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C. D. Jäkel

Institut für Theoretische Physik, Universität Wien, Austria

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On the Uniqueness of the Equilibrium State for an Interacting Fermion Gas at High Temperatures and Low Densities

Christian D. Jäkel [1]
Institut für Theoretische Physik
Universität Wien

Abstract. Starting from bounded local perturbations we release the spatial cutoff and prove the uniqueness of the KMS state at high temperatures and low densities for the continuous fermion system with pair-interaction proposed by Narnhofer and Thirring.

[1] Present address: II. Inst. f. Theor. Phys., Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, e-mail: i02jak@dsyibm.desy.de

1. Introduction

Rigorous thermodynamics is (and will remain for years to come) a difficult but relevant issue. While spin systems have been studied in some detail, not much is known about continuous systems. Only recently a C*-dynamical model for fermions with pair-interactions was established by Narnhofer and Thirring [1,2]. It is the aim of this article to prove that this system has exactly one equilibrium state at high temperatures and low densities.

Let us shortly review the model and its setting:

In the algebraic framework of quantum statistical mechanics [3-7] a nonrelativistic fermion system is described by a net of local observables,

$$(I \times \mathcal{O}) \to (I, \mathcal{A}(\mathcal{O})), \tag{1}$$

I an open time intervall, $\mathcal{O}$ a bounded, open region in $\mathbb{R}^3$, and $\mathcal{A}(\mathcal{O})$ the corresponding C^*-algebra. For mathematical convenience this net is embedded in its C*-closure

$$(\mathbb{R}, \mathcal{A}) := \overline{\bigcup_{(I,\mathcal{O})\subset\mathbb{R}^4} (I, \mathcal{A}(\mathcal{O}))}. \tag{2}$$

The *algebra of observables* $\mathcal{A}$ is identified with the self-adjoint, gauge invariant elements of a *field algebra* $\mathcal{F}$ [8]. If we have only one type of fermions, then the unique (up to *-isomorphisms) C^*-algebra generated by $\mathbb{1}$ and $a(f)$, $f \in L^2(\mathbb{R}^3, \mathbb{C})$, $f \to a(f)$ antilinear, satisfying the *canonical anticommutation relations*,

$$\{a(f), a(g)\} = 0, \qquad \{a(f), a^*(g)\} = (f|g)\,\mathbb{1}, \qquad f, g \in L^2(\mathbb{R}^3, \mathbb{C}), \tag{3}$$

is the standard choice for $\mathcal{F}$. Operators smeared out with coherent wavefunctions[1] of the form

$$f_{q,p} := (2\pi)^{-3/4} e^{-(x-q)^2/2+ipq} \in L^2(\mathbb{R}^3, \mathbb{C}) \tag{4}$$

provide a convenient coordinate system in $\mathcal{F}$. Heuristically $a_z := a(f_z)$ (resp. a_z^*) can be interpreted as the removal (resp. creation) of a particle concentrated around $z = (q,p)$ in phase space $T^*(\mathbb{R}^3)$. This is consistent with the interpretation of $\rho_z = a_z^* a_z$ as a density operator in phase space.

The state[2] independent dynamics of a net of local observables $(\mathbb{R}, \mathcal{A})$ is provided by a time-evolution automorphism τ. For fermions with pair-interactions such a "C*-dynamical model", $(\mathbb{R}, \mathcal{A}, \tau)$, has recently been constructed by Narnhofer and Thirring [1,2]. We give the precise formulation:

Theorem 1.1. *Let τ° denote the free time evolution automorphism, $\tau_t^\circ(a(f)) = a(e^{-i\Delta t} f)$, and consider the following sequence of automorphisms,*

$$\tau_t^n(a) := \tau_t^\circ(a) + \sum_{k=1}^{\infty} \frac{i^k}{k!} \int\limits_{[0,t]^k} d(t_1 ... t_k) \;\; [\tau_{t_k}^\circ(v_n), [..., [\tau_{t_1}^\circ(v_n), \tau_t^\circ(a)]...]], \qquad a \in \mathcal{F}, \tag{5}$$

where the interaction $v_n \in \mathcal{A}$ is given by[3]

$$v_n = \int_n d^3q d^3q' d^3p d^3p' \, a_{q,p}^* a_{q',p'}^* V(|q-q'|, |p-p'|) a_{q',p'} a_{q,p}, \tag{6}$$

