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About Loop States in Supergravity

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Abstract

The Wilson loop functionals in terms of Ashtekar's variables were the first (formal) solutions to the quantized hamiltonian constraint of canonical gravity. Here it is shown that the same functionals also solve the supergravity constraints and some evidence is presented that they are artificially generated by multiplying the constraints by the metric determinant, which has become a widely accepted procedure. Using the same method in 2+1 dimensional gravity and supergravity leads to wrong results, e.g. 2+1 gravity is no longer a purely topological theory. As another feature of the densitized constraints it turns out that the classical theory desribed by them is not invariant under space time diffeomorphisms.

In the main part of this paper we will focus on the comparison of the metric and connection representation of supergravity. Both will be derived

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using Ashtekar's variables [1], which simplify the construction of the canonical theory, even in the metric formalism, considerably. For the N=1 theory in 3+1 dimensions the canonical treatment 'a la Dirac' [2] in the metric (or vierbein) representation leads to first class constraints which in the quantized version become inhomogeneous second order differential equations [3] with non-analytic coefficients, whereas in the connection representation they are polynomials in the canonically conjugate variables and homogeneous in the momentum variables [4].

But, as for the connection representation of pure gravity, one has to multiply the supersymmetry constraint by the determinant of the spatial metric to obtain a polynomial expression. However, in supergravity this multiplication does not only produce a 'densitized' Wheeler DeWitt operator but also the diffeomorphism constraint becomes a density of weight 1, because both are generated by the commutator of the supersymmetry constraint with its conjugate.

We will see that the densitized supersymmetry constraints are formally solved by the *same* Wilson loop functionals which are known to solve the Wheeler DeWitt constraint of pure gravity [5]. In contrast to pure gravity, however, a single loop functional now solves *all* constraints without considering functionals that depend on knot classes only, i.e. the state functionals are no longer invariant under spatial diffeomorphism.

Though this result comes out when dealing with supergravity, it can be reproduced for pure gravity, too, just by dropping all the fermionic quantities from the action and the constraints. When defining the polynomial constraints in this way, a single loop functional again becomes a solution to *all* constraints. Obviously, multiplying the diffeomorphism constraint with extra vierbein factors destroys the invariance of the theory under spatial coordinate transformations. In fact, for the classical theory we will explicitly see that the invariance under space time diffeomorphisms is lost, if we use the densitized hamiltonian constraint *and* allow the metric to be singular.

Another property of the loop functionals in supergravity is that they are purely bosonic states, i.e. they do not depend on the fermionic variables. There has been a discussion about the existence of such states [6, 7, 8, 9]. For the metric representation it was shown in [8] that purely bosonic states do not exist, because they cannot fulfill one of the supersymmetry constraints. This suggests that the Wilson loop states are nothing but solutions artificially generated by multiplying the constraints with the metric determinant. Note

that the solutions themselves are annihilated by this determinant.

To make all these arguments more precise, in the second part we will discuss N=2 supergravity in 2+1 dimensions, which is completely soluble both on the classical and the quantum level. It will be shown that the metric and connection representation are equivalent. However, to define the connection representation properly, no extra factors of the dreibein are needed to make the constraints polynomial, and both the metric and the connection representation are able to deal with singular metrics.

But, if one starts from the four dimensional densitized constraints and reduces them to three dimensions, or, equivalently, if one multiplies the hamiltonian constraint by the metric determinant in the same way as for the 3+1 dimensional theory, one obtains a different set of constraints, the 'densitized connection representation'. In this representation, one again finds the loop states and the quantum theory becomes completely different. In particular, there are now infinitly many states even if the spatial topology is trivial and one requires the state to be invariant under diffeomorphisms (which does not follow from the constraints!), whereas in the 'correct' quantum theory there are only finitely many classical degrees of freedom, i.e the wave functional depends on finitely many variables only, and there is only one state for trivial topology.

1 N=1 supergravity in four dimensions

The canonical formalism for N=1 supergravity in terms of Ashtekar's new variables has been worked out in [4]. We will use slightly different notation here to make the results formally as similar as possible to the corresponding results for the three dimensional theory below.

The first order action for N=1 supergravity is usually written as [10, 11]

$$I'[E,\Omega,\psi,\bar{\psi}] = \int \mathrm{d}^4x\, L'_{\mathrm{EH}} + L'_{\mathrm{RS}}, \tag{1.1}$$

where

$$\begin{aligned} L'_{\mathrm{EH}} &= \tfrac{1}{2} E E_A{}^M E_B{}^N R_{MN}{}^{AB}[\Omega], \\ L'_{\mathrm{RS}} &= \mathrm{i}\varepsilon^{MNPQ}(\bar{\psi}_M \sigma_N D_P \psi_Q - D_M \bar{\psi}_N \sigma_P \psi_Q) \end{aligned} \tag{1.2}$$

are the Einstein Hilbert and Rarita Schwinger action, respectively. The notation is as follows. Indices from the beginning of the alphabet always denote flat tangent space vectors, those from the middle of the alphabet curved space time indices. The vierbein $E_A{}^M$ (or its inverse $E_M{}^A$) thus has the flat index A running from 0 to 3, raised and lowered by the lorentzian metric $\eta_{AB} = \mathrm{diag}(-,+,+,+)$, and the curved index M, taking the values $\mathrm{t}, \mathrm{x}, \mathrm{y}, \mathrm{z}$. E is the determinant of $E_M{}^A$ and ε^{MNPQ} the Levi Civita tensor density with $\varepsilon^{\mathrm{txyz}} = 1$.

The spin connection Ω_{MAB} defines the covariant derivative of a tangent space vector with flat index

$$D_M V_A = \partial_M V_A + \Omega_M{}^A{}_B V^B. \tag{1.3}$$

The field strength or curvature of Ω is given by

$$R_{MNAB}[\Omega] = \partial_M \Omega_{NAB} - \partial_N \Omega_{MAB} + \Omega_{MA}{}^C \Omega_{NCB} - \Omega_{NA}{}^C \Omega_{MCB}. \tag{1.4}$$

We introduce Ashtekar's variables by mapping the $4 \cdot 6$ components of the real $\mathbf{so}(3,1)$ spin connection Ω_{MAB} onto $4 \cdot 3$ complex components of the $\mathbf{so}(3,\mathsf{C})$ connection A_{Ma}, $a = 1,2,3$. The mapping is given by

$$A_{Ma} = J_{aAB} \Omega_M{}^{AB}, \qquad A^*_{Ma} = J^*_{aAB} \Omega_M{}^{AB}, \tag{1.5}$$

where the coefficients J_{aAB} and J^*_{aAB} form a basis of selfdual and antiselfdual antisymmetric tensors, i.e.

$$J_{aAB} = -\tfrac{\mathrm{i}}{2} \varepsilon_{AB}{}^{CD} J_{aCD}, \qquad J^*_{aAB} = \tfrac{\mathrm{i}}{2} \varepsilon_{AB}{}^{CD} J^*_{aCD}, \tag{1.6}$$

where ε^{ABCD} is the flat Levi Civita tensor with $\varepsilon^{0123} = -\varepsilon_{0123} = 1$. The J-symbols are complete and orthonormal

$$\begin{aligned} J_{aAB} J_a{}^{CD} + J^*_{aAB} J^{*CD}_a &= \delta^{[C}_{[A} \delta^{D]}_{B]} = \tfrac{1}{2} \delta^C_A \delta^D_B - \tfrac{1}{2} \delta^D_A \delta^C_B, \\ J_{aAB} J_b{}^{AB} = J^*_{aAB} J^{*AB}_b &= \eta_{ab}, \qquad J_{aAB} J^{*AB}_b = 0, \end{aligned} \tag{1.7}$$

where $\eta_{ab} = \delta_{ab}$ is the 'spatial' component of the flat metric (remember that $a = 1,2,3$ whereas $A = 0,1,2,3$). Thus the relation (1.5) can be inverted to give

$$\Omega_{MAB} = J_{aAB} A_{Ma} + J^*_{aAB} A^*_{Ma}. \tag{1.8}$$

An explicit representation for J is

$$J_{aAB} = \tfrac{\mathrm{i}}{2}\eta_{aA}\delta^0_B - \tfrac{\mathrm{i}}{2}\eta_{aB}\delta^0_A - \tfrac{1}{2}\varepsilon^0{}_{aAB}. \tag{1.9}$$

Here we used that the range of the small indices is just a subset of that of the big indices. Another useful property of J is that it provides a four dimensional representation of $\mathbf{so}(3)$, which commutes with its conjugate

$$\begin{aligned} J_{aA}{}^B J_{bB}{}^C &= -\tfrac{1}{4}\eta_{ab}\delta^C_A + \tfrac{1}{2}\varepsilon_{abc}J_{cA}{}^C, \\ J^*_{aA}{}^B J^*_{bB}{}^C &= -\tfrac{1}{4}\eta_{ab}\delta^C_A + \tfrac{1}{2}\varepsilon_{abc}J^*_{cA}{}^C, \\ J^*_{aA}{}^B J_{bB}{}^C &= J_{bA}{}^B J^*_{aB}{}^C, \end{aligned} \tag{1.10}$$

Thus A_{Ma} is a $\mathbf{so}(3,\mathsf{C})$ connection acting on selfdual tensors $T_{AB} = J_{aAB}T_a$ as

$$D_M T_a = \partial_M T_a + \varepsilon_{abc}A_{Mb}T_c \tag{1.11}$$

and its field strength is given by

$$F_{MNa}[A] = \partial_M A_{Na} - \partial_N A_{Ma} + \varepsilon_{abc}A_{Mb}A_{Nc}. \tag{1.12}$$

