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Periodic–Orbit Theory of the Number Variance $\Sigma^2(L)$ of Strongly Chaotic Systems [1]

by

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Abstract

We discuss the number variance $\Sigma^2(L)$ and the spectral form factor $F(\tau)$ of the energy levels of bound quantum systems whose classical counterparts are strongly chaotic. Exact periodic–orbit representations of $\Sigma^2(L)$ and $F(\tau)$ are derived which explain the breakdown of universality, i. e., the deviations from the predictions of random–matrix theory. The relation of the exact spectral form factor $F(\tau)$ to the commonly used approximation $K(\tau)$ is clarified. As an illustration the periodic–orbit representations are tested in the case of a strongly chaotic system at low and high energies including very long–range correlations up to $L = 700$. Good agreement between "experimental" data and theory is obtained.

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I Introduction

Today it is commonly accepted that chaos in classical dynamics is a generic property of complex systems [1]. The most striking property of deterministic chaos is the sensitive dependence on initial conditions such that neighbouring trajectories in phase space separate at an exponential rate. As a result, the long–time behaviour of a strongly chaotic system is unpredictable.

In recent years much work has been devoted to the question [1] whether this well established phenomenon of classical chaos manifests itself also in the quantum world in an analogous phenomenon which could be called "quantum chaos". By this we mean the following: Given a classical dynamical system which is strongly chaotic, is there any manifestation in the corresponding quantal system which betrays its chaotic character? Since the quantum mechanical time–evolution is almost periodic for bound conservative systems, there exists no quantum chaos that manifests itself in the long–time behaviour of quantum mechanics.

Instead one can search for fingerprints of classical chaos in the quantal *spectrum* of a classically chaotic system. Thus one can seek to understand the properties of the energy levels $\{E_n\}$ and the eigenfunctions $\{\Psi_n(\vec{x})\}$, i. e., one studies the stationary states in quantum mechanics which concern the most extreme situation $t = \infty$ since these states never change. (In this paper we are exclusively considering bound conservative systems whose quantum Hamiltonians have only discrete spectra.) Ideally, classically chaotic systems should excel in a random behaviour of these fingerprints that qualify the systems to be called "chaotic" also in quantum mechanics, whereas classically integrable systems should show completely different spectral fluctuations.

Ten years ago it has been conjectured [2] that generic quantum systems, whose classical counterparts are strongly chaotic, show in their quantal spectra statistical correlations according to the predictions of random–matrix theory [1, 3], i. e., that the correlations fall in a few universality classes in dependence on certain symmetry properties of the classical system. First numerical studies seemed to demonstrate the validity of the conjecture in the case of the Sinai billiard [2] and the stadium billiard [4]. During the last years it became clear, however, that the predictions of random–matrix theory agree only for short– and medium–range correlations, whereas the long–range correlations are non–universal and depend on the specific system as well as on the energy range considered. Already in 1985, Berry [5] carried out a semiclassical analysis based on Gutzwiller's periodic–orbit theory [1] which suggested that one of the commonly studied spectral statistics, the so–called Dyson–Mehta spectral rigidity $\Delta_3(L)$ [6], should saturate for large L, i. e., $\Delta_3(L) \to \Delta_\infty$ for $L \to \infty$, in contrast to the logarithmic increase predicted by random–matrix theory. ($\Delta_3(L)$ measures the mean–square fluctuations in the number of energy levels in an energy range containing on the average L levels.) Non–universal long–range correlations occur due to the important role played by the short periodic orbits. Berry's analysis relies on a model for the so–called semiclassical spectral form factor $K(\tau)$; its behaviour for large τ–values determines the short–range correlations, whereas the long–range correlations are given by very small τ–values. Later [7] the semiclassical model for $K(\tau)$ was successfully applied to calculate the number variance $\Sigma^2(L)$ of the non–trivial zeros of the Riemann zeta function. (The precise definition of $\Sigma^2(L)$ will be given in section II.) Until now it is completely unknown whether the mysterious Riemann zeros have anything to do with the eigenvalues of a self–adjoint operator which could be interpreted as Schrödinger operator of a bound dynamical system with a chaotic classical limit. Thus there is the great challenge to demonstrate for a genuine dynamical system that periodic–orbit theory correctly describes the observed non–universal long–range correlations of the two–point statistics.

In this paper we present a consistent periodic–orbit analysis of the spectral form factor and

the number variance for a class of strongly chaotic systems which have the unique property that Gutzwiller's trace formula [1] is not just a semiclassical approximation but rather exact since it is identical to the famous Selberg trace formula [8]. We are thus in the fortunate situation that we can derive an *exact* periodic–orbit representation of the form factor and the number variance without any need to employ semiclassical approximations which are in general difficult to control. Furthermore, we are working with absolutely convergent periodic–orbit sums and thus avoid the notorious convergence problems which are usually encountered when employing the divergent Gutzwiller formula. Having an exact periodic–orbit expression, our only approximation consists in a truncation of the periodic–orbit sums which is, however, unavoidable since the length spectrum of the periodic orbits is only known up to a maximal length $\mathcal{L}$. Our results demonstrate clearly that the non–universal long–range correlations can be quantitatively described by the short periodic orbits.

The spectral rigidity and the number variance measure essentially the same two–point correlations. It is known [9], however, that the rigidity is approximately a monotonously increasing function of L which saturates slowly, while the number variance saturates fast (i.e., reaches already for relatively small L–values the average plateau height), but then shows non–universal oscillations around the average plateau height. Thus the number variance appears to be more interesting and is also more difficult to be described theoretically, and this is the reason why we have decided to discuss in this paper exclusively the number variance. In an earlier paper [10] we have already shown how a saturation of the spectral rigidity can be obtained from periodic–orbit theory.

Our paper is organized as follows. In section II the number variance $\Sigma^2(L)$ is defined and its relation to the exact spectral form factor $F(\tau)$ is given via an integral transformation which has been derived in random–matrix theory. Then an inverse integral equation is derived which expresses $F(\tau)$ in terms of $d\Sigma^2(L)/dL$ and which will be useful in section IV. Furthermore, the asymptotic expansion of $F(\tau)$ for $\tau \to \infty$ is obtained which shows that $F(\tau) \to 1$ in the limit $\tau \to \infty$, a result which has previously been obtained by employing a semiclassical sum rule [11]. In section III we derive an exact representation of the number variance in terms of a new form factor called "pre–form factor" which can be expressed as an integral over the two–point function. A certain approximation of this integral clarifies the relation of the exact form factor $F(\tau)$ to the commonly used form factor $K(\tau)$. Section IV contains one of our main results of this paper that is an exact periodic–orbit representation of the spectral form factor $F(\tau)$. This representation is used in section V to calculate the number variance of a strongly chaotic system in a large L–range. The theoretical results are shown for three different energy ranges and turn out to be in good agreement with the "exact" results obtained from a numerical solution of the Schrödinger equation. Section VI contains a discussion of the results obtained in this paper. Finally, we describe in the Appendix how the remainder terms of the form factor are calculated.

II The number variance $\Sigma^2(L)$ and the spectral form factor $F(\tau)$

The *number variance* $\Sigma^2(L)$ is defined as $(L > 0)$

$$\Sigma^2(L) \;=\; \Sigma^2(L,\overline{E}) \;:=\; \Big\langle\, (\, n(L,E) - L\,)^2 \,\Big\rangle_{\overline{E},\Delta} \quad , \tag{1}$$

where $n(L,E)$ is the number of quantal levels E_n in the energy interval $[E - \frac{L}{2\overline{d}(E)}, E + \frac{L}{2\overline{d}(E)}]$ and $\overline{d}(E)$ is the mean level density. The brackets $\langle \ldots \rangle_{\overline{E},\Delta}$ denote an average near the energy $\overline{E}$

of interest in an energy range from $\overline{E} - \frac{\Delta}{2}$ to $\overline{E} + \frac{\Delta}{2}$ such that enough quantal levels are taken into account in order to give a significant statistical measure. This is usually enforced by the condition $1/\overline{d}(E) \ll \Delta \ll \overline{E}$. Then the expectation value of $n(L,E)$ is $\langle n(L,E)\rangle_{\overline{E},\Delta} = L$.

In the framework of random–matrix theory, the number variance $\Sigma^2(L,\overline{E})$ is independent of the energy $\overline{E}$ because the Gaussian and Circular random–matrix ensembles are stationary over the energy spectrum [12, 13]. The definition of the number variance $\Sigma^2(L)$ used in random–matrix theory differs from definition (1) in the kind of averaging. Instead of averaging over an energy range, an average over an ensemble at fixed energy is employed. However, in [13] the ergodic behaviour of the Gaussian ensemble has been studied, i. e., the equivalence of the ensemble average and the spectral average, and it turns out that all fluctuation measures, which are deriveable from k–point functions, are locally ergodic for the Gaussian and Circular ensemble. This result is in sharp contrast to the behaviour of chaotic systems, for which it is known that only the short–range correlations are universal and stationary. This can be understood in the framework of Gutzwiller's periodic–orbit theory [1] from which one infers that the long–range correlations are determined by the lower part of the length spectrum of periodic orbits which differs not only from system to system, but also leads to an energy–dependent behaviour [5].

In random–matrix theory, the number variance $\Sigma^2(L)$ is related to the so–called *spectral form factor* $F(\tau)$ by means of the integral transformation [6]

$$\Sigma^2(L) \;=\; \frac{2}{\pi^2}\;\int_0^\infty\; d\tau\;\frac{\sin^2(\pi L\tau)}{\tau^2}\;F(\tau) \tag{2}$$

with $F(\tau) = 1 - b(\tau)$. Here $b(\tau)$ is the Fourier transform of the two–level cluster function $Y_2(r)$ [14]

$$b(\tau) \;=\; \int_{-\infty}^{\infty}\; dr\;Y_2(r)\;e^{i2\pi\tau r} \quad . \tag{3}$$

To derive an exact periodic–orbit expression of the form factor $F(\tau)$, it is convenient to work with the inverse of the integral equation (2). To this purpose we differentiate eq.(2) with respect to L which gives

$$\frac{d}{dL}\,\Sigma^2(L) \;=\; \frac{2}{\pi}\;\int_0^\infty\; d\tau\;\sin(2\pi L\tau)\frac{F(\tau)}{\tau} \quad . \tag{4}$$

Inversion of this Fourier sine–transform yields the desired inverse integral equation

$$F(\tau) \;=\; 2\pi\tau\;\int_0^\infty\; dL\;\sin(2\pi\tau L)\frac{d}{dL}\Sigma^2(L) \quad . \tag{5}$$

This integral equation allows to infer the asymptotic behaviour of $F(\tau)$ for $\tau \to \infty$ from the short–range behaviour, $L \to 0$, of the number variance $\Sigma^2(L)$. Integrating the following integral twice by parts under the assumptions $f(L), f'(L) \in L^1(0,\infty)$, yields

$$\int_0^\infty\; dL\;\sin(2\pi\tau L)\;f(L) \;=\; \frac{1}{2\pi\tau}f(0) \;-\; \frac{1}{4\pi^2\tau^2}\int_0^\infty\; dL\;\sin(2\pi\tau L)\;f''(L) \quad . \tag{6}$$

This gives a recursion formula which determines the asymptotic behaviour of $F(\tau)$ for $\tau \to \infty$

$$F(\tau) \;=\; \frac{d}{dL}\Sigma^2(L)|_{L=0} \;-\; \frac{1}{4\pi^2\tau^2}\;\frac{d^3}{dL^3}\Sigma^2(L)|_{L=0} \;+\; O(\frac{1}{\tau^4}) \quad . \tag{7}$$

The required small–L behaviour of the number variance is determined by $\Sigma^2(L) = L + O(L^2)$. The reason is that for $L \to 0$ the probability of finding an eigenvalue in a randomly chosen interval $[E - L/2\overline{d}(E), E + L/2\overline{d}(E)]$ is L whereas the probability of finding no eigenvalue is $1 - L$. In the first case one obtains $n(L) - L = 1 - L$ and in the second case $n(L) - L = -L$ and therefore one gets for the expectation value $\langle\, (n(L) - L)^2 \,\rangle = \Sigma^2(L) = L + O(L^2)$. From this universal property one obtains the asymptotic expansion

$$F(\tau) \;=\; 1 \;-\; \frac{1}{4\pi^2\tau^2}\,\frac{d^3}{dL^3}\Sigma^2(L)|_{L=0} \;+\; O(\frac{1}{\tau^4}) \quad . \tag{8}$$

In random–matrix theory one derives for the Gaussian orthogonal ensemble (GOE) the following exact result of the spectral form factor [3]

$$F_{\rm GOE}(\tau) \;=\; \begin{cases} 2\tau - \tau\ln(1+2\tau) & , \quad 0 \le \tau \le 1 \\ 2 - \tau\ln\left(\frac{2\tau+1}{2\tau-1}\right) & , \quad \tau \ge 1 \end{cases} \tag{9}$$

which yields in particular $F_{\rm GOE}(\tau) = 2\tau + O\left(\,\tau^2\,\right)$ for $\tau \le \frac{1}{2}$, $F_{\rm GOE}(\tau) = 1 - \frac{1}{12\tau^2} + O\left(\,\frac{1}{\tau^4}\,\right)$ for $\tau \ge 1$, $F_{\rm GOE}(\frac{1}{2}) = 1 - \frac{1}{2}\ln 2 = 0.653\ldots$ and $F_{\rm GOE}(1) = 2 - \ln 3 = 0.901\ldots$.