Then $\tau := \mathrm{s} - \lim_{n\to\infty} \tau^n$ — where $\mathrm{s} - \lim$ denotes[4] *the strong (=pointwise) limit in $\mathcal{B}(\mathcal{A})$ — exists for all $V(|r|, |s|) \in L^1(\mathbb{R}^6, d^3r d^3s) \cap C(\mathbb{R}^6)$ as a strongly continuous one parameter group of automorphisms of $\mathcal{F}$.*

---

1 They are characterised by minimal uncertainty with respect to the Heisenberg relation $\Delta x \Delta p \geq 1/2$ and form a total set in $L^2(\mathbb{R}^3)$. Therefore $\mathbb{1}$ and $a_z := a(f_z)$, $z \in \mathbb{R}^6$ generate the unique CAR algebra over $L^2(\mathbb{R}^3, \mathbb{C})$.

2 States are defined as positive, normalized linear functionals over $\mathcal{A}$.

3 The integration runs over the region $|q|, |q'|, |p|, |p'| \leq n$.

4 $\mathcal{B}(\mathcal{A})$ denotes the Banach space of bounded linear operators from $\mathcal{A}$ to $\mathcal{A}$.

The time evolution τ is Galilei invariant and exists without reference to a special representation. In principle one could therefore tackle problems of nonequilibrium thermodynamics. But work in this direction seems difficult and much remains to be done. As a first result in this direction we note [9-10] that $(\mathbb{R}, \mathcal{A}, \tau)$ is *mixing*,

$$\lim_{t\to\infty} \|a\tau_t(b)\| = \|a\| \, \|b\|, \qquad \forall a, b \in \mathcal{A}. \tag{7}$$

This means that the system behaves totally chaotic: every information gets lost since all propositions eventually become compatible. This clearly excludes the possibility of quasi-periodic observables. It is unknown whether or not the system is in addition *asymptotically Abelian*,

$$\lim_{t\to\infty} \| [a, \tau_t(b)] \| = 0, \qquad \forall a, b \in \mathcal{A}. \tag{8}$$

This would mean that, as time passes, the disturbance due to a measurement diffuses so widely that local observables are not affected at much later times. Roughly speaking, the system would act classically on a macroscopic timescale.

The existence of equilibrium states for the Narnhofer-Thirring model has been established in [2]. The algebraic framework provides an excellent framework for a rigorous discussion of equilibrium states: When the dynamical law is changed by a local perturbation, which is slowly switched on and slowly switched off again, then an equilibrium state returns to its original form at the end of this procedure. This condition of *adiabatic invariance* is expressed [11] by the stability requirement [12],

$$\lim_{t\to\infty} \int_{-t}^{t} dt\, \omega([a, \tau_t(b)]) = 0, \qquad \forall a, b \in \mathcal{A}. \tag{9}$$

In a pioniering work Haag, Kastler and Trych-Pohlmeyer [12] showed that the characterisation (9) of an equilibrium state leads to a sharp mathematical criterion, first encountered by Haag, Hugenholtz and Winnink [13] and more implicitly by Kubo [14], Martin and Schwinger [15]:

Definition. Let $(\mathbb{R}, \mathcal{A}, \tau)$ be a C*-dynamical sytem in the sense of [3]. The state ω over $\mathcal{A}$ is defined to be a (τ, β)-*KMS state* for some $\beta \in \mathbb{R} \cup \{\pm\infty\}$, if

$$\omega(ab) = \omega(b\tau_{i\beta}(a)) \tag{10}$$

for all a, b in a norm dense, τ-invariant *-subalgebra of $\mathcal{A}_\tau$, where $\mathcal{A}_\tau \subset \mathcal{A}$ denotes the set of analytic elements for τ.

