DEUTSCHES ELEKTRONEN-SYNCHROTRON

DESY 94-028
March 1994


# Eigenvalues of the Laplacian and of the Hecke Operators for PSL(2,$\mathbb{Z}$)

G. Steil

*II. Institut für Theoretische Physik, Universität Hamburg*

ISSN 0418-9833

NOTKESTRASSE 85 - 22603 HAMBURG

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX,
send them to (if possible by air mail):

| | |
|---|---|
| **DESY**<br>**Bibliothek**<br>**Notkestraße 85**<br>**22603 Hamburg**<br>**Germany** | **DESY-IfH**<br>**Bibliothek**<br>**Platanenallee 6**<br>**15738 Zeuthen**<br>**Germany** |

DESY 94-028
March 1994

ISSN 0418-9833

# Eigenvalues of the Laplacian and of the Hecke operators for PSL(2,$\mathbb{Z}$)

Gunther Steil

II. Institut für Theoretische Physik, Universität Hamburg
Luruper Chaussee 149, 22761 Hamburg, Germany

**Abstract**

A new method is described to compute with high accuracy a large number of eigenvalues and eigenfunctions (Maaß wave forms) of the Laplacian and of the Hecke operators for the modular group. It relies essentially on the theory of Hecke operators. The results of the computations confirm some important conjectures from number theory, namely Ramanujan–Petersson, Sato–Tate, and the conjecture that the discrete spectrum of the Laplacian be simple. Examples of the numerical data are included as a reference. The algorithm can be generalized to other non-cocompact but cofinite arithmetic groups, like Picard group PSL(2, $\mathbb{Z}[i]$) and Hecke triangle groups $\Gamma(\sqrt{2})$ and $\Gamma(\sqrt{3})$.



## 1 Introduction and summary

The spectrum of the hyperbolic Laplacian for the modular group PSL(2, $\mathbb{Z}$) seems to be one of the most interesting in modern number theory and the formalism of the Selberg trace formula, cf. [15, 21, 35]. While the continuous part of the spectrum can be described rather explicitly in terms of the Eisenstein series, less is known about its discrete part whose existence is not even a trivial fact [9, 23, 31]. In line with the increasing power of computers, in the past some 15 years the idea of getting numerical insight into this and related (dicrete) spectra attracted more and more interest, indicating as well a creeping motion of new "technical" techniques towards hard mathematics. Lately physicists got involved into this business by the discourse of quantum chaos.

The relevant references concerning the computational work in the case of the modular group are [6, 7, 8, 10, 13, 14, 16, 17, 18, 19, 20, 34, 36, 37, 42]; reviews can be found in [15, 37].

In this paper we present a numerical method for computing with high accuracy a large number of eigenvalues and eigenfunctions (Maaß wave forms) of the Laplacian and of the Hecke operators for the modular group. We compute simultaneously eigenvalues and square-summable eigenfunctions of

$$-\Delta f(z) = \lambda f(z) = \left(\tfrac{1}{4} + R^2\right) f(z), \tag{1}$$

$$\mathrm{T}_p f(z) = c_p f(z), \qquad p \text{ prime}, \tag{2}$$

subject to the automorhy condition

$$f(\gamma z) = f(z), \qquad \gamma \in \mathrm{PSL}(2,\mathbb{Z}), \ \ \mathrm{Im}\, z > 0. \tag{3}$$

The Laplacian is given by

$$\Delta = y^2(\partial_x^2 + \partial_y^2), \tag{4}$$

for details and the definition of the Hecke operators $\mathrm{T}_p$ see section 2 below. The eigenvalues $c_p$ of the Hecke operators enter the Fourier expansion (20) of $f(z)$ as coefficients. Coefficients with non-prime index are polynomials in the $c_p$'s. To facilitate language we occasionally shall refer to the eigenvalues of the Laplacian simply as eigenvalues and to those of the Hecke operators as coefficients.

The algorithm takes as starting point the Fourier expansion of the eigenfunctions with the coefficients expressed as polynomials in the prime coefficients $c_p$. This fulfils eq. (1) and part of the invariance condition (3). The $c_p$'s are determined as functions of $R$ from the nonlinear system that arises from writing

down explicitely eq. (2) for the respective $p$. The eigenvalues are found iff the remaining conditions (3) are fulfilled. This proceeding can also be applied to other systems for which Hecke operators with similar properties exist. This is the case for the subgroups (so-called Hecke triangle groups) $\Gamma(\sqrt{m})$, $m = 2, 3$, of the modular group [11], and for the Picard group $\mathrm{PSL}(2, \mathbb{Z}[i])$ which corresponds to a 3-dimensional system [22]. For the latter, we have been computing the first eigenvalues recently. Results will be reported in future paper.

Most of the cited papers deal with the determination of the eigenvalues $R$ (this is the usual parameter) as solutions of eqs. (1) and (3). The first eigenvalues have been computed by Cartier and Haas [7, 13]. Major progress is due to Hejhal [16] who was the first to go beyond $R \approx 25$ (so far only 20 correct eigenvalues have been known). More than one thousand eigenvalues were respectively found in [8, 34, 37]; the first two papers consider only odd eigenfunctions, the latter developed the present method.

The determination of eigenfunctions essentially consists in finding the appropriate Fourier coefficients. Stark was the first to recognize that equation (2) could be useful in this respect [36]. Apart from this, only Hejhal considers coefficients/eigenfunctions in two recent papers [17, 19]. Huntebrinker [20] presents finite element approximations of eigenfunctions.

Our computations cover the ranges $0 < R < 350$ and $500 < R < 510$ which include about 10000 eigenvalues (tables 4, 5 show examples). The completeness was checked by comparison of the results with a generalized Weyl's law (cf. figure 1). Additionally, a couple of large eigenvalues around $R = 1000$, 2000, and 4000 have been found, the largest of which lies above the $1.3 \times 10^6$th eigenvalue (table 1). The spectrum is simple in the whole range, but nearly degeneracies occur frequently (in the language of quantum chaos the spectrum is Poissonian, see [4, 5]).

The Sato–Tate conjecture asserts that the $c_p$'s of any eigenfunction should be distributed according to Wigner's semicircle law. To get numerical hints on its validity we prepared extensive lists of $c_p$'s for a couple of Maass wave forms which include up to 30000 coefficients. They clearly confirm the conjecture (fig. 2). (See also [17].)

Keeping $p$ fixed, it is seen (fig. 3) that the distribution of the Fourier coefficients $c_p$ agree with the predictions of a theorem of Sarnak [32].

The Ramanujan–Petersson conjecture states

$$|c_p| \leq 2. \tag{5}$$



We have not found any violation of this bound among more than 300000 coefficients.

The results are also of interest in the context of quantum chaos (see e.g. [12] for general informations) where the eigenvalues of the Laplacian correspond to the energy levels of a quantum mechanical particle sliding freely on the modular domain. The classical counterpart of this system has a chaotic geodesic flow [2, 28]. Nevertheless, it turned out [3, 4, 34, 37] that the statistical properties of the spectrum look more like those of an integrable system. At reasonable high energies the spectrum shows Poissonian fluctuations, i.e. fluctuations similar to those of a sequence of equidistributed random numbers, whereas most (time-reversal invariant) chaotic systems exhibit GOE statistics as described by the random matrix theory [29]. The origin of this exceptional behaviour lies in the arithmetic structure of the modular group. For this and related systems which are based on arithmetic groups the notion of arithmetic chaos was recently introduced. For detailed informations consult [3, 4, 5, 33].

## 2 Preliminaries

In this section we shortly collect well-known basic notions and facts which are necessary to construct the algorithm in the following section. A more detailed presentation and many references to the original literature can be found e.g. in Terras' book [39].