III An exact representation of the number variance and its relation to the approximate spectral form factor $K(\tau)$

In this section we derive our first exact representation of the number variance and of the spectral form factor and show how the exact form factor $F(\tau)$ is related to the commonly used approximate form factor $K(\tau)$.

In contrast to definition (1) which considers $\Sigma^2(L, \overline{E})$ in terms of the energy spectrum $\{E_n\}$, we shall consider from now on the spectrum in terms of the momentum $p = \sqrt{2mE}$, i. e., in the form $\{p_n\}$, where the positive root has to be taken. The corresponding number variance will be denoted by $\Sigma^2(L, \overline{p})$ with the obvious meaning in analogy to (1). Since the periodic–orbit theory has the classical action as primary variable, which is for two–dimensional billiards proportional to the momentum p, the use of the momentum instead of the energy E is appealing and will simplify the following derivations.

The spectral staircase $\mathcal{N}(p)$ counts the number of quantal levels from the ground state up to momentum p, i.e., $\mathcal{N}(p) := \#\{p_n | 0 \le p_n \le p\}$. Then the level density is given by $d(p) = d\mathcal{N}(p)/dp$ and $\overline{d}(p)$ describes its mean. The mean spectral staircase is then given by $\overline{\mathcal{N}}(p) = \int_0^p dp'\; \overline{d}(p')$. Let us consider the interval $[p_a, p_b]$ centred at p, i. e., $p_a = p - \frac{\Delta p}{2}$ and $p_b = p + \frac{\Delta p}{2}$, where $\Delta p = \Delta p(L, p)$ is determined by the condition that the expected number of levels in this interval is L

$$\overline{\mathcal{N}}(p_b) - \overline{\mathcal{N}}(p_a) \;=\; L \quad . \tag{10}$$

A Taylor expansion at p yields

$$\overline{\mathcal{N}}(p_b) - \overline{\mathcal{N}}(p_a) \;=\; \Delta p\,\overline{d}(p) \;+\; \frac{(\Delta p)^3}{24}\overline{\mathcal{N}}'''(p) \;+\; \ldots \;=\; L \tag{11}$$

and thus $\Delta p = \frac{L}{\overline{d}(p)}$ for $\overline{d}(p) \gg \frac{(\Delta p)^2}{24}|\overline{\mathcal{N}}'''(p)|$. To compute the number variance, we have to consider the expression

$$n(L,p) - L \;=\; \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp' \;\Big(d(p') - \overline{d}(p')\Big) \quad . \tag{12}$$

In the framework of *periodic–orbit theory* one can express $d_{\rm osc}(p) := d(p) - \overline{d}(p)$ by a periodic–orbit sum which is given by [1]

$$d_{\rm osc}(p) \;=\; \frac{1}{2\pi\hbar} \sum_{\{l_n\}} \sum_{k=-\infty}^{\infty}{}' A_{n,k} \;\; e^{\frac{i}{\hbar}pkl_n} \qquad , \qquad A_{n,k} = \frac{g_n\, l_n\, \chi_n^k}{\sqrt{|2 - {\rm Tr} M_n^{|k|}|}} \quad , \tag{13}$$

where l_n, g_n, $\chi_n \in \{\pm 1\}$ and M_n denote the length of a primitive periodic orbit, its multiplicity, its phase factor and monodromy matrix, respectively. The phase factor $\chi_n = \exp(-i\pi\nu_n/2)$ is determined by ν_n, where the classical quantity ν_n denotes the maximal number of conjugate points along a primitive periodic orbit plus twice the number of reflections on those parts of the boundary where Dirichlet boundary conditions are required [1] in the case of billiards. The prime at the k–sum counting positive and negative traversals denotes that the term with $k = 0$ has to be excluded. (In this paper we are exclusively considering bound conservative systems which are strongly chaotic, i.e., all periodic orbits are isolated and unstable.)

Before we shall insert the periodic–orbit expression (13), it is worthwhile to transform the number variance

$$\Sigma^2(L,\overline{p}) \;=\; \left\langle \left(\int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp'\; d_{\rm osc}(p') \right)^2 \right\rangle_{\overline{p},\Delta} \tag{14}$$

in the following way

$$\begin{aligned}
\Sigma^2(L,\overline{p}) \;&=\; \left\langle \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp_1 \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp_2\; d_{\rm osc}(p_1)\, d_{\rm osc}(p_2) \right\rangle_{\overline{p},\Delta} \\
&=\; \left\langle \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp_1 \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp_2 \int_{-\infty}^{\infty} d\xi\; \delta(\xi - p_2 + p_1) \right. \\
&\qquad\qquad \left. \times\, d_{\rm osc}\left(\frac{p_1+p_2}{2} - \frac{\xi}{2}\right)\, d_{\rm osc}\left(\frac{p_1+p_2}{2} + \frac{\xi}{2}\right) \right\rangle_{\overline{p},\Delta} \\
&=\; \left\langle \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp_1 \int_{p-\frac{\Delta p}{2}}^{p+\frac{\Delta p}{2}} dp_2 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{dT}{2\pi} e^{iT(\xi - p_2 + p_1)} \right. \\
&\qquad\qquad \left. \times\, d_{\rm osc}\left(\frac{p_1+p_2}{2} - \frac{\xi}{2}\right)\, d_{\rm osc}\left(\frac{p_1+p_2}{2} + \frac{\xi}{2}\right) \right\rangle_{\overline{p},\Delta} \\
&=\; \left\langle \int_0^{\Delta p} dy \int_{-\infty}^{\infty} \frac{dT}{\pi} \cos(Ty) \int_{-\infty}^{\infty} d\xi\; e^{iT\xi} \int_{p-\frac{\Delta p}{2}+\frac{y}{2}}^{p+\frac{\Delta p}{2}-\frac{y}{2}} dx \right. \\
&\qquad\qquad \left. \times\, d_{\rm osc}(x - \frac{\xi}{2})\, d_{\rm osc}(x + \frac{\xi}{2}) \right\rangle_{\overline{p},\Delta} \quad ,
\end{aligned} \tag{15}$$

where in the last step the new variables $x = \frac{p_1+p_2}{2}$ and $y = p_2 - p_1$ have been introduced. (Notice that Δp is given by $L/\overline{d}(p)$, whereas the parameter Δ is a fixed number with $1/\overline{d}(p) \ll \Delta \ll \overline{p}$.) Now let us introduce the "*pre–form factor*"

$$\Phi(T,p,\rho) \;:=\; \int_{-\infty}^{\infty} d\xi\; e^{iT\xi}\, \frac{1}{\rho} \int_{p-\frac{\rho}{2}}^{p+\frac{\rho}{2}} dx\; d_{\rm osc}(x - \frac{\xi}{2})\, d_{\rm osc}(x + \frac{\xi}{2}) \quad . \tag{16}$$

Then we obtain for the number variance

$$\Sigma^2(L,\overline{p}) \;=\; \left\langle \int_{-\infty}^{\infty} \frac{dT}{\pi} \int_0^{\Delta p} d\rho \;\rho\; \cos\left(T(\Delta p-\rho)\right)\; \Phi(T,p,\rho) \right\rangle_{\overline{p},\Delta} \quad . \tag{17}$$

In the next step we would like to interchange the T–integration and the momentum–averaging. Since the cosine–term depends on the momentum p via the relation $\Delta p = L/\overline{d}(p)$ and thus on the averaging, we introduce a new dimensionless variable τ by $T = 2\pi\overline{d}(p)\tau$ and arrive then at the following *exact representation of the number variance* ($\lambda = \overline{d}(p)\rho$)

$$\Sigma^2(L,\overline{p}) \;=\; 4\int_0^{\infty} d\tau \int_0^{L} d\lambda\;\lambda\;\cos(2\pi(L-\lambda)\tau) \left\langle \frac{1}{\overline{d}(p)}\,\Phi\left(2\pi\overline{d}(p)\tau, p, \frac{\lambda}{\overline{d}(p)}\right)\right\rangle_{\overline{p},\Delta} \quad . \tag{18}$$

In order to derive a periodic–orbit representation, one only has to insert the periodic–orbit expression of the pre–form factor (16) which in turn is obtained by inserting the periodic–orbit formula (13) into equation (16). Here we are not writing down this representation since we shall derive an alternative exact representation in the next section which turns out to be more convenient for our purpose in this paper. Instead we would like to demonstrate how the exact representation (18) is related to the approximate representation which is commonly used in the literature. To this end we introduce the *two–point function*

$$k(T,p) \;:=\; \int_{-\infty}^{\infty} d\xi\; e^{iT\xi}\, d_{\rm osc}(p-\frac{\xi}{2})\, d_{\rm osc}(p+\frac{\xi}{2}) \tag{19}$$

in terms of which the pre–form factor (16) can be expressed as

$$\Phi(T,p,\rho) \;=\; \frac{1}{\rho}\int_{p-\frac{\rho}{2}}^{p+\frac{\rho}{2}} dx\; k(T,x) \quad . \tag{20}$$

Inserting this relation into eq.(18) leads to the alternative *exact representation of the number variance*

$$\Sigma^2(L,\overline{p}) \;=\; 4\int_0^{\infty} d\tau \int_0^{L} d\lambda\;\cos(2\pi(L-\lambda)\tau) \left\langle \int_{p-\frac{\lambda}{2\overline{d}(p)}}^{p+\frac{\lambda}{2\overline{d}(p)}} dx\; k\left(2\pi\overline{d}(p)\tau, x\right)\right\rangle_{\overline{p},\Delta} \quad . \tag{21}$$

The x–integration in (21) can be viewed as a rectangular averaging of the two–point function over the interval $x \in \left[p-\frac{\lambda}{2\overline{d}(p)}, p+\frac{\lambda}{2\overline{d}(p)}\right]$. It follows from the mean–value theorem for integrals that there exists a value $\widehat{x} = \widehat{x}(p,\lambda) \in \left[p-\frac{\lambda}{2\overline{d}(p)}, p+\frac{\lambda}{2\overline{d}(p)}\right]$ such that

$$\Sigma^2(L,\overline{p}) \;=\; 4\int_0^{\infty} d\tau \int_0^{L} d\lambda\;\lambda\;\cos(2\pi(L-\lambda)\tau) \left\langle \frac{1}{\overline{d}(p)}\,k\left(2\pi\overline{d}(p)\tau, \widehat{x}\right)\right\rangle_{\overline{p},\Delta} \quad . \tag{22}$$

In order to see the connection with the commonly used form factor $K(\tau)$, we simply replace $\widehat{x}$ by p in eq.(22); the validity of this replacement will be discussed below. Then the λ–integration in eq.(22) can be carried out leading to

$$\Sigma^2(L,\overline{p}) \;\simeq\; \frac{2}{\pi^2}\int_0^{\infty} d\tau\;\frac{\sin^2(\pi\tau L)}{\tau^2} \left\langle \frac{1}{\overline{d}(p)}\,k(2\pi\overline{d}(p)\tau, p)\right\rangle_{\overline{p},\Delta} \quad . \tag{23}$$

The corresponding approximation of the form factor $F(\tau,\overline{p})$ follows from a comparison with equation (2)

$$\begin{aligned} F(\tau,\overline{p}) \;\simeq\; K(\tau) \;&:=\; \left\langle \frac{1}{\overline{d}(p)}\, k(2\pi\overline{d}(p)\tau,p) \right\rangle_{\overline{p},\Delta} \\ &=\; \int_{-\infty}^{\infty} d\xi \left\langle \frac{e^{i2\pi\overline{d}(p)\tau\xi}}{\overline{d}(p)}\, d_{\rm osc}(p-\frac{\xi}{2})\, d_{\rm osc}(p+\frac{\xi}{2}) \right\rangle_{\overline{p},\Delta} \\ &=\; \int_{-\infty}^{\infty} d\eta\; e^{i2\pi\tau\eta} \left\langle \frac{d_{\rm osc}\left(p-\frac{\eta}{2\overline{d}(p)}\right)}{\overline{d}(p)}\, \frac{d_{\rm osc}\left(p+\frac{\eta}{2\overline{d}(p)}\right)}{\overline{d}(p)} \right\rangle_{\overline{p},\Delta} \end{aligned} \tag{24}$$

and turns out to be identical to the commonly used spectral form factor $K(\tau)$ which has been studied in particular by Berry [5, 7, 11], apart from the fact that our derivation has been carried out for the momentum p instead of the energy E. This derivation thus avoids the problem of dealing with $d_{\rm osc}(E)$ for negative energies E, i. e. for imaginary momentum p, for which no convergent periodic–orbit expression exists.

