\mu(VV^*) = 1$. The orthonormality is verified by summing $V^*\alpha_g(VV^*)V$ over the group, which implies $\sum_{g\neq e}(V^*V_g)(V^*V_g)^* = 0$ and therefore $V^*V_g = \delta_{ge}$. (2.9) follows from the definitions.

Next, we discuss the reducibility of endomorphisms. Let $\sigma : M \to M$ be an endomorphism of a type *III* factor M. It is reducible iff the relative commutant $\sigma(M)' \cap M$ is non-trivial. Every projection $e \in \sigma(M)' \cap M$ corresponds to a sub-endomorphism σ_e defined as follows. Pick an isometry $w \in M$ such that $ww^* = e$. Then

$$\sigma_e(\cdot) := w^*\sigma(\cdot)w.$$

By construction, w is an intertwiner : $\sigma_e \to \sigma$. If

$$1 = \sum_s e_s \qquad \text{and} \qquad e_s = \sum_i e_s^i$$

are the central partition of unity in $\sigma(M)' \cap M$, and partitions of e_s into minimal projections in $\sigma(M)' \cap M$, respectively, then the corresponding sub-endomorphisms $\sigma_s^i = w_s^{i*}\sigma(\cdot)w_s^i$ fall into inner equivalence classes (sectors) labelled by s, while $i = 1, \ldots n_s$ counts the multiplicity of the sector $[s]$ within σ. It is possible and convenient to choose $\sigma_s^i = \sigma_s$ independent of i, and to write

$$\sigma \simeq \bigoplus_s n_s\, \sigma_s.$$

For reducible inclusions one defines $Index(N \subset M)$ to be the *minimal* index [10], and calls $d(\sigma) := \sqrt{Index[\sigma(M) \subset M]}$ the *dimension* of an endomorphism. The dimension is additive and multiplicative:

$$d(\sigma) = \sum_s n_s d(\sigma_s) \qquad \text{and} \qquad d(\sigma\varrho) = d(\sigma)d(\varrho).$$

Proposition 2: Let $A \subset B$ be as in Prop. 1. Denote by $\varrho : A \to A$ the restriction of γ to A. Then

$$\varrho(A) \subset \gamma(B) \subset A \subset B \tag{2.10}$$

is (the beginning of) a Jones tower. In particular, all inclusions are irreducible with index λ and $d(\gamma) = d(\varrho) = \lambda$. Both $\gamma(B) \subset B$ and $\varrho(A) \subset A$ are reducible (unless $\lambda = 1$):

$$(a) \quad \varrho \simeq \bigoplus_s N_s \; \varrho_s \qquad \text{and} \qquad (b) \quad \gamma \simeq \bigoplus_t M_t \; \gamma_t \,, \tag{2.11}$$

either decomposition containing the identity as a sub-endomorphism with multiplicity 1.

Let $W \in A$ be the intertwiners : $\varrho_s \to \varrho$ according to the above decomposition theory for ϱ, and $V \in B$ as in Prop. 1. The map $W \mapsto \psi := W^*V$ is bijective onto the linear space $\mathcal{H}_s \subset B$ of operators satisfying

$$\psi a = \varrho_s(a)\psi \qquad (a \in A). \tag{2.12}$$

$\mathcal{H}_s$ is an N_s-dimensional Hilbert space of isometries with scalar product $\psi^*\psi' \in A' \cap B = \mathbb{C}$.

In the quantum field theoretical context, A are local observables $\mathcal{A}(\mathcal{O})$, B are local fields $\mathcal{B}(\mathcal{O})$, and ϱ_s are (localized) endomorphisms of $\mathcal{A}$ representing the superselection sectors [2], see Sect. 3. We have thus established the existence of intertwining fields for all sectors contained in ϱ, and conversely all sectors which have intertwining fields in $\mathcal{B}$ are contained in ϱ.

Proof of the last statements of Prop. 2: The map $W \mapsto \psi$ is injective since $\mu(\psi\psi^*) = \lambda^{-1}W^*W > 0$. It is inverted by $\psi \mapsto W := \lambda\mu(V\psi^*) \in A$. The stated intertwining properties of ψ resp. W are readily checked.

The following specification of Prop. 2 in the group symmetric case is easily obtained with group theoretical arguments.