It follows from (1.5) that this field strength is related to the curvature R_{MNAB} by

$$F_{MNa} = J_{aAB}R_{MN}{}^{AB}, \qquad F^*_{MNa} = J^*_{aAB}R_{MN}{}^{AB}, \tag{1.13}$$

or

$$R_{MNAB} = J_{aAB}F_{MNa} + J^*_{aAB}F^*_{MNa}. \tag{1.14}$$

With the last formula the Einstein-Hilbert action splits into a part holomorphic in A and its conjugate. So after some algebra, which uses the self-duality of J, L'_{EH} becomes the real part of

$$L_{\mathrm{EH}} = -\tfrac{\mathrm{i}}{2}\varepsilon^{MNPQ}\,E_M{}^A E_N{}^B\,J_{aAB}F_{PQa}[A]. \tag{1.15}$$

As in [4] we represent the gravitinos as complex 2-component Graßmann valued spinors, transforming in the selfdual complexified $\mathbf{su}(2)$ representation of the Lorentz group. But we will not write out the spinor indices explicitly. Instead, we introduce the four hermitian two by two matrices σ_A with

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.16}$$

Matrices with curved indices are obtained by multiplication with a vierbein: $\sigma_M = E_M{}^A \sigma_A$. Note that the algebra of these matrices is given by the J-symbols

$$\sigma_A \sigma_a = 2\mathrm{i} J_{aAB} \sigma^B, \qquad \sigma_a \sigma_A = -2\mathrm{i} J^*_{aAB} \sigma^B. \tag{1.17}$$

The covariant derivative of a spinor χ then reads

$$D_M \chi = \partial_M \chi - \tfrac{\mathrm{i}}{2} A_{Ma}\, \sigma_a \chi. \tag{1.18}$$

A Dirac conjugate spinor, which is defined by $\bar\chi = \chi^\dagger(\mathrm{i}\sigma_0) = \mathrm{i}\chi^\dagger$, transforms under the complex conjugate or antiselfdual representation and its covariant derivative is

$$D_M \bar\chi = \partial_M \bar\chi + \tfrac{\mathrm{i}}{2} A^*_{Ma}\, \bar\chi \sigma_a. \tag{1.19}$$

Using the properties of J it is straightforward to show that $\mathrm{i}\bar\chi\sigma_A\chi$ is real and transforms as a vector. Note that $\bar\chi\chi$ is not a scalar but the 0-component of a vector.

With these definitions we see immediately that also the Rarita Schwinger action splits into a part holomorphic in A and its conjugate, thus L'_{RS} is the real part of

$$L_{\mathrm{RS}} = 2\mathrm{i}\varepsilon^{MNPQ}\, \bar\psi_M \sigma_N D_P \psi_Q. \tag{1.20}$$

The total action I' is given as the real part of $I[E, A, \psi, \bar\psi] = \int \mathrm{d}^4x\, L_{\mathrm{EH}} + L_{\mathrm{RS}}$. As this is a holomorphic function of A, the equations of motion for A are the same as those for Ω in $I'[E, \Omega, \psi, \bar\psi]$, which directly follows from the Cauchy Riemann differential equations for holomorphic functions. As shown in [4], the complex action becomes real (up to a total derivative) if the equations of motion for A_{Ma} are satisfied and thus I and I' imply the same equation for all fields. Therefore we can take the complex action as starting point for the canonical quantization.

The metric representation

Though our lagrangian

$$L = \mathrm{i}\varepsilon^{MNPQ}\big(-\tfrac{1}{2} E_M{}^A E_N{}^B J_{aAB} F_{PQa} + 2\,\bar\psi_M \sigma_N D_P \psi_Q\big) \tag{1.21}$$

is written in terms of Ashtekar's variables, it is still possible to reproduce the metric (or vierbein) representation of D'Eath [3]. With the help of Ashtekar's

variables and the complex action L, however, the derivation of the constraint operators becomes considerably simpler. Thus we will show this construction briefly.

As we shall not use the variables A_{Ma} as canonical variables, we use 1.5 order formalism here, i.e. A_{Ma} is a function of the other fields ${E_M}^A$ and ψ_M, which is defined by its equations of motion. We already know that these equations are the same as those for the real lagrangian L'. The solutions are

$$A_{Ma}[E,\psi] = J_{aAB}\Omega_M{}^{AB}, \tag{1.22}$$

where Ω_{MAB} is implicitly given by

$$\partial_{[M}{E_{N]}}^A + \Omega_{[M}{}^A{}_B{E_{N]}}^B - \bar{\psi}_{[M}\sigma^A\psi_{N]} = 0. \tag{1.23}$$

The last equation is just the usual torsion equation obtained by differentiating L' with respect to Ω_{MAB}.

We now split the space time spanned by the coordinates $M = \mathrm{t},\mathrm{x},\mathrm{y},\mathrm{z}$ into a spatial hypersurface spanned by $m = \mathrm{x},\mathrm{y},\mathrm{z}$ and a coordinate t which serves as the canonical time variable and which we assume to be a globally defined coordinate. By inserting this into the lagrangians we obtain

$$\begin{aligned}
L_{\mathrm{EH}} &= -\mathrm{i}\varepsilon^{mnp}{E_m}^A{E_n}^B J_{aAB}F_{\mathrm{t}pa} - \mathrm{i}\varepsilon^{mnp}{E_\mathrm{t}}^A{E_m}^B J_{aAB}F_{npa} \\
&= 2\mathrm{i}\varepsilon^{mnp}\partial_\mathrm{t}{E_m}^A\,{E_n}^B J_{aAB}A_{pa} \\
&\quad - \mathrm{i}\varepsilon^{mnp}D_p({E_m}^A{E_n}^B J_{aAB})\,A_{\mathrm{t}a} - \mathrm{i}\varepsilon^{mnp}{E_\mathrm{t}}^A{E_m}^B J_{aAB}F_{npa} \\
L_{\mathrm{RS}} &= 2\mathrm{i}\varepsilon^{mnp}(\bar{\psi}_\mathrm{t}\sigma_m D_n\psi_p - \bar{\psi}_m\sigma_\mathrm{t}D_n\psi_p \\
&\qquad + \bar{\psi}_m\sigma_n D_\mathrm{t}\psi_p - \bar{\psi}_m\sigma_n D_p\psi_\mathrm{t}).
\end{aligned} \tag{1.24}$$

Here we have integrated by parts the Einstein Hilbert term to get a lagrangian without second order time derivatives. Note that A_{Ma} is a function of ${E_M}^A$ and ψ_M and the derivatives of ${E_M}^A$. As already mentioned, the imaginary part of L is a total derivative, thus in fact we have a real action. A general discussion of the canonical formalism for a lagrangian of this type is given in [12].

We can read off the momenta of ${E_m}^A$ and ψ_m. They are

$$P_A{}^m = \frac{\delta L}{\delta(\partial_\mathrm{t}{E_m}^A)} = 2\mathrm{i}\varepsilon^{mnp}J_{aAB}{E_n}^B A_{pa},$$

$$\bar{\pi}^m = \frac{\delta L}{\delta(\partial_{\rm t}\psi_m)} = -2{\rm i}\varepsilon^{mnp}\,\bar{\psi}_n\sigma_p. \tag{1.25}$$

Spinor derivatives always act from the left, which produces the extra sign in the equation for $\bar{\pi}^m$. As there are no 'mixing momenta', i.e. the canonical variables split into configuration variables ${E_m}^A$ and ψ_m and momentum variables ${P_A}^m$ and $\bar{\pi}^m$, there is no need to compute dirac brackets. Instead, we can read off the correct brackets directly from

$$\{{E_m}^A, {P_B}^n\} = \delta^A_B\delta^n_m, \qquad \{\psi_m, \bar{\pi}^n\} = -\delta^n_m\mathbf{1}. \tag{1.26}$$

Note the sign of the fermionic Poisson bracket, which has to be chosen such that the brackets reproduce the correct equations of motion. $\mathbf{1}$ is the two by two unity matrix.

The constraints are obtained directly from L by differentiating with respect to the Lagrange multipliers ${E_{\rm t}}^A$, $\psi_{\rm t}$ and $\bar{\psi}_{\rm t}$. They are

$$\begin{aligned}
\mathcal{H}_A &= -{\rm i}\varepsilon^{mnp}{E_m}^B J_{aAB}F_{npa} - 2{\rm i}\varepsilon^{mnp}\,\bar{\psi}_m\sigma_A D_n\psi_p,\\
\mathcal{S} &= 2{\rm i}\varepsilon^{mnp}\sigma_m D_n\psi_p,\\
\bar{\mathcal{S}} &= 2{\rm i}\varepsilon^{mnp}D_m(\bar{\psi}_n\sigma_p),
\end{aligned} \tag{1.27}$$

where $\mathcal{H}_A$ is the combined Wheeler DeWitt constraint together with the generators of spatial diffeomorphism. This combination is useful in supergravity because it is just the commutator of the supersymmetry constraints $\mathcal{S}$ and $\bar{\mathcal{S}}$.

As A_{ma} appears here, they are rather complicated functions of the canonical fields. Note, however, that all Lorentz covariant derivatives contain A_{ma} and not its conjugate, because ψ_m as well as $\bar{\psi}_m\sigma_n$ transform under the self-dual representation of the Lorentz group.