Let us now discuss the validity of the substitution of $\hat{x}$ by p. In general the point $\hat{x}$ in eq.(22) lies anywhere in the interval $p\left[1-\frac{\lambda}{2p\overline{d}(p)}, 1+\frac{\lambda}{2p\overline{d}(p)}\right]$ and is not equal to p. At first sight it seems that the approximation is meaningful for short–range correlations, $L \ll 2p\overline{d}(p)$, since $0 \le \lambda \le L$ in eq.(22). But a closer look reveals that this is only a necessary condition. To see this let us insert the periodic–orbit expression of $d_{\rm osc}(p)$, eq. (13), in the x–integral in eq.(21), whereby one obtains a multiple series over periodic orbits. A single summand of the series reads (besides of pre–factors)

$$A_{n,k}\, A_{n',k'} \int_{p-\frac{\lambda}{2\overline{d}(p)}}^{p+\frac{\lambda}{2\overline{d}(p)}} dx\; e^{\frac{i}{\hbar}(kl_n+k'l_{n'})x} \;=\; \frac{\lambda}{\overline{d}(p)}\, A_{n,k}\, A_{n',k'}\; e^{\frac{i}{\hbar}(kl_n+k'l_{n'})p}\; \frac{\sin\left(\frac{\lambda}{2\hbar\overline{d}(p)}(kl_n+k'l_{n'})\right)}{\frac{\lambda}{2\hbar\overline{d}(p)}(kl_n+k'l_{n'})} \quad . \tag{25}$$

The approximation leading to eq.(23) consists in evaluating the integrand in the above integral at $x=p$ and thus leads to the same expression as in eq.(25) apart from the last factor, i. e., the approximation in a single summand of the multiple series consists in replacing the term $\frac{\sin z}{z}$ by 1 which is only justified under the restriction

$$L\,|kl_n + k'l_{n'}| \;\ll\; 2\hbar\overline{d}(p) \quad , \tag{26}$$

since $0 \le \lambda \le L$. Thus the approximation can be justified only for very short lengths such that $|kl_n + k'l_{n'}|$ is small enough (if in addition one assumes that the relevant contribution comes from small repetition numbers $|k|$ and $|k'|$). However, the periodic–orbit sum runs over the full unbounded length spectrum $\{l_n\}$, and thus almost all terms in the sum are dubiously approximated. The approximation for $F(\tau)$, eq.(24), is therefore based on a weak foundation. In the next section we shall derive an exact periodic–orbit expression of $F(\tau,\overline{p})$.

IV An exact periodic–orbit representation of the spectral form factor $F(\tau, \overline{p})$

In the last section we used a representation of $\Sigma^2(L, \overline{p})$ in terms of $d_{\rm osc}(p)$, see eq.(14). Here we present an alternative derivation which starts with the periodic–orbit expression of the spectral staircase $\mathcal{N}(p) = \overline{\mathcal{N}}(p) + \mathcal{N}_{\rm osc}(p)$, where $\overline{\mathcal{N}}(p)$ denotes the mean behaviour and $\mathcal{N}_{\rm osc}(p)$ describes the fluctuations around the mean. Using a Gaussian smoothing the periodic–orbit expression reads [10]

$$\mathcal{N}^{\varepsilon}_{\rm osc}(p) \;=\; \frac{1}{\pi} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{A_{n,k}}{k l_n} \sin\left(\frac{p k l_n}{\hbar}\right) e^{-\left(\frac{\varepsilon k l_n}{2\hbar}\right)^2} \quad , \tag{27}$$

where $\varepsilon > 0$ is the smoothing parameter. (Here, the k–sum runs only over the positive integers.) The Gaussian smoothing has the advantage that all steps in the following derivation of the periodic–orbit expression of $F(\tau)$ can be carried out with an absolutely convergent sum. Since the product of two absolutely convergent sums is again absolutely convergent, the multiple periodic–orbit sums occuring in the following derivation are always well defined. In the limit $\varepsilon \to 0$ the unsmoothed theory is recovered.

We start with

$$n(L,p) - L \;=\; \mathcal{N}\left(p + \frac{\Delta p}{2}\right) - \mathcal{N}\left(p - \frac{\Delta p}{2}\right) - L \quad . \tag{28}$$

Because Δp is determined by $\overline{\mathcal{N}}\left(p + \frac{\Delta p}{2}\right) - \overline{\mathcal{N}}\left(p - \frac{\Delta p}{2}\right) = L$, we have

$$n_{\varepsilon}(L,p) - L \;=\; \mathcal{N}^{\varepsilon}_{\rm osc}\left(p + \frac{\Delta p}{2}\right) - \mathcal{N}^{\varepsilon}_{\rm osc}\left(p - \frac{\Delta p}{2}\right) \quad . \tag{29}$$

Inserting (27) and using the addition theorem of the sine–function gives

$$n_{\varepsilon}(L,p) - L \;=\; \frac{1}{\pi} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{A_{n,k}}{k l_n} e^{-\left(\frac{\varepsilon k l_n}{2\hbar}\right)^2} 2 \sin\left(\frac{\Delta p k l_n}{2\hbar}\right) \cos\left(\frac{p k l_n}{\hbar}\right) \quad . \tag{30}$$

This gives immediately the following periodic–orbit expression of the number variance

$$\Sigma^2_{\varepsilon}(L, \overline{p}) \;=\; \frac{4}{\pi^2} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \sum_{\{l_{n'}\}} \sum_{k'=1}^{\infty} \frac{A_{n,k}\, A_{n',k'}}{k l_n\, k' l_{n'}} e^{-\frac{\varepsilon^2}{4\hbar^2}\left((k l_n)^2 + (k' l_{n'})^2\right)} \times \left\langle \sin\left(\frac{\Delta p k l_n}{2\hbar}\right) \sin\left(\frac{\Delta p k' l_{n'}}{2\hbar}\right) \cos\left(\frac{p k l_n}{\hbar}\right) \cos\left(\frac{p k' l_{n'}}{\hbar}\right) \right\rangle_{\overline{p},\Delta} \quad . \tag{31}$$

With the help of the integral representation

$$\sin(ax)\, \sin(ay) \;=\; \int_0^{\infty} dT\, \frac{\sin^2(aT)}{T^2}\, R(T,x,y) \quad ,$$

$$R(T,x,y) \;=\; T^2 \left\{ \delta\left(T + \frac{x+y}{2}\right) + \delta\left(T - \frac{x+y}{2}\right) - \delta\left(T + \frac{x-y}{2}\right) - \delta\left(T - \frac{x-y}{2}\right) \right\} \tag{32}$$

it is easy to see that the number variance (31) satisfies the integral equation (2) and thus the spectral form factor can be derived from equation (5). To obtain the exact periodic–orbit

expression of $F(\tau,\overline{p})$, one simply has to insert $\frac{d}{dL}\Sigma^2(L,\overline{p})$ into the integral equation (5). With $\Delta p = \frac{L}{\overline{d}(p)}$ one derives

$$\frac{d}{dL}\,\Sigma^2_\varepsilon(L,\overline{p}) \;=\; \frac{4}{\pi^2\hbar}\sum_{\{l_n\}}\sum_{k=1}^{\infty}\sum_{\{l_{n'}\}}\sum_{k'=1}^{\infty}\frac{A_{n,k}\,A_{n',k'}}{k'l_{n'}}\,e^{-\frac{\varepsilon^2}{4\hbar^2}\left((kl_n)^2+(k'l_{n'})^2\right)}$$
$$\times\left\langle\frac{1}{\overline{d}(p)}\,\cos\left(\frac{\Delta pkl_n}{2\hbar}\right)\,\sin\left(\frac{\Delta pk'l_{n'}}{2\hbar}\right)\,\cos\left(\frac{pkl_n}{\hbar}\right)\,\cos\left(\frac{pk'l_{n'}}{\hbar}\right)\right\rangle_{\overline{p},\Delta}\quad. \tag{33}$$

$F(\tau,\overline{p})$ follows then from eq.(5)

$$F_\varepsilon(\tau,\overline{p}) \;=\; \frac{8\tau}{\pi\hbar}\sum_{\{l_n\}}\sum_{k=1}^{\infty}\sum_{\{l_{n'}\}}\sum_{k'=1}^{\infty}\frac{A_{n,k}\,A_{n',k'}}{k'l_{n'}}\,e^{-\frac{\varepsilon^2}{4\hbar^2}\left((kl_n)^2+(k'l_{n'})^2\right)}$$
$$\times\left\langle I\left(2\pi\tau\overline{d}(p),\frac{k'l_{n'}}{2\hbar},\frac{kl_n}{2\hbar}\right)\cos\left(\frac{pkl_n}{\hbar}\right)\,\cos\left(\frac{pk'l_{n'}}{\hbar}\right)\right\rangle_{\overline{p},\Delta} \tag{34}$$

with

$$I(a,b,c) \;:=\; \int_0^\infty dx\;\sin(ax)\sin(bx)\cos(cx) \tag{35}$$
$$\;=\; \frac{\pi}{4}\,\left[\,-\delta(a+b-c)+\delta(b+c-a)+\delta(c+a-b)-\delta(a+b+c)\,\right]\quad, \tag{36}$$

because of $\int_0^\infty dx\cos(ax)=\pi\delta(a)$. This leads to

$$F_\varepsilon(\tau,\overline{p}) \;=\; \frac{1}{\pi\hbar}\sum_{\{l_n\}}\sum_{k=1}^{\infty}\sum_{\{l_{n'}\}}\sum_{k'=1}^{\infty}\frac{A_{n,k}\,A_{n',k'}}{k'l_{n'}}\,e^{-\frac{\varepsilon^2}{4\hbar^2}\left((kl_n)^2+(k'l_{n'})^2\right)}$$
$$\times\left\langle\left[\,\delta\left(\overline{d}(p)-\frac{kl_n+k'l_{n'}}{4\pi\tau\hbar}\right)-\delta\left(\overline{d}(p)-\frac{kl_n-k'l_{n'}}{4\pi\tau\hbar}\right)\right.\right.$$
$$\left.\left.+\delta\left(\overline{d}(p)+\frac{kl_n-k'l_{n'}}{4\pi\tau\hbar}\right)\,\right]\,\cos\left(\frac{pkl_n}{\hbar}\right)\,\cos\left(\frac{pk'l_{n'}}{\hbar}\right)\right\rangle_{\overline{p},\Delta}\quad, \tag{37}$$

where $\overline{d}(p)\geq 0$ is assumed. This is our *periodic–orbit representation of the spectral form factor*, and we would like to emphasize that its derivation is exact. Only the periodic–orbit expression of $\mathcal{N}_{\rm osc}(p)$ itself is a semiclassical approximation in general. However, in the case of billiards which tessellate the hyperbolic plane, the periodic–orbit expression is even exact, since in this case the generalized Gutzwiller trace formula is identical to the exact Selberg trace formula [8]. In the next section we shall apply our representation (37) of $F(\tau)$ in the case of a strongly chaotic system belonging to the exact case.