Proposition 2′: Let $A = B^\alpha$ be as in Prop. 1′. Then

$$(a) \quad \varrho \simeq \bigoplus_r dim(r)\varrho_r \qquad \text{and} \qquad (b) \quad \gamma \simeq \bigoplus_{g \in G} \alpha_g \tag{2.13}$$

where in (a) the sum extends over the unitary irreducible representations τ^r (of dimension $dim(r)$) of the symmetry group G. For every sector $[r]$ and an orthonormal basis of intertwiners $W_r^i : \varrho_r \to \varrho$ in A, the corresponding charged fields $\psi^i = \sqrt{|G|/dim(r)} \cdot (W_r^i)^* V \in \mathcal{H}_r \subset B$ are a complete set of orthonormal isometries:

$$(i) \quad \psi^{i*}\psi^j = \delta_{ij} \qquad (ii) \quad \sum_{i=1}^{dim(r)} \psi^i \psi^{i*} = 1 \tag{2.14}$$

which transform linearly in the representation τ^r:

$$\alpha_g(\psi^i) = \sum_{j=1}^{dim(r)} \psi^j \cdot \tau^r_{ji}(g) \tag{2.15}$$

and implement the sectors:

$$\varrho_r(a) = \sum_{i=1}^{dim(r)} \psi^i \, a \, \psi^{i*}. \tag{2.16}$$

The reader will find the sketch of a constructive proof in the appendix.

With Prop. 1′ we have recovered the Cuntz algebra (= C^* algebra generated by a complete set of orthonormal isometries) of intertwining fields which implement the sectors. In the context of superselection sectors in local quantum field theory, it was first derived in [15], and is the basic tool for the reconstruction [4] of the gauge symmetric field algebra from the sector structure of the observables.

If one looks through the argument for the completeness relation (2.14(ii)) (see the appendix) one finds that it owes its specific form to the linear transformation law (2.15). Namely, by our general construction, for every $\psi \in \mathcal{H}_s$ the "average" of $\psi\psi^*$ is a multiple of unity:

$$\mu(\psi\psi^*) = \lambda^{-1} \cdot W^*W \propto 1. \tag{2.17}$$

On the other hand, introduce the joint range projection of the Hilbert space of isometries $\mathcal{H}_s$ in terms of any orthonormal basis $\psi^i \in \mathcal{H}_s$

$$E_s := \sum_{i=1}^{N_s} \psi^i \psi^{i*}. \tag{2.18}$$

Now, under the *linear* group action (2.15), the average $\mu(\psi\psi^*)$ is a multiple of $\sum_i \psi^i\psi^{i*}$, hence $E_s = 1$. In the general case, however, (2.17) is just an abstract relation in B rather than an algebraic relation among the operators $\psi \in \mathcal{H}_s$, while E_s will be a non-trivial projection in $\varrho_s(A)' \cap B$. This "defect" $E_s < 1$ is related to the mismatch between the multiplicity N_s and the dimension $d(\varrho_s)$ in (2.11(a)):

Proposition 3: If $A \subset B$ has finite index, then the multiplicities N_s in (2.11(a)) are not larger than the dimensions $d(\varrho_s)$:

$$N_s \leq d(\varrho_s). \tag{2.19}$$

Equality holds iff the defect projection E_s is unity.

Proof: For $\psi^i \in \mathcal{H}_s$ as above, the operators $X_i := \gamma(\psi^i)$ are a system of orthogonal intertwiners in A

$$X_i \varrho(a) = \varrho\varrho_s(a) X_i \, .$$

Hence the N_s projections $X_i X_i^* \in \varrho\varrho_s(A)' \cap A$ correspond to N_s disjoint sub-endomorphisms equivalent to ϱ contained in $\varrho\varrho_s$. The additivity and multiplicativity of dimensions implies

$$N_s \cdot d(\varrho) \leq d(\varrho\varrho_s) = d(\varrho)d(\varrho_s)$$

where equality holds iff $\sum_i X_i X_i^*$ is a partition of unity. Since $d(\varrho) = \lambda$ is finite, the inequality (2.19) follows, and since $\sum_i X_i X_i^* = \gamma(E_s)$, equality holds iff $E_s = 1$.

In the group symmetry case, we conclude $d(\varrho_r) = N_r = dim(r)$ and $\varrho\varrho_s \simeq d(\varrho_s) \cdot \varrho$. The dimension formula for finite groups, $|G| = \sum_r dim(r)^2$, is recovered from (2.13). It becomes also clear from Prop. 3 that sectors with non-integer dimension $d(\varrho_s)$ must suffer the defect. Finally, the defect is related to the depth [5] of the inclusion $A \subset B$: suppose the defect to be absent in all sectors ϱ_s. Then together with ϱ_s all sub-endomorphisms of $\varrho_s\varrho_r$ are implemented, and by Prop. 2 are already contained in ϱ. Then $\gamma(B) \subset A$ and $A \subset B$ have depth 2 and $A \subset B$ is given by the action of a Hopf algebra [13].

The failure of Prop. 3 at infinite index will be exemplified in Sect. 6.