Let

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R},\, y > 0\} \tag{6}$$

be the Poincaré upper half plane which is equipped with the Riemannian metric

$$ds^2 = y^{-2}\left(dx^2 + dy^2\right). \tag{7}$$

The (hyperbolic) Laplace operator and the volume element associated with $ds^2$ are

$$\Delta = y^2(\partial_x^2 + \partial_y^2), \tag{8}$$

$$dz = y^{-2}dx\,dy. \tag{9}$$

$\mathcal{H}$ has constant negative Gaussian curvature $K = -1$ and is a model for hyperbolic geometry in two dimensions. Geodesics are semicircles and straight lines orthogonal to the real axis.

The group of orientation preserving isometries is

$$G = \mathrm{PSL}(2,\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2} \mid ad - bc = 1 \right\} / \{\pm I\} \tag{10}$$

acting on $\mathcal{H}$ by (fractional) linear transformations

$$gz = \frac{az+b}{cz+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \tag{11}$$

Both the volume element and the Laplacian are invariant under the action of $G$.

The modular group

$$\Gamma = \mathrm{PSL}(2,\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} \mid ad - bc = 1 \right\} / \{\pm I\} \tag{12}$$

is one of the simplest discrete subgroups of $G$. Its standard fundamental domain, which is together with appropriate boundary conditions a realisation of the quotient $\Gamma\backslash\mathcal{H}$, is the modular domain

$$\mathcal{F} = \{ z \in \mathcal{H} \,|\, |z| \geq 1,\ |x| \leq 1/2 \}. \tag{13}$$

$\Gamma$ is generated by the translation $z \mapsto z+1$ and the inversion $z \mapsto -1/z$ identifying opposite sides of $\partial\mathcal{F}$. $\Gamma$ is co-finite since $\mathrm{vol}(\Gamma\backslash\mathcal{H}) = \int_{\mathcal{F}} dz = \pi/3 < \infty$, but not co-compact since $\mathcal{F}$ has a cusp at $i\infty$. Functions on $\Gamma\backslash\mathcal{H}$ can be considered as automorphic functions on $\mathcal{H}$ which have to satisfy $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$.

Let $L^2(\Gamma\backslash\mathcal{H})$ be the Hilbert space of square summable automorphic functions $(f,f) = \int_{\mathcal{F}} f(z)\overline{g(z)}\, dz < \infty$. $\Delta$ is essentially self-adjoint on $L^2(\Gamma\backslash\mathcal{H})$ and therefore has a unique self-adjoint extension [9]. We will not distinguish between both of them in the following. The spectrum of $\Delta$ splits $L^2(\Gamma\backslash\mathcal{H})$ into the orthogonal sum [23, 31]

$$L^2(\Gamma\backslash\mathcal{H}) = \mathit{Eisenstein} \oplus \mathit{Constant} \oplus \mathit{Cusp} \tag{14}$$

*Eisenstein* is spanned by the Eisenstein series and contains the continuous spectrum of $\Delta$ which extents from $\frac{1}{4}$ to $\infty$. Functions in *Eisenstein* are even functions in $x$.

*Constant* contains the constant eigenfunctions associated with the trivial eigenvalue $\lambda_0 = 0$ which exists since $\mathcal{F}$ has finite hyperbolic area.

We are concerned with the third part *Cusp* of the spectral decomposition. It contains the discrete spectrum of $\Delta$ which is embedded in the continuous spectrum, and which is supposed but not proven to be simple.



Let the Hecke operators $\mathrm{T}_n$, $n \in \mathbb{N}$, be defined as [25, 26]

$$\mathrm{T}_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n,\ d>0 \\ b \bmod d}} f\left(\frac{az+b}{d}\right) \tag{15}$$

and let the parity operator be

$$\mathrm{P} f(z) = f(-\overline{z}). \tag{16}$$

Then,

$$\{\Delta, \mathrm{T}_n, \mathrm{P}\}_{n\in\mathbb{N}} \tag{17}$$

forms a commutative family of self-adjoint operators on $L^2(\Gamma\backslash\mathcal{H})$. Let $(f_j)_{j\in\mathbb{N}}$ be a orthogonal basis of *Cusp* consisting of simultaneous eigenfunctions of this family. The $f_j$'s are called Maaß wave forms or cusp forms. We describe in the following the properties of these functions which could be deduced by the application of the different operators. Occasionally we will drop the indices of individual cusp forms and related quantities.

The cusp forms fall into two parity classes containing even and odd functions with respect to P and satisfying Neumann ($\frac{\partial f}{\partial n} = 0$) and Dirichlet ($f = 0$) boundary conditions on $\partial\mathcal{F} \cup \{x = 0\}$, respectively. Accordingly, we will speak about even and odd eigenvalues. The fulfilment of the boundary conditions is equivalent to automorphy in each parity class.

By definition cusp forms are eigenfunctions of the Laplacian,

$$\Delta f_j + \lambda_j f_j = \Delta f_j + \left(\tfrac{1}{4} + R_j^2\right) f_j = 0, \tag{18}$$

and we may order the $f_j$'s by the size of their eigenvalues

$$\frac{1}{4} < \frac{3\pi^2}{2} < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \tag{19}$$

with $\lambda_j \to \infty$ as $j \to \infty$. The lower bound $3\pi^2/2$ [30] allows to write $\lambda_j = \frac{1}{4} + R_j^2$ with positive $R_j$ which henceforth will be called eigenvalue as well.

Formal solutions of (18) can be obtained by a separation ansatz and the employment of periodicity in $x$. This leads to the Fourier expansion

$$f_j(z) = \sum_{n=1}^{\infty} c_n^j \sqrt{y}\, \mathrm{K}_{iR_j}(2\pi n y) \cos(2\pi n x) \tag{20}$$

for even cusp forms while in the odd case sines replace the cosines. It can be shown that the first Fourier coefficient never vanishes. Otherwise one would deduce from

relations like (25) that all other coefficients have to vanish, too. The rest of the paper will be dealing with so-called normalized cusp forms whose first coefficient is fixed to be

$$c_1 = 1. \tag{21}$$

We remark that normalized cusp forms do not have an $L^2$-norm of one.

$K_{iR}(x)$ denotes the modified Bessel function of the third kind [1, 41] which is real for real $x$ and $R$. For fixed $R$, $K_{iR}(x)$ oscillates about 0 in the interval $x \in (0, R)$ with a period approximately proportional to $x$ and an amplitude of order $\sqrt{2\pi/R}\exp(-\pi R/2)$. The last extremum is positive and the following turning point lies slightly below $x = R$. For $x > R$, $K_{iR}(x)$ remains positive and dies off roughly exponentially as $x \to \infty$. Consequently, for numerical purposes one can find a value $x_{\max} = x_{\max}(R, \varepsilon) > R$ and take $K_{iR}(x) = 0$ for $x > x_{\max}$ with an error being smaller than any desired $\varepsilon > 0$.

Series (20) is absolutely convergent on $\mathcal{H}$ due to the exponential decrease of the Bessel function and the estimate [40]

$$|c_n| \le d(n) n^{1/4}, \tag{22}$$

where $d(n)$ counts the number of divisors of $n$.

Letting the Hecke operators act on normalized cusp forms given by the Fourier expansion (20), their eigenvalues turn out to be the Fourier coefficients,

$$T_n f(z) = c_n f(z) \qquad \forall\, n \in \mathbb{N}. \tag{23}$$

Since $T_n$ is self-adjoint $c_n \in \mathbb{R}$ for all $n \in \mathbb{N}$ causing normalized cusp forms to be real functions. More explicitly, equation (23) with the definition (15) reads for $n = p$ prime

$$c_p f(z) = \frac{1}{\sqrt{p}} \left( \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + f(pz) \right). \tag{24}$$

The coefficients are not all independent but obey the Hecke relations

$$c_m c_n = \sum_{d|(m,n)} c_{mn/d^2}. \tag{25}$$

E.g., for $n \in \mathbb{N}$, $p$ prime, one has

$$c_{np} = c_n c_p - c_{n/p} \tag{26}$$

where $c_{n/p}$ has to be taken as zero if $p$ is not a divisor of $n$. It follows that the coefficients $c_n$ are polynomials in the $c_p$'s with $p \le n$.