V Application to a strongly chaotic system

The chaotic system, for which we want to study the spectral form factor $F(\tau,\overline{p})$ and the number variance $\Sigma^2(L,\overline{p})$, is a conservative Hamiltonian system with two degrees of freedom which classically consists of a point particle of mass m sliding freely on compact hyperbolic surfaces. The surfaces to be considered are compact Riemann surfaces $\mathcal{F}$ of constant negative Gaussian

curvature with genus $g = 2$, i. e., they have the topology of a sphere with two handles. Due to the Gauß–Bonnet theorem, $\mathrm{Area}(\mathcal{F}) = 4\pi(g-1)$, the area of such a surface is $\mathrm{Area}(\mathcal{F}) = 4\pi$. The sphere with two handles can be cut so that one obtains an octagon with geodesic edges where opposite sides must be identified leading to periodic boundary conditions. A given octagon is mapped into the Poincaré disc $\mathcal{D}$ which consists of the interior of the unit circle in the complex z–plane ($z = x_1 + ix_2$) endowed with the hyperbolic metric

$$g_{ij} = \frac{4}{(1-x_1^2-x_2^2)^2}\delta_{ij} \ , \ i,j = 1,2 \tag{38}$$

corresponding to constant negative Gaussian curvature $K = -1$. (This fixes the length scale.) The hyperbolic distance $d(z,z')$ between two points z and z' is given by

$$\cosh d(z,z') \ := \ 1 + \frac{2|z-z'|^2}{(1-|z|^2)(1-|z'|^2)} \quad . \tag{39}$$

The classical motion (geodesic flow on $\mathcal{F}$) is determined by the *Hamiltonian* $H = \frac{1}{2m} p_i g^{ij} p_j$, $p_i \ = \ m\, g_{ij}\, dx^j/dt$. The geodesics are circles intersecting the boundary of the Poincaré disc $\mathcal{D}$ perpendicularly. Since the system is invariant under time reversal, the periodic orbits are twofold degenerate, in general, leading to an asymptotic mean multiplicity $\overline{g} = 2$. The classical system is a K–system and thus strongly chaotic.

The quantum mechanical system is governed by the *Schrödinger equation*

$$-\Delta\Psi_n(z) \ = \ E_n\Psi_n(z) \quad , \quad \Delta = \frac{1}{4}(1-x_1^2-x_2^2)^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) \quad , \tag{40}$$

where units $\hbar = 2m = 1$ are used from now on. The periodic boundary conditions are realized by identifying the points z and $z' \equiv b(z)$,

$$b(z) := \frac{\alpha z + \beta}{\beta^* z + \alpha^*} \ , \ |\alpha|^2 - |\beta|^2 = 1 \ , \tag{41}$$

where the "boosts"

$$b \ = \ \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \ \in \mathrm{SU(1,1)}/\{\pm\mathbf{1}\}$$

are chosen such that they map a given edge onto the opposite edge. Four boosts are sufficient for the description of a given octagon. These four boosts and their inverses are the generators of the Fuchsian group Γ which tessellates the Poincaré disc $\mathcal{D}$. The solutions of the Schrödinger equation (40) have to obey the *periodic boundary conditions*

$$\Psi_n(z) = \Psi_n(\, b(z)\,) \quad \text{for all} \quad b \in \Gamma \quad , \tag{42}$$

and are normalized according to

$$\int\!\!\int_{\mathcal{F}} dx_1\, dx_2\, \frac{4}{(1-x_1^2-x_2^2)^2}\, \Psi_m^*(z)\Psi_n(z) \ = \ \delta_{mn} \quad . \tag{43}$$

The energy spectrum of the Schrödinger equation (40) is discrete, $0 = E_0 < E_1 \leq E_2 \leq \ldots$. For more details of this model, see [1, 15, 10, 9, 16, 17, 18, 19]. For an arbitrarily chosen asymmetric octagon we have numerically computed the quantal energy levels. (This is the same octagon which has been studied in our earlier papers [16, 17, 18, 19].) Since the Schrödinger equation

(40) and the boundary of an octagon are invariant under the parity operation $z \to -z$, the quantal levels can be classified according to their parity, where we will concentrate on the positive parity class $\Psi_n^{(+)}(-z) = +\Psi_n^{(+)}(z)$. The quantal energy levels of positive parity have been computed in the two energy intervals $[0, 2250]$ and $[9000, 12000]$, allowing the study of the energy dependence of the level statistics. Since the mean level spacing is $\overline{\Delta E} = 2$, these energy ranges correspond to the range from the ground state to the 1125^{th} level and from the 4500^{th} to the 6000^{th} level, respectively. The short–range correlations in the energy spectrum are in accordance with the GOE prediction of random–matrix theory. As an example, in figure 1 the nearest–neighbour level spacing $P(0, s)$ for the two considered energy ranges is shown in comparison with the GOE prediction. Very good agreement is observed. For a study of the long–range correlations one could employ the higher level spacings $P(k, s)$, $k > 0$ [9], but it is more convenient to use the number variance $\Sigma^2(L)$ to which this paper is devoted.

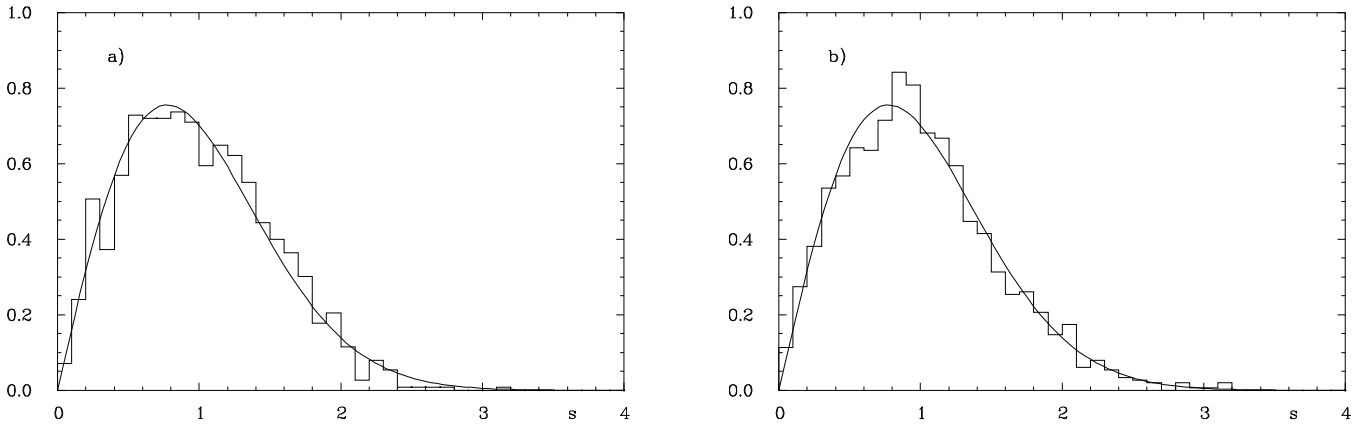

Figure 1: The nearest–neighbour level spacing $P(0, s)$ is shown for the strongly chaotic system discussed in the text. a) shows the low–energy range, $E \in [0, 2250]$, whereas b) shows the high–energy range, $E \in [9\,000, 12\,000]$. Both histograms are in good agreement with the GOE prediction of random–matrix theory displayed as the smooth curves.

Let us apply our periodic–orbit expression of the spectral form factor $F(\tau, \overline{p})$ for the computation of the number variance $\Sigma^2(L, \overline{p})$ in the case of an asymmetric octagon. The mean level density $\overline{d}(p)$ of an desymmetrized hyperbolic octagon is given by [17]

$$\overline{d}(p) \;=\; |p| + O\left(e^{-\pi|p|}\right) \quad . \tag{44}$$

(Notice that the relation between energy E and momentum p reads for an octagon $E = p^2 + \frac{1}{4}$.) The fact that the remainder term vanishes exponentially with $|p| \to \infty$ is special to our system. In generic two–dimensional billiards there occurs also a term proportional to $\sqrt{p}$ according to the circumference term in Weyl's law. However, the hyperbolic octagons are defined by periodic boundary conditions and thus have no "geometric" boundary.

At this point we have to specify the kind of averaging around $\overline{p}$ in eq.(37). Among the many possibilities, e. g., a rectangular averaging, we choose a Gaussian averaging, i. e.,

$$\langle\, f(p)\,\rangle_{\overline{p},\Delta} \;:=\; \int_{-\infty}^{\infty} dp\, \frac{1}{\sqrt{2\pi}\Delta}\, e^{-\frac{(p-\overline{p})^2}{2\Delta^2}}\, f(p) \quad , \tag{45}$$

where Δ gives the momentum range around $\overline{p}$ for which $F(\tau, \overline{p})$ is considered, $1/\overline{p} \ll \Delta \ll \overline{p}$. The Gaussian averaging has the great advantage that one can perform the limit $\varepsilon \to 0$ as will become clear below.

Now consider first for convenience the term arising from the first δ–function in (37), where the sum of lengths of periodic orbits occurs, and denote it by $F_1(\tau,\overline{p})$. The remaining contributions from the other two δ–functions involving the difference of lengths, denoted by $F_2(\tau,\overline{p})$, are discussed below. With the averaging (45) and the level density (44) one immediately obtains

$$F_{1,\varepsilon}(\tau,\overline{p}) \;=\; \frac{1}{\pi\Delta\sqrt{2\pi}} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \sum_{\{l_{n'}\}} \sum_{k'=1}^{\infty} \frac{A_{n,k}\, A_{n',k'}}{k' l_{n'}}\; e^{-\frac{\varepsilon^2}{4}\,((kl_n)^2+(k'l_{n'})^2)}$$

$$\times \left[\; e^{-\frac{1}{2\Delta^2}\left(\frac{kl_n+k'l_{n'}}{4\pi\tau}-\overline{p}\right)^2} \;+\; e^{-\frac{1}{2\Delta^2}\left(\frac{kl_n+k'l_{n'}}{4\pi\tau}+\overline{p}\right)^2} \;\right] \cos\left(\frac{kl_n+k'l_{n'}}{4\pi\tau} kl_n\right) \cos\left(\frac{kl_n+k'l_{n'}}{4\pi\tau} k'l_{n'}\right) \quad . \tag{46}$$

Here one clearly observes the advantage of the Gaussian averaging (45) since the limit $\varepsilon \to 0$ can now be performed because the Gaussian of width $\Delta > 0$ enforces convergence. After carrying out the limit, one ends up with the final result

$$F_1(\tau,\overline{p}) \;=\; \frac{1}{\sqrt{2\pi}\pi\Delta} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \sum_{\{l_{n'}\}} \sum_{k'=1}^{\infty} \frac{A_{n,k}\, A_{n',k'}}{k' l_{n'}}$$

$$\times \left[\; e^{-\frac{1}{2\Delta^2}\left(\frac{kl_n+k'l_{n'}}{4\pi\tau}-\overline{p}\right)^2} \;+\; e^{-\frac{1}{2\Delta^2}\left(\frac{kl_n+k'l_{n'}}{4\pi\tau}+\overline{p}\right)^2} \;\right] \cos\left(\frac{kl_n+k'l_{n'}}{4\pi\tau} kl_n\right) \cos\left(\frac{kl_n+k'l_{n'}}{4\pi\tau} k'l_{n'}\right) \quad . \tag{47}$$

To arrive at this result, the only approximation consisted in neglecting the remainder term of $\overline{d}(p)$, which, however, vanishes exponentially, see eq.(44). Note that the coefficients $A_{n,k}$ are for hyperbolic octagons explicitly given by

$$A_{n,k} \;=\; \frac{g_n l_n \chi_n^k}{2\sinh\left(\frac{|k|l_n}{2}\right)} \quad ,$$

since all periodic orbits are direct hyperbolic and have identical Lyapunov exponents $\lambda_n = 1$.