3 The relation to statistics

We have so far considered a single pair of von Neumann algebras $A \subset B$. A local quantum field theory is given by a local net, i.e. an assignment of an algebra of local observables $\mathcal{A}(\mathcal{O})$ to every space-time region $\mathcal{O}$ such that local algebras at space-like distance commute. These algebras are type *III* factors for typical bounded ("double cone") regions [16]. In the case of chiral current algebras, the local algebras $\mathcal{A}_I$ are assigned to the intervals I of the circle. $\mathcal{A}_I$ are also type *III* factors [7, 17]. The total algebra $\mathcal{A}$ of observables is the C^* algebra generated by all its local subalgebras.

A theory is covariant with respect to the space-time symmetry group (the Poincaré group or, on the circle, the Möbius group) if the latter acts by automorphisms on the local net, i.e.

$$\alpha_x(\mathcal{A}(\mathcal{O})) = \mathcal{A}(x\mathcal{O}). \tag{3.1}$$

A representation is covariant if the automorphisms α_x are implemented by unitary operators in the Hilbert space. The spectrum condition refers to the positivity of the generators of the translations in a given representation.

Let us now turn to a subtheory. By this we mean a pair of local nets $\mathcal{A}$ and $\mathcal{B}$ such that for every region

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{O}), \tag{3.2}$$

and the covariance automorphisms $\alpha_x^{(\mathcal{B})}$ of $\mathcal{B}$ coincide on $\mathcal{A}$ with $\alpha_x^{(\mathcal{A})}$ (we shall thus drop the distinction).

The analysis of Sect. 2 then applies to every single inclusion (3.2). It is important to note that the canonical endomorphism γ certainly, and the conditional expectation μ possibly depends on $\mathcal{O}$. There arise thus consistency problems when extending the maps μ, γ, ϱ globally. These problems are not subject of the present article. Let us therefore – rather than deriving it from some more general principles – make the certainly very reasonable assumption that μ is consistently defined for all local algebras (i.e. $\mu^{(\mathcal{O}_1)}|_{\mathcal{B}(\mathcal{O}_2)} = \mu^{(\mathcal{O}_2)}$ if $\mathcal{O}_2 \subset \mathcal{O}_1$) and commutes with α_x. Furthermore, we shall assume that the vacuum state ω_0 of $\mathcal{B}$ is the unique Poincaré invariant state. Then ω_0 is also invariant under the conditional expectation:

$$\omega_0 \circ \alpha_x = \omega_0 = \omega_0 \circ \mu \ . \tag{3.3}$$

We fix $\mathcal{O}_0$ and apply Prop. 1 and 2 to the inclusion $\mathcal{A}(\mathcal{O}_0) \subset \mathcal{B}(\mathcal{O}_0)$. In particular, V is an isometry in $\mathcal{B}(\mathcal{O}_0)$. Then we extend γ globally by (2.3):

$$\gamma(b) := \lambda \cdot \mu(VbV^*) \qquad (b \in \mathcal{B}). \tag{3.4}$$

By locality, (1.6), and (2.2), $\varrho := \gamma|_{\mathcal{A}}$ is an endomorphism of $\mathcal{A}$ localized in $\mathcal{O}_0$, namely it acts trivially on $a \in \mathcal{A}(\mathcal{O}')$ if $\mathcal{O}'$ is space-like separated from $\mathcal{O}_0$. The same is not true for $\gamma \in End(\mathcal{B})$, nor does γ map $\mathcal{B}(\mathcal{O})$ into $\mathcal{A}(\mathcal{O})$ unless $\mathcal{O}$ contains $\mathcal{O}_0$.

The operators $\psi \in \mathcal{H}_s \subset \mathcal{B}(\mathcal{O}_0)$ are local intertwiners for the sub-endomorphisms ϱ_s of ϱ which are also localized in $\mathcal{O}_0$.

Next, let ϱ correspond to a covariant representation of $\mathcal{A}$. Then [2] there is a unitary cocycle $x \mapsto U_x \in \mathcal{A}$

$$U_{xy} = \alpha_x(U_y)U_x \tag{3.5}$$

such that $\varrho_x := Ad_{U_x} \circ \varrho = \alpha_x \varrho \alpha_x^{-1}$ is equivalent to ϱ and localized in $x\mathcal{O}_0$. Then also the sub-endomorphisms ϱ_s are covariant (with cocycle U_x^s). We claim:

Lemma: $$\alpha_x(\psi) = U_x^s \cdot \psi \qquad (\psi \in \mathcal{H}_s). \tag{3.6}$$

Proof: It is easily checked that $(U_x^s)^* \alpha_x(\psi)$ are again intertwiners for ϱ_s and therefore belong to $\mathcal{H}_s$. Thus the map $(x, \psi) \mapsto (U_x^s)^* \alpha_x(\psi)$ is a finite dimensional unitary representation of the Poincaré or Möbius group on $\mathcal{H}_s$. Since every such representation is trivial, one gets (3.6).

