R/94/9
DESY 94-092
hep-th/9406015
June, 1994

Massive (p,q)-supersymmetric sigma models revisited

G. Papadopoulos

II. Institute for Theoretical Physics
University of Hamburg
Luruper Chaussee 149
22761 Hamburg

and

P. K. Townsend

DAMTP
Silver Street
University of Cambridge

ABSTRACT

We recently obtained the conditions on the couplings of the general two-dimensional massive sigma-model required by (p,q)-supersymmetry. Here we compute the Poisson bracket algebra of the supersymmetry and central Noether charges, and show that the action is invariant under the automorphism group of this algebra. Surprisingly, for the (4,4) case the automorphism group is always a subgroup of $SO(3)$, rather than $SO(4)$. We also re-analyse the conditions for (2,2) and (4,4) supersymmetry of the zero torsion models without assumptions about the central charge matrix.

1. Introduction

In a recent work [1] we investigated the restrictions imposed by (p,q) supersymmetry on the general two-dimensional supersymmetric sigma model with scalar potential. Omitting fermions, the action is

$$I = \int dxdt \left[(g+b)_{ij} \partial_{\ddagger} \phi^i \partial_{=} \phi^j - V(\phi) \right] , \tag{1.1}$$

where ϕ is a map from the two-dimensional Minkowski space-time, with light-cone co-ordinates $(x^{\ddagger}, x^{=})$, into the target manifold $\mathcal{M}$ with metric g. The two-form b is a locally-defined potential for a globally-defined 'torsion' three-form H with components $H_{ijk} = \frac{3}{2}\partial_{[i} b_{jk]}$, and V is the scalar potential. Since V contains no derivatives its presence requires a coupling constant m with dimensions of mass; for this reason we refer to models with $V \neq 0$ as 'massive' sigma-models.

Here we shall be interested in sigma-models with at least (1,1) supersymmetry for which the scalar potential takes the form [2]

$$V = \frac{1}{4} m^2 g^{ij} (u - X)_i (u - X)_j , \tag{1.2}$$

where X is a (possibly zero) Killing vector field on $\mathcal{M}$ and u is a one-form on $\mathcal{M}$ whose exterior derivative du is determined by X and H via the formula

$$X^k H_{kij} = \partial_{[i} u_{j]} . \tag{1.3}$$

Additional supersymmetries further restrict the scalar potential by imposing conditions on X and u. To find the weakest possible conditions one must allow for the action of a possible total of pq central charges on the sigma-model fields. These 'off-shell' central charges are associated with pq mutually-commuting Killing vector fields $\{Z_{I'I}; I = 0, 1, \ldots, p-1,\ I' = 0, 1, \ldots q-1\}$ of which X is one. Each of

them is paired with a one-form $u^{I'I}$ satisfying

$$Z^k_{I'I} H_{kij} = \partial_{[i} u^{I'I}_{j]} \ , \tag{1.4}$$

in precise analogy with (1.3). Requiring that the (1,1)-supersymmetric sigma-model action be invariant under additional supersymmetry transformations leads to conditions on the vector-field/one-form pairs $\{Z_{I'I}, u^{I'I}\}$ which we obtained in our previous work [1] and which we summarise in the following section.

We follow this with a presentation of the Noether charges associated with the supersymmetry and central charge transformations. We then show that closure of the algebra of Poisson brackets of these charges requires slightly stronger conditions conditions than those obtained from consideration of the transformations alone, at least if the flat two-dimensional spacetime is assumed to be Minkowski spacetime. Specifically, whereas closure of the algebra of transformations was found to require certain functions to be constants, closure of the Poisson bracket algebra requires these constants to vanish.

One purpose of this paper is to extend these results to include the transformations associated with the automorphism group of the supersymmetry algebra with central charges. By this we mean the subgroup of the automorphism group of the supersymmetry algebra without central charges that *leaves invariant the matrix of central charges*. Thus, in our usage, which we believe to be standard in the context of supersymmetry, central charges are central not only in the supersymmetry algebra but also in the extension of this algebra by its automorphism group. For the (2,2) models this group is a subgroup of $SO(2)$ and for the (4,4) models it is a subgroup of $SO(3)$ (rather than $SO(4)$). We show that the automorphism group can be realized in terms of transformations of the sigma model fields.