# 3 The algorithm

Consider the Fourier expansion (20). Let $R > 0$ and

$$\vec{c}_p = (c_2, c_3, c_5, \ldots) \tag{27}$$

be any real vector whose components fulfil (22). Using (26) $\vec{c}_p$ determines the vector

$$\vec{c} = (1, c_2, c_3, c_4, \ldots) = \vec{c}(\vec{c}_p) \tag{28}$$

of "fake" coefficients that could be inserted into the Fourier expansion. The resulting function $f(z) = f(z; \vec{c}_p, R)$ is square summable over $\mathcal{F}$ and an eigenfunction of the Laplacian on $\mathcal{H}$. But generally it is not an eigenfunction of $\Delta$ on $\Gamma\backslash\mathcal{H}$, i.e. not a cusp form, since it is not necessarily $\Gamma$-invariant.

$\Gamma$ is generated by the translation $T(z) = z+1$ and the reflection $S(z) = -1/z$. The Fourier expansion automatically assures translation invariance $f(z) = f(z+1)$. Hence the only remaining condition for $f$ to be a cusp form is

$$f(z) = f\left(-\frac{1}{z}\right) \quad \text{for all} \quad z \in \mathcal{H}. \tag{29}$$

An equivalent requirement would be the imposition of the correct (Dirichlet or Neumann) boundary conditions on the lower part $|z| = 1$ of $\partial\mathcal{F}$.

The basic idea is the following. Let $z_0 \in \mathcal{F}$ be fixed and

$$\Phi_z(R) = f(z; \vec{c}_p, R) - f(-1/z; \vec{c}_p, R). \tag{30}$$

Then by scanning the $R$-axis we are looking for zeroes $\tilde{R}$ of $\Phi_{z_0}(R)$ which are candidates for eigenvalues. The correct eigenvalues are picked up by testing $\Phi_z(\tilde{R})$ to vanish (numerically) as a function of $z \in \mathcal{H}$. The corresponding $f$ will be a cusp form.

The main question in this approach is how to determine $\vec{c}_p$ for the actual $R$. We do this with the help of the eigenvalue equations of the Hecke operators (24) which hold for any $z \in \mathcal{H}$ and any cusp form. Fixing $z = \hat{z} \in \mathcal{F}$ we interpret (24) for any $R > 0$ as an equation for the unknown $c_p$. Since $f$ shall be $\Gamma$-invariant one is allowed to write

$$c_p f(\hat{z}) = \frac{1}{\sqrt{p}} \left( \sum_{j=0}^{p-1} f(z_j^p) + f(z_p^p) \right) \tag{31}$$

where $z_p^p = p\hat{z}$ and $z_j^p \in \mathcal{F}$ denotes the point in the modular domain which is $\Gamma$-equivalent to $(\hat{z}+j)/p$. The point $z_j^p$ is easily found by translations into the strip

$|\mathrm{Re}\, z| \le 1/2$ and reflections by $z \mapsto -1/z$ in turn. This choice minimizes the number of terms needed to compute the Fourier series to a prescribed accuracy. Since the Bessel function $\mathrm{K}_{iR}(x)$ is a monotonically decreasing function of $x$ above $x = R$, the convergence of (20) is better the larger $y$ is. The modular domain and its translates just contain the points with largest imaginary part of all $\Gamma$–equivalent points.

Truncating the Fourier series and exchanging summations yields the polynomial, i.e. nonlinear, system of equations for the prime coefficients

$$0 = \sum_{n=1}^{N} c_n(\vec{c}_p)\Big[c_p SKC(1,n) - SKC(p,n)\Big], \qquad (32)$$
$$p = 2,3,5,\ldots,P, \quad \pi(N) = \pi(P)$$

where the constants

$$\begin{aligned} SKC(1,n) &= \sqrt{\hat{y}}\,\mathrm{K}_{iR}(2\pi n\hat{y})\cos(2\pi n\hat{x}), \\ SKC(p,n) &= \frac{1}{\sqrt{p}}\sum_{j=0}^{p}\sqrt{y_j^p}\,\mathrm{K}_{iR}(2\pi n y_j^p)\cos(2\pi n x_j^p) \end{aligned} \qquad (33)$$

contain numerical data.

This system is solved with Newton's method. Since one does not know in advance where to find its solutions it turns out to be effective to generate starting values for the Newton iteration randomly. In more detail a typical step of the calculations looks like follows. System (32) is nonlinear and therefore one generally has a couple of solutions $\vec{c}_p^{\,\mathrm{old}}$ for a given $R = R^{\mathrm{old}}$. Going to the next $R = R^{\mathrm{new}} = R^{\mathrm{old}} + \Delta R$ one takes each $\vec{c}_p^{\,\mathrm{old}}$ as a starting value for a Newton iteration. Each time the iteration converges one tests if

$$\Phi_{z_0}(R^{\mathrm{old}})\Phi_{z_0}(R^{\mathrm{new}}) \le 0 \qquad (34)$$

holds. In this case one expects a zero of $\Phi_{z_0}$ in the interval $[R^{\mathrm{old}}, R^{\mathrm{new}}]$ which we detect by a modified secant method or a modified Regula Falsi. If such a zero $R_0$ is found one has to check with different points $z \neq z_0$ whether $\Phi_z(R_0) = 0$ or not. In the former case one has hunted up an eigenvalue, and the corresponding cusp form is given by its Fourier expansion.

However, since it is impossible to know if the list of solutions of (32) is complete for the actual $R^{\mathrm{new}}$, the next step is very important as well as for getting the whole procedure started. We try to find new solutions by choosing randomly starting



values for the iteration, i.e. typically $c_p \in [-2p^{1/4}, 2p^{1/4}]$ or $c_p \in [-2,2]$ due to (22) and Ramanujan–Petersson conjecture (5), respectively.

With a single run one does usually not find all eigenvalues in a given interval. To circumvent this problem there is still the freedom of choosing the parameter $\hat{z} \in \mathcal{F}$ in eqs. (32) and (33). Since $\hat{z}$ carries no a priori information about the eigenvalues one can hope for the statistical independence of the solutions found with distinct $\hat{z}$. For "macroscopically" different $\hat{z}$ this seems to be true, and by doing several runs with different values of $\hat{z}$ we were able to complete the list of eigenvalues.

## Some technical remarks

From Weyl's law (cf. (40,41)) one expects a mean distance between successive even or odd eigenvalues of about $12/R$. The grid on which one searches for sign changes of $f(z) - f(-1/z)$ should therefore have a (nonconstant) width of $\Delta R = \pm\frac{1}{F}\left(\frac{12}{R}\right)$ with $F > 1$. Good values of $F$ turn out to be typically $10 < F < 30$.

The number $N$ of random $\vec{c}_p$'s at each step of the computations was mostly held between 10 and 30. Both quantities, $F$ and $N$, have to be coordinated since small values worsen the number of found eigenvalues in a single run and large values can increase the running time without an appropriate gain in the output of eigenvalues.

As an example we mention in more detail three of our runs in the interval $R \in [100, 250]$, the parity was odd. We took $\hat{z} = 0.2 + i1.1$, $0.2 + i1.15$, and $0.3 + i1.25$. The $R$-grid was of width $\frac{1}{15}\left(\frac{12}{R}\right)$, the number of random $\vec{c}_p$ was 30 and each $c_p \in [-2.2, 2.2]$. To yield an accuracy of 13 digits demands about 11 prime coefficients at $R = 100$ and 19 at $R = 250$. The maximum number of simultaneously found solutions of (32) increases from about 20 to nearly 40. Each run found about $70 - 80\%$ of the eigenvalues and all three together between 95% and 100%. The portion of found eigenvalues in a single run is slightly decreasing with $R$. At $R \approx 500$ we got typically about 50%.