Locality of $\mathcal{B}$ implies that $\alpha_x(\psi)$ and ψ commute when $x\mathcal{O}_0$ is space-like to $\mathcal{O}_0$. Therefore $(i, j = 1, \ldots N_s)$

$$U_x^s \psi^i \psi^j = \psi^j U_x^s \psi^i = \varrho_s(U_x^s)\psi^j \psi^i.$$

Now, $U_x^{s*} \varrho_s(U_x^s)$ is the statistics operator $\varepsilon_s = \varepsilon(\varrho_s, \varrho_s)$ [2], so locality reads

$$\varepsilon_s \, \psi^i \psi^j = \psi^j \psi^i. \tag{3.7}$$

When there is no defect as discussed in Prop. 3, this can be solved for ε_s:

$$\varepsilon_s = \sum_{ij} \psi^j \psi^i \psi^{j*} \psi^{i*} \tag{3.8}$$

implying $\varepsilon_s^2 = 1$, i.e. permutation group statistics as expected for sectors implemented by local fields. In contrast, when there is a defect, one only gets

$$\varepsilon_s^2 \psi^i \psi^j = \psi^i \psi^j, \tag{3.9}$$

i.e. the non-trivial joint range projection of $\psi^i \psi^j$ is an eigenprojection of the monodromy operator with eigenvalue 1. On the other hand, the monodromy operator is diagonalized by the projections for the decomposition of $\varrho_s^2 \simeq \bigoplus \varrho_k$ into its irreducible components [2]. Thus, although $\psi^i \psi^j$ are intertwiners for ϱ_s^2, they do not project to intertwiners for irreducible sub-endomorphisms ϱ_k unless the monodromy eigenvalue

$$\frac{\kappa(\varrho_k)}{\kappa(\varrho_s)^2} \stackrel{!}{=} 1. \tag{3.10}$$

Here κ is the statistics phase [2] related to the (fractional) spin of a sector. Namely, every candidate intertwiner $T^* \psi^i \psi^j \in \mathcal{B}$ (with $\mathcal{A} \ni T : \varrho_k \to \varrho_s^2$) for a subsector ϱ_k violating (3.10) must vanish. This immediately explains the "truncation" of the fusion rules (1.1) as compared to the intrinsic fusion rules (1.2).

Actually, one can prove more, thereby restricting the branching rules for the vacuum representation in the first place. (3.10) is then a trivial consequence.

Proposition 4: All subsectors of ϱ have statistics phase $\kappa(\varrho_s) = 1$.

Proof: The "master" equation underlying (3.7):

$$\varepsilon_\varrho V V = V V, \tag{3.11}$$

where $\varepsilon_\varrho = \varepsilon(\varrho, \varrho)$, obtains in the same way as (3.7). Introduce isometric intertwiners in A: $W := \lambda^{1/2} \mu(V) : id \to \varrho$ and $R := \gamma(V) W : id \to \varrho^2$, and observe $V^* R = \lambda^{-1/2} V$. Define the left-inverse $\phi_\varrho(a) := R^* \varrho(a) R$ and compute $d(\varrho)\phi_\varrho(\varepsilon_\varrho) = \lambda \mu(V^* \varepsilon_\varrho V)$. Next, compute ($\hat{R} \equiv \lambda^{1/2} R$)

$$V^* \varepsilon_\varrho V = \hat{R}^* V \varepsilon_\varrho V = \hat{R}^* \varrho(\varepsilon_\varrho) V V = \varrho(\hat{R}^*) \varepsilon_\varrho^* V V = \varrho(\hat{R}^*) V V = V \hat{R}^* V = V V^*.$$

The equality in the middle comes from the theory of statistics [2]. Therefore,

$$\mathcal{K}_\varrho := d(\varrho)\phi_\varrho(\varepsilon_\varrho) = 1. \tag{3.12}$$

Thus ϕ_ϱ is the standard left-inverse of ϱ, and since the spectrum of $\mathcal{K}_\varrho$ is given by the statistics phases $\kappa(\varrho_s)$ of all subsectors of ϱ [2], the claim follows.

The lemma and the constraint (3.10) derived from it, as well as Prop. 4 fail if $A \subset B$ has infinite index (see Sect. 6).

4 The generalized symmetry

An alternative way to show the completeness ($2.14(ii)$) of the charged isometries in the group symmetric case relies on the gauge principle: since the defect projection $E_s \in \varrho_s(A)' \cap B$ is gauge invariant by the linear transformation law (2.15), it is actually contained in A and therefore must be a scalar.