Finally, we also presented in [1] an analysis of the conditions required by (p,p) supersymmetry in massive models without torsion, which were first considered by Alvarez-Gaumé and Freedman [3]. We argued in section (7) of [1] that no generality would be lost if the Killing vector matrix $Z_{I'I}$ of the (p,p) models were assumed

to be diagonal and we deduced the consequences for V under this assumption. Our argument was based on the fact that V is invariant under $SO(p) \times SO(q)$ transformations of the matrix $Z_{I'I}$ which, if $p = q$, can be used to diagonalize it. We implicitly assumed that such an $SO(p) \times SO(q)$ transformation could be effected by some redefinition of the fermion fields. Unfortunately, this turns out not to be true so our previous results for the potentials of the (p,p) models without torsion must be considered as special cases of a possibly more general result. Another purpose of this paper is to determine the form of the potential V for these models without making any assumptions about the matrix $Z_{I'I}$ of Killing vectors. We present this more general analysis in section (4). Our new result for the (2,2) models without torsion is indeed more general than the result of [1] and is complete agreement with eq. 50 of [3][$\star$. We consider this to be a useful check on our results for the general (p,q) case with torsion. Our previous conclusions concerning the massive (4,4) models without torsion remain unchanged but we show here that the automorphism group of the supersymmetry algebra of these models is always $SO(3)$ (whereas not all massive (2,2) models are $SO(2)$ invariant).

2. Massive supersymmetric sigma-models

In the presence of an off-shell central charge standard (1,1) superspace methods are inapplicable. The most economical formulation of the general (p,q)-supersymmetric sigma model is in terms of (1,0) superfields. The (1,0)-superspace action of the general (1,1)-supersymmetric model is a functional of the bosonic (1,0)-superfields ϕ^i and the fermionic (1,0)-superfields ψ^i, and takes the form [2]

$$S = \int d^2x d\theta^+ \left\{ D_+\phi^i \partial_= \phi^j (g_{ij} + b_{ij}) + i\psi_-^i \nabla_+^{(-)} \psi_-^j \, g_{ij} + im\,(u_i - X_i)\psi_-^i \right\} \ , \quad (2.1)$$

$\star$ The corresponding result of [1] was in apparent agreement with eq. 53 of [3]; there appears to be a transcription error between eqs. 50 and 53 of [3].

where m is a mass parameter and $\nabla^{(\pm)}$ is the covariant derivative with connection

$$\Gamma^{(\pm)k}_{ij} = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \pm H_{ij}{}^k \ , \tag{2.2}$$

i.e. H_{ijk} is the torsion of the connection of $\nabla^{(+)}$. We refer to [1] for details of the superspace conventions. The action (2.1) is invariant under the superfield transformations

$$\begin{aligned} \delta_\epsilon \phi^i &= -\frac{i}{2} D_+ \epsilon_= D_+ \phi^i + \epsilon_= \partial_{\mp} \phi^i \\ \delta_\epsilon \psi^i_- &= -\frac{i}{2} D_+ \epsilon_= D_+ \psi^i_- + \epsilon_= \partial_{\mp} \psi^i_- \ , \end{aligned} \tag{2.3}$$

for x-independent (1,0)-superfield $\epsilon_=$. The constant $(D_+\epsilon_=)|$ is the anticommuting parameter of the manifest (1,0) supersymmetry[$\star$. The action (2.1) is also invariant under the transformations

$$\begin{aligned} \delta_\zeta \phi^i &= D_+ \zeta \psi^i_- + m \zeta X^i \\ \delta_\zeta \psi^i_- &= -i D_+ \zeta \partial_= \phi^i + m \zeta \partial_j X^i \psi^j_- \end{aligned} \tag{2.4}$$

for x-independent bosonic (1,0)-superfield parameter ζ. The constant $\zeta|$ is the transformation generated by the Killing vector X while the anticommuting constant $(D_+\zeta)|$ is the parameter of (0,1) supersymmetry.