For the evaluation of the Bessel functions we have been using either a routine of Hejhal and Bombieri [16], or our own routine which is based on asymptotic formulae, as proposed in [8], combined with a Miller algorithm [38] for certain arguments. The latter one is faster without loss of accuracy, but in any case the Bessel function calls are the most time consuming part of the algorithm besides the linear algebra associated with Newton's method. For this reason it was nec-

essary to reduce drastically the Bessel function calls by interpolation techniques. For small and moderate indices ($R < 250$, say) a Lagrange interpolation in $R$ for each argument $2\pi n y_j^p$ works well. For (fixed) large $R$ a Newton interpolation relative to the argument is to be prefered.

Most of the computations were carried out on IBM RS/6000 workstations.

# 4 Discussion of the algorithm

The idea to use Hecke operators in connection with computations concerning Maass wave forms is mentioned by Stark [36]. He proposed to compute Fourier coefficients by a direct iteration of equation (24) or slight rearrangements of it, e.g.

$$c_n^{k+1} f^k(z) = \mathrm{T}_n f^k(z) \tag{35}$$

where $f^k(z)$ is given by the $k$-th iterate $\{c_n^k\}_{n\in\mathbf{N}}$. Although he was able to detect the first even eigenvalue, this method generally does not work since one cannot assure contraction of (35). It is easy to find examples in which, even if starting close to a correct eigenvalue and the corresponding $\vec{c}$, the iteration is not converging to this solution. Examples of this effect can be found in [17].

The first method to find a larger number of eigenvalues stems from Hejhal [16]; it makes use of the Hecke relations, too. The basic idea is to write down explicitly the equation $f(z) - f(-1/z) = 0$ for different points $z \in \mathcal{H}$ where $f$ is given by its Fourier expansion (20) and to deduce a system of linear equations which is solved for the unknown (not only prime) coefficients. The eigenvalues are found by varying $R$ and by the requirement that the Hecke relations be fulfilled. The main drawback of this method is its lack of stability which is sufficient to find the eigenvalues but not sufficient to get out correctly all the coefficients involved. The reason of this behaviour is the ill-conditioning of the established matrix. (To avoid this problem Hejhal & Arno [17] went back to the ideas of Stark's iterative method to compute the coefficients of certain cusp forms from given approximations. In spite of some tricky improvements their method has still a kind of experimental stage. It deserves some supervision and is not well suited for detecting unknown solutions.)

Our method does not have these problems with stability and convergence. At first sight the introduction of the nonlinear system (32) might seem to increase the problems at hand, but this is not true. The problem of convergence is simply



solved by application of Newton's method which assures locally quadratic convergence. Its use is made easier by the fact that the solutions one is looking for are contained in a finite box of dimension (see (22))

$$[-2\cdot 2^{1/4}, 2\cdot 2^{1/4}] \times [-2\cdot 3^{1/4}, 2\cdot 3^{1/4}] \times [-2\cdot 5^{1/4}, 2\cdot 5^{1/4}] \times \ldots \tag{36}$$

The Newton method also reduces the nonlinear problem to solving linear equations. But in this case the relevant matrix of the derivative is much better conditioned, at least partly since differentiating (32) with respect to $\vec{c}_p$ produces a term $f(\hat{z})$ in each element of the diagonal which dominates (as long as it stays away from zero) the much smaller off-diagonal elements of the last columns.

The reduced matrix dimensions of $\pi(N)\times\pi(N)$, in the nonlinear case, instead of $(N-1)\times(N-1)$, in the linear case, compensates for the effort of computing multiple solutions. Recall the expense of a Gauss elimination is roughly proportional to the third power of the matrix dimension. In the extreme case of $R \approx 4000$ where $N \approx 825$ and $\pi(N) \approx 145$ this leads to a factor of $(825/145)^3 \approx 184$.

The ambiguity of the non-linear system allows to find more than a single eigenvalue in one mesh of the $R$-grid. This is good for two reasons. First it makes it possible to detect without complications degenerated eigenvalues, if existing. However, this possibility has not yet occurred. Secondly, the $R$-grid has not to be finer than the distance between successive eigenvalues. This is important in view of the fact that the eigenvalues tend to prefer small distances, level clustering in the parlance of quantum chaos, as was shown in [4, 34]. Table 3 clearly demonstrates the effect of nearly degenerate eigenvalues in the most extreme cases. It is visible even at small $R$: until recently [16] the two odd eigenvalues $23.20^+$ and $23.26^+$ were not known correctly. They are missing in [7, 18], and [10] displays somehow diplomatically the mean 23.23. Generally, the separation of such close lying solutions is difficult for any algorithm based on linear equations, whereas the presented method is not affected by this behaviour.

To explore the limits of the algorithm we have done a couple of test runs at $R = 1000$, 2000, and 4000. In either case eigenvalues were found although the running time is increasing enormously. The results are presented in the tables 1 and 2.

| even | | odd | $n \approx$ | |
|---|---|---|---|---|
| 1000.02199511676 | 1000.27230927167 | 1000.17795675640 | 39155 | (even) |
| 1000.06366726075 | 1000.28599362215 | 1000.32119881281 | 40561 | (odd) |
| 1000.11061578997 | 1000.29288170050 | 1000.41037141913 | | |
| 1000.11952236414 | 1000.36237116994 | 1000.47009730337 | | |
| 1000.14318460208 | 1000.37009864881 | 1000.52984253891 | | |
| 1000.17315583652 | 1000.39238529136 | 1000.59811856215 | | |
| 1000.19228308221 | 1000.39691060345 | 1000.72636359908 | | |
| 1000.25974630314 | | | | |
| 2000.03261184360 | | 2000.01919434933 | 160982 | (even) |
| 2000.05158832762 | | 2000.04473789000 | 164235 | (odd) |
| 2000.06353658232 | | 2000.05785938212 | | |
| 2000.12752840208 | | 2000.08253608295 | | |
| 2000.14541255499 | | | | |
| 4000.01102588972 | | 4000.05723539020 | 653973 | (even) |
| 4000.01746322179 | | 4000.05981212797 | 661361 | (odd) |
| | | 4000.06807822332 | | |
| | | 4000.10329568733 | | |

Table 1: Large eigenvalues $R_n$ for the modular group (not complete). The third column exhibits the approximate value of $n$. Displayed are the respective values of $N^{\pm}_{\mathrm{mean}}(1/4+R^2)$, for $R = 1000,\ 2000$, and 4000, cf. eqs(40),(41).