An additional constraint follows from the definition of ${P_m}^A$ as a function of the velocities $\partial_t{E_m}^A$ (which are implicit in A_{pa} in (1.25)). Of course, this constraint is just the equation of motion for $A_{{\rm t}a}$, i.e. the t-component of the torsion equation. The derivative of L with respect to $A_{{\rm t}a}$ is

$$-{\rm i}\varepsilon^{mnp}D_m({E_n}^A{E_p}^B J_{aAB}) + \varepsilon^{mnp}\,\bar{\psi}_m\sigma_n\sigma_a\psi_p = 0 \tag{1.28}$$

Writing out the covariant derivative explicitly and using (1.10) we get the Lorentz constraint

$$\mathcal{L}_a = -{\rm i}\varepsilon^{mnp}\partial_m({E_n}^A{E_p}^B J_{aAB}) - {J_{aA}}^B{E_m}^A{P_B}^m + \tfrac{\rm i}{2}\bar{\pi}^m\sigma_a\psi_p \tag{1.29}$$

To simplify the notation, we now introduce a 'selfdual densitized dreibein' defined by

$$\tilde{e}_a{}^p = -\varepsilon^{mnp} E_m{}^A E_n{}^B J_{aAB}. \tag{1.30}$$

Note that this is complex and obeys certain reality conditions which we are not interested in. Its determinant $\tilde{e} = \det \tilde{e}_a{}^m$ is given by the determinant of the three metric $g_{mn} = E_m{}^A E_{nA}$, thus $\tilde{e}_a{}^m$ can be inverted and divided by the determinant to give an inverse dreibein e_{am} which defines the spatial metric via $g_{mn} = e_{ma} e_{na}$. A straightforward calculation shows that these g_{mn} are in fact the spatial components of the four metric[1] $G_{mn} = E_{mA} E_n{}^A$. The dreibein also defines a spatial $\mathbf{so}(3, \mathbb{C})$ spin connection ω_{ma} by

$$\partial_{[m} e_{n]a} + \varepsilon_{abc} \omega_{[mb} e_{n]c} = \nabla_{[m} e_{n]a} = 0, \tag{1.31}$$

where ∇ denotes the full covariant derivative on the spatial hypersurface with respect to the dreibein e_{ma}. As all derivatives appearing here are antisymmetric the Christoffel connection can be omitted. We can also define the full covariant derivative of the spinor ψ_n and the momentum $\bar{\pi}^m$, both transforming in the selfdual representation. The combinations not including the Christoffel connection are (note that $\bar{\pi}^m$ is a density of weight 1)

$$\nabla_{[m} \psi_{n]} = \partial_{[m} \psi_{n]} - \tfrac{\mathrm{i}}{2} \omega_{[ma}\, \sigma_a \psi_{n]}, \qquad \nabla_m \bar{\pi}^m = \partial_m \bar{\pi}^m + \tfrac{\mathrm{i}}{2} \omega_{ma}\, \bar{\pi}^m \sigma_a. \tag{1.32}$$

Using this the Lorentz constraint can be written as

$$\mathcal{L}_a = \mathrm{i} \partial_m \tilde{e}_a{}^m - J_{aA}{}^B E_m{}^A P_B{}^m + \tfrac{\mathrm{i}}{2}\, \bar{\pi}^m \sigma_a \psi_m \tag{1.33}$$

We now want to quantize the theory in the E-ψ-representation. The wave functional Ψ thus depends on $E_m{}^A$ and ψ_m, and the operators have to satisfy

$$[\widehat{P_B{}^n}, \widehat{E_m{}^A}] = \mathrm{i}\hbar \delta^A_B \delta^n_m, \qquad [\widehat{\psi_m}, \widehat{\bar{\pi}^n}] = \mathrm{i}\hbar \delta^n_m \mathbf{1}. \tag{1.34}$$

The simplest choice for $\hat{P}$ and $\hat{\bar{\pi}}$ would be $\mathrm{i}\hbar\delta/\delta E$ and $\mathrm{i}\hbar\delta/\delta\psi$, respectively. But because of the $\partial_m \tilde{e}_a{}^m$ term in $\mathcal{L}_a$, which then becomes a multiplicator, the Lorentz constraint would fail to generate proper Lorentz transformations on the wave functional. To avoid this, one has to start from a more

[1] If one uses a gauge fixed vierbein with $E_m{}^0 = 0$, i.e. the timelike unit covector pointing along the t axis, the selfdual dreibein becomes real and equal to the spacelike part of the vierbein.

general representation given by

$$\widehat{P_A{}^m} = \mathrm{i}\hbar\frac{\delta}{\delta E_m{}^A} - \mathrm{i}\frac{\delta G[E]}{\delta E_m{}^A}, \qquad \widehat{\tilde{\pi}_{m\alpha}} = \mathrm{i}\hbar\frac{\delta}{\delta\psi_{m\alpha}}, \tag{1.35}$$

where $G[E]$ is a functional of the fields $E_m{}^A$ and $\alpha = 1, 2$ denote the spinor index. We will write out the spinor index only if its contraction is not simply given by matrix multiplication. With this representation the quantized Lorentz constraint reads

$$\widehat{\mathcal{L}_a} = \mathrm{i}\partial_m\tilde{e}_a{}^m - \mathrm{i}J_{aA}{}^B E_m{}^A\Big(\hbar\frac{\delta}{\delta E_m{}^B} - \frac{\delta G[E]}{\delta E_m{}^B}\Big) + \tfrac{1}{2}\hbar\,(\sigma_a\psi_m)_\alpha\frac{\delta}{\delta\psi_{m\alpha}}. \tag{1.36}$$

Now assume that G is given as a function of the $\tilde{e}_m{}^a$. Then we have

$$\begin{aligned}\mathrm{i}J_{aA}{}^B E_m{}^A\frac{\delta G}{\delta E_m{}^B} &= -2\mathrm{i}\varepsilon^{mnp}J_{aA}{}^B J_{bBC}E_m{}^A E_n{}^C\frac{\delta G}{\delta\tilde{e}_b{}^p}\\ &= -\mathrm{i}\varepsilon^{mnp}\varepsilon_{abc}J_{cAC}E_m{}^A E_n{}^C\frac{\delta G}{\delta\tilde{e}_b{}^p}\\ &= \mathrm{i}\varepsilon_{abc}\tilde{e}_c{}^m\frac{\delta G}{\delta\tilde{e}_b{}^m}.\end{aligned} \tag{1.37}$$

This obviously cancels against the $\partial_m\tilde{e}_a{}^m$ term if the derivative of G gives the spatial spin connection ω_{mb}. Such a functional, however, is well known and serves as a generating functional for the canonical transformation from the metric to the connection representation [13, 14]. The functional has the simple form

$$G[E] = \tfrac{1}{2}\int \mathrm{d}^3x\,\varepsilon^{mnp}e_{ma}\partial_n e_{pa}, \tag{1.38}$$

however, the dependence of G on $\tilde{e}_a{}^m$ and thus $E_m{}^A$ is highly nonlinear because of the inverse dreibein e_{ma} in its definition.

The explicit operator for $P_A{}^m$ is now given by

$$\widehat{P_A{}^m} = \mathrm{i}\hbar\,\frac{\delta}{\delta E_m{}^A} + 2\mathrm{i}\varepsilon^{mnp}J_{aAB}E_n{}^B\omega_{pa}. \tag{1.39}$$

It is therefore useful to introduce the quantity

$$Q_A{}^m = 2\mathrm{i}\varepsilon^{mnp}J_{aAB}E_n{}^B(A_{pa} - \omega_{pa}), \tag{1.40}$$

which is represented by the quantum operator

$$\widehat{Q_a{}^m} = \mathrm{i}\hbar \frac{\delta}{\delta E_m{}^A}. \tag{1.41}$$

The Lorentz constraint now generates (selfdual) Lorentz transformations on the wave functional:

$$\widehat{\mathcal{L}_a} = -\mathrm{i}\hbar J_a{}_A{}^B E_m{}^A \frac{\delta}{\delta E_m{}^B} + \tfrac{1}{2}\hbar \left(\sigma_a \psi_m\right)_\alpha \frac{\delta}{\delta \psi_{m\alpha}}. \tag{1.42}$$

To obtain a real constraint $\mathcal{L}_A{}^B$ in the SO(3, 1) representation of the Lorentz group, one has to compute the real part of $\mathcal{L}_a J_a{}_A{}^B$. As $P_A{}^m$ itself is not real (because of the complex lagrangian), this becomes a more involved but straightforward calculation. The result is, as expected,

$$\widehat{\mathcal{L}_A{}^B} = -\mathrm{i}\hbar E_m{}^{[B} \frac{\delta}{\delta E_m{}^{A]}} + \tfrac{1}{2} J_a{}_A{}^B \, \hbar \left(\sigma_a \psi_m\right)_\alpha \frac{\delta}{\delta \psi_{m\alpha}}. \tag{1.43}$$

The supersymmetry constraints of [3] are now easily reproduced. For $\mathcal{S}$ we get

$$\begin{aligned}
\mathcal{S} &= 2\mathrm{i}\varepsilon^{mnp}\, \sigma_m D_n \psi_p \\
&= 2\mathrm{i}\varepsilon^{mnp}\, \sigma_m \nabla_n \psi_p + \varepsilon^{mnp}\, \sigma_m \sigma_a \psi_p \left(A_{na} - \omega_{na}\right) \\
&= 2\mathrm{i}\varepsilon^{mnp}\, \sigma_m \nabla_n \psi_p + 2\mathrm{i}\varepsilon^{mnp} E_m{}^A J_{aAB}\, \sigma^B \psi_p \left(A_{na} - \omega_{na}\right) \\
&= 2\mathrm{i}\varepsilon^{mnp}\, \sigma_m \nabla_n \psi_p - \sigma^A \psi_m \, Q_A{}^m.
\end{aligned} \tag{1.44}$$

and the same calculation for $\bar{\mathcal{S}}$ yields

$$\begin{aligned}
\bar{\mathcal{S}} &= 2\mathrm{i}\varepsilon^{mnp}\, D_m(\bar{\psi}_n \sigma_p) \\
&= -\nabla_m \bar{\pi}^m + \bar{\psi}_m \sigma^A \, Q_A{}^m.
\end{aligned} \tag{1.45}$$