Let us turn to $F_2(\tau,\overline{p})$, i. e. the other two δ–functions in (37). With the exception of the factor $\frac{1}{k'l_{n'}}$, the summand in (37) is symmetrical under interchanging the pairs $(k,l_n) \leftrightarrow (k',l_{n'})$. Thus one can interchange in the third δ–function the pairs if $\frac{1}{k'l_{n'}}$ is substituted by $\frac{1}{kl_n}$ which leads to

$$F_{2,\varepsilon}(\tau,\overline{p}) \;=\; \frac{1}{\pi} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \sum_{\{l_{n'}\}} \sum_{k'=1}^{\infty} \frac{k'l_{n'}-kl_n}{kl_n\, k'l_{n'}}\, A_{n,k}\, A_{n',k'}\; e^{-\frac{\varepsilon^2}{4}\,((kl_n)^2+(k'l_{n'})^2)}$$

$$\times \left\langle\; \delta\left(\overline{d}(p) - \frac{kl_n-k'l_{n'}}{4\pi\tau}\right) \cos(pkl_n)\, \cos(pk'l_{n'}) \right\rangle_{\overline{p},\Delta} \quad . \tag{48}$$

Again we use the Gaussian averaging (45) and the level density (44) which yields the periodic–orbit expression

$$F_{2,\varepsilon}(\tau,\overline{p}) \;=\; \frac{1}{\sqrt{2\pi}\pi\Delta} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \sum_{\{l_{n'}\}} \sum_{\substack{k'=1 \\ k'l_{n'}<kl_n}}^{\infty} \frac{k'l_{n'}-kl_n}{kl_n\, k'l_{n'}}\, A_{n,k}\, A_{n',k'}\; e^{-\frac{\varepsilon^2}{4}\,((kl_n)^2+(k'l_{n'})^2)}$$

$$\times \left[\; e^{-\frac{1}{2\Delta^2}\left(\frac{kl_n-k'l_{n'}}{4\pi\tau}-\overline{p}\right)^2} \;+\; e^{-\frac{1}{2\Delta^2}\left(\frac{kl_n-k'l_{n'}}{4\pi\tau}+\overline{p}\right)^2} \;\right] \cos\left(\frac{kl_n-k'l_{n'}}{4\pi\tau} kl_n\right) \cos\left(\frac{kl_n-k'l_{n'}}{4\pi\tau} k'l_{n'}\right) \quad . \tag{49}$$

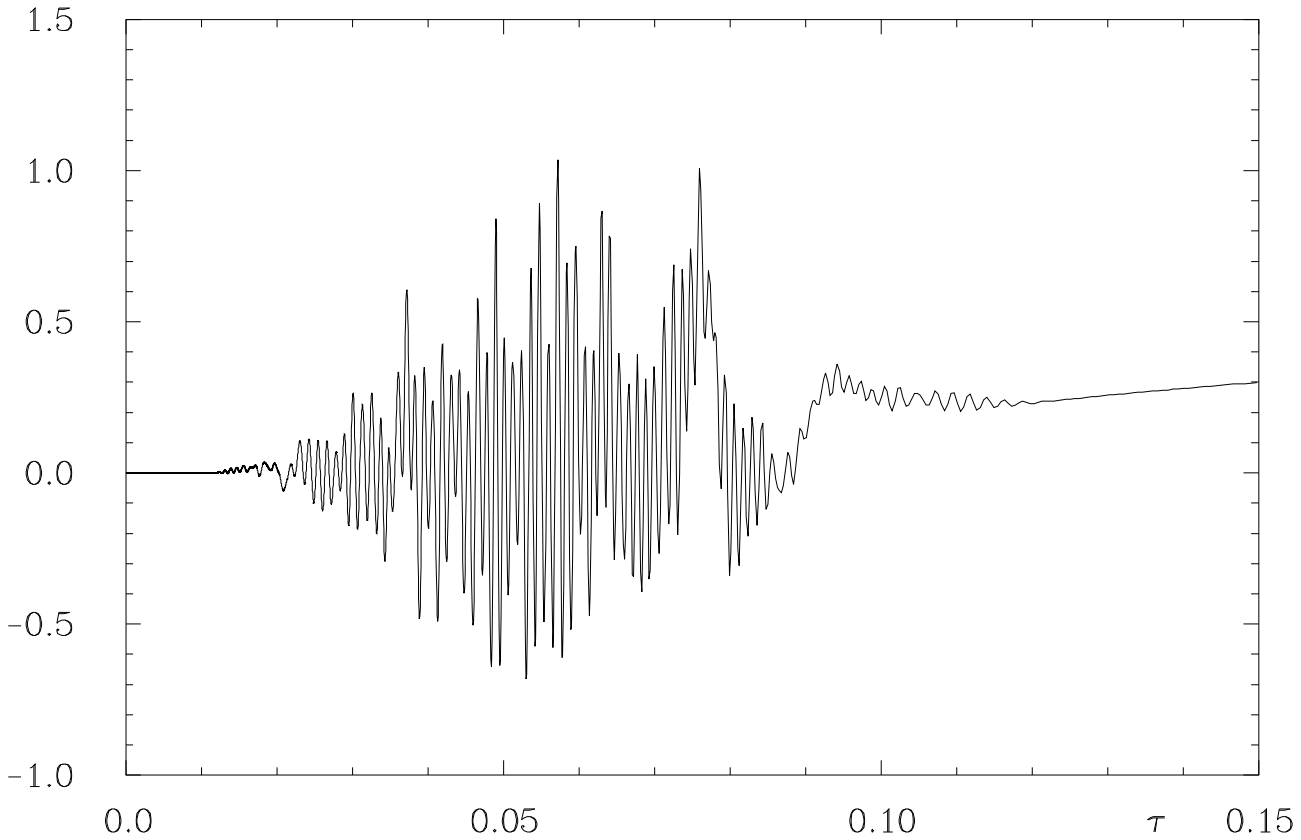

Figure 2: The spectral form factor $F_1(\tau,\overline{p})$ is displayed for $\overline{p}=20$ and $\Delta=2.5$. The periodic–orbit sum (47) has been evaluated up to the cut–off length $\mathcal{L}=11$; in addition, the remainder terms (59) and (63) have been included.

With $F(\tau,\overline{p}) = F_1(\tau,\overline{p}) + F_2(\tau,\overline{p})$ it is now possible in principle to compute the number variance $\Sigma^2(L,\overline{p})$ for the hyperbolic octagon using the integral representation (2). However, since we know only the lower part of the length spectrum of the periodic orbits, $F(\tau,\overline{p})$ can only be computed for small values of τ, $\tau \le \widehat{\tau}$, and thus we have to approximate $F(\tau,\overline{p})$ for $\tau > \widehat{\tau}$ by an appropriate ansatz. Since $F(\tau,\overline{p})$ goes asymptotically to one, see equation (8), what is really required is only an approximation in the intermediate range $\widehat{\tau} < \tau \le \tau^\star$ where $\tau^\star > 1$. Because the large–τ behaviour corresponds to the short–range behaviour of the number variance $\Sigma^2(L,\overline{p})$, which is in reasonable accordance with the GOE expectation of random–matrix theory, it seems reasonable to employ the GOE spectral form factor $F_{\rm GOE}(\tau)$, see eq.(9), for $\tau > \widehat{\tau}$. This leads to the equation

$$\Sigma^2(L,\overline{p}) \;\simeq\; \frac{2}{\pi^2}\int_0^{\widehat{\tau}} d\tau\, \frac{\sin^2(\pi L\tau)}{\tau^2}\, F(\tau,\overline{p}) \;+\; \frac{2}{\pi^2}\int_{\widehat{\tau}}^{\infty} d\tau\, \frac{\sin^2(\pi L\tau)}{\tau^2} F_{\rm GOE}(\tau) \quad . \tag{50}$$

One can rewrite (50) as follows. Extend the second integral in (50) from 0 to ∞, which yields $\Sigma^2_{\rm GOE}(L)$, and subtract the added part from the first integral. This immediately gives

$$\Sigma^2(L,\overline{p}) \;\simeq\; \Sigma^2_{\rm GOE}(L) \;+\; \frac{2}{\pi^2}\int_0^{\widehat{\tau}} d\tau\, \frac{\sin^2(\pi L\tau)}{\tau^2}\; \left(F(\tau,\overline{p}) - F_{\rm GOE}(\tau) \right) \tag{51}$$

with

$$\Sigma^2_{\rm GOE}(L) \;=\; \frac{2}{\pi^2}\,\Big\{\,\ln(2\pi L)+\gamma+1+\frac{1}{2}{\rm Si}^2(\pi L)-\frac{\pi}{2}{\rm Si}(\pi L)-\cos(2\pi L)$$

$$-{\rm Ci}(2\pi L)+\pi^2 L\left(1-\frac{2}{\pi}{\rm Si}(2\pi L)\right)\Big\}\quad . \qquad (52)$$

Equation (51) displays nicely the main properties of the number variance: i) In the short–range limit $L\to 0$, the integral in eq.(51) gives a negligible contribution and thus $\Sigma^2\simeq\Sigma^2_{\rm GOE}$ shows a universal behaviour as described by random–matrix theory. ii) However, in the medium– and long–range regime the integral in (51) cannot be neglected and leads to the well–known non–universal, energy–dependent behaviour which depends on the individual properties of the classical system encoded in the length spectrum of the periodic orbits.

Using equation (51) we have computed the number variance $\Sigma^2(L,\overline{p})$ as will be discussed in the following. Since our numerical computations show that $F_2(\tau,\overline{p})$ oscillates so fast that it has practically no influence on the number variance using eq.(51), we will concentrate on $F_1(\tau,\overline{p})$ which alone seems to determine $\Sigma^2(L,\overline{p})$ with sufficient accuracy. The extremely fast oscillations in $F_2(\tau,\overline{p})$ are due to the fact that differences of lengths occur instead of sums as it is the case for $F_1(\tau,\overline{p})$. Figure 2 shows $F_1(\tau,\overline{p})$ for $\overline{p}=20$ and $\Delta=2.5$. In the periodic–orbit summation all periodic orbits with length l_n shorter than the cut–off length $\mathcal{L}=11$ have been taken into account. The contribution of the periodic orbits with $l_n>\mathcal{L}$ has been taken into account by the two remainder terms (59) and (63) which are derived in the Appendix. For the understanding of figure 2 it may be helpful to consider three special values of τ. The value of τ at which two lengths l_n and $l_{n'}$ contribute, can be inferred from the Gaussian in (47) or from the delta functions in (37). The first value of τ at which periodic orbits contribute is determined by the shortest periodic orbit having length $l_1=1.97639\ldots$ yielding $\tau_1=\frac{l_1}{2\pi\overline{p}}$ which is in our case $\tau_1\simeq 0.0157$. The next τ–value to be considered is $\tau_2=\frac{l_1+\mathcal{L}}{4\pi\overline{p}}$ which is the largest τ at which the periodic–orbit contributions are completely taken into account provided the length spectrum is complete up to $\mathcal{L}$. For $\mathcal{L}=11$ one obtains $\tau_2\simeq 0.0516$. The largest τ–value up to which the considered length spectrum contributes is given by $\tau_3=\frac{\mathcal{L}}{2\pi\overline{p}}\simeq 0.0875$. Figure 2 indeed shows that the amplitude of the oscillations decreases between τ_2 and τ_3. The monotonous behaviour for larger τ–values is determined by the remainder terms (59) and (63). We would like to emphasize that if one would have enough computational power to evaluate the multiple periodic–orbit sum for much larger $\mathcal{L}$–values, the oscillations would not be damped in the way as figure 2 suggests.