In the general case, an explicit formula like (2.15) is lacking. The gauge transformations α_g – which by ($2.13(b)$) are the irreducible components in the decomposition of the canonical endomorphism – are replaced by the endomorphisms γ_t ($2.11(b)$). The latter will not act linearly on the intertwiner spaces $\mathcal{H}_s$ of charged field operators. Yet, it seems natural to consider the action of γ_t on B as the generalized symmetry.

The questions arise what sort of a generalized group γ_t form, and in which sense A consists of invariant quantities. We shall not elaborate on this program here, but only include three remarks to illustrate our conception of (gauge) symmetry: everything that is apt to characterize the position of a subalgebra in an algebra (the gauge invariant quantities among all fields) is a good candidate for a generalized symmetry.

1.) The underlying structure to γ_t and all sub-endomorphisms of their products is described by the even part of the bipartite graph associated with the inclusion $A \subset B$ as discussed by Ocneanu [5]. The odd part of the graph is obtained if one writes the canonical endomorphism in the form

$$\gamma = \sigma\bar{\sigma} \qquad \text{where} \quad \sigma(B) = A. \tag{4.1}$$

The odd part of the graph then comprises all subsectors of $\gamma^n\sigma$. In (4.1), σ is an irreducible endomorphism of B (since $A \subset B$ is irreducible), and $\bar{\sigma}$ is a conjugate endomorphism uniquely determined up to inner equivalence by the requirements that $\bar{\sigma}$ is irreducible and $\sigma\bar{\sigma}$ contains the identity.

2.) If it is only known that all irreducible sub-endomorphisms γ_t of γ in ($2.11(b)$) have dimension 1, i.e. are automorphisms, then one can show [12, 18] that the latter may be (uniquely) chosen within their inner equivalence classes such that they form a group of order λ pointwise preserving A. In other words, it is possible to recover the symmetry group G of $A = B^\alpha$ from its canonical endomorphism $\gamma : B \to A$. The argument is simple: since for every t, $\gamma_t^{-1}\gamma = \gamma_t^{-1}\sigma \circ \bar{\sigma}$ again contains the identity, one concludes that $\gamma_t^{-1}\sigma$ is conjugate to $\bar{\sigma}$ and therefore equivalent to σ, and so is $\gamma_t\sigma$. It follows that $\gamma_t^{-1}\gamma$ and $\gamma_t\gamma$ contain the same sectors as γ, so the sector multiplication of γ_t must form a group. Now, if $\gamma_t\sigma = Ad_{U_t} \circ \sigma$, then $\alpha_t := Ad_{U_t^*} \circ \gamma_t$ preserve $A = \sigma(B)$ pointwise and satisfy the composition law of the group up to inner conjugation with some cocycle in B. Since the cocycle commutes with A it is scalar and the inner conjugation is trivial.

3.) There is a generalized "gauge principle" which allows to recover (up to inner conjugation) the observables A from the knowledge of the endomorphisms $\gamma_t \in End(B)$

together with their multiplicities. Namely, according to (2.11(b)) one may construct $\gamma \in End(B)$ which has the abstract properties characterizing canonical endomorphisms. E.g., there is a pair of isometric intertwiners $V : id \to \gamma$ and $W : \gamma \to \gamma^2$ satisfying the identities [13]

$$\begin{array}{c} V^*W = \lambda^{-1/2} = \gamma(V^*)W \\ \gamma(W)W = WW, \qquad \gamma(W^*)W = WW^*. \end{array} \tag{4.2}$$

¿From this one recovers the conditional expectation $\mu(\cdot) := W^*\gamma(\cdot)W$ and the observables $A := \mu(B) =$ fixpoints of μ. Conversely, μ, V, γ are the data of Prop. 1 associated with $A \subset B$, and $W = \lambda^{1/2}\mu(V)$.

These remarks justify our proposal for the generalized symmetry underlying the inclusion $A \subset B$.

5 The index of coset models

Let us now turn to the specific coset models of conformal quantum field theory. It is known [7] that for current algebras as well as for the algebras of the energy-momentum tensor on the circle, Haag duality holds in the vacuum representation, i.e. the local von Neumann algebras associated with complementary intervals I and $I^c = S^1 \setminus I$ are each other's commutants:

$$\pi_0(\mathcal{A}_I) = \pi_0(\mathcal{A}_{I^c})'. \tag{5.1}$$

In other representations, Haag duality will be violated in general. Consider the inclusion (locality!)

$$\pi(\mathcal{A}_I) \subset \pi(\mathcal{A}_{I^c})'. \tag{5.2}$$

If $\pi = \pi_0 \circ \varrho$ is given by an endomorphism of $\mathcal{A}$ which is localized in I, i.e. $\varrho(a') = a'$ for $a' \in \mathcal{A}_{I^c}$, then due to (5.1), (5.2) turns into

$$\varrho(\mathcal{A}_I) \subset \mathcal{A}_I\,. \tag{5.3}$$

The index of the inclusions (5.2), (5.3) is given by the square of the statistical dimension $d(\pi)^2 \equiv d(\varrho)^2$ of the endomorphism ϱ [11], i.e., the notion of statistical dimension for localized endomorphisms of the C^* algebra $\mathcal{A}$ and the notion of dimension for endomorphisms of a local von Neumann algebra $\mathcal{A}_I$ as introduced in Sect. 2 coincide.