The quantum operator for $\mathcal{S}$ is obtained simply by inserting the operator for $Q_A{}^m$:

$$\widehat{\mathcal{S}} = 2\mathrm{i}\varepsilon^{mnp}\, \sigma_m \nabla_n \psi_p - \mathrm{i}\hbar \sigma^A \psi_m \frac{\delta}{\delta E_m{}^A}, \tag{1.46}$$

which is exactly the same as the operator given in equation (4.6) of [3]. To write down the operator for $\bar{\mathcal{S}}$ we have to give an explicit representation for

$\bar{\psi}_m$, because $\bar{\mathcal{S}}$ does not depend on $\bar{\psi}_m$ via $\bar{\pi}^m$ only. We have to invert the relation (1.25), thus we have to find a matrix D_{mq} with

$$\varepsilon^{mnp}\,\sigma_p D_{mq} = \delta^n_q \mathbf{1}, \tag{1.47}$$

which then gives us (see [3] for an explicit expression for D_{mn})

$$\bar{\psi}_m = \tfrac{\mathrm{i}}{2}\bar{\pi}^n D_{nm}, \qquad \hat{\bar{\psi}}_m = -\tfrac{1}{2}\hbar\frac{\delta}{\delta\psi_n}\,D_{nm}. \tag{1.48}$$

Inserting this we can also reproduce the conjugate constraint (equation (4.10) of [3]), which reads

$$\hat{\bar{\mathcal{S}}} = -\mathrm{i}\hbar\nabla_m\Big(\frac{\delta}{\delta\psi_m}\Big) - \tfrac{\mathrm{i}}{2}\hbar^2\,\frac{\delta}{\delta\psi_n}\,D_{nm}\sigma^A\,\frac{\delta}{\delta {E_n}^A}. \tag{1.49}$$

The representations of the remaining constraints $\mathcal{H}_A$ are rather cumbersome and we will not give them here. $\mathcal{H}_A\sigma^A$ is the bracket of $\mathcal{S}$ with $\bar{\mathcal{S}}$. Thus every solution to the supersymmetry and Lorentz constraint is always a solution to all constraints, if we define the operator ordering for $\mathcal{H}_A$ by

$$\widehat{\mathcal{H}_A}\sigma^A = [\,\hat{\mathcal{S}}\,,\hat{\bar{\mathcal{S}}}\,] \tag{1.50}$$

There are some ansätze for solutions of the supersymmetry constraints in this representation [6, 7, 8, 9], but so far no exact solution is known. See, however, [8], where it is shown that the constraints cannot be solved by a purely bosonic state, i.e. a wave functional that does not depend on the fermionic variables.

It is reasonable that there are no purely bosonic states, because the wave functional should be invariant under supersymmetry transformations and thus it cannot depend on the bosonic configuration only, as long as there are no bosonic fields that are invariant under supersymmetry. For the densitized constraints in the connection representation, however, we will find purely bosonic solutions and this may be interpreted as a first hint that there is 'something wrong' with the loop states.

The connection representation

Again we start with the lagrangian

$$L = \mathrm{i}\varepsilon^{MNPQ}(\,-\tfrac{1}{2}{E_M}^A{E_N}^B J_{aAB}F_{PQa} + 2\,\bar{\psi}_M\sigma_N D_P\psi_Q\,), \tag{1.51}$$

but now we will use first order formalism. Thus the independent variables are ${E_M}^A$, A_{Ma}, $\bar{\psi}_M$ and ψ_M. The momenta of A_{ma} and ψ_m are

$$\frac{\delta L}{\delta(\partial_{\rm t} A_{ma})} = -{\rm i}\varepsilon^{mnp} {E_n}^A {E_p}^B J_{aAB} = {\rm i}\tilde{e}_a{}^m,$$
$$\frac{\delta L}{\delta(\partial_{\rm t} \psi_m)} = -2{\rm i}\varepsilon^{mnp}\bar{\psi}_n\sigma_p = \bar{\pi}^m, \tag{1.52}$$

leading to the Poisson brackets

$$\{\tilde{e}_a{}^m, A_{nb}\} = {\rm i}\eta_{ab}\delta^m_n, \qquad \{\psi_m, \bar{\pi}^n\} = -\delta^n_m\,\mathbf{1}. \tag{1.53}$$

The constraints are the same as before, except that now the Lorentz generator comes out as the derivative of L with respect to the Lagrange multipliers $A_{{\rm t}a}$:

$$\begin{aligned}
\mathcal{H}_A &= -{\rm i}\varepsilon^{mnp} {E_m}^B J_{aAB} F_{npa} - 2{\rm i}\varepsilon^{mnp}\,\bar{\psi}_m\sigma_A D_n\psi_p,\\
\mathcal{L}_a &= {\rm i}D_m\tilde{e}_a{}^m + \tfrac{\rm i}{2}\bar{\pi}^m\sigma_a\psi_m,\\
\mathcal{S} &= 2{\rm i}\varepsilon^{mnp}\sigma_m D_n\psi_p,\\
\bar{\mathcal{S}} &= -D_m\bar{\pi}^m.
\end{aligned} \tag{1.54}$$

In Ashtekar's representation the variables A_{ma} should appear as multiplication operators and $\tilde{e}_a{}^m$ is replaced by a differential operator. To quantize the constraints it is therefore necessary to write them as polynomials in $\tilde{e}_a{}^m$. $\mathcal{L}'_a = \mathcal{L}_a$ and $\bar{\mathcal{S}}' = \bar{\mathcal{S}}$ are polynomials already, so only $\mathcal{S}$ has to be multiplied by a suitable function of ${E_n}^A$ to obtain a polynomial $\mathcal{S}'$. The hamiltonian constraint is then obtained by taking the bracket of $\bar{\mathcal{S}}'$ with $\mathcal{S}'$, which, of course, gives again a polynomial. It turns out that we have to define

$$\mathcal{S}' = \tfrac{1}{2}\sigma^C\, E\sigma^{\rm t}\,\sigma_C\,\mathcal{S}\,, \tag{1.55}$$

where $E\sigma^{\rm t}$ is the 'densitized upper t component' of the curved Pauli matrices, which depends on the 'lower spatial components' ${E_m}^A$ of the vierbein only. Explicitly we have

$$E\sigma^{\rm t} = \tfrac{1}{6}\varepsilon^{mnp}\varepsilon_{ABCD}\,\sigma^A\,{E_m}^B{E_n}^C{E_p}^D. \tag{1.56}$$

Note that this is the same transformation which was found to make the 2+1 dimensional matter coupled supersymmetry constraint in [15] a polynomial in the canonically conjugate variables.

With the help of the polynomial $\mathcal{S}'$ we now define a new combination of the hamiltonian and diffeomorphism constraints as a matrix $\mathcal{K}' = \{\mathcal{S}', \bar{\mathcal{S}}'\}$. It is then equal to $\mathcal{H}_A \sigma^A$, multiplied by the matrix in (1.55), and up to a term proportional to $\mathcal{S}$. So the complete set of constraints is

$$\begin{aligned}
\mathcal{K}' &= \tfrac{1}{2}\mathrm{i} F_{mna}\, \tilde{\sigma}^m \tilde{\sigma}^n \sigma_a - 2\mathrm{i}\varepsilon_{abc}\, \sigma_a D_{[m}\psi_{n]}\, \tilde{e}_b{}^m\, \bar{\pi}^n \sigma_c, \\
\mathcal{L}'_a &= \mathrm{i} D_m \tilde{e}_a{}^m + \tfrac{\mathrm{i}}{2} \bar{\pi}^m \sigma_a \psi_m, \\
\mathcal{S}' &= -2\tilde{\sigma}^m \tilde{\sigma}^n D_{[m}\psi_{n]}, \\
\bar{\mathcal{S}}' &= -D_m \bar{\pi}^m,
\end{aligned} \tag{1.57}$$

where $\tilde{\sigma}^m = \tilde{e}_a{}^m \sigma_a$. Note that this is not the four dimensional curved Pauli matrix with upper index.

Again, to solve the quantized constraints, we only need to solve $\mathcal{L}'_a$, $\mathcal{S}'$ and $\bar{\mathcal{S}}'$. Choosing the representation

$$\widehat{\tilde{e}_a{}^m} = \hbar \frac{\delta}{\delta A_{ma}}, \qquad \widehat{\bar{\pi}^p} = \mathrm{i}\hbar \frac{\delta}{\delta \psi_m}, \tag{1.58}$$

they become exactly those given in [4]:

$$\begin{aligned}
\widehat{\mathcal{L}'_a} &= \mathrm{i}\hbar D_m \Big(\frac{\delta}{\delta A_{ma}}\Big) + \tfrac{1}{2}\hbar\, (\sigma_a \psi_m)_\alpha \frac{\delta}{\delta \psi_{m\alpha}}, \\
\widehat{\mathcal{S}'} &= -2\mathrm{i}\hbar^2 \varepsilon_{abc}\, \sigma_c D_m \psi_n \frac{\delta}{\delta A_{ma}} \frac{\delta}{\delta A_{nb}}, \\
\widehat{\bar{\mathcal{S}}'} &= -\mathrm{i}\hbar D_m \Big(\frac{\delta}{\delta \psi_m}\Big).
\end{aligned} \tag{1.59}$$

Given a wave functional $\Psi[A_{ma}, \psi_m]$, the constraints $\mathcal{L}'_a$ and $\bar{\mathcal{S}}'$ just generate local Lorentz and supersymmetry transformations, respectively. Thus they require Ψ to be invariant under $\delta A_{ma} = D_m \lambda_a$, $\delta \psi_m = \frac{\mathrm{i}}{2} \lambda^a \sigma_a \psi_m$ and under the chiral supersymmetry transformation $\delta \psi_m = D_m \epsilon$, which does not act on A_{ma}.