| $R = 4000.01102588972$ (even) | | | | | |
|---|---|---|---|---|---|
| $p$ | $c_p$ | $p$ | $c_p$ | $p$ | $c_p$ |
| 2 | +1.86152900987 | 229 | +0.40240262889 | 523 | −0.79130478848 |
| 3 | −0.92703892921 | 233 | −0.91005983320 | 541 | −0.81280846921 |
| 5 | −1.49515928669 | 239 | +1.45447815199 | 547 | −0.29048332219 |
| 7 | +1.41609304830 | 241 | +1.84183990712 | 557 | −0.07804495400 |
| 11 | +0.59083640633 | 251 | +1.77696920305 | 563 | +0.13778840313 |
| 13 | −1.10371440498 | 257 | +1.22804101784 | 569 | +0.07778930576 |
| 17 | −1.81591719288 | 263 | −1.38486170210 | 571 | +0.83186815755 |
| 19 | −0.00059943179 | 269 | +0.79760601225 | 577 | +0.25429506830 |
| 23 | −1.15691899782 | 271 | +1.44260220951 | 587 | −0.18669290000 |
| 29 | −1.93030329169 | 277 | −1.35250477123 | 593 | +0.68261695040 |
| 31 | −1.09508769312 | 281 | −0.89566315809 | 599 | −0.52751449102 |
| 37 | −0.63275087480 | 283 | +1.15067558493 | 601 | −0.21482133252 |
| 41 | +0.24155719241 | 293 | −0.73790889097 | 607 | +0.05605604815 |
| 43 | +0.13153011095 | 307 | +0.82646033248 | 613 | −0.14807826405 |
| 47 | +0.98917826888 | 311 | −1.10717669769 | 617 | +1.58718124320 |
| 53 | +0.23986793193 | 313 | −0.76681186015 | 619 | +1.08120349554 |
| 59 | +0.79542391321 | 317 | −1.81745120782 | 631 | +0.20040709253 |
| 61 | +1.02319296879 | 331 | +0.07898804822 | 641 | −1.09300402439 |
| 67 | +0.61248808946 | 337 | −0.14490104335 | 643 | +0.59335255754 |
| 71 | −0.22374550490 | 347 | −1.20233805528 | 647 | +1.53232568969 |
| 73 | −0.24139477345 | 349 | +0.79698998800 | 653 | +0.62721920109 |
| 79 | −0.19410698510 | 353 | −1.06738973040 | 659 | −0.24096680598 |
| 83 | −0.39407408315 | 359 | −1.49803981313 | 661 | −1.37369501480 |
| 89 | −1.20504335370 | 367 | +0.86493750133 | 673 | −0.09724230159 |
| 97 | +1.76194709700 | 373 | +0.21241186162 | 677 | +0.51406521141 |
| 101 | +0.62445217323 | 379 | +0.20422929697 | 683 | +0.39332271020 |
| 103 | +1.61285055953 | 383 | +0.04775783212 | 691 | −0.06666354459 |
| 107 | −1.09753341225 | 389 | +0.50992150601 | 701 | +1.37975283234 |
| 109 | −1.50818375918 | 397 | −1.50877637775 | 709 | −0.25867275495 |
| 113 | −0.69043125185 | 401 | −1.08194363005 | 719 | −0.68643969888 |
| 127 | −0.14430197494 | 409 | −0.69953202889 | 727 | −1.78283036350 |
| 131 | +0.17148692012 | 419 | −0.00160379742 | 733 | +0.97904688103 |
| 137 | −0.37454999405 | 421 | +1.68814255660 | 739 | +0.63799172079 |
| 139 | −1.01094537096 | 431 | +0.48269642557 | 743 | −0.62877041130 |
| 149 | +1.31945062199 | 433 | −1.01139767455 | 751 | +0.14043116498 |
| 151 | +1.42546578251 | 439 | +0.73170342766 | 757 | +0.25119631130 |
| 157 | −0.97276137210 | 443 | −0.13530325375 | 761 | +0.63129565363 |
| 163 | −0.63386438223 | 449 | +0.11891073872 | 769 | −1.44770863042 |
| 167 | +1.55524394856 | 457 | +0.29680219504 | 773 | +1.39689981865 |
| 173 | −1.92304601702 | 461 | +1.71162158491 | 787 | −0.39594362338 |
| 179 | −0.69946345685 | 463 | −1.67320503010 | 797 | −1.75571847264 |
| 181 | −0.75181511422 | 467 | −1.28404594592 | 809 | −0.94014648438 |
| 191 | +1.13777924414 | 479 | +0.20388074569 | 811 | +0.57098050887 |
| 193 | −0.82577940186 | 487 | +1.24343585378 | 821 | −1.57416488537 |
| 197 | −1.71902309526 | 491 | +1.57921023359 | 823 | −1.49389838667 |
| 199 | −1.44573363611 | 499 | −0.25964442752 | 827 | +0.77284508214 |
| 211 | +0.48218884999 | 503 | +1.86280891281 | 829 | −1.68885757885 |
| 223 | −0.21103202962 | 509 | −0.97220977121 | | |
| 227 | +1.45212225278 | 521 | −1.36012101331 | | |

Table 2: Fourier coefficients of a normalized even cusp form ($c_1 = 1$).

| | $R$ | $dR$ | | $R$ | $dR$ |
|---|---|---|---|---|---|
| even | 242.6673326133 | $\frac{1}{2668}$ | odd | 231.7806243370 | $\frac{1}{5543}$ |
| | 242.6673535416 | | | 231.7806342045 | |
| | 271.7958204649 | 6.47 | | 348.8593978542 | 6.43 |
| | 272.1148275386 | | | 349.0893379077 | |

Table 3: Successive eigenvalues $R \in (0, 350)$ with the most extreme relative distances $dR$. $dR$ is measured in units of the mean expected distance given by Weyl's law, eqs. (40),(41).

| + | 0 | 450 | 2141 | 9325 |
|---|---|---|---|---|
| 1 | 13.7797513518907389 | 123.8331309255 | 250.0142918545 | 500.0388246192 |
| 2 | 17.7385633810573779 | 123.8727236673 | 250.1570226403 | 500.0480426360 |
| 3 | 19.4234814708282552 | 124.1395466897 | 250.1715412434 | 500.0664609711 |
| 4 | 21.3157959402037188 | 124.2126870734 | 250.1998488441 | 500.0759968907 |
| 5 | 22.7859084941898736 | 124.3508437282 | 250.2205188129 | 500.1139410029 |
| 6 | 24.1123527298408640 | 124.4979173823 | 250.2941562035 | 500.1380623518 |
| 7 | 25.8262437127091584 | 124.5181798749 | 250.3230632640 | 500.1417069147 |
| 8 | 26.1520854492213488 | 124.5823857987 | 250.3600047846 | 500.1519712977 |
| 9 | 27.3327080831494099 | 124.7999715417 | 250.5126017731 | 500.2147562702 |
| 10 | 28.5307476929183202 | 124.8986908677 | 250.5215749611 | 500.2322981577 |
| 11 | 28.8633943539222970 | 124.9944384466 | 250.6110083423 | 500.2712342591 |
| 12 | 30.4106788046542568 | 125.0368588880 | 250.6264347412 | 500.2835512456 |
| 13 | 31.5265821967902692 | 125.3138401770 | 250.6309088536 | 500.3275946928 |
| 14 | 31.5662754117538871 | 125.3475585710 | 250.7200641698 | 500.3528235961 |
| 15 | 32.5081177599090003 | 125.5239875728 | 250.7482615783 | 500.3595282663 |
| 16 | 32.8911702135103197 | 125.6736019295 | 250.8287516940 | 500.3944749497 |
| 17 | 34.0278842001002688 | 125.8964729376 | 250.8494850031 | 500.4081702583 |
| 18 | 34.4562715330310283 | 126.0187785113 | 251.0096635387 | 500.4304849651 |
| 19 | 35.5023497713694011 | 126.0663817855 | 251.0206514583 | 500.4854369799 |
| 20 | 35.8416764325828391 | 126.1139945203 | 251.0595380115 | 500.5042750580 |
| 21 | 36.6775529931452006 | 126.2504058608 | 251.0926894563 | 500.5387670349 |
| 22 | 36.8563494959215496 | 126.3135693958 | 251.1131940447 | 500.5551887337 |
| 23 | 37.8250722905930271 | 126.3211487082 | 251.2710099800 | 500.5729304251 |
| 24 | 38.3032761524950115 | 126.3797686713 | 251.2884826640 | 500.5887269688 |
| 25 | 39.1680849679279492 | 126.6407541406 | 251.3219346949 | 500.6672979560 |

Table 4: Even eigenvalues $R_n$ of the Laplacian for $\mathrm{PSL}(2, \mathbb{Z})$. The index $n$ indicates the $n$-th even eigenvalue, and is the sum of the numbers at the very top and the very left of the respective eigenvalue.