Now, denote the pairs of algebras of the coset models (1.8), (1.9) by $\mathcal{A} \subset \mathcal{B}$. Clearly, the inclusion holds also for the local algebras: $\mathcal{A}_I \subset \mathcal{B}_I$. Let π be a positive-energy representation of $\mathcal{B}$ and $\pi_|$ the restriction of π to $\mathcal{A}$. The following idea is due to Wassermann [7]. Consider the chain of inclusions

$$\pi_|(\mathcal{A}_I) \overset{\alpha}{\subset} \pi(\mathcal{B}_I) \overset{\beta}{\subset} \pi(\mathcal{B}_{I^c})' \overset{\alpha'}{\subset} \pi_|(\mathcal{A}_{I^c})'. \tag{5.4}$$

The index of (α) is both independent of the interval I by conformal covariance and independent of the representation π since all representations are locally equivalent. We may therefore define $Index(\alpha)$ as the "index of the subtheory" $\mathcal{A} \subset \mathcal{B}$.

Since the index of a subfactor equals the index of its commutant, (α) and (α') have the same index. The index of (β) is given by the statistical dimension of π, $d(\pi)^2$, and the index of the total inclusion $(\alpha' \circ \beta \circ \alpha)$ is given by the statistical dimension of the reducible representation $\pi_| \simeq \bigoplus_s N_s \pi_s$ of $\mathcal{A}$, $d(\pi_|)^2$. One can solve for the index of $\mathcal{A} \subset \mathcal{B}$:

$$Index[\mathcal{A} \subset \mathcal{B}] = \frac{d(\pi_|)}{d(\pi)} = \frac{\sum_s N_s d(\pi_s)}{d(\pi)}. \tag{5.5}$$

Clearly, finite index requires finite reducibility of $\pi_|$.

In coset models, the branching rules for the decomposition of a representation of $\mathcal{B}$ upon restriction to $\mathcal{A}$ are well known, and the statistical dimensions of the involved representations can be computed from the chiral partition functions $\chi_\pi(\beta) = Tr_\pi \exp(-2\pi\beta L_0)$ as the "asymptotic dimensions" $d_{\rm as}(\pi) := \lim \chi_\pi(\beta)/\chi_0(\beta)$ in the high-temperature limit $\beta \searrow 0$. (Actually, the coincidence of the asymptotic and statistical dimensions has been established only in a few special cases (e.g., see [7]), but is widely believed to hold in general. Note that the independence of (5.5) of the representation π is manifest with the asymptotic dimension.)

The following table lists a few examples with the branching rules for the vacuum representation (the sectors of current algebras are denoted by the corresponding isospin, while the sectors of $W = Vir(c)$ are given in the minimal model nomenclature) and the evaluation of (5.5):

$$\begin{array}{ll}
SU(2)_4 \subset SU(3)_1 : & [0] \to [0] + [2] \\
 & Index = 1 + 1 = 2 \\
SU(2)_{10} \subset SO(5)_1 : & [0] \to [0] + [3] \\
 & Index = 1 + \sin\tfrac{7\pi}{12} / \sin\tfrac{\pi}{12} = 3 + \sqrt{3} \\
SU(2)_2 \otimes Vir(\tfrac{1}{2}) \subset SU(2)_1 \otimes SU(2)_1 : & [0]\otimes[0] \to [0]\otimes[(1,1)] + [1]\otimes[(1,3)] \\
 & Index = 1\cdot 1 + 1 \cdot 1 = 2 \\
SU(2)_3 \otimes Vir(\tfrac{7}{10}) \subset SU(2)_2 \otimes SU(2)_1 : & [0]\otimes[0] \to [0]\otimes[(1,1)] + [1]\otimes[(1,3)] \\
 & Index = 1\cdot 1 + 2\cos\tfrac{\pi}{5} \cdot 2\cos\tfrac{\pi}{5} \\
 & \qquad = 4\cos^2\tfrac{\pi}{10}.
\end{array} \tag{5.6}$$

As a by-product, we draw an interesting conclusion from the first and third entries of this list. In both cases, the subtheory $\mathcal{A}$ is contained in the subtheory of $\mathbb{Z}_2$-invariants of $\mathcal{B}$, where the action of $\mathbb{Z}_2$ is the involutive Lie group automorphism of $SU(3)$ associated with the symmetric space $SU(3)/SU(2)$, lifted to the current algebra, and the "flip" of the two $SU(2)_1$ tensor factors, respectively. Now, since the index of $\mathcal{A} \subset \mathcal{B}$ is 2, $\mathcal{A}$ must actually coincide with the invariant subtheory. Thus, these two examples fall into the class described by a global gauge group. One would not have expected this result in terms of the respective Kac-Moody modes j^a_n.