In contrast to the metric representation, there are now purely bosonic solutions to the constraints. In fact, they are just *the same* formal solutions as those found in [5] for bosonic gravity, namely the Wilson loop functionals

$$T_\eta = \mathrm{Tr}\, \mathcal{P} \exp \tfrac{\mathrm{i}}{2} \oint_\eta \mathrm{d}s\, \dot{\eta}^m(s)\, A_{ma}(\eta(s)) \sigma_a \tag{1.60}$$

where $\eta(s)$ is a smooth, non-intersecting loop in the spatial hypersurface, and $\dot{\eta}^m(s)$ denotes its tangent vector. As is well known, when computing the second derivative of T_η with respect to A_{ma} and A_{nb} one obtains an expression symmetric in m, n. Thus T_η is a solution to all the constraints, because the second derivative in $\mathcal{S}'$ is antisymmetrized, and T_η also solves $\mathcal{L}'_a$ and $\bar{\mathcal{S}}'$, because it is Lorentz invariant and does not depend on ψ_m.

There are some questions arising here. The first is: Where do these purely bosonic solutions come from? They were not present in the metric formalism. As was realized shortly after the discovery of the loop solutions for pure gravity, they are annihilated by the determinant of the spatial metric and therefore somehow represent states with singular metric [16]. So far this is not a serious problem, as Ashtekar's action and the constraints are polynomials in the canonical variables and are thus able to handle singular metrics.

But if the solutions represent states with singular metric, another problem arises: As we multiplied the constraints by $E\sigma^t$, which is a third order polynomial of the inverse vierbein $E_m{}^A$, it seems that these solutions are not solutions to the original constraints, but they are simply annihilated by this factor. Note that the factors of $\tilde{e}_a{}^m$ are ordered *to the right* in $\mathcal{S}'$, thus the extra factors act on the wave functional first.

A second problem is that these solutions also solve the constraints for pure gravity. If we drop all fermions from the constraints, we are left with $\mathcal{L}'_a$ and $\mathcal{K}'$ only, and $\mathcal{K}'$ again is proportional to the antisymmetrized second derivative with respect to A_{ma}. The solutions do not 'see' the factor F_{mna} in $\mathcal{K}'$, because they are already annihilated by $\tilde{e}_a{}^{[m}\tilde{e}_b{}^{n]}$. Thus we can replace the curvature by the Rarita Schwinger field strength $D_{[m}\psi_{n]}$ without changing the solutions, and this replaces $\mathcal{K}'$ by $\mathcal{S}'$, thus switches from pure gravity to supergravity.

This problem is somehow related to that arising when constructing loop solutions for matter coupled gravity or when adding a cosmological constant; e.g. there are extended Wilson loops solving the Wheeler DeWitt constraint of gravity coupled to scalar fields [17], which do not 'see' the mass of these fields, nor do the loop states depend on the cosmological constant. It seems that they are just artificially generated by the extra factors of the vierbein (or the dreibein in the gauge fixed version often used) and do not correspond to solutions of the Wheeler DeWitt equation in its original metric representation.

Obviously, this problem is closely related to the question how to treat singular metrics, because multiplication of an equation by a term that may become zero can change its solutions drastically. We will come to this problem again after discussing the three dimensional theory

There is still another feature of the loop solutions for supergravity. In the usual connection representation of pure gravity, the loops are annihilated by the Wheeler DeWitt constraint but not by the diffeomorphism constraints, simply because they are not invariant under spatial diffeomorphism. One usually solves this problem by defining a loop representation and requiring the wave functional depending on the knot class of the loop only, which then is invariant under diffeomorphisms.

But for supergravity we saw that T_η solves all constraints, though it is not invariant under diffeomorphisms. To make this problem more explicit, observe that the Wheeler DeWitt and diffeomorphism constraints are both obtained from $\mathcal{H}_A$: The diffeomorphism constraint is given by $\mathcal{D}'_m = E_m{}^A \mathcal{H}_A$ and the Wheeler DeWitt constraint is $\mathcal{C}' = EE^{\mathrm{t}A}\mathcal{H}_A$.

On the other hand, if we are allowed to obtain the Wheeler DeWitt operator by multiplying $\mathcal{H}_A$ by a third order function of the vierbein, there is no reason why this should not be allowed for the diffeomorphism constraint. This is exactly what we have done when using (1.59) as the constraint algebra for supergravity. Doing the same for pure gravity, the loop functionals become solutions to *all* constraints, too: there is no need to consider functionals on knot classes only in the loop representation, as every arbitrary function which has support on smooth loops only is a solution.

We will discuss all these problems in more detail in the last section.

2 N=2 supergravity in three dimensions

Let us now discuss the three dimensional theory, because some of the problems concerning the loop solutions can be made more explicit here as the theory can be solved exactly. We use the same notation as in [7]. The bosonic variables for N=2 supergravity are the dreibein $e_\mu{}^a$, $a = 0, 1, 2$ the flat and $\mu = \mathrm{t}, \mathrm{x}, \mathrm{y}$ the curved index, and the spin connection $A_{\mu a}$ with curvature

$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - \varepsilon_{abc} A_\mu{}^b A_\nu{}^c, \tag{2.1}$$

where the flat indices are raised by the lorentzian metric $\eta_{ab} = \mathrm{diag}(-,+,+)$, and $\varepsilon^{012} = -\varepsilon_{012} = 1$. The gravitinos are again represented by 2-component complex spinors ψ_μ. As the Lorentz group $\mathrm{SO}(2,1)$ has a real spinor representation $\mathrm{SL}(2,\mathsf{R})$, we choose real gamma matrices

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

obeying

$$\gamma_a \gamma_b = \eta_{ab} \mathbf{1} - \varepsilon_{abc} \gamma^c. \tag{2.3}$$

The Lorentz covariant derivative of a vector v_a, a spinor χ and its conjugate $\bar{\chi} = \chi^\dagger(\mathrm{i}\gamma_0)$ reads

$$\begin{aligned} D_\mu v_a &= \partial_\mu v_a - \varepsilon_{abc} A_\mu{}^b v^c, \\ D_\mu \chi &= \partial_\mu \chi + \tfrac{1}{2} A_\mu{}^a \gamma_a \chi, \\ D_\mu \bar{\chi} &= \partial_\mu \bar{\chi} - \tfrac{1}{2} A_\mu{}^a \bar{\chi} \gamma_a. \end{aligned} \tag{2.4}$$

The lagrangian is

$$L = \tfrac{1}{2} \varepsilon^{\mu\nu\rho} e_\mu{}^a F_{\nu\rho a} + 2\varepsilon^{\mu\nu\rho} \bar{\psi}_\mu D_\nu \psi_\rho. \tag{2.5}$$

Splitting space time into space and time, i.e. splitting the index μ into t and $i = \mathrm{x}, \mathrm{y}$, the configuration variables become A_{ia} and ψ_i with their momenta

$$\frac{\delta L}{\delta(\partial_{\mathrm{t}} A_{ia})} = \varepsilon^{ij} e_j{}^a, \qquad \frac{\delta L}{\delta(\partial_{\mathrm{t}} \psi_i)} = -2\varepsilon^{ij} \bar{\psi}_j. \tag{2.6}$$

where $\varepsilon^{\mathrm{xy}} = \varepsilon_{\mathrm{xy}} = 1$. The Poisson brackets read

$$\{A_{ia}, e_j{}^b\} = \varepsilon_{ij} \delta_a^b, \qquad \{\psi_i, \bar{\psi}_j\} = \tfrac{1}{2} \varepsilon_{ij} \, \mathbf{1}. \tag{2.7}$$

The lagrange multipliers $e_{\mathrm{t}}{}^a$, $A_{\mathrm{t}a}$, $\bar{\psi}_{\mathrm{t}}$ and ψ_{t} generate the constraints

$$\begin{aligned} \mathcal{H}_a &= \tfrac{1}{2} \varepsilon^{ij} F_{ija}, \\ \mathcal{L}_a &= \varepsilon^{ij} D_i e_{ja} - \varepsilon^{ij} \bar{\psi}_i \gamma_a \psi_j, \\ \mathcal{S} &= 2\varepsilon^{ij} D_i \psi_j, \\ \bar{\mathcal{S}} &= 2\varepsilon^{ij} D_i \bar{\psi}_j. \end{aligned} \tag{2.8}$$

Note that their structure is similar to (1.54), but they are much simpler, e.g. only $\mathcal{L}_a$ contains the dreibein. In particular, there are no operator ordering ambiguities and thus no anomalies in the algebra. In quantum theory it is sufficient to solve $\mathcal{L}_a$, $\mathcal{S}$ and $\bar{\mathcal{S}}$, as $\{\mathcal{S}, \bar{\mathcal{S}}\} = -\mathcal{H}_a\gamma^a$.

The connection representation

Here we briefly review the main results of [7]. The operators in the connection representation are

$$\widehat{e_i^a} = -\mathrm{i}\hbar\varepsilon_{ij}\frac{\delta}{\delta A_{ja}}, \qquad \hat{\bar{\psi}}_{i\alpha} = \tfrac{\mathrm{i}}{2}\hbar\varepsilon_{ij}\frac{\delta}{\delta\psi_{j\alpha}}. \tag{2.9}$$

The constraints split into a set of multiplication operators $\widehat{\mathcal{H}_a}$ and $\widehat{\mathcal{S}}$, and first order differential operators $\widehat{\mathcal{L}_a}$ and $\widehat{\bar{\mathcal{S}}}$, which generate local Lorentz and chiral supersymmetry transformations ($\delta\psi_i = D_i\epsilon$) on the wave functional $\Psi[A_{ia}, \psi_i]$.