| − | 0 | 1500 | 3500 | 10000 |
|---|---|---|---|---|
| 1 | 9.53369526135355756 | 200.2518741643 | 301.0302010452 | 501.9847360659 |
| 2 | 12.1730083246796778 | 200.2580950447 | 301.0402448107 | 502.0222609522 |
| 3 | 14.3585095182598128 | 200.3958010057 | 301.1187593097 | 502.0391021629 |
| 4 | 16.1380731715210306 | 200.4227411463 | 301.1736468811 | 502.1150314774 |
| 5 | 16.6442592018998199 | 200.5245258224 | 301.2186015631 | 502.1177585828 |
| 6 | 18.1809178345307039 | 200.5997034279 | 301.2236844511 | 502.1273121331 |
| 7 | 19.4847138547410134 | 200.6198039845 | 301.2591944152 | 502.1332352300 |
| 8 | 20.1066946825523129 | 200.6499781469 | 301.2888877222 | 502.1534067708 |
| 9 | 21.4790575447485450 | 200.6550046228 | 301.2941914205 | 502.1711402021 |
| 10 | 22.1946739775723304 | 200.8146803730 | 301.3496957296 | 502.1946528131 |
| 11 | 23.2013961812270875 | 200.9185589240 | 301.4655904599 | 502.1980076715 |
| 12 | 23.2637115379396382 | 200.9444508522 | 301.5452135943 | 502.2593584926 |
| 13 | 24.4197154423254233 | 200.9995646841 | 301.5764451197 | 502.2920024737 |
| 14 | 25.0508548507616351 | 201.0851848471 | 301.5768016298 | 502.2924299865 |
| 15 | 26.0569177606664270 | 201.1369300987 | 301.5878738659 | 502.3506265321 |
| 16 | 26.4469964180468775 | 201.2018927851 | 301.5909436571 | 502.4040281720 |
| 17 | 27.2843840116822503 | 201.2694635188 | 301.6298575970 | 502.4645005738 |
| 18 | 27.7759207017968351 | 201.2995037852 | 301.7093530841 | 502.4766515547 |
| 19 | 28.5102777031464807 | 201.3653199545 | 301.7205684940 | 502.5094637752 |
| 20 | 29.1375875578395346 | 201.4125363106 | 301.8205764778 | 502.5108813650 |
| 21 | 29.5463881241460127 | 201.4210768703 | 301.8497937810 | 502.5135527742 |
| 22 | 30.2790484991474997 | 201.4707419985 | 301.9332581402 | 502.5310764428 |
| 23 | 30.4043270540436687 | 201.5938041875 | 302.0021004356 | 502.5370504927 |
| 24 | 31.0565339621067000 | 201.7198256463 | 302.0078365485 | 502.5566016515 |
| 25 | 31.9161824709196205 | 201.7836083392 | 302.0507893299 | 502.5634889499 |

Table 5: Odd eigenvalues $R_n$ of the Laplacian for $\mathrm{PSL}(2, \mathbb{Z})$. The index $n$ is determined as in table 4.

## 5 Eigenvalues of the Laplacian

We have been computing all eigenvalues up to $R = 350$ (4401 even and 4776 odd) and between 500 and 510 (395 even and 410 odd). We have not found any degeneracies and the values are in agreement with the results of other authors we know (the references are given in the introduction). As mentioned before we computed additionally eigenvalues up to $R \approx 4000$, a region that lies above the $1.3 \times 10^6$th eigenvalue.

The precision that could be obtained is mainly restricted by the accuracy of the available Bessel function routine. Most of our runs were done with an accuracy of 13 to 15 digits. For the smaller $R$ values of tables 4 and 5 we have enhanced the accuracy to 18 digits using the 10 bytes extended arithmetic of a PC.

We exhibit a couple of examples of our eigenvalues in the tables 4, 5, 1. They are chosen so as to contain the regions of former results, especially those of Hejhal [16].

The completeness was checked by comparison with a generalized Weyl's law which describes the asymptotic behaviour of the spectral staircase ($\lambda = \frac{1}{4} + R^2$)

$$N(\lambda) = \#\{\lambda_j \leq \lambda\}. \tag{37}$$

$N(\lambda)$ can be written as a sum

$$N(\lambda) = N_{\text{mean}}(\lambda) + N_{\text{fluct}}(\lambda) \tag{38}$$

where $N_{\text{mean}}(\lambda)$ is a smooth function which describes a mean value of the number of eigenvalues below $\lambda$, and the fluctuating part $N_{\text{fluct}}(\lambda)$ is given by a function that oscillates about zero

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \int_0^\lambda N_{\text{fluct}}(s)ds = 0. \tag{39}$$

From Selberg's trace formula one can deduce [24, 27] the Weyl formulae ($\lambda \to \infty$)

$$N^+_{\text{mean}}(\lambda) = \frac{\lambda}{24} - \frac{3}{4\pi}\sqrt{\lambda}\ln\lambda + \frac{6 + 4\ln\pi - \ln 2}{4\pi}\sqrt{\lambda} - \frac{13}{144} + \frac{3}{32\pi}\frac{\ln\lambda}{\lambda} + O\left(\frac{1}{\lambda}\right) \tag{40}$$

$$N^-_{\text{mean}}(\lambda) = \frac{\lambda}{24} - \frac{1}{4\pi}\sqrt{\lambda}\ln\lambda - \frac{3\ln 2 - 2}{4\pi}\sqrt{\lambda} + \frac{23}{144} + \frac{1}{32\pi}\frac{\ln\lambda}{\lambda} + O\left(\frac{1}{\lambda}\right) \tag{41}$$

where the upper indices indicate the parity. Figure 1 shows the numerical analogue to $N_{\text{fluct}}$,

$$\delta_n = N_{\text{mean}}(\lambda_n) - \left[N(\lambda_n) - 1/2\right], \tag{42}$$

indicating the oscillatory behaviour about 0. As an example we took the first 1000 eigenvalues of even parity. The bracket in (42) is equal to $n - 1/2$ which is the middle of the step at the $n$-th eigenvalue (if the spectrum is simple). If an eigenvalue $\lambda_N$ is missing, all subsequent $n \geq N$ are underestimated by 1. Accordingly, $\delta_n$ is shifted by 1 for all $n \geq N$. By searching eigenvalues around such jumps we have completed our list. It should be noted that $N_{\text{mean}}(\lambda)$ describes the mean very well from the very beginning, in spite of its asymtotic character.



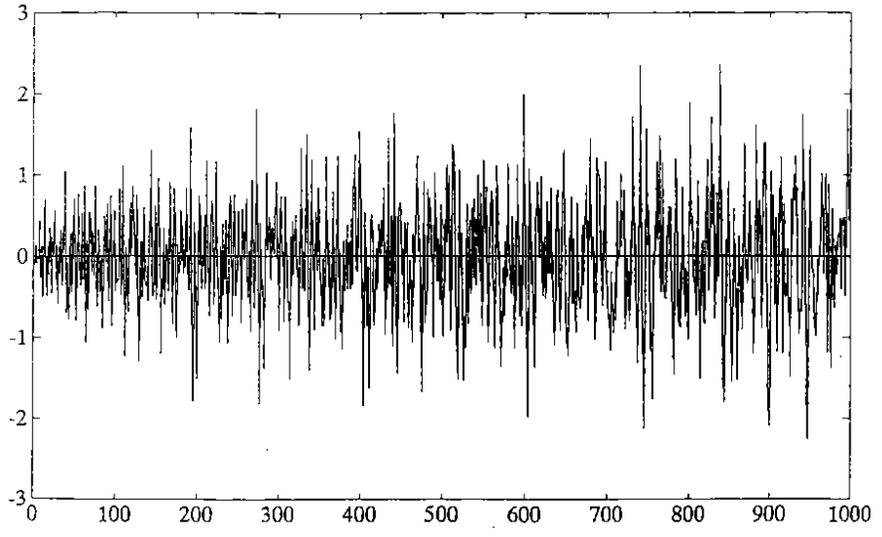


Figure 1: Difference $\delta_n$ of the spectral staircase $N(\lambda)$ to Weyl's law for the first 1000 even eigenvalues $\lambda_n$ as a function of $n$.

# 6 Eigenvalues of the Hecke operators

It is known that the eigenvalues of the Hecke operators, i.e. the Fourier coefficients of the normalized cusp forms, obey the bounds [32, 40]

$$|c_p| \leq 2\left(p^{1/5} + p^{-1/5}\right), \tag{43}$$

$$|c_p| \leq 2p^{1/4}, \tag{44}$$

a fact we have been exploiting in the algorithm. From a number theoretical point of view the first estimate is the better one because of the smaller exponent but the second one delivers smaller numerical bounds for moderate $p$. The Ramanujan-Petersson conjecture asserts an improvement to

$$|c_p| \leq 2. \tag{45}$$

All about 320000 prime coefficients we know agree with this conjecture, being so far the most extensive (numerical) test.