Thus equilibrium states are characterised[3] by a real parameter β, which has the meaning of inverse temperature. The following result [4; 5.4.25] shows how the chemical potential $\mu \in \mathbb{R}$ arises:

Lemma 1.2. *Let $\beta \in \mathbb{R}/\{0\}$ and ω_β be an extremal (τ, β)-KMS state over $\mathcal{A}$. Furthermore let $t \to \gamma_t$ denote the group of gauge automorphisms of $\mathcal{A}$, $\gamma_t(a(f)) = a(e^{-it}f)$. Each extremal τ-invariant extension $\phi_{\beta,\mu}$ of ω_β to $\mathcal{F}$ is a KMS state at value β for the *-automorphism $t \to \tau_t\gamma_{t\mu}$, where $\mu \in \mathbb{R}$ is uniquely determined by $\phi_{\beta,\mu|\mathcal{A}} =: \omega_{\beta|\mathcal{A}}$.*

The chemical potential μ is related to the particle density, according to

$$\rho(\mu) = \left(\frac{4\pi n^3}{3}\right)^{-1} \int_n d^3q \int d^3p \;\; \omega_{\beta,\mu}(a^*_{q,p}a_{q,p}) \in \mathbb{R}^+_\circ. \tag{11}$$

We can therefore speak of the set $S_{T,\rho}$ of equilibrium states for a given temperature T and particle density ρ. This paper is entirely devoted to the high-temperature-low-density regime where one expects that droplets'

[3] The temperature T is defined as $T{=}d\epsilon/d\sigma$; where ϵ (resp. σ) denotes the energy (resp. entropy) density. For the free fermion gas both can be easily computed and one finds $\beta{=}1/kT$, where k is the Boltzmann constant.

disappear and the interacting equilibrium state differs only slightly from the equilibrium state for the free time evolution. Nothing will be said about KMS states at low temperatures or high densities, where phase transitions may occur.

We conclude this introduction with an identity that will be explored in detail in a more specialized setting in the next chapter.

Theorem 1.3. *Let $\mathcal{A}_\tau \cap \mathcal{A}_{\tau^\circ}$ form a dense set in $\mathcal{A}$ and $\omega_{\beta,\mu}$ be a (τ,β,μ)-KMS state. Then*

$$\omega_{\beta,\mu}\big((a+\tau^\circ_{i\beta}\gamma_{i\beta\mu}(a))b\big) = \omega_{\beta,\mu}\big(\{b,\tau^\circ_{i\beta}\gamma_{i\beta\mu}(a)\}\big) + \omega_{\beta,\mu}\big(b(\tau_{i\beta}\gamma_{i\beta\mu} - \tau^\circ_{i\beta}\gamma_{i\beta\mu})(a)\big), \tag{12}$$

for all $a \in \mathcal{A}_\tau \cap \mathcal{A}_{\tau^\circ}$.

The second term on the right-hand side of the equation (12) can be expressed as a power series in β, without constant term, and iteration of the resulting equation will allow us to deduce the high temperature behaviour of our system. In fact, if $\tau = \tau^\circ$, then iteration of the recursion relation

$$\omega^\circ_{\beta,\mu}\big(a^*(1+e^{-\beta(\mathbf{H}_\circ-\mu)}f_1)a^*(f_2)\dots a(f_{2n})\big) = \omega^\circ_{\beta,\mu}\big(\{a^*(f_2)\dots a(f_{2n}), a(e^{-\beta(\mathbf{H}_\circ-\mu)}f)\}\big), \tag{13}$$

expresses the quasifree n-point function in terms of its two point functions ([4], p.49). In the interacting case (12) this idea can now be combined with the perturbation expansion (5) in the coupling constant for τ.

Proof. Applying the KMS relation we find

$$\begin{aligned}\omega_{\beta,\mu}(ab) &= \omega_{\beta,\mu}(b(\tau_{i\beta}\gamma_{i\beta\mu} - \tau^\circ_{i\beta}\gamma_{i\beta\mu})(a)) + \omega_{\beta,\mu}(b\tau^\circ_{i\beta}\gamma_{i\beta\mu}(a)) \\ &= \omega_{\beta,\mu}(b(\tau_{i\beta}\gamma_{i\beta\mu} - \tau^\circ_{i\beta}\gamma_{i\beta\mu})(a)) - \omega_{\beta,\mu}(\tau^\circ_{i\beta}\gamma_{i\beta\mu}(a)b) + \omega_{\beta,\mu}(\{b,\tau^\circ_{i\beta}\gamma_{i\beta\mu}(a)\}).\end{aligned} \tag{14}$$