The solution to the constraints is given as follows. $\widehat{\mathcal{H}_a}\Psi = 0$ together with $\widehat{\mathcal{S}}\,\Psi = 0$ implies that Ψ has support only on those fields A_{ia}, ψ_i which have vanishing curvature $F_{ija} = 0$ and vanishing supercurvature $D_{[i}\psi_{j]} = 0$. To simplify the notation we introduce a matrix valued connection field $A_i = \frac{1}{2}A_{ia}\gamma^a$. The covariant derivative of a spinor then reads $D_i\phi = \partial_i\phi + A_i\phi$.

The complete set of curvature free pairs A_i, ψ_i is parametrized by a $\mathrm{SL}(2,\mathsf{R})$ matrix field g and a spinor filed ϕ on the covering manifold of the spatial 2-surface, which are subject to certain relations between their values on points mapped to the same point of the 2-surface (see [7]). These conditions are in fact equivalent to the requirement that

$$A_i[g] = g^{-1}\partial_i g, \qquad \psi_i[g,\phi] = g^{-1}\partial_i\phi \tag{2.10}$$

are single valued on the 2-surface. Obviously, $A_i[g]$ and $\psi_i[g,\phi]$ are curvature free.

A complete set of solutions to $\widehat{\mathcal{H}_a}\Psi = 0$ and $\widehat{\mathcal{S}}\,\Psi = 0$ now reads

$$\Phi_{g,\phi}[A_i,\psi_i] = \prod_x \delta(g^{-1}\partial_i g - A_i)\prod_x \delta(\partial_i\phi - g\psi_i). \tag{2.11}$$

This is in fact an overcomplete set as e.g. g and $g_0 g$, where g_0 is a constant group element, or ϕ and $\phi + \phi_0$ with constant ϕ_0, produce the same wave

functional. Nevertheless, a wave function can now be written as

$$\Psi_F[A_i,\psi_i] = \int \mathrm{D}g\mathrm{D}\phi\, F[g,\phi]\, \Phi_{g,\phi}[A_i,\psi_i], \tag{2.12}$$

where the functional integral runs over all fields g and ϕ subject to the restrictions mentioned above, and $F[g,\phi]$ is an arbitrary functional. The measure we assume to be invariant under multiplication of g with an arbitrary matrix field $g \mapsto gh$ and under addition of a spinor field $\phi \mapsto \phi + \chi$.

The constraints $\widehat{\mathcal{L}_a}$ and $\widehat{\bar{\mathcal{S}}}$ now impose additional conditions on F: $\widehat{\mathcal{L}}\Psi = 0$ requires that Ψ is invariant under Lorentz transformations. As can be verified, the change of $\Phi_{g,\phi}$ under a Lorentz transformation with parameter $\lambda = \frac{1}{2}\lambda^a\gamma_a$, i.e.

$$\delta A_i = \partial_i\lambda - [\lambda, A_i], \qquad \delta\psi_i = -\lambda\psi_i, \tag{2.13}$$

can be compensated by changing g as $\delta g = g\lambda$. Similarly, a supersymmetry transformation generated by $\bar{\mathcal{S}}$ with parameter ϵ is given by

$$\delta\psi_i = D_i\epsilon, \tag{2.14}$$

and the variation of $\Phi_{g,\phi}$ can be compensated by $\delta\phi = g\epsilon$. So we have to require that $F[g,\phi]$ is invariant under Lorentz and supersymmetry transformations

$$\delta g = g\lambda, \qquad \delta\phi = g\epsilon \tag{2.15}$$

to obtain the full solution to all constraints.

For a 2-surface which is homeomorphic to R^2 these restrictions imply the F is constant, as every field configuration can be transformed into any other by (2.15). But remember that λ and ϵ are single valued fields on the 2-surface whereas g and ϕ are fields on the covering manifold. For nontrivial topologies there remain finitely many degrees of freedom for F, called the moduli, which cannot be gauged away by (2.15). For a more detailed discussion we again refer to [7] and references therein.

The metric representation

We will now show that the complete solution to all the constraints can also be given in the metric representation, and that it is equivalent to the connection representation as it should be, because it is a quantum theory with finitely many degrees of freedom and thus all representations should be equivalent.

The transformation to the metric (or dreibein) representation is obtained in two steps. First we introduce a 'pseudo' metric representation, which is simply the Fourrier transform of the connection representation. As in the four dimensional theory this is not the usual metric representation, because of an extra multiplicator term in the Lorentz constraint. The original metric representation is then obtained by an additional canonical transformation.

In the pseudo metric representation the operators are just given by interchanging the multiplication and differentiation operators. The wave functional now depends on ${e_i}^a$ and $\bar{\psi}_i$, the operators are

$$\widehat{A_{ia}} = -\mathrm{i}\hbar\varepsilon_{ij}\frac{\delta}{\delta {e_j}^a}, \qquad \widehat{\psi_{i\alpha}} = -\tfrac{\mathrm{i}}{2}\hbar\varepsilon_{ij}\frac{\delta}{\delta\bar{\psi}_{j\alpha}}, \tag{2.16}$$

and the constraints read

$$\begin{aligned}
\widehat{\mathcal{H}_a} &= \varepsilon^{ij}\partial_i\widehat{A_{ja}} - \tfrac{1}{2}\varepsilon^{ij}\varepsilon_{abc}\widehat{A_i{}^b}\widehat{A_j{}^c},\\
\widehat{\mathcal{L}_a} &= \varepsilon^{ij}\partial_i e_{ja} - \varepsilon^{ij}\varepsilon_{abc}{e_i}^b\widehat{A_j{}^c} - \varepsilon^{ij}\bar{\psi}_i\gamma_a\widehat{\psi_j},\\
\widehat{\mathcal{S}} &= 2\varepsilon^{ij}(\partial_i\widehat{\psi_j} + \widehat{A_i}\,\widehat{\psi_j}),\\
\widehat{\bar{\mathcal{S}}} &= 2\varepsilon^{ij}(\partial_i\bar{\psi}_j - \bar{\psi}_j\,\widehat{A_i}).
\end{aligned} \tag{2.17}$$

The complete solutions to these differential equations can now be given by Fourrier transforming the solutions for the connection representation. This procedure is rather simple because we just have to replace the δ-functions in $\Phi_{g,\phi}$ by the corresponding exponentials. We define

$$\Phi_{g,\phi}[{e_i}^a,\bar{\psi}_i] = \exp\Big(\frac{\mathrm{i}}{\hbar}\int \mathrm{d}^2x\ \varepsilon^{ij}\mathrm{Tr}(g^{-1}\partial_i g\gamma_j) - 2\varepsilon^{ij}\bar{\psi}_i g^{-1}\partial_j\phi\Big). \tag{2.18}$$

Then, using

$$\frac{\delta\Phi_{g,\phi}}{\delta {e_i}^a} = -\frac{\mathrm{i}}{\hbar}\varepsilon^{ij}\mathrm{Tr}(g^{-1}\partial_j g\gamma_a)\,\Phi_{g,\phi} \quad \Rightarrow \quad \widehat{A_i}\,\Phi_{g,\phi} = g^{-1}\partial_i g\,\Phi_{g,\phi} \tag{2.19}$$

and

$$\frac{\delta\Phi_{g,\phi}}{\delta\bar{\psi}_j} = -\frac{2\mathrm{i}}{\hbar}\varepsilon^{ij}g^{-1}\partial_j\phi\,\Phi_{g,\phi} \quad \Rightarrow \quad \widehat{\psi_i}\,\Phi_{g,\phi} = g^{-1}\partial_i\phi\,\Phi_{g,\Phi}, \tag{2.20}$$

one immediately finds that $\widehat{\mathcal{H}_a}\Phi = 0$ and $\widehat{\mathcal{S}}\,\Phi = 0$. A straightforward calculation then shows that acting on Φ with $\widehat{\mathcal{L}_a}$ or $\widehat{\bar{\mathcal{S}}}$ again gives a variation that can be absorbed by a suitable transformation of the type (2.15). As in the connection representation we define the state functional by

$$\Psi_F[e_i{}^a, \bar{\psi}_i] = \int \mathrm{D}g\mathrm{D}\phi\, F[g,\phi]\, \Phi_{g,\phi}[e_i{}^a, \bar{\psi}_i], \tag{2.21}$$

where F again is a functional that is invariant under the transformations (2.15). Thus every state in the connection representation corresponds to an equivalent state in the pseudo metric representation and vice versa.

Observe that the states are constructed such that the constraints, though they are second order differential operators, do not need any regularization. Of course, the regularization is 'hidden' in the definition of Ψ as a functional integral over the fields g and ϕ, whose measure has to be regulated somehow.

The final step from this pseudo metric to the usual dreibein representation is similar to the procedure used in the four dimensional theory to obtain a Lorentz constraint that generates proper Lorentz transformations on the wave functional. Note that though Ψ_F is a annihilated by $\widehat{\mathcal{L}_a}$, is is not invariant under Lorentz transformations.