With this approximation to the spectral form factor $F_1(\tau,\overline{p})$ we obtain from (51) the number variance $\Sigma^2(L,\overline{p})$ shown in figure 3. The upper full curve shows $\Sigma^2(L,\overline{p})$ for the values $\overline{p}=102.3$ and $\Delta=2$ measuring the statistical properties in the high–energy range $E\in[9000,12000]$ where $\widehat{\tau}=0.019$ is used. The middle and the lower full curves correspond to $\overline{p}=40$ and $\Delta=1.5$, and $\overline{p}=20$ and $\Delta=2.5$, respectively, where $\widehat{\tau}=0.15$ is used. The dashed curve represents the GOE prediction of random–matrix theory. The dots show the results obtained directly from the energy spectrum, where we have generated 5000 points p which are Gaussian distributed with the corresponding values of $\overline{p}$ and Δ and have then averaged the 5000 variances. Thus we have employed the same averaging for both the periodic–orbit theory and the direct evaluation from the energy spectrum. Figure 3 demonstrates how successful the length spectrum up to $\mathcal{L}=11$ can predict the true number variance $\Sigma^2(L,\overline{p})$. The small discrepancies are solely due to the cut–off of the multiple sum in the numerical computation since this was the only approximation. In other words, to resolve much finer details in the number variance, the information contained in the length spectrum of the long orbits with lengths $l_n>\mathcal{L}$ is necessary. Figure 4 gives an

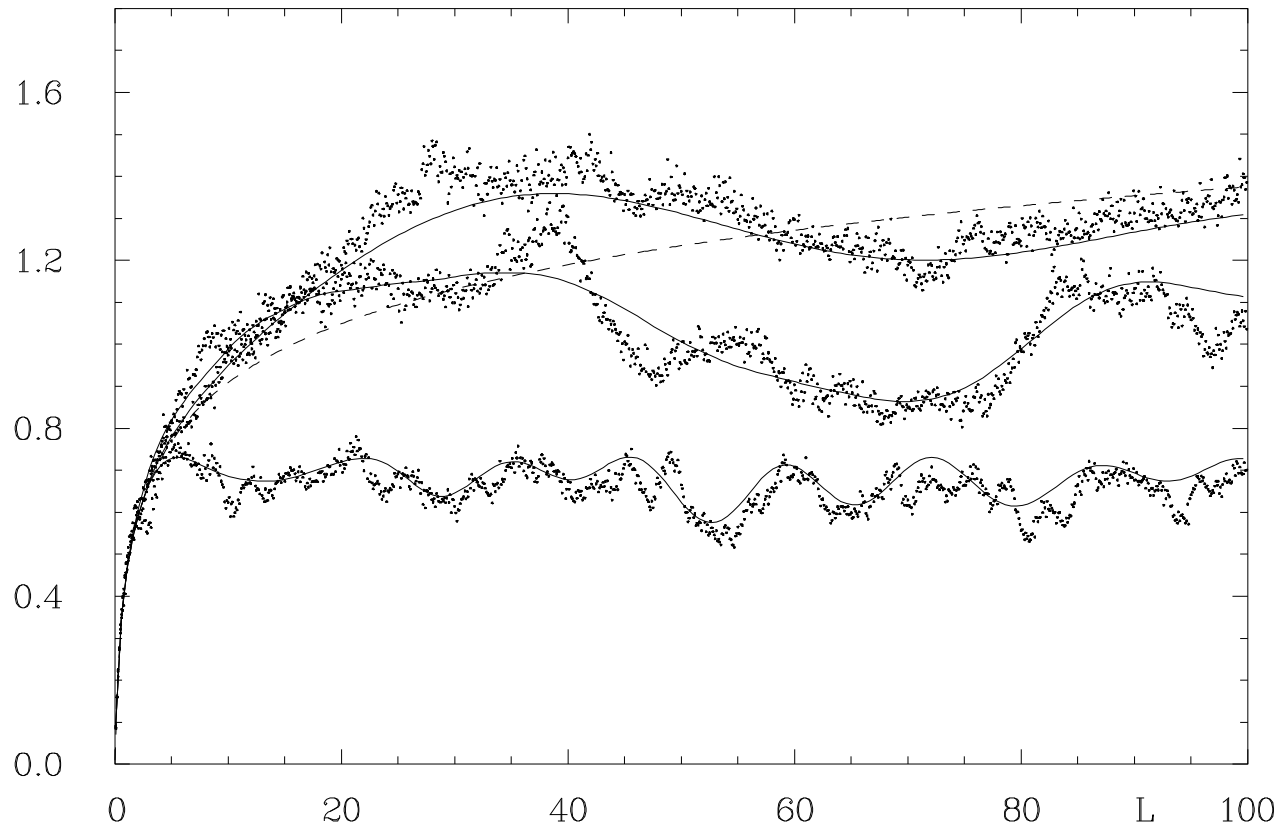

Figure 3: The number variance $\Sigma^2(L,\overline{p})$ is shown in comparison with the predictions of the periodic–orbit theory (full curves) obtained from $F_1(\tau,\overline{p})$ for the three cases $\overline{p}=20$, $\Delta=2.5$ (lower curve), $\overline{p}=40$, $\Delta=1.5$ (middle curve) and $\overline{p}=102.3$, $\Delta=2$ (upper curve). In addition, the GOE prediction of random–matrix theory is displayed as a dashed curve.

impression of the quality with which our formula describes the number variance even for the "very–long–range" correlations comprising as much as 700 levels.

Finally we would like to turn to the so–called *diagonal approximation* which states that it suffices to incorporate only the diagonal terms occuring in the periodic–orbit sum (47), i.e., the terms $k=k'$ and $n=n'$. To test this approximation we computed only the diagonal sum in eq.(47) up to the cut–off length $\mathcal{L}$ and corrected this cut–off by the remainder term (63). The number variance $\Sigma^2(L,\overline{p})$ obtained from this diagonal approximation is shown as a dashed curve for $\overline{p}=20$ and $\Delta=2.5$ in figure 5. In addition figure 5 shows the corresponding result already displayed in figure 3. One observes that the diagonal approximation does not lead to completely wrong results. However, in comparison with the result obtained by keeping the multiple sum one concludes that the full theory is much better.

VI Discussion

One of the main motivations to study the energy–level statistics of quantum systems, whose classical counterparts are strongly chaotic, come from the hope that it may be possible to identify unique fingerprints of classical chaos in certain spectral fluctuations. In this paper we have concentrated on a particular statistic, the number variance $\Sigma^2(L)$, which has played an

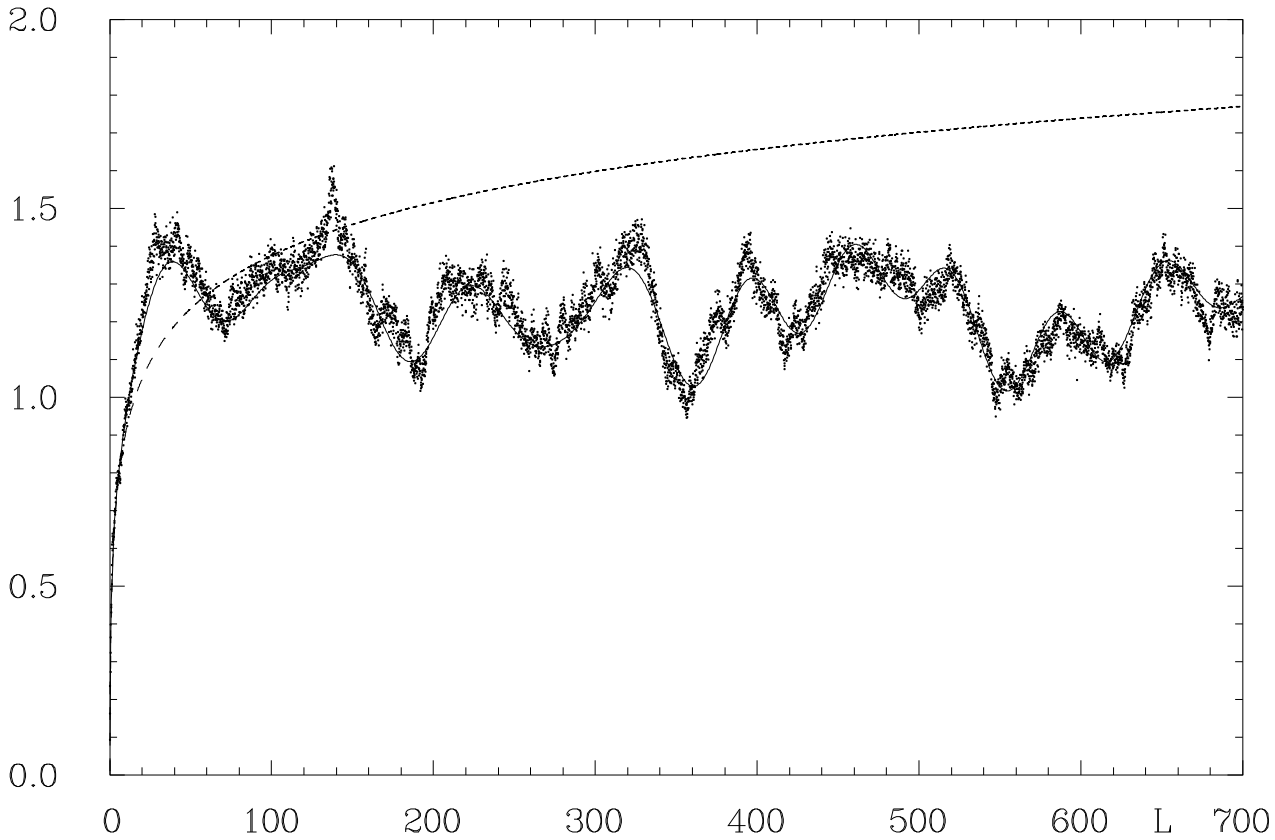

Figure 4: The number variance $\Sigma^2(L,\overline{p})$ is shown for $\overline{p} = 102.3$, $\Delta = 2$ in a large L–interval in comparison with the prediction of the periodic–orbit theory (full curve) obtained from $F_1(\tau,\overline{p})$. In addition, the GOE prediction of random–matrix theory is displayed as a dashed curve.

important role since the early applications of random–matrix theory in nuclear physics as a representative statistic that is bilinear in the spectral density.

In random–matrix theory $\Sigma^2(L)$ is defined as in eq.(1), but with the important difference that the brackets $\langle \ldots \rangle$ denote an average *at fixed energy* over an *ensemble* of "systems" (random matrices chosen from a given ensemble reflecting the presence or absence of certain symmetries) which is in the same spirit as the use of Gibbs ensembles in statistical mechanics. Taking the average over an infinite number of systems, it is intuitively to be expected that individual properties are washed out and thus $\Sigma^2(L)$ defined in this way exhibits some *universal property* of the level fluctuations of the considered ensemble. This is indeed the case, and one finds, e. g. in the case of the Gaussian orthogonal ensemble (GOE), the universal formulae (9) and (52) for $F_{\rm GOE}(\tau)$ and $\Sigma^2_{\rm GOE}(L)$, respectively. Modelling a complex system with *many* degrees of freedom, like a big compound nucleus, by random–matrix theory seems therefore quite reasonable at least for short–range correlations.

The situation is, however, completely different once we are considering small systems having only a few degrees of freedom as it is the case in the theory of quantum chaos where practically all systems which have been investigated until now had only *two* degrees of freedom. The idea, of course, is that the spectral fluctuations of a small quantum system may be nevertheless very stochastic if only its classical counterpart is strongly chaotic. In contrast to the applications in nuclear physics, it is, however, not plausible at all that this idea should turn out to be true and, in particular, why the statistics should be universal.