and by linearity and reordering we find (12). □

2. High Temperature Expansion

The KMS condition connects the equilibrium state with the time evolution, allowing us to derive a high temperature expansion for equilibrium states[4]. The basic strategy is to controll the n-point functions $\{\omega_{\beta,\mu}(a^*_{q_1,p_1}\dots a^*_{q_n,p_n}a_{q_{n+1},p_{n+1}}\dots a_{q_{2n},p_{2n}});\ q_1,\dots,p_{2n}\in\mathbb{R}^3; n\in\mathbb{N}\}$ by considering them as bounded functions over the index set $\{q_1,\dots,p_{2n}\in\mathbb{R}^3; n\in\mathbb{N}\}$. More precisely, let $\mathcal{X}$ denote the Banach space of bounded complex antisymmetric functions Γ: $\bigoplus_{n=1}^{\infty}\mathbb{R}^{12n} \to l^\infty(\mathbb{N})$, equipped with the supremum norm. If $\Omega_{\beta,\mu}$ denotes the family $\{\omega_{\beta,\mu}(a^*_{q_1,p_1}\dots a_{q_{2n},p_{2n}});\ q_1,\dots,p_{2n}\in\mathbb{R}^3; n\in\mathbb{N}\}$, it follows that $\Omega_{\beta,\mu}\in\mathcal{X}$ and $\|\Omega_{\beta,\mu}\| = 1$. The identity (12), which is obeyed by KMS states, can be translated into an inhomogeneous integral equation for $\Omega_{\beta,\mu}$,

$$(\mathbb{1} - \mathbf{K}_{\beta,\mu} - \mathbf{L}_{\beta,\mu})\Omega_{\beta,\mu} = \Gamma^\circ_{\beta,\mu}, \tag{15}$$

where $\Gamma^\circ_{\beta,\mu}$ seperates out the free two point function, $\mathbf{K}_{\beta,\mu}$ is defined by the free time evolution and $\mathbf{L}_{\beta,\mu}$ takes into account the difference between the free and the interacting time evolution. Hence $\Omega_{\beta,\mu}$ is uniquely determined, and

$$\Omega_{\beta,\mu} = \sum_{n=0}^{\infty}(\mathbf{K}_{\beta,\mu}+\mathbf{L}_{\beta,\mu})^n\Gamma^\circ_{\beta,\mu}, \tag{16}$$

whenever $\|\mathbf{K}_{\beta,\mu}+\mathbf{L}_{\beta,\mu}\| < 1$. Since we know that (τ,β,μ)-KMS states exist for all densities μ and temperatures β, in the case of a unique $\Omega_{\beta,\mu}$ the correspondence

$$\omega_{\beta,\mu}(a^*_{q_1,p_1}\dots a^*_{q_n,p_n}a_{q_{n+1},p_{n+1}}\dots a_{q_{2n},p_{2n}}) = \Omega_{\beta,\mu}(q_1,\dots,p_n,q_{n+1},\dots,p_{2n}), \tag{17}$$

is one to one, and defines a unique (τ,β,μ)-KMS state.

[4] Similar ideas have been applied to spin systems in the past. For the benefit of the reader familiar with spin systems we closely follow the presentation in [4].

The details are are as follows. Let $\Gamma = \{\Gamma_n; n \in \mathbf{N}\} \in \mathcal{X}$, then we define[5]

$$\Gamma^\circ_{\beta,\mu}(q_1,\ldots,p_{2n}) := \delta_{1,n}(f_{q_1,p_1}|\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}}f_{q_2,p_2}),$$

$$(\mathbf{K}_{\beta,\mu}\Gamma)_n(q_1,\ldots,p_{2n}) := \sum_{l=1}^{n}(\delta_{1,n}-1)^{n-l+1}(h_l|\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}}f_{q_1,p_1})\Gamma_{n-1}(q_2,\ldots,p_{n+l-1},q_{n+l+1},\ldots,p_{2n}),$$

$$(\mathbf{L}_{\beta,\mu}\Gamma)_n(q_1,\ldots,p_{2n}) := \sum_{k=0}^{\infty}\int d^3q_1^{(k)}\ldots d^3p_{2k}^{(k)}\ \Upsilon_k(q_1,\ldots,p_{2n};q_1^{(k)},\ldots,p_{2k}^{(k)})\ \Gamma_k(q_1^{(k)},\ldots,p_{2k}^{(k)}), \tag{18}$$