To obtain an invariant state functional, we have to use another operator for the connection, i.e.

$$\widehat{A_{ia}} = -\mathrm{i}\hbar\varepsilon_{ij}\,\frac{\delta}{\delta e_j{}^a} + \varepsilon_{ij}\,\frac{\delta G[e]}{\delta e_j{}^a}. \tag{2.22}$$

Again, G has to be chosen such that its contribution to $\widehat{\mathcal{L}_a}$ cancels against the derivative of the dreibein:

$$\varepsilon_{abc} e_i{}^b \frac{\delta G[e]}{\delta e_{ic}} = \varepsilon^{ij}\partial_i e_{ja}. \tag{2.23}$$

A solution to this equation is

$$G[e] = \tfrac{1}{2}\int \mathrm{d}^3x\, \varepsilon^{ij} g^{kl} \varepsilon_{abc}\, e_i{}^a e_k{}^b\, \partial_j e_l{}^c, \tag{2.24}$$

where g^{kl} is the inverse of the two dimensional metric $g_{kl} = e_l{}^a e_{ka}$. The derivative of G with respect to the dreibein will be called ω_{ia}. Note that this can not be interpreted as the spin connection for which the dreibein

becomes covariantly constant: we are dealing here with a *drei*bein on a *two* dimensional surface, i.e. it is not invertable and the requirement that the torsion $\partial_{[i}e_{j]a} - \varepsilon_{abc}\omega_{[i}{}^{b}e_{j]}{}^{c}$ vanishes does not imply that the full covariant derivative of $e_i{}^a$ vanishes. The new operator for A_{ia} now becomes

$$\widehat{A_{ia}} = \omega_{ia} - \mathrm{i}\hbar\varepsilon_{ij}\,\frac{\delta}{\delta e_j{}^a}. \tag{2.25}$$

Inserting this into the constraints we obtain

$$\begin{aligned}
\widehat{\mathcal{L}_a} &= -\mathrm{i}\hbar\varepsilon_{abc}e_i{}^b\,\frac{\delta}{\delta e_{ic}} - \tfrac{1}{2}\hbar\bar{\psi}_i\gamma_a\,\frac{\delta}{\delta\bar{\psi}_i},\\
\widehat{\mathcal{S}} &= \mathrm{i}\hbar\nabla_i\Big(\frac{\delta}{\delta\bar{\psi}_i}\Big) - \tfrac{1}{2}\hbar^2\varepsilon_{ij}\gamma_a\,\frac{\delta}{\delta\bar{\psi}_i}\,\frac{\delta}{\delta e_j{}^a},\\
\widehat{\bar{\mathcal{S}}} &= 2\varepsilon^{ij}\,\nabla_i\bar{\psi}_j + \mathrm{i}\hbar\,\bar{\psi}_i\gamma^a\,\frac{\delta}{\delta e_i{}^a},
\end{aligned} \tag{2.26}$$

where ∇_i again denotes the covariant derivative with respect to ω_{ia}, i.e.

$$\nabla_{[i}\bar{\psi}_{j]} = \partial_{[i}\bar{\psi}_{j]} - \tfrac{1}{2}\omega_{[ia}\,\bar{\psi}_{j]}\gamma^a \tag{2.27}$$

For the Wheeler DeWitt operator we get a slightly cumbersome expression which we will not give explicitly here: it is still the commutator of $\widehat{\bar{\mathcal{S}}}$ with $\widehat{\mathcal{S}}$. The constraints are now formally the same as those for the four dimensional theory, except that they do not contain the inverse vierbein

The complete solutions to these constraints are now given by multiplying the old solutions with the exponential of the functional G, i.e. replacing Φ by

$$\tilde{\Phi}_{g,\phi}[e_i{}^a,\bar{\psi}_i] = \exp\Big(-\frac{\mathrm{i}}{\hbar}G[e_i{}^a]\Big)\,\Phi_{g,\phi}[e_i{}^a,\bar{\psi}_i]. \tag{2.28}$$

Inserting $\tilde{\Phi}$ instead of Φ into (2.21) then gives the solution to the constraints in the metric representation.

2.1 The densitized connection representation

The connection representation for the three dimensional theory discussed above is not directly related to that of the four dimensional theory. In particular, because of the simple structure of the constraints, it is not necessary

to multiply the constraints by extra factors of the dreibein to obtain polynomial expressions.

The situation is different when we consider, e.g., the dimensionally reduced version of the four dimensional N=1 theory, which contains additional matter fields, but whose graviton and gravitino action is the same as that of the N=2 theory discussed here. In [15] we found that for the matter coupled theory we also have to multiply one of the supersymmetry constraints by an extra factor of $e\gamma^{\mathrm{t}} = -\frac{1}{2}\varepsilon^{ij}\varepsilon_{abc}\gamma^a e_i{}^b e_j{}^c$ to get polynomial constraints, which directly corresponds to the definition (1.55).

Thus the question arising here is what happens if we start with the constraints given in [15] for the matter coupled theory and just drop all the matter terms. We then obtain

$$\begin{aligned}
\mathcal{L}'_a &= \varepsilon^{ij} D_i e_{ja} - \varepsilon^{ij}\bar{\psi}_i\gamma_a\psi_j, \\
\mathcal{S}' &= -\varepsilon^{ij}\varepsilon^{kl}\varepsilon_{abc} e_k{}^a e_l{}^b\,\gamma^c D_i\psi_j, \\
\bar{\mathcal{S}}' &= 2\varepsilon^{ij} D_i\bar{\psi}_j.
\end{aligned} \tag{2.29}$$

They are just the three dimensional versions of (1.57) and can be constructed in the same way, i.e. multiplying $\mathcal{S}$ by $e\gamma^{\mathrm{t}}$ and then defining $\mathcal{H}'_a$ as the bracket of $\bar{\mathcal{S}}'$ with $\mathcal{S}'$. A priori there is no reason why these constraints could not be regarded as the canonical constraints of 2+1 supergravity, if (1.57) describes 3+1 supergravity. They are nothing but their three dimensional counterparts, and assumed we didn't know that there is a much simpler expression for them, we had to use them to quantize the 2+1 theory.

The quantized versions of the Lorentz and supersymmetry constraints now become

$$\begin{aligned}
\widehat{\mathcal{L}'_a} &= \mathrm{i}\hbar D_i\Big(\frac{\delta}{\delta A_{ia}}\Big) - \tfrac{\mathrm{i}}{2}\hbar\,(\gamma_a\psi_i)_\alpha\,\frac{\delta}{\delta\psi_{i\alpha}}, \\
\widehat{\mathcal{S}'} &= -\hbar^2\varepsilon^{ij}\varepsilon_{abc}\,\gamma^c D_i\psi_j\,\frac{\delta}{\delta A_{ia}}\,\frac{\delta}{\delta A_{jb}}, \\
\widehat{\bar{\mathcal{S}}'} &= -\mathrm{i}\hbar\,D_i\Big(\frac{\delta}{\delta\psi_i}\Big).
\end{aligned} \tag{2.30}$$

Their structure is again similar to (1.59) and, in fact, we can also give

the Wilson loop solutions to them:

$$T_\eta = \mathrm{Tr}\,\mathcal{P}\exp\tfrac{1}{2}\oint_\eta \mathrm{d}s\,\dot\eta^i(s)\,A_{ia}(\eta(s))\gamma^a, \tag{2.31}$$

which is obtained from the 3+1 version just by replacing the σ_a-matrices by γ_a. Again, $\bar{\mathcal{S}}'$ and $\mathcal{L}'_a$ are solved by T_η because it is Lorentz invariant and does not depend on the gravitino. If the loop is smooth and does not have self intersections, the remaining constraints are solved because of the same symmetry arguments as in 3+1 dimensions, thus every smooth loop embedded in the spatial surface yields a state functional.

This is a completely different result than that obtained above, where we found that the state is described by a function F that depends on finitely many degrees of freedom, the moduli, only. Here the general solution is given as an arbitrary superposition of T_η's. In particular, the constraints do not imply that the curvature vanishes and thus different loops really produce different wave functionals.

At first sight, the problem seems to be that we are dealing with single loops here instead of equivalence classes of loops under diffeomorphisms. We already mentioned this problem for the four dimensional theory: it arises because we also multiplied the diffeomorphism constraint by a dreibein factor. The usual way to obtain the diffeomorphism and Wheeler DeWitt constraint from $\mathcal{H}_a$ would in this case be $\mathcal{D}'_i = e_i{}^a\mathcal{H}_a$ and $\mathcal{C}' = \frac{1}{2}\varepsilon_{abc}\varepsilon^{ij}e_i{}^a e_j{}^b\mathcal{H}^c$.

Then $\mathcal{D}'_i$ in fact generates diffeomorphisms on the 2-surface and requires Ψ to be invariant, and $\mathcal{C}'$ is automatically solved for every solution of $\mathcal{S}'$ and $\bar{\mathcal{S}}'$. Doing this, however, the result is still totally different. Using the loop representation, the wave function becomes a function on the generalized knot classes of the 2-surface. But there are still infinitly many[2], even if the topology is trivial, whereas for the metric or 'undensitized' connection representation there is only one state Ψ_F with constant F.

3 Discussion

In the 2+1 case described above it is rather obvious that the loop solutions are artificially generated by multiplying the constraints with a factor, which

[2]Of course, there are no real 'knots' on a 2-surface, but there are infinitly many link classes, given e.g. by different numbers of disconnected loops.

is in principle the determinant of the spatial metric. As this factor appears to the right in the quantized constraints and already annihilates the loop functionals, it is not clear what they have to do with the original theory described by (2.8). Here we cannot argue that the 'polynomialized' primed constraints can deal with singular metrics and thus we get new solutions representing singular metrics: the constraints were polynomial from the beginning and the unprimed constraints can handle singular metrics, too.

To make this argument more precise, consider the classical theory described by the primed and unprimed constraints. If we allow the dreibein to be singular, there are solutions to the primed constraints which do not solve the unprimed ones. Just take the trivial example ${e_i}^a = 0, \psi_i = 0$, but A_{ia} not curvature free.