All (p,q)-supersymmetric sigma models with $p,q \geq 1$ are special cases of the (1,1)-supersymmetric model. The additional supersymmetries simply impose further restrictions on the sigma model couplings and the geometry of $\mathcal{M}$. In the massless case these restrictions are long-established [4, 5]. For example, an additional p-1 left-handed supersymmetries requires the existence of p-1 complex structures I_r on $\mathcal{M}$ that are covariantly constant with respect to the connection $\Gamma^{(+)}$, and that the metric g of $\mathcal{M}$ be hermitian with respect to them. In the case that $p = 4$, closure of the algebra of supersymmetry transformations requires in addition that the complex structures I_r $(r = 1,2,3)$ obey the algebra of imaginary

$\star$ The vertical bar indicates the $\theta = 0$ component of a superfield.

unit quaternions. Similarly, an additional q-1 right-handed supersymmetries requires the existence of q-1 complex structures J_s on $\mathcal{M}$, but in this case the complex structures J_s are covariantly constant with respect to the connection $\Gamma^{(-)}$. The metric g must be Hermitian with respect to all complex structures.

The (1,0)-superfield transformations for extended[† (p,0) and (0,q) transformations are most conveniently presented in terms of the 'covariantized' fermion variation

$$\Delta\psi_-^i \equiv \delta\psi_-^i + \delta\phi^j \Gamma^{(-)\,i}_{jk}\psi_-^k \ . \tag{2.5}$$

These transformations involve the complex structures I_r and J_s and several of the Killing-vector/one -form pairs $\{Z_{I'I}, u^{I'I}\}$. For consistency with [1] we adopt the notation

$$\begin{aligned} Z_{00} &= X \quad u^{00} = u \\ Z_{0r} &= Z_r \quad u^{0r} = v_r \\ Z_{s0} &= Y_s \quad u^{s0} = w_s \\ Z_{sr} &= Z_{sr} \quad u^{sr} = v_{sr} \ . \end{aligned} \tag{2.6}$$

The extended (p,0) transformations are

$$\begin{aligned} \delta_\eta\phi^i &= i\eta_-^r I_r{}^i{}_j(\phi) D_+\phi^j \\ \Delta_\eta\psi_-^i &= \frac{1}{2}\eta_-^r \hat{I}_r{}^i{}_j(\phi)\mathcal{S}^j + \frac{im}{2}\eta_-^r (Z_r - v_r)^i \end{aligned} \tag{2.7}$$

where $\mathcal{S}^i = 0$ is the ψ_-^i field equation and the (p-1) parameters η_-^r are anticommuting constants. The extended (0,q) transformations are included in[‡

$$\begin{aligned} \delta_\kappa\phi^i &= D_+\kappa^s J_s{}^i{}_j\psi_-^j + m\kappa^s Y_s^i(\phi) \\ \Delta_\kappa\psi_-^i &= iD_+\kappa^s J_s{}^i{}_j\partial_=\phi^j + D_+\kappa^s J^i_{sm} H^m{}_{nl} J_s^n{}_j J^l_{sk}\psi_-^j\psi_-^k + m\kappa^s \nabla_j^{(+)} Y_s{}^i\psi_-^j \end{aligned} \tag{2.8}$$

where the parameters κ^r are x-independent bosonic (1,0) superfields. The (q-1)

† Our use of the adjective 'extended' indicates that we exclude the (1,0) and (0,1) transformations already considered.

‡ These transformations can be shown to be equivalent to those of [1] by using the various conditions derived in that reference, in particular the vanishing of the Nijenhuis tensor.

anticommuting constants $(D_+\kappa^r)|$ are the parameters of extended (0,q) supersymmetry. The constants $(\kappa^r)|$ are the parameters for transformations generated by the Killing vector fields Y_r.

Invariance of the action and closure of the algebra of supersymmetry transformations requires that each of the Killing vector fields $Z_{I'I}$ leave invariant the complex structures I_r and J_s and the torsion three-form H. It is also required that

$$Z_{I'I}\cdot u^{J'J}+Z_{J'J}\cdot u^{I'I}=0 \qquad \begin{cases} I'=J'=0, \quad I=J=0,1,\ldots,p-1 \\ I=J, \quad I'=J'=0,1,\ldots,q-1 \end{cases} \tag{2.9}$$