where $h_l := f_{z_{n+l}} - \sum_{m=1}^{l}(f_{z_{n+m}}|f_{z_{n+l}})f_{z_{n+m}}$, using the fact that in expressions of the form $a(f_1)\ldots a(f_n)$ only the orthogonal part contributes. The functions $\Upsilon_k \in L^1(\mathbf{R}^{12k}, d^3q_1^{(k)}\ldots d^3p_{2k}^{(k)})$ are pretty implicitly defined by

$$\sum_{k=0}^{\infty}\int d^3q_1^{(k)}\ldots d^3p_{2k}^{(k)}\ \Upsilon_k(q_1,\ldots,p_{2n};q_1^{(k)},p_1^{(k)},\ldots,q_{2k}^{(k)},p_{2k}^{(k)})\ a^*_{q_1^{(k)},p_1^{(k)}}\ldots a^*_{q_k^{(k)},p_k^{(k)}}a_{q_{k+1}^{(k)},p_{k+1}^{(k)}}\ldots a_{q_{2k}^{(k)},p_{2k}^{(k)}} =$$

$$= \sum_{i=1}^{n}\frac{(-)^{i+1}}{n}a^*_{q_1,p_1}\ldots a^*_{q_{i-1},p_{i-1}}a^*_{q_{i+1},p_{i+1}}\ldots a_{q_{2n},p_{2n}}\times$$

$$\times\sum_{m=1}^{\infty}\frac{(-)^m}{m!}\lim_{n\to\infty}\int_{[0,\beta]^m} d(\beta_1\ldots\beta_m)\ \ [\tau^\circ_{\beta_m}(v_n),[\ldots,[\tau^\circ_{\beta_1}(v_n),a^*\left(\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}}f_{q_i,p_i}\right)]\ldots]] \tag{19}$$

This obviously leads to an expansion of the functions Υ_k in powers of β. Up to order three they have been explicitly computed in [16]. If $\mathbf{L}_{\beta,\mu}$ vanishes, then $\mathbf{K}^n_{\beta,\mu}$ generates the quasifree n-point functions from the two-point function $\Gamma^\circ_{\beta,\mu}$, and the following result tells us via equation (16) and (17) that there is only one (τ,β,μ)-KMS state for given β and μ.

Lemma 2.1. *Let* $\mathbf{H}_\circ$ *be a positive selfadjoint operator acting on* $L^2(\mathbf{R}^3, d^3x)$. *Then*

$$\|\mathbf{K}_{\beta,\mu}\| \le (1+e^{-\beta\mu})^{-1}, \tag{20}$$

for all $\beta \in \mathbf{R}^+$, $\mu \in \mathbf{R}$.

Proof. We have to evaluate $\|\mathbf{K}_{\beta,\mu}\| = \sup_{\|\Gamma\|=1}\|\mathbf{K}_{\beta,\mu}\Gamma\|$. Let $\{h_l\}_{l\in\mathbb{N}} \supset \{h_l\}_{l\in(1,\ldots,n)}$ be a set of orthonormal functions, then

$$\begin{aligned}\sup_{\|\Gamma\|=1}\|\mathbf{K}_{\beta,\mu}\Gamma\|^2 &= \sup_{\|\Gamma\|=1}\ \sup_{n\in\mathbb{N}/\{1\}}\ \sup_{(q_i,p_i)\in\mathbb{R}^6}\left|\sum_{l=1}^{n}(h_l|\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}}f_{q_1,p_1})\Gamma_{n-1}(q_2,\ldots,p_{n+l-1},q_{n+l+1},\ldots,p_{2n})\right|^2\\ &\le \sup_{(q_1,p_1)\in\mathbb{R}^6}\ \sum_{l=1}^{\infty}\left|(h_l|\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}}f_{q_1,p_1})\right|^2\\ &\le \sup_{(q,p)\in\mathbb{R}^6}\ \left\|\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}}f_{q,p}\right\|_2^2\\ &\le (1+e^{-\beta\mu})^{-2}.\end{aligned} \tag{21}$$

□

Thus uniqueness of the KMS state is ensured if $\|\mathbf{L}_{\beta,\mu}\| < (1+e^{\beta\mu})^{-1}$. Introducing an additional momentum cutoff, the norm of $\mathbf{L}_{\beta,\mu}$ will be bounded in the next section.

[5] We will show in Appendix A that the following definition reproduces (12) in the form given in (15).