Thus the primed constraints do not describe the classical theory defined by the lagrangian (2.5), if we allow singular metrics, though the lagrangian itself does not contain the inverse metric. A lagrangian which directly leads to the primed constraint can be written in the same way as one usually writes the Einstein Hilbert lagrangian in Ashtekars variables with the time component of the dreibein split into a lapse and shift function. One has to replace the Lagrange multipliers ${e_{\mathrm{t}}}^a$ by ${e_{\mathrm{t}}}^a = n^i {e_i}^a + n\varepsilon^{abc}\varepsilon^{ij}e_{ib}e_{jc}$, and a similar replacement for $\bar{\psi}_t$, but let us drop the fermions now because the following arguments also apply for pure gravity.

Differentiation with respect to n and n^i then directly yields the primed diffeomorphism and hamiltonian constraints. But the action is no longer invariant under space time diffeomorphisms. The transformation of the variables n, n^i and ${e_i}^a$ under the full 2+1 dimensional diffeomorphisms involve the inverse of the metric determinant, as n is a density of weight -1.

The argument can be transfered to the four dimensional theory as well. The lagrangian in term of Ashtekar's variables we started from (see (1.21)) was polynomial in ${E_M}^A$, thus it is well defined for singular metrics.[3] Again, to obtain the primed constraints directly, one has to introduce a lapse and shift function by

$$ {E_{\mathrm{t}}}^A = N^m {E_m}^A - \tfrac{1}{6} N \varepsilon^A{}_{BCD} \varepsilon^{mnp} {E_m}^B {E_n}^C {E_p}^D . \qquad (3.1) $$

[3]Observe that also the real first order Einstein Hilbert action is polynomial. But when treating it canonically there arise second class constraints making the dirac brackets non-polynomial. It is this fact and not the polynomial action that simplifies the canonical treatment with Ashtekar's variables.

Thus there are two different actions for 3+1 gravity, one given by (1.21) and the other one by inserting (3.1) and using N, N^m instead of ${E_{\rm t}}^A$ as the primary field variables. Both actions induce the same equations of motion for non-singular metrics, and both are well defined for (different kinds of) singular metrics.

As in the 2+1 theory the equations of motion are different for the different actions if the metric becomes singular. It seems that there is a priori no reason why one lagrangian can be better than the other, because we do not know the 'correct' equations of motion. But there *is* a crucial difference between the two lagrangians: Only one of them is invariant under the full diffeomorphism group of space time. If one requires the invariance of general relativity under all diffeomorphisms *and* allows singular metrics, one has to use the action (1.21). We can conclude that out of the following three properties of classical canonical gravity only two can be realized simultaneously:

- invariance under space time diffeomorphism,
- polynomial constraints,
- singular metrics allowed.

This classical argument is in agreement with our conclusion concerning the loop solutions in the quantized theory. We saw that, if we also 'densitize' the diffeomorphism constraint, which came out automatically in supergravity, then the loops became solutions to all constraints without considering wave functions that depend on the knot class only. Thus the invariance under spatial diffeomorphisms is destroyed by multiplying the corresponding constraint by extra factors of the vierbein. Now we saw that in the classical theory the invariance under space time diffeomorphisms is lost when introducing the lapse function N of weight -1, i.e. when going over from the unprimed to the primed constraints. Assuming that the arguments above apply to the quantized theory as well, we must conclude that quantum gravity described by the loop states is not invariant under space time diffeomorphisms, because it explicitly needs the polynomial constraints and singular metrics. This is rather unsatisfactory, because invariance under space time diffeomorphisms is one of the first principles of general relativity: in fact, it *is* 'general relativity'.

So what is the conclusion out of this? Let us first note that the problem does not arise with the use of Ashtekar's variables. They can be eliminated by

going back to the real action with ${E_M}^A$ and Ω_{MAB} as primary fields. All the arguments concerning the classical theory apply to this action as well, except that there is no 'reason' to introduce a densitized lapse function because this would not simplify the canonical constraints considerable.

The problem arises when one has to give up one of the three properties listed above. For practical reasons it is most convenient to drop general invariance, because in the canonical formulation the *manifest* invariance is lost anyway. But, as already mentioned, this should be the last principle given up when dealing with general relativity. If one insists on 3+1 diffeomorphism invariance and singular metrics, then one has to start with the unprimed constraints and one is not allowed to multiply them by factors which may become zero. In quantum theory it is even more inappropriate to order these extra factors *to the right*.

On the other hand, if singular metrics are not allowed, then the different actions and different sets of constraints are equivalent. Nevertheless there is still a problem with the operator ordering. Note here that it is the ordering with the extra factors appearing *to the left* in the Wheeler DeWitt operator which leads to a closed algebra of constraints, whereas there is an anomaly in the algebra admitting the loop solutions (see [18] for a discussion of different factor orderings).

This is an interesting result, because the 'Chern Simons' state found in [18] solves the constraints in the opposite factor ordering (and with a cosmological constant) and it does not correspond to a singular metric. In fact, one doesn't have to make use of the ${\tilde{e}_a}^m$ factors in the Wheeler DeWitt constraints to show that the Chern Simons form is a solution. It really 'solves' the non-polynomial constraints, where the quotation marks shall indicate that the problem is how to define the quantized versions of the non-polynomial constraints properly.

These properties of the Chern Simons solution are easily proofed. As they only exist for a non-vanishing cosmological constant, we add to L

$$\begin{aligned} \tfrac{1}{2}\Lambda E &= -\tfrac{1}{48}\,\Lambda\,\varepsilon^{MNPQ}\varepsilon_{ABCD}\,{E_M}^A{E_N}^B{E_P}^C{E_Q}^D \\ &= -\tfrac{\mathrm{i}}{12}\,\Lambda\,\varepsilon^{MNPQ}J_{aAB}J_{aCD}{E_M}^A{E_N}^B{E_Q}^C{E_P}^D, \end{aligned} \tag{3.2}$$

where we used that $J_{a[AB}J_{aCD]} = -\frac{\mathrm{i}}{4}\varepsilon_{ABCD}$. Now the diffeomorphism and hamiltonian constraints become

$$\mathcal{H}_A = -\mathrm{i}{E_m}^B J_{aAB}(\varepsilon^{mnp}F_{npa} - \tfrac{1}{3}\Lambda{\tilde{e}_a}^m). \tag{3.3}$$

Obviously, this constraint has a simple solution in the connection representation, namely the exponential of the Chern Simons form, as its derivative yields the field strength:

$$\Psi = \exp\Big(\frac{1}{\hbar\Lambda}\int \mathrm{d}^3x\, \varepsilon^{mnp}(3A_{ma}\partial_n A_{pa} - \varepsilon^{abc}A_{ma}A_{nb}A_{pc})\Big) \tag{3.4}$$

obeys

$$\Big(\varepsilon^{mnp}F_{npa} - \tfrac{1}{3}\hbar\Lambda, \frac{\delta}{\delta A_m{}^a}\Big)\Psi = 0. \tag{3.5}$$

Of course, there are some problems with this functional, too; e.g. it is not clear whether it is a normalizable state functional. But let us ignore these problems for the moment and discuss only the properties related to the operator ordering.

We see that Ψ is annihilated by $\widehat{\mathcal{H}_A}$, though this is not yet defined properly, because it is not clear how to represent $E_m{}^A$ in the connection representation. But the solution does not depend on how it is defined.

Thus the question arises whether one can make sense out of the non-polynomial expression $\mathcal{H}_A$ in the connection representation. A possible answer is the following. The action of $\widehat{C\mathcal{H}_A}$ on Ψ should give zero for every function $C[E_m{}^A]$ that makes $C\mathcal{H}_A$ a polynomial in $\tilde{e}_a{}^m$. Of course, this only makes sense if the extra factors always appear to the left of $\mathcal{H}_A$. This would just lead to the polynomial Wheeler DeWitt and diffeomorphism constraints with the dreibein factors ordered to the left, as there are only four independent choices for C that make $C\mathcal{H}_A$ a polynomial. So the Chern Simons functional is a solution of this type, whereas the Wilson loops are not.

Maybe this procedure to treat a non-polynomial constraint is as close as possible to the classical theory which is invariant under the complete 3+1 diffeomorphism group. Thus there are more reasons to prefer the operator ordering where the dreibein appears to the left than just the fact that only then the quantum algebra closes.

Finally we should emphasize that, even if we do not use the polynomialized constraints, Ashtekar's variables still simplify the canonical treatment in a way that may have been overlooked because the polynomial form of the constraints seems to be their 'simplest' form and only this has been studied so far.

But there is still a difference between the constraint $\mathcal{H}_A$ in terms of Ashtekar's variables given in (3.3) and the original Wheeler DeWitt oper-

ator constructed in the metric representation [19, 20], which contains the curvature scalar of the spatial metric and thus the inverse metric. However, the only non-polynomial term in $\mathcal{H}_A$ is the vierbein component $E_m{}^A$, which is not the inverse but a kind of 'square root' of the canonical variable $\tilde{e}_a{}^m$ (see the definition (1.30)), and a square root of a differential operator is sometimes less a problem then to define its inverse; a well know example is the Dirac operator as 'square root' of the Klein Gordon operator.

So the question is whether it is possible to define an operator $\widehat{E_m{}^A}$ in the connection representation such that

$$[\widehat{E_m{}^A}, \widehat{E_n{}^B}] = 0, \qquad [\varepsilon^{mnp}\widehat{E_m{}^A}\widehat{E_n{}^B}J_{aAB}, A_{qb}] = -\hbar\delta^p_q\eta_{ab}. \tag{3.6}$$

Then $\mathcal{H}_A$ becomes well defined and the Chern Simons functional would be a well defined solution to the constraints which correspond to the space time diffeomorphism invariant and singular metric allowing version of Einstein gravity.

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