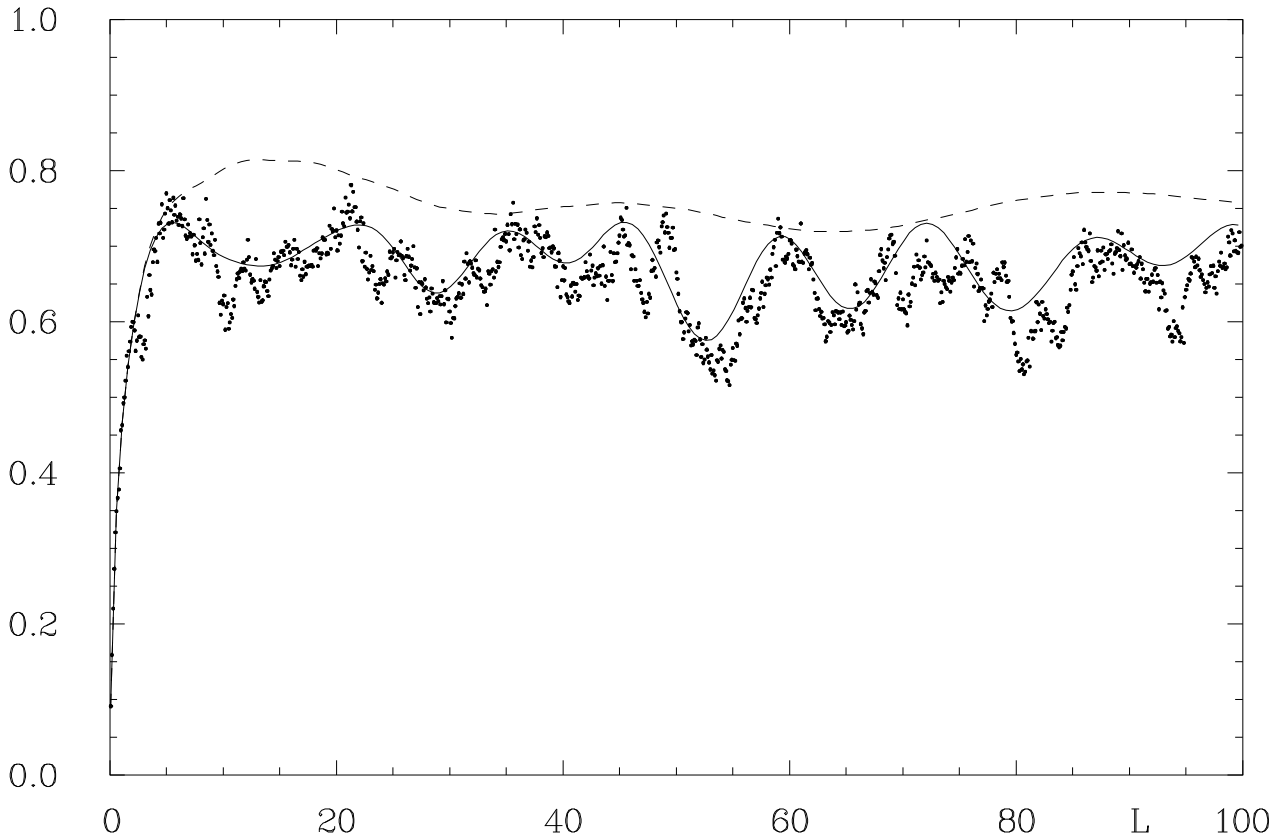

Figure 5: The number variance $\Sigma^2(L,\overline{p})$ is shown for $\overline{p} = 20$, $\Delta = 2.5$ and compared with the predictions of periodic–orbit theory obtained from $F_1(\tau,\overline{p})$ using only the diagonal sum (dashed curve) and using the multiple sum (full curve) as in figure 3.

To test this idea in small quantum systems, it would seem most natural to employ again the definition of $\Sigma^2(L)$ used in random–matrix theory and to consider therefore an *ensemble* of chaotic systems in analogy with the "nuclear data ensemble" studied in nuclear physics. One would hope that taking the average over an ensemble of systems is already sufficient to wash out non–universal properties of the individual members of the ensemble such as the lengths of the shortest periodic orbits. To carry out this program requires, however, to solve the Schrödinger equation not only for a single system, but for a sufficiently large number of quantal systems. Since the computation of energy levels of a quantum system with classically chaotic counterparts is non–trivial, this definition of the number variance needs much work and computer time. Probably this is the main reason why the average over a "quantum chaos ensemble" has never been employed – apart from our own work on the "Hadamard–Gutzwiller ensemble" published in [9]. In that paper we studied the energy–level statistics of an ensemble of 30 different systems (octagons) which were exactly of the type of the system discussed in the present paper in section V. Our detailed analysis showed [9] that the short–range correlations are governed by the GOE predictions, but that the level statistics describing correlations over greater level distances deviate from the GOE predictions. The observed breakdown of universality could be explained as a simple consequence of the fact that the spectral rigidity $\Delta_3(L)$ saturates at a finite value Δ_∞ for $L \to \infty$.

In the search for quantum chaos carried out during the last decade, the definition (1) has been used, i. e., instead of considering an ensemble of systems, a *single* system has been

studied. In other words, instead of averaging over many different spectra at fixed energy, one rather considers a single quantum spectrum $\{E_n\}$, but then takes the average over many energy intervals near the energy of interest. Implicitly one assumes that the energy average mimics somehow the ensemble average in random–matrix theory. Since we have seen [9] that even an ensemble average is not sufficient to produce a universal behaviour at all scales, there is no reason to expect that the energy average employed in (1) is able to lead to universality either. Nevertheless, the universality conjecture put forward in [2] has played an important role by emphasizing the importance of energy–level statistics in seeking for quantum manifestations of classical chaos. One may speculate how research in quantum chaology would have been changed, if the early numerical tests [2, 4] of the GOE prediction would not have been limited to short–range correlations ($L \leq 15$), but rather would have been carried out at larger L–values and thus would have led to an early discovery of the breakdown of universality.

Since the number variance violates the universal laws of random–matrix theory, it is clear that there do not exist universal fingerprints of classical chaos that manifest themselves in the two–point statistics. It thus seems desirable to look for another quantity that clearly distinguishes quantum systems with chaotic classical limits from those with integrable ones. Very recently it has been conjectured [20] that the following quantity

$$W(p) \; := \; \frac{\mathcal{N}_{\rm osc}(p)}{\sqrt{\Delta_\infty(p)}} \tag{53}$$

possesses a limit distribution for $p \to \infty$ with zero mean and unit variance which is absolutely continuous with respect to Lebesgue measure on the real line with a density $f(w)$. Furthermore, if the corresponding classical limit is strongly chaotic, it is conjectured that $f(w)$ is a Gaussian,

$$f(w) \; = \; \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2}w^2} \quad , \tag{54}$$

whereas a classical integrable system leads to a non–Gaussian density $f(w)$. (In eq.(53) $\mathcal{N}_{\rm osc}(p)$ denotes the fluctuation part of the spectral staircase $\mathcal{N}(p)$ as in eq.(27), while $\Delta_\infty(p) := \lim_{L\to\infty} \Delta_3(L,p)$ is the saturation value of the rigidity as a function of the momentum of interest (called $\overline{p}$ in the previous part of this paper).) Since a Gaussian limit distribution for $W(p)$ can be viewed as the validity of a central limit theorem for the spectral fluctuations, the content of the conjecture is that classically strongly chaotic systems have *maximally random quantum spectra*. Numerical tests of the conjecture carried out for three strongly chaotic systems [20] give firm evidence in favor of it. At present it seems that the validity of this conjecture (together with a similar one for eigenstates [19, 20]) provides the best *definition* of quantum chaos.

Although the number variance cannot serve as a good definition of chaos in a quantum system, there remains the important problem to *understand* its behaviour, in particular the *breakdown of universality*. The present paper has been devoted to this problem.

For the ensemble average employed in random–matrix theory, one can relate $\Sigma^2(L)$ to the two–level cluster function via the integral transformation (2). It is commonly taken for granted that this implies automatically that an analogous relation holds also true for the number variance defined in eq.(1) involving an energy average. All studies, which we are aware of, start with eq.(2) and, furthermore, use the semiclassical form factor $K(\tau)$ defined in (24) in terms of the two–point function. In section III we have shown how $K(\tau)$ is obtained from the exact representation of $\Sigma^2(L)$. We have pointed out that the approximation leading to $K(\tau)$ cannot rigorously be justified since the necessary condition (26) cannot be fulfilled for all periodic

orbits. This is a direct consequence of ergodicity which implies that the length spectrum of periodic orbits is unbounded. At this point it has been important to employ the periodic–orbit sum (13) for the level density which at present constitutes our only method to incorporate the complex structure of phase space in the case of strongly chaotic systems. Actually, the situation is even more subtle if one takes into account that the periodic–orbit sum (13) is divergent.

In section IV we have demonstrated that there is no need at all to make approximations and to work with divergent series, but instead one can start with the absolutely convergent periodic–orbit expression (27) for the spectral staircase. We have explicitly shown that expression (31) for $\Sigma^2(L)$ satisfies the integral equation (2). This has enabled us to use the inverse integral equation (5) to derive the exact and absolutely convergent periodic–orbit expression (37) for the spectral form factor.

The main purpose of section V has been to illustrate that the periodic–orbit formulae derived in this paper can really be used to calculate the number variance of a strongly chaotic system. Since the length spectrum of periodic orbits is only known up to a largest length $\mathcal{L}$, the periodic–orbit sums have been truncated, and the sums over the unknown part of the length spectrum have been approximated by integrals using Huber's law. The resulting remainder terms have been discussed in detail in the Appendix.

Fig.2 shows clearly the complex structure of the spectral form factor at small τ–values which is determined by the lower part of the length spectrum of periodic orbits and is responsible for the non–universal behaviour of the number variance for long–range correlations. Since the form factor has to go to one in the limit $\tau \to \infty$ on general grounds, see the derivation in section II, there remains an unknown region $\widehat{\tau} < \tau \leq \tau^\star$ with $\widehat{\tau} = 0.15$ or $\widehat{\tau} = 0.019$, respectively, and $\tau^\star > 1$ where the available information on periodic orbits is not sufficient to allow a computation of $F(\tau)$. To get a plausible interpolation between small and large τ–values, we have used the GOE form factor (9) as a first approximation for $\tau > \widehat{\tau}$, a strategy which has already been advocated by Berry [5, 7]. Under this assumption we have obtained formula (51) which has been the basis of our numerical computations of the number variance displayed in figs.3–5. Formula (51) can be rewritten as follows

$$\Sigma^2(L,\overline{p}) \;\simeq\; \Sigma^2_{\rm GOE}(L) + \Sigma^2_{\rm POT}(L,\overline{p}) \quad , \tag{55}$$

where the second term denoted by POT is given by the integral in eq.(51) and represents the non–universal contribution to the number variance as derived from periodic–orbit theory (POT). Eq.(55) exhibits nicely the violation of universality, i. e., the deviation from the GOE prediction. For all three energies $\overline{E} = \overline{p}^2 + \frac{1}{4}$ considered, i. e., $\overline{E} = 400.25, 1600.25$ and $10\,465.54$, agreement with the GOE prediction (see the dashed curve in fig.3) is observed only for the short–range correlations $L < 5 - 10$. Figs.3–5 demonstrate clearly that the number variance shows a "*quasi–saturation effect*", i. e., a saturation decorated by characteristic oscillations whose global structure is well described by periodic–orbit theory. Even very–long range correlations comprising as much as 700 levels are well reproduced as can be seen in fig.4.

Appendix: The remainder terms

In the case of a numerical evaluation of the multiple sums occuring in $F_1(\tau,\overline{p})$, eq.(47), one can of course take into account only a finite part of the length spectrum $\{l_n\}$. Thus one computes the sums with the length spectrum $\{l_n\}$ truncated at a given cut–off length $\mathcal{L}$. The remaining part is taken into account by approximating the sums by integrals using Huber's law [18]. Since asymptotically only the summation with $k = k' = 1$ contributes, we have to consider

instead of the quadruple sum a double sum over $\{l_n\}$ and $\{l_{n'}\}$. To compute the remainder term correctly one has to consider two cases, as will become clear below.