Table 2 exhibits as a reference eigenvalues of the Hecke operators for a large even $R$. This is equivalent to the numerical knowledge of the corresponding eigenfunction.

We turn now to statistical properties of the $c_p$'s. A result of Sarnak [32] shows that not too much coefficients can violate the Ramanujan–Petersson conjecture: Consider the sequence $\left(c_p^j\right)_{j\in\mathbb{N}}$, $p$ prime, of coefficients with fixed index and ordered by increasing corresponding eigenvalues. Then they are equidistributed with respect to

$$d\mu_p(x) = \begin{cases} \dfrac{(p+1)\sqrt{4-x^2}dx}{2\pi\left[\,(p^{1/2}+p^{-1/2})^2 - x^2\,\right]}, & |x|<2 \\ 0, & \text{otherwise.} \end{cases} \tag{46}$$

Equidistribution of a sequence $(x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ with respect to $d\lambda$ means

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(x_j) = \int_{\mathbb{R}} f(x)d\lambda(x) \quad \text{for all} \quad f\in C_c(\mathbb{R}). \tag{47}$$

The densities $d\mu_p(x)$ are by no means trivial and show a strong dependence upon $p$ for small values of $p$. This delivers an additional overall test of the correctness of our results. Figure 3 compares the distribution of the numerically given coefficients with the prediction for a couple of $p$'s. One recognizes a good agreement.

As $p\to\infty$, $d\mu_p(x)$ converges to the so-called semicircle distribution

$$d\mu(x) = \begin{cases} \dfrac{1}{2\pi}\sqrt{4-x^2}dx, & |x|<2 \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

The question arises if one is allowed to reverse the limits $j\to\infty$ and $p\to\infty$ in considering $\left(c_p^j\right)_{j\in\mathbb{N},\,p\in\mathbb{P}}$. If this were the case the so-called Sato–Tate conjecture would be true which states that the prime coefficients of a single cusp form $\left(c_p^j\right)_{p\in\mathbb{P}}$ are equidistibuted with respect to $d\mu$. Since this conjecture is unproved it is interesting to get some numerical hints on its validity. First studies in this direction were done in [17] where the respective first five even and odd cusp forms and two even ones of larger eigenvalue ($R=47.9^+$, $R=125.31^+$) are considered. With the exception of the last one, all corresponding $c_p$'s with $p<10000$ have been computed whose number is 1229. To improve the statistical significance we have done similar computations including more coefficients and larger $R$. The examples are listed in table 6. Figure 2 shows the numerical data together with



| | $R$ | parity | # $c_p$ | $p\leq$ |
|---|---|---|---|---|
| a. | 9.533695261354 | odd | 30000 | 350377 |
| b. | 13.77975135189 | even | " | " |
| c. | 99.86259201648 | even | " | " |
| d. | 249.9355630821 | even | " | " |
| e. | 502.0222609522 | odd | 10000 | 104729 |
| f. | aggregation of a. to e. | | 130000 | |

Table 6: Eigenvalues for which the Sato–Tate conjecture is checked in figure 2.

the semicircle law. The agreement is very good and a clear confirmation of the Sato–Tate conjecture.

The method at hand is to take (24) as an explicit equation for $c_p$ whenever $p$ exceeds the truncation index $N$ of the Fourier series (20), as proposed in [36]. The crucial point is the increasing computing time as $R$ and $p$ get large since each $c_p$ requires an effort of about $\frac{1}{2}Np$ Bessel function evaluations. Here, a factor of $\frac{1}{2}$ is introduced due to the fact that one can restrict the evaluations to those arguments for which $2\pi n y_j^p < x_{\max}$. The points $y_j^p$ are assumed to be randomly distributed over the modular domain with respect to $dz$ as $p\to\infty$ [36], and for that reason $n=N$ is only needed for the lowest lying points in $\mathcal{F}$. To save cpu time it is best first to tabulate the Bessel function in the whole needed range of its argument and then to compute it by means of a simple interpolation.

## Acknowledgements

I would like to thank J. Bolte and F. Steiner for reading the manuscript and for many helpful hints.

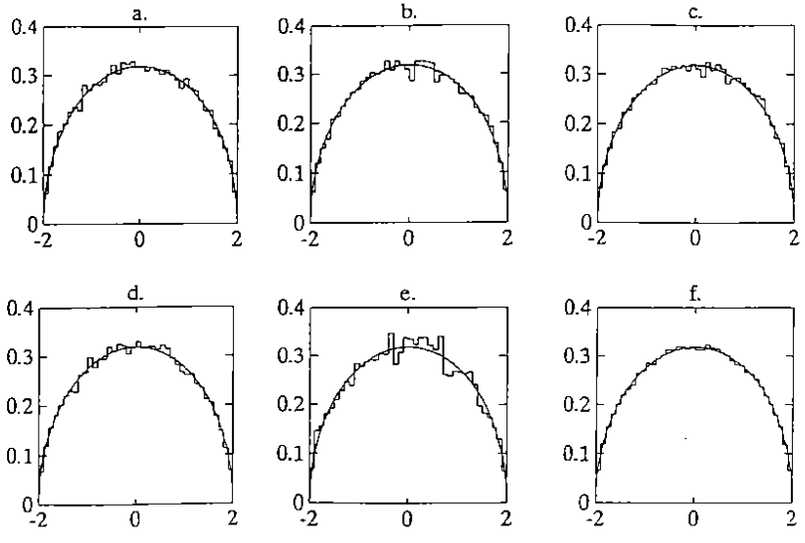


Figure 2: Distribution of prime coefficients (Hecke eigenvalues) of certain cusp forms in comparison to the semicircle law as predicted by the Sato–Tate conjecture. The small letters correspond to table 6.



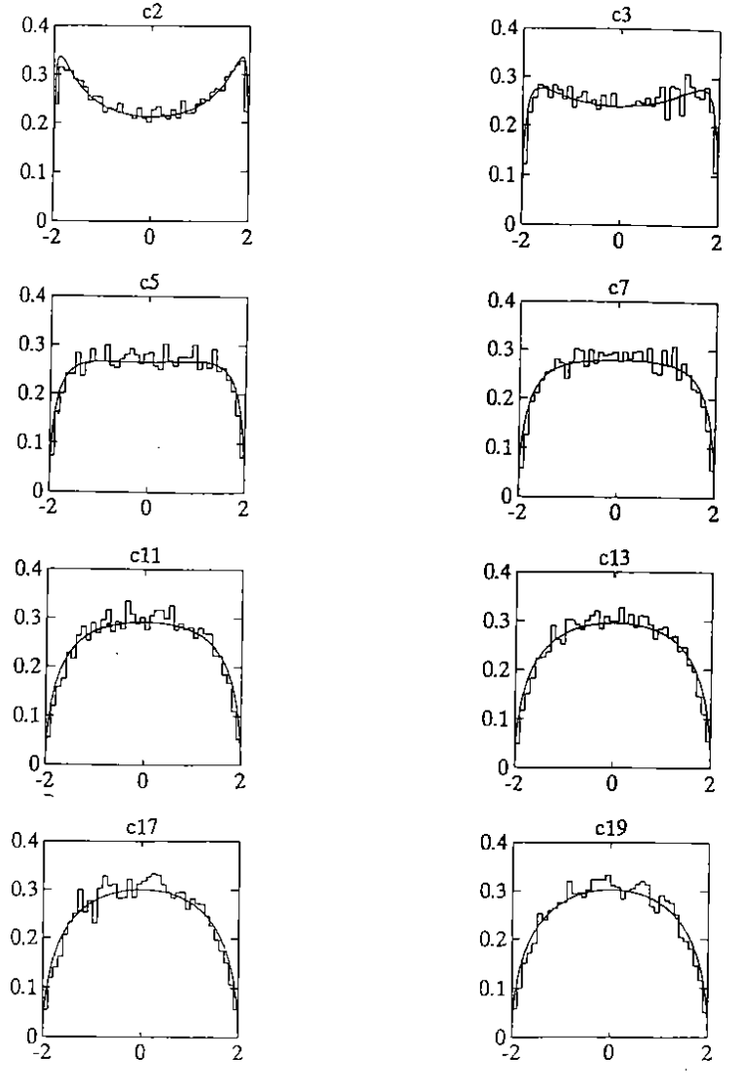


Figure 3: Collected $c_p$'s of cusp forms with $R < 350$ in comparison with the asymtotic distributions (46).