Actually, closure of the algebra of supersymmetry transformations was shown in [1] to imply only the weaker condition $Z_{I'I}\cdot u^{J'J}+Z_{J'J}\cdot u^{I'I}=$ const. but consideration of the Poisson bracket algebra of supersymmetry charges, which we discuss in the following section, shows that constants some of these constants, those of (2.9), must vanish if the two-dimensional spacetime is Minkowski, as assumed here. For simplicity, we will take all constants

$$Z_{I'I}\cdot u^{J'J}+Z_{J'J}\cdot u^{I'I}=0. \tag{2.10}$$

The most important of the remaining restrictions imposed by (p,q) supersymmetry can now be summarized by the following set of relations[§ between the Killing vector fields $Z_{I'I}$ and their associated one-forms $u^{I'I}$:

$$\begin{aligned} (Z_r+v_r)_i+I_r{}^k{}_i(X+u)_k&=0 \\ (Y_s-w_s)_i+J_s{}^k{}_i(X-u)_k&=0 \\ (Z_{sr}+v_{sr})_i+I_r{}^k{}_i(Y_s+w_s)_k&=0 \\ (Z_{sr}-v_{sr})_i+J_s{}^k{}_i(Z_r-v_r)_k&=0. \end{aligned} \tag{2.11}$$

It was shown in [1] that the scalar potential V of the general (p,q)-supersymmetric sigma model can be expressed as the length of any one of the vectors $Z_{I'I}\pm u^{I'I}$. It follows from (2.10) and (2.11) that these vectors all have the same length.

§ Correcting a typographical error in [1].

3. The algebra of charges

The supersymetry and central charges associated to the symmetries summarized in the previous section are most conveniently expressed in terms of the physical component fields. For models with at least (1,1) supesymmetry these are

$$\phi^i = \phi^i| \qquad \lambda^i_+ = D_+\phi^i| \qquad \psi^i_- = \psi^i_-| \ , \tag{3.1}$$

where the vertical bar indicates the $\theta^+ = 0$ component of a superfield. Performing the θ^+ integration and eliminating auxiliary fields produces the component action

$$\begin{aligned} S = \int d^2x \big\{ \partial_{\mp}\, \phi^i \partial_= \phi^j (g_{ij} + b_{ij}) + i g_{ij} \lambda^i_+ \nabla^{(+)}_= \lambda^j_+ - i g_{ij} \psi^i_- \nabla^{(-)}_{\mp} \psi^j_- \\ - \frac{1}{2} \psi^k_- \psi^l_- \lambda^i_+ \lambda^j_+ R^{(-)}_{ijkl} + m \nabla^{(-)}_i (u - X)_j \lambda^i_+ \psi^j_- - V(\phi) \big\} \ , \end{aligned} \tag{3.2}$$

where $V = \frac{m^2}{4} g^{ij} (u - X)_i (u - X)_j$. The total energy of a sigma model field configuration is

$$E = \frac{1}{2} \int dx [g_{ij} \partial_t \phi^i \partial_t \phi^j + g_{ij} \partial_x \phi^i \partial_x \phi^j + V(\phi) + \text{ fermions }] \tag{3.3}$$

and the total momentum is

$$P = \int dx [g_{ij} \partial_t \phi^i \partial_x \phi^j + \text{fermions }] \ . \tag{3.4}$$

The fermion contributions will not be needed for what follows so we omit them. Note that the torsion term in the action does not contribute to the energy. The conserved currents associated with the Killing vectors $Z^{I'I}$ are

$$J^{I'I}_\mu = \partial_\mu \phi^i Z^{I'I}_i - \varepsilon_\mu{}^\nu \partial_\nu \phi^i u^{I'I}_i + \text{fermions} \ , \tag{3.5}$$

where the second term in the current is due to the torsion term in the action. The corresponding charges are

$$Q_{I'I} = \int dx [Z^{I'I}{}_i \partial_t \phi^i + u^{I'I}{}_i \partial_x \phi^i + \text{fermions}]. \tag{3.6}$$

Observe that since $u^{I'I}$ is defined up to the derivative of a scalar the Noether charges are defined only up the addition of a topological charge. In particular if

the Killing vector field $Z^{I'I}$ vanishes the corresponding charge does not necessarily vanish but rather becomes a topological charge. These charges must of course be taken into account in a determination of the automorphism group of the supersymmetry algebra; they will therefore be an important ingredient in our discussion of this matter in section 5.