3. The Narnhofer-Thirring Model

The knowledge of truely interacting theories like the Narnhofer-Thirring model is still scant and we can so far only release the spatial cutoff, such that v_n becomes

$$v_s^{cut} = \int_s d^3q d^3q' \int_{p_{max}} d^3p d^3p' \; a^*_{q,p} a^*_{q',p'} V(|q-q'|, |p-p'|) a_{q',p'} a_{q,p}. \tag{22}$$

We have to evaluate $\|\mathbf{L}_{\beta,\mu}\| = \sup_{\|\Gamma\|=1} \|\mathbf{L}_{\beta,\mu}\Gamma\|$. By definition,

$$\begin{aligned} \sup_{\|\Gamma\|=1} \|\mathbf{L}_{\beta,\mu}\Gamma\| &\leq \sup_{\|\Gamma\|=1} \sup_{(q_1,\dots p_{2n})\in\mathbb{R}^{12n}} \sum_{k=0}^{\infty} \left| \int d^3q_1^{(k)} \dots d^3p_{2k}^{(k)} \; \Upsilon_k(q_1,\dots,p_{2n}; q_1^{(k)},\dots,p_{2k}^{(k)}) \; \Gamma_k(q_1^{(k)},\dots,p_{2k}^{(k)}) \right| \\ &\leq \sup_{\|\Gamma\|=1} \sup_{(q_1,\dots p_{2n})\in\mathbb{R}^{12n}} \sum_{k=0}^{\infty} \int d^3q_1^{(k)} \dots d^3p_{2k}^{(k)} \left| \Upsilon_k(q_1,\dots,p_{2n}; q_1^{(k)},\dots,p_{2k}^{(k)}) \right|, \end{aligned} \tag{23}$$

Let us now have a look at the definition (19) of the functions Υ_k. If we rewrite

$$\lim_{s\to\infty} \int_{[0,\beta]^m} d(\beta_1...\beta_m) \;\; [\tau^\circ_{\beta_m}(v_s^{cut}), [..., [\tau^\circ_{\beta_1}(v_s^{cut}), a^*\left(\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}} f_{q_1,p_1}\right)]...]] \tag{24}$$

in terms of anticommutators, then every new order in the coupling constant introduces a product of two creation and two annihilation operators. In the first order there are two nonvanishing anticommutators for

$$\int d^3q d^3q' \int_{p_{max}} d^3p d^3p' \; V(|q-q'|, |p-p'|) \; \tau^\circ_{\beta_1}(a^*_{q,p} a^*_{q',p'}) [\tau^\circ_{\beta_1}(a_{q',p'} a_{q,p}), a^*\left(\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}} f_{q_1,p_1}\right)\}, \tag{25}$$

and we are left with two products of two creation and one annihilation operators. If we expand the next commutator into anticommutators, each of the 2+1 creation and annihilation operators has two non vanishing anticommutators with the interaction. Therefore in second order we have $2\cdot(2+1)\cdot 2$ terms of length (= number of creation and annihilation operators) $5 = 2\cdot 2+1$. By the same line of arguments the number of nonvanishing terms in third order is given by $2\cdot(2+1)\cdot 2\cdot(2\cdot 2+1)\cdot 2$. Taking into account the $1/m!$ factor from the definition (19), we find that the number of terms can be estimated by

$$\frac{2\cdot(2+1)\cdot 2\cdot(2\cdot 2+1)\cdot\ldots\cdot 2\cdot(m\cdot 2+1)\cdot 2}{1\cdot 2\cdot 3\cdot\ldots\cdot m} = 2^{m+1}\cdot(2+1)\cdot(2+\frac{1}{2})\cdot\ldots\cdot(2+\frac{1}{m}) \leq 2^{m+1}3^m, \tag{26}$$

for arbitrary order m. As far as the normal ordering is concerned, we have to (anti-)commute $2m$ creation operators from the m-th order of the interaction to the left side of the products. This can be done with only one nonvanishing anticommutator, because, as we pointed out before, in expressions of the form $a(f_1)\ldots a(f_n)$ only the orthogonal part contributes[6]. While the normal ordering introduces new combinatorial coefficients, the substantial part in finding bounds for $\|\mathbf{L}_{\beta,\mu}\|$ is unaffected by the details of this normal ordering, so we will not dwell on this tedious point. The total number of terms is bounded by

$$2^{m+1}3^m \times 2.(2m) < 4 \times 8^m. \tag{27}$$

Now we have to re-express all the time evolved operators in terms of the original ones. For example,

$$\tau^\circ_{\beta_1}(a_{q,p}) = \int d^3\hat{q} d^3\hat{p} \; (f_{q,p}|e^{\mathbf{H}_\circ\beta_1} f_{\hat{q},\hat{p}}) \; a_{\hat{q},\hat{p}}. \tag{28}$$

[6] This fact will be illustrated in Appendix B.