Another example of this type, although of infinite index, is the energy-momentum tensor theory $Vir(c = 1)$ contained in the $SU(2)_1$ current algebra, which in fact coincides with the subtheory of invariants under the global $SU(2)$ symmetry [19].

The last entry in the list (5.6) is an inclusion with index of the "rigid" Jones form $4\cos^2\frac{\pi}{n}$.

6 Discussion and outlook

We have seen that some non-trivial sectors of a subtheory $H_l \otimes W$ and *a forteriori* of H_l alone are in fact generated by local fields of the theory G_k. How would one have guessed the extension G_k, if the physical observables are chosen to be a given H_l theory? In particular, what is the role of the coset theory which completely decouples from the given H_l theory? Let us begin with a simpler question: Why have models like the last two entries in our list (5.6) escaped the systematic search in [1] for local extensions of $SU(2)$ current algebras?

The answer is of course that the currents of the extending G_k theory are not primary with respect to the energy-momentum tensor $T^{(H)}$ of the original theory, but only with respect to the energy-momentum tensor $T^{(G)}$ which according to (1.10) is obtained by including the decoupling coset degrees of freedom. The proper space-time covariance is only implemented by the latter.

For the general problem of the existence of local charged fields generating sectors of the observables H_l, however, an extension

$$H_l \subset G_k \tag{6.1}$$

rather than (1.9) is perfectly admissible. This incites us to take a glimpse at infinite index. Namely, the subtheory inclusion (6.1) is reducible with infinite index whenever W in (1.9) is non-trivial, i.e. $T^{\rm coset} \neq 0$. The branching of the vacuum representation comes with infinite multiplicities, clearly falsifying Prop. 3. One also readily checks in models that Prop. 4 and the phase condition (3.10) are violated. This is possible since the the argument leading to (3.6) fails. The infinite-dimensional multiplicity space $\mathcal{H}_s$ carries an *infinite*-dimensional "internal" representation of the Möbius group by unitaries in W which modifies (3.6) without affecting the intertwining property of ψ for the sectors of the observables H_l. This fact is the abstract version of the coset Sugawara formula (1.10). The internal part of the covariant transformation law for ψ accounts for $\kappa(\varrho_s) \neq 1$ and for the violation of (3.10).

Thus, the reason for the introduction of the coset W-algebra is the demand (principle?) of *local* and *covariant* charged fields in the sector-generating algebra $\mathcal{B}$. The latter then automatically also contains the decoupled coset algebra.

We suggest that this sort of "principle" could be an alternative to the search for a

generalized gauge principle which imposes some linear transformation law and braided commutation relations on definitely non-local and therefore *a priori* unobservable objects. In contrast, our approach attempts to find *local* fields which are by definition unobservable ($\notin \mathcal{A}$) with the measuring apparatus we have equipped our observer with, but which admit the option of becoming observable with an upgraded apparatus ("breaking the symmetry" by switching on "magnetic fields").

Actually, there *is* a way to observe the decoupled coset degrees of freedom. Since gravity couples to the total energy-momentum density, an observer in the H_l world could detect the excess $T^{\rm coset}$ of the total energy-momentum tensor $T^{(G)}$ beyond the observable energy-momentum tensor $T^{(H)}$ through a gravitational effect ("dark matter"). Clearly, this argument is highly speculative. It seems not to apply in four dimensions, where it is known that all sectors of the algebra of observables are generated (actually implemented) by charged fields (e.g. fermions) [4] which are accounted for in the standard energy-momentum tensor. Yet, ignoring this objection, one could pursue a dark matter scenario like the above. It would entail that gravitational effects belong to a different concept of observability than the one given in terms of local quantum measurements. Thus quantum gravity would have to stand on a very different footing than ordinary quantum field theory.

We observe that infinite index inclusions like (6.1) can give rise to "anyonization" of braid group statistics. Namely, a given level k theory is always diagonally embedded into the tensor product of k level 1 theories (cf. (1.8)), and all sectors of the former are obtained by restriction of the sectors of the latter. But all sectors of level 1 theories for Lie groups of Dynkin type A, D, E are simple (anyonic) sectors. They are described by (explicitly known) localized automorphisms and can be implemented by unitary operators which satisfy anyonic commutation relations. Thus, these are models where one can further extend the local field algebra $\mathcal{B}$ into an anyonic field algebra $\mathcal{F}$ such that all sectors of $\mathcal{A}$ arise in the vacuum representation of $\mathcal{F}$ and the corresponding intertwining isometries in $\mathcal{F}$ satisfy anyonic commutation relations.