## References


[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.

[2] E. Artin, *Ein mechanisches System mit quasiergodischen Bahnen*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität **3** (1924), 170–175.

[3] E. Bogomolny, B. Georgeot, M.-J. Giannoni, and C. Schmit, *Chaotic billiards generated by arithmetic groups*, Phys. Rev. Lett. **69** (1992), 1477–1480.

[4] J. Bolte, G. Steil, and F. Steiner, *Arithmetical chaos and violation of universality in energy level statistics*, Phys. Rev. Lett. **69** (1992), 2188–2191.

[5] J. Bolte, *Some studies on arithmetical chaos in classical and quantum mechanics*, Int. J. Mod. Phys. **B7** (1993), 4451–4553.

[6] P. Cartier, *Some numerical computations relating to automorphic functions*, in: Computers in Number Theory (A. Atkin and B. Birch, eds.), Academic Press, London and New York, 1971, pp. 37–48.

[7] ——, *Analyse numérique d'un problème de valeurs propres a haute précision [application aux fonctions automorphes]*, Preprint, IHES, 1978.

[8] A. Csordás, R. Graham, and P. Szépfalusy, *Level statistics of a noncompact cosmological billiard*, Phys. Rev. **A44** (1991), 1491–1499.

[9] L. Faddeev, *Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobačevskiĭ plane*, Transactions of the Moscow Math. Soc. **17** (1967), 357–386.

[10] V. Golovčanskiĭ and M. Smotrov, *The first few eigenvalues of the Laplacian on the fundamental domain of the modular group*, Preprint, Far Eastern Scientific Center, Academy of Science USSR, Wladiwostok, 1982, In Russian.

[11] ——, *Small eigenvalues of the Laplacian on* $\Gamma\backslash\mathcal{H}_3$ *for* $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$, Preprint, Bielefeld, 1991.

[12] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer, New York, 1990.

[13] H. Haas, *Numerische Berechnung der Eigenwerte der Differentialgleichung* $-\Delta u = \lambda y^{-2} u$ *für ein unendliches Gebiet im* $\mathbb{R}^2$, Diploma thesis, Universität Heidelberg, Institut für Angewandte Mathematik, 1977.

[14] D. A. Hejhal, *Some observations concerning eigenvalues of the Laplacian and Dirichlet L-Series*, in: Recent Progress in Analytic Number Theory (H. Halberstam and C. Hooley, eds.), vol. 2, Academic Press, 1981, pp. 95–110.





[15] ——, *The Selberg Trace Formula for* $\mathrm{PSL}(2,\mathbb{R})$, vol. 2, Springer Lecture Notes in Mathematics **1001** (1983).

[16] ——, *Eigenvalues of the Laplacian for* $\mathrm{PSL}(2,\mathbb{Z})$*: Some new results and computational techniques*, in: International Symposium in Memory of Hua Loo-Keng (S. Gong, Q. Lu, Y. Wang, and L. Yang, eds.), vol. 1, Science Press and Springer, 1991, pp. 59–102.

[17] D. Hejhal and S. Arno, *On Fourier coefficients of Maass waveforms for* $\mathrm{PSL}(2,\mathbb{Z})$, Math. Comp. **61** (1993), 245–267, Lehmer Memorial Volume.

[18] D. A. Hejhal and B. Berg, *Some new results concerning eigenvalues of the non-Euclidean Laplacian for* $\mathrm{PSL}(2,\mathbb{Z})$, Technical Report No. 82–172, University of Minnesota, 1982.

[19] D. A. Hejhal and B. Rackner, *On the topography of Maass waveforms for* $\mathrm{PSL}(2,\mathbb{Z})$*: Experiments and heuristics*, Supercomputer Inst. Research Rep. UMSI 92/162, University of Minnesota, 1992.

[20] W. Huntebrinker, *Numerische Bestimmung von Eigenwerten des Laplace-Operators auf hyperbolischen Räumen mit adaptiven Finite-Element-Methoden*, Bonner Mathematische Schriften 225, 1991.

[21] H. Iwaniec, *Non-holomorphic modular forms and their applications*, in: Modular Forms (R. A. Rankin, ed.), Ellis Horwood, Chichester, 1984, pp. 157–196.

[22] M. I. Knopp, *On the cuspidal spectrum of the arithmetic Hecke groups*, Math. Comp. **61** (1993), 269–275.

[23] T. Kubota, *Elementary Theory of Eisenstein Series*, Kodansha, Tokyo and Halsted Press, New York, 1973.

[24] N. Kuznecov, *The distribution of norms of primitive hyperbolic classes of the modular group and asymptotic formulas for the eigenvalues of the Laplace-Beltrami operator on a fundamental region of the modular group*, Soviet Math. Doklady **19** (1978), 1053–1056.

[25] H. Maaß, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141–183.

[26] ——, *Lectures on Modular Functions of one Complex Variable*, Tata Institute of Fundamental Research, Bombay 1964, Springer, Berlin Heidelberg New York Tokyo, Revised 1983.

[27] C. Matthies and F. Steiner, *Selberg's* $\zeta$ *function and the quantization of chaos*, Phys. Rev. A44 (1991), R7877–7880;
——, DESY report in preparation.

[28] F. Mautner, *Geodesic flows on symmetric Riemann spaces*, Ann. Math. **65** (1957), 416–431.

[29] M. Mehta, *Random Matrices*, Revised and enlarged second edition, Academic Press, San Diego, 1991.

[30] W. Roelcke, *Über die Wellengleichung bei Grenzkreisgruppen erster Art*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Mathematisch-naturwissenschaftliche Klasse, Jahrgang 1953/1955, Springer, Heidelberg, 1956.

[31] ——, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene*, Math. Ann. **167** (1966), 292–337 and **168** (1967), 261–324.

[32] P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators*, in: Analytic Number Theory and Diophantine Problems (A. Adolphson, J. Conrey, A. Ghosh, and R. Yager, eds.), Proceedings of a Conference at Oklahoma State University 1984, Birkhäuser, Boston Basel Stuttgart, 1987, pp. 321–331.

[33] ——, *Arithmetic Quantum Chaos*, Expanded version of the Schur lectures Tel-Aviv 1992 and Blythe lectures Toronto 1993, Preprint, Princeton, 1993.

[34] C. Schmit, *Triangular billiards on the hyperbolic plane: Spectral properties*, Preprint IPNO/TH 91-68, 1991.

[35] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.

[36] H. Stark, *Fourier coefficients of Maass waveforms*, in: Modular Forms (R. A. Rankin, ed.), Ellis Horwood, Chichester, 1984, pp. 263–269.

[37] G. Steil, *Über die Eigenwerte des Laplaceoperators und der Heckeoperatoren für* $\mathrm{SL}(2,\mathbb{Z})$, Diploma thesis, Universität Hamburg, 1992.

[38] N. M. Temme, *On the numerical evaluation of the modified Bessel function of the third kind*, J. Comp. Phys. **19** (1975), 324–337.

[39] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications*, vol. I, Springer, New York Berlin Heidelberg Tokyo, 1985.

[40] M.-F. Vignéras, *Quelque remarque sur la conjecture* $\lambda_1 \geq 1/4$, in: Séminaire de Théorie des Nombres, Paris 1981–82 (M.-J. Bertin, ed.), Birkhäuser, Boston Basel Stuttgart, 1983, pp. 321–343.

[41] G. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 2. edition, 1958.

[42] A. M. Winkler, *Cusp forms and Hecke groups*, Journal für die reine und angewandte Mathematik **386** (1988), 187–204.