The momentum cutoff introduced in (22) makes it easy to bound these integrals,

$$\int d^3\hat{q} d^3\hat{p}\, |(f_{q,p}| e^{\mathbf{H}_\circ \beta_1} f_{\hat{q},\hat{p}})| = \left(\frac{(2\pi)^2(1+\beta_1)}{(1+2\beta_1)}\right)^{3/2} \exp\left(\frac{\beta_1}{1+2\beta_1} p^2\right)$$
$$\leq C_1(p_{max}). \tag{29}$$

It remains to find a bound for the first anticommutator: For all $q,p \in \mathbf{R}^3$ and $0 \leq |\beta_1| \leq \beta < \infty$, we find

$$\int d^3q' \int_{p_{max}} d^3p' \, |(f_{q',p'}| \frac{e^{-\mathbf{H}_\circ(\beta-\beta_1)}}{e^{-\beta\mu} + e^{-\mathbf{H}_\circ \beta}} f_{q,p})| \leq \left|\left|\frac{e^{-\mathbf{H}_\circ(\beta-\beta_1)}}{e^{-\beta\mu} + e^{-\mathbf{H}_\circ\beta}}\right|\right| \frac{4\pi^2 p_{max}^3}{3} \int d^3q' d^3x \, |f_{q',p'}(x) f_{q,p}(x)|$$
$$\leq \frac{4\pi^2 p_{max}^3}{3} \int d^3q' e^{-\frac{1}{4}(q-q')^2} := C_2(p_{max}). \tag{30}$$

Thus $\|\mathbf{L}_{\beta,\mu}\| \leq C_2 \sum_{m=0}^\infty 4 \times 8^m \beta^m C_1^m \|V\|_1^m$, and this proves our main result:

Theorem 3.1. *Let $C_2 < 1/8$. For the Narnhofer-Thirring model with a momentum cutoff (22) there exists one and only one KMS state for $\beta < \beta_{max}(\mu)$, μ fixed, where the dependence of $\beta_{max}(\mu)$ on μ is given through*

$$1 = \frac{1}{1+e^{-\beta_{max}\mu}} + \frac{4C_2}{1 - 8\beta_{max} C_1 \|V\|_1}. \tag{31}$$

We believe that the momentum cutoff can be removed, but we were not able to settle this question. For a discussion we refer the reader to [16].

Appendix A

We will now motivate the definition (18) and show how the inhomogeneous integral equation (15) can be derived from the identity (12). Let us first consider the case n=1. Then (12) reads as follows

$$\omega_{\beta,\mu}\big(a^*(1+e^{-\beta(\mathbf{H}_\circ-\mu)} f_1) a^*(f_2)\big) = \{a^*(f_2), a(e^{-\beta(\mathbf{H}_\circ-\mu)} f_1)\} + \omega_{\beta,\mu}\big(a^*(f_2)(\tau_{i\beta}\gamma_{i\beta\mu} - \tau^\circ_{i\beta}\gamma_{i\beta\mu})(a^*(f_1))\big)$$
$$= (f_2| e^{-\beta(\mathbf{H}_\circ-\mu)} f_1) + \omega_{\beta,\mu}\big(a^*(f_2)(\tau_{i\beta}\tau^\circ_{-i\beta} - id)(a^*(e^{-\beta(\mathbf{H}_\circ-\mu)} f_1))\big). \tag{32}$$