Finally, we include a remark answering a question raised at this conference by S. Majid. For the mathematical theory of subfactors, there is a duality between the inclusions $A \subset B$ and $\gamma(B) \subset A$ and between the associated "paragroups" [5]. In particular, the same structure theory applies for $\varrho = \gamma|_A$ which is the canonical endomorphism for the latter inclusion and γ which is equivalent to the restriction of ϱ to $\gamma(B)$. However, when extended to nets of local algebras $\mathcal{A} \subset \mathcal{B}$ this balance will fail: namely in order to guarantee ϱ to be a localized endomorphism of the subtheory we had to require the compatibility of the conditional expectation with the space-time symmetries α_x. Now, on one hand this implies the existence of statistics, and therefore the semigroup generated by the subsectors of ϱ is abelian. On the other hand, the above compatibility is not preserved by duality: descending the Jones tower, the conditional expectation $\nu : A \to \gamma(B)$ is given by $\nu(a) = \gamma(V^*aV) = \lambda\mu(VV^*aVV^*)$. So the semigroup generated by γ_t need not be commutative, and it definitely isn't in the case of a non-abelian symmetry group. If one wants to

maintain the mathematical balance between the "symmetry" paragroup (generated by γ_t) and the "superselection" paragroup (generated by ϱ_s) in physics, either one has to abandon localization, or the symmetry has to be commutative (in the weak sense of composition of equivalence classes). Is this a hint at some unknown "principle" which could explain the phenomenological fact that all exact superselection rules observed in Nature are due to abelian symmetries?

Appendix

We sketch how Prop. 2′ follows from Prop. 2 by entirely group theoretical arguments.

First, the decomposition of γ is just (2.9). In order to determine the decomposition of ϱ, we must compute the commutant $\varrho(A)' \cap A$. The map $g \mapsto |G| \cdot \mu(V\alpha_g(V^*))$ maps G into the commutant $\varrho(A)' \cap A$:

$$\begin{aligned}\mu(V\alpha_g(V^*)) \cdot \varrho(a) &= \mu(V\alpha_g(V^*)\varrho(a)) = \mu(V\alpha_g(V^*\varrho(a))) = \mu(V\alpha_g(aV^*)) = \\ &= \mu(Va\alpha_g(V^*)) = \mu(\varrho(a)V\alpha_g(V^*)) = \varrho(a) \cdot \mu(V\alpha_g(V^*)).\end{aligned}$$

By invariance of μ: $\mu \circ \alpha_g = \mu$, and (2.2(i)), this map extends linearly to a $*$-homomorphism from the group algebra $\mathbb{C}G$ into $\varrho(A)' \cap A$, which is actually an isomorphism. Now, $\mathbb{C}G$ is a direct sum of matrix rings corresponding to the representations τ^r of G of dimension $dim(r)$. There are therefore matrix units $X^r_{ik} \in \mathbb{C}G$ which reduce the left regular representation of G:

$$g\ X^r_{ik} = \sum_j \overline{\tau^r_{ji}(g)} \cdot X^r_{jk}\,.$$

Consequently, the operators

$$a^r_{ik} := |G| \cdot \mu(V\alpha_{X^r_{ik}}(V^*))$$

are algebraically matrix units in $\varrho(A)' \cap A$ and can be written in the form

$$a^r_{ik} = W^i_r (W^k_r)^*$$

with orthonormal isometries $W^i_r \in A$ yielding ϱ_r and the multiplicities as in (2.13(a)). Next, (2.2(i)) yields

$$\phi^{i*}_r := V^* W^i_r = V^* a^r_{ik} W^k_r = \alpha_{X^r_{ik}}(V^*) W^k_r$$

for every k, implying the linear transformation law (2.15) for $\psi \propto \phi$. By (2.15), the numerical matrix $(\phi^{i*}_r \phi^j_r)_{ij}$ commutes with the irreducible representation $\tau^r(g)$ and therefore is a multiple of δ_{ij}. Also by (2.15), the average on the left-hand-side of

$$|G|\mu(\phi^i_r \phi^{j*}_r) = (W^i_r)^* |G| \mu(VV^*) W^j_r = (W^i_r)^* W^j_r = \delta_{ij}$$

can be computed:

$$\sum_{g \in G} \alpha_g(\phi^i_r \phi^{j*}_r) = \delta_{ij} \frac{|G|}{dim(r)} \sum_{k=1}^{dim(r)} \phi^k_r \phi^{k*}_r.$$

This establishes the completeness relations (together with the missing normalization of $\psi \propto \phi$), such that finally (2.12) entails (2.16).

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