At first, we consider in eq.(47) the case $l_{n'} < \mathcal{L}$ where we want to sum up the contributions with $l_n > \mathcal{L}$,

$$F_1^{\mathcal{L}}(\tau,\overline{p}) \;:=\; \frac{1}{\sqrt{2\pi}\,\pi\Delta} \sum_{\substack{\{l_{n'}\}\\ l_{n'}<\mathcal{L}}} \sum_{\substack{\{l_n\}\\ \mathcal{L}<l_n}} \frac{A_{n,1}\,A_{n',1}}{l_{n'}}$$

$$\times\; e^{-\frac{1}{2\Delta^2}\left(\frac{l_n+l_{n'}}{4\pi\tau}-\overline{p}\right)^2} \cos\left(\frac{l_n+l_{n'}}{4\pi\tau}l_n\right)\cos\left(\frac{l_n+l_{n'}}{4\pi\tau}l_{n'}\right) \quad . \qquad (56)$$

Here the second exponential in the square bracket of eq.(47) has been dropped, since it gives only an exponentially small contribution. Now, $A_{n,1}$ is asymptotically approximated by $A_{n,1} \simeq \overline{g} l_n e^{-l_n/2}$, since the case of positive parity is considered where all phase factors are $\chi_n = +1$. In the next step, the sum over $l_n > \mathcal{L}$ is replaced by the integral

$$\sum_{\substack{\{l_n\}\\ \mathcal{L}<l_n}} \;\simeq\; \int_{\mathcal{L}}^{\infty} dl\, \frac{e^l}{\overline{g}\, l} \quad . \qquad (57)$$

Then one obtains

$$F_1^{\mathcal{L}}(\tau,\overline{p}) \;\simeq\; \frac{1}{\sqrt{2\pi}\,\pi\Delta} \sum_{\substack{\{l_{n'}\}\\ l_{n'}<\mathcal{L}}} \frac{A_{n',1}}{l_{n'}} \int_{\mathcal{L}}^{\infty} dl\; e^{l/2}$$

$$\times\; e^{-\frac{1}{2\Delta^2}\left(\frac{l+l_{n'}}{4\pi\tau}-\overline{p}\right)^2} \cos\left(\frac{l+l_{n'}}{4\pi\tau}l\right)\cos\left(\frac{l+l_{n'}}{4\pi\tau}l_{n'}\right) \quad . \qquad (58)$$

An elementary computation leads to

$$F_1^{\mathcal{L}}(\tau,\overline{p}) \;\simeq\; \frac{1}{4\pi\sqrt{2}\Delta} \sum_{\substack{\{l_{n'}\}\\ l_{n'}<\mathcal{L}}} \frac{A_{n',1}}{l_{n'}}$$

$$\times\; \mathrm{Re}\left[\frac{e^{c+\nu^2/\mu}\mathrm{erfc}\left(\frac{\nu}{\sqrt{\mu}}+\mathcal{L}\sqrt{\mu}\right)}{\sqrt{\mu}} + \frac{e^{c+\mathrm{Re}(\nu)^2/\mu^\star}\mathrm{erfc}\left(\frac{\mathrm{Re}(\nu)}{\sqrt{\mu^\star}}+\mathcal{L}\sqrt{\mu^\star}\right)}{\sqrt{\mu^\star}}\right] \qquad (59)$$

with

$$\mu = \frac{1}{32\pi^2\tau^2\Delta^2} - \frac{i}{4\pi\tau} \quad , \quad \nu = \frac{\frac{l_{n'}}{4\pi\tau}-\overline{p}}{8\pi\tau\Delta^2} - \frac{1}{4} - \frac{i l_{n'}}{4\pi\tau} \quad , \quad c = -\frac{\left(\frac{l_{n'}}{4\pi\tau}-\overline{p}\right)^2}{2\Delta^2} + \frac{i l_{n'}^2}{4\pi\tau} \quad . \qquad (60)$$

Next one has to consider the contribution of the lengths with $l_{n'} > \mathcal{L}$ and $l_n \geq l_1$. To deal correctly with the double sum we consider the diagonal contribution, i. e., $n' = n$, separately. The diagonal sum deserves a special consideration because each summand in the periodic–orbit sum contains then a factor $A_{n,k}^2$ which is positive even in the general case where the phase factors χ_n take on both signs $\{\pm 1\}$ with equal probability. This is in sharp contrast to the subdiagonals where the factor $A_{n,k}A_{n',k'}$, $n \neq n'$, takes on both signs in the case of stochastic phase factors, and it is plausible that the diagonal sum dominates all the subdiagonals. This argumentation can, however, only be valid for not too large values of τ, since $F(\tau,\overline{p})$ goes

to one for $\tau \to \infty$ whereas the diagonal sum is proportional to τ, see eq.(63) below. For $\tau \simeq 1/\overline{g}$ the subdiagonals must therefore conspire in a miraculous way to cause a saturation of $F(\tau, \overline{p})$. At present there is no way to derive this behaviour from the length spectrum, and it seems probable that a proof must incorporate very subtle properties of the distribution of the lengths and their phase factors. A computation, where we have assumed a length spectrum with Poisson distributed lengths, i.e., a "length spacing" given by $P(k,s) = \frac{1}{k!}s^k e^{-s}$, with $s = (l_{n+1+k} - l_n)e^{l_n}/(\overline{g}l_n)$, does not lead to a satisfactory description. So it seems to us that very subtle additional properties must play a crucial role at $\tau > 1/\overline{g}$.

In the case of a hyperbolic octagon with positive parity, all phase factors are $\chi_n = +1$ and we show in figure 6 the diagonal sum $n = n'$ in comparison with the subdiagonal $n+1 = n'$ and with the next–subdiagonal $n+2 = n'$ for the parameters $\overline{p} = 20$ and $\Delta = 2.5$. The diagonal sum is already for very small τ proportional to $\overline{g}\tau$ ($\overline{g} = 2$). However, already at slightly larger τ, the subdiagonal is as important as the diagonal. The same is true for the next–subdiagonal shortly thereafter. This next–subdiagonal takes on negative values for $\tau \simeq 0.03$ so that it then can partly cancel the contribution of its precursor. Only the summation over all contributions should yield the correct behaviour. In the following we assume that the diagonal remainder term correctly takes into account for $\tau \ll 1/\overline{g}$ the contribution from the multiple sum even in the case where all phase factors are equal to $\chi_n = 1$.

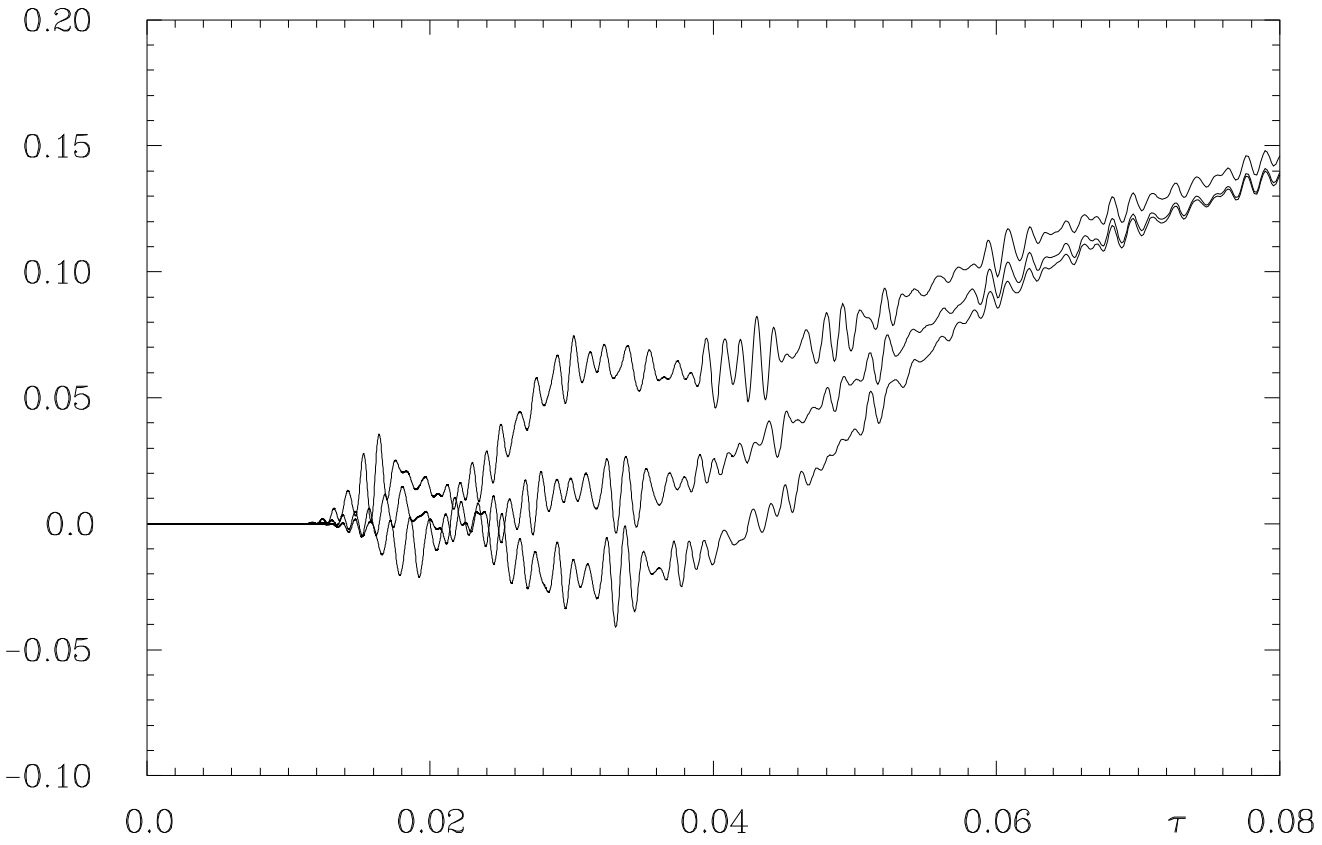

Figure 6: The diagonal sum of $F_1(\tau, \overline{p})$, eq.(47), is shown for $\overline{p} = 20$ and $\Delta = 2.5$ in comparison with the subdiagonal sum and the next–subdiagonal sum, respectively. At $\tau \simeq 0.04$ the diagonal sum is represented by the upper most curve followed by the subdiagonal sum. The lowest curve belongs to the next–subdiagonal.

The remainder of the diagonal contribution is given by

$$F_D^{\mathcal{L}}(\tau,\overline{p}) \;:=\; \frac{1}{\sqrt{2\pi}\pi\Delta} \sum_{\substack{\{l_{n'}\} \\ l_{n'}>\mathcal{L}}} \frac{A_{n',1}^2}{l_{n'}}\, e^{-\frac{1}{2\Delta^2}\left(\frac{l_{n'}}{2\pi\tau}-\overline{p}\right)^2} \cos^2\left(\frac{l_{n'}^2}{2\pi\tau}\right) \quad . \tag{61}$$

Again, $A_{n',1}$ is asymptotically approximated and the sum is replaced by its corresponding integral (57) leading to

$$F_D^{\mathcal{L}}(\tau,\overline{p}) \;\simeq\; \frac{\overline{g}}{\sqrt{2\pi}\pi\Delta} \int_{\mathcal{L}}^{\infty} dl\; e^{-\frac{1}{2\Delta^2}\left(\frac{l}{2\pi\tau}-\overline{p}\right)^2} \cos^2\left(\frac{l^2}{2\pi\tau}\right) \quad . \tag{62}$$

A short calculation gives

$$F_D^{\mathcal{L}}(\tau,\overline{p}) \;\simeq\; \frac{\overline{g}\tau}{2}\,\mathrm{erfc}\left(\frac{\frac{\mathcal{L}}{2\pi\tau}-\overline{p}}{\sqrt{2}\Delta}\right) \;+\; \frac{\overline{g}e^{-p^2/2\Delta}}{8\pi\sqrt{2}\Delta}\;\mathrm{Re}\left[\frac{1}{\sqrt{\mu}}\,e^{\rho^2/4\mu}\,\mathrm{erfc}\left(\frac{\rho}{2\sqrt{\mu}}+2\mathcal{L}\sqrt{\mu}\right)\right] \quad , \tag{63}$$

where μ is given by (60) and $\rho = \frac{\overline{p}}{4\pi\tau\Delta^2}$. The first term gives the leading asymptotic contribution. For $\tau \gg \mathcal{L}/2\pi\overline{p}$ the erfc–function approaches its asymptotic value 2 and the leading term is approximately $F_D^{\mathcal{L}}(\tau,\overline{p}) \sim \overline{g}\tau$.

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