Therefore by linearity and the replacement of f_1 by $(1+e^{-\beta(\mathbf{H}_\circ-\mu)})^{-1} f_{q_1,p_1}$ one finds

$$\Omega_{\beta,\mu}(q_1,\ldots,p_2) = (f_{q_1,p_1}| \frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}} f_{q_2,p_2}) - \sum_{m=1}^\infty \frac{(-)^m}{m!} \lim_{n\to\infty} \int_{[0,\beta]^m} d(\beta_1 \ldots \beta_m) \times$$
$$\times\, \omega_{\beta,\mu}\Big(a_{q_2,p_2}\ [\tau^\circ_{\beta_m}(v_n), [\ldots, [\tau^\circ_{\beta_1}(v_n), a^*\Big(\frac{e^{-\beta(\mathbf{H}_\circ-\mu)}}{1+e^{-\beta(\mathbf{H}_\circ-\mu)}} f_{q_1,p_1}\Big)]\ldots]]\Big)$$
$$= \Gamma^\circ_{\beta,\mu}(q_1,\ldots,p_2) + \mathbf{L}_{\beta,\mu}\Omega_{\beta,\mu}(q_1,\ldots,p_2). \tag{33}$$

Let us now set $z = (q,p)$ and consider the case n=2. Applying the same procedure we find

$$\Omega_{\beta,\mu}(z_1,\ldots,z_4) = \Gamma^\circ_{\beta,\mu}(z_1,z_3)\Omega_{\beta,\mu}(z_2,z_4) - \Gamma^\circ_{\beta,\mu}(z_1,z_4)\Omega_{\beta,\mu}(z_2,z_3) + \mathbf{L}_{\beta,\mu}\Omega_{\beta,\mu}(z_1,\ldots,z_4)$$
$$= \mathbf{K}_{\beta,\mu}\Omega_{\beta,\mu}(z_1,\ldots,z_4) + \mathbf{L}_{\beta,\mu}\Omega_{\beta,\mu}(z_1,\ldots,z_4). \tag{34}$$

A short moment of reflection will convince the reader that the second line holds true for arbitrary n. Together with the Kronecker symbols in the definition of (18), we find $\Omega_{\beta,\mu} = \Gamma^\circ_{\beta,\mu} + \mathbf{K}_{\beta,\mu}\Omega_{\beta,\mu} + \mathbf{L}_{\beta,\mu}\Omega_{\beta,\mu}$, and (15) is a consequence of linearity.

Appendix B

We will now show that terms of the form

$$\{a^*(f_1)\dots a^*(f_n), a(g)\}, \tag{35}$$

produce only one single operator. Let us consider the case $n = 3$. If the functions f_1, f_2, f_3 are linear dependent then (35) vanishes. The same holds true if f_1, f_2, f_3, g are linear independent. Now assume that g, e_1, e_2 forms an orthogonal basis for the linear space spanned by f_1, f_2, f_3. Then

$$\begin{aligned}
&\{a^*((g|f_1)g + (e_1|f_1)e_1 + (e_2|f_1)e_2)a^*((g|f_2)g + (e_1|f_2)e_1 + (e_2|f_2)e_2)a^*((g|f_3)g + (e_1|f_3)e_1 + (e_2|f_3)e_2), a(g)\} \\
&= \quad (g|f_1)(e_1|f_2)(e_2|f_3)\{a^*(g)a^*(e_1)a^*(e_2), a(g)\} - (g|f_1)(e_2|f_2)(e_1|f_3)\{a^*(g)a^*(e_1)a^*(e_2), a(g)\} \\
&\quad -(e_1|f_1)(g|f_2)(e_2|f_3)\{a^*(g)a^*(e_1)a^*(e_2), a(g)\} + (e_1|f_1)(e_2|f_2)(g|f_3)\{a^*(g)a^*(e_1)a^*(e_2), a(g)\} \\
&\quad +(e_2|f_1)(g|f_2)(e_1|f_3)\{a^*(g)a^*(e_1)a^*(e_2), a(g)\} - (e_2|f_1)(e_1|f_2)(g|f_3)\{a^*(g)a^*(e_1)a^*(e_2), a(g)\} \\
&= \quad \Big((g|f_1)(e_1|f_2)(e_2|f_3) - (g|f_1)(e_2|f_2)(e_1|f_3) - (e_1|f_1)(g|f_2)(e_2|f_3) \\
&\qquad + (e_1|f_1)(e_2|f_2)(g|f_3) + (e_2|f_1)(g|f_2)(e_1|f_3) - (e_2|f_1)(e_1|f_2)(g|f_3)\Big) \times a^*(e_1)a^*(e_2). \qquad (36)
\end{aligned}$$

The same line of arguments holds true for arbitrary $n \in N$.

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