The supersymmetry charges of the (p,q) supersymmetric sigma model can be found by standard manipulations from the supersymmetry transformations. The results are as follows. The (1,0)-supersymmetry charge is

$$S_+ = \int dx [g_{ij}\partial_{\mp}\phi^i\lambda_+^j - \frac{i}{2}m(u-X)_i\psi_-^i] \ . \tag{3.7}$$

The (0,1)-supersymmetry charge is

$$S_- = \int dx [ig_{ij}\partial_{=}\phi^i\psi_-^j + \frac{1}{3}\psi_-^i\psi_-^j\psi_-^k H_{ijk} + \frac{m}{2}(X+u)_i\lambda_+^i] \ . \tag{3.8}$$

The extended (p,0)-supersymmetry charges are

$$S_+^r = \int dx [I_{r\,ij}(\partial_{\mp}\phi^i\lambda_+^j - iH^i{}_{kl}\lambda_+^j\lambda_+^k\lambda_+^l) - i\frac{m}{2}(v_r - Z_r)_i\psi_-^i] \ . \tag{3.9}$$

The extended (0,q)-supersymmetry charges are

$$S_-^s = \int dx [iJ_{s\,ij}\partial_{=}\phi^i\psi_-^j + \frac{1}{3}H_{mnl}J_s{}^m{}_iJ_s{}^n{}_jJ_s{}^l{}_k\psi_-^i\psi_-^j\psi_-^k + \frac{m}{2}(Y_s+w_s)_i\lambda_+^i]. \tag{3.10}$$

To calculate the Poisson bracket algebra of the above charges, one must first re-express them in terms of the fields ϕ, λ, ψ (and their spatial derivatives) and the corresponding conjugate momenta which follow in the usual way from the action (3.2). We omit the details of this step and simply present the result of the subsequent calculation of the Poisson Brackets. Firstly,

$$\{S_+, S_+\} = 2(E+P), \quad \{S_+^r, S_+^s\} = 2\delta_{rs}(E+P), \quad \{S_+, S_+^r\} = 0 \ . \tag{3.11}$$

One does not expect central charges to appear in these anticommutators because their presence is forbidden by two-dimensional Lorentz invariance. However, the

calculation shows that there are Lorentz non-invariant central charges proportional to the volume of space; a typical such charge is $\int dx\, X \cdot u$. As we remarked earlier, closure of the algebra of supersymmetry transformations implies that the integrand is constant, so the charge is infinite if space is infinite, as it is for two-dimensional Minkowski spacetime. Under these circumstances these constants must vanish[$\star$ and the Poisson bracket algebra of the (p,0)-supersymmetry charges is as given in (3.11). Under the same circumstances the algebra of Poisson brackets of the (0,q)-supersymmetry charges is

$$\{S_-, S_-\} = 2(E-P), \quad \{S_-^r, S_-^s\} = 2\delta_{rs}(E-P), \quad \{S_-, S_-^s\} = 0\ . \tag{3.12}$$

Lorentz-invariant central charges can appear in the Poisson brackets of the (p,0)-supersymmetry charges with the (0,q) ones, and indeed we find that

$$\begin{aligned} \{S_+, S_-\} &= mQ_{oo}, \quad \{S_+, S_-^s\} = mQ_{so}, \\ \{S_-, S_+^r\} &= mQ_{or}, \quad \{S_+^r, S_-^s\} = mQ_{sr}\ . \end{aligned} \tag{3.13}$$

Finally, the Poisson brackets of $Q_{I'I}$ with any supersymmetry charge vanishes. Thus we have now obtained the Noether charges of the general massive (p,q)-supersymmetric sigma-model and verified that they realize the algebra of (p,q) supersymmetry with central charges.

$\star$ They need not vanish if space is compact. Analogous central charges are important in the context of supersymmetric extended objects in toroidal spacetimes [6].

4. Automorphism symmetries

All of the transformations summarized in section 2 have parameters that are x-independent (1,0) superfields except those of the extended (p,0) transformations for which the parameters η_-^r were (anticommuting) constants. If $m = 0$ this restriction is unnecessary and the action remains invariant if η_-^r is also an x-independent superfield. The symmetry with parameters $(D_+\eta_-^r)|$ is an $SO(N)$ rotation of the fermions $(D_+\phi^i)|$, where $N = 2$ if $p = 2$ and $N = 3$ if $p = 4$. There is also an $SO(M)$ invariance of the form

$$\delta_\xi \psi_-^i = \xi^s R_s{}^i{}_j \psi_-^j \ , \tag{4.1}$$

for constant parameters ξ_s, provided that the tensors R_s satisfy

$$(R_s)_{(ij)} = 0 \qquad \nabla_i^{(-)}(R_s)_{kl} = 0 \tag{4.2}$$

and

$$[R_r, R_s] = \sum_t \varepsilon_{rst} R_t \ . \tag{4.3}$$

The commutator of (4.1) with the (0,1)-supersymmetry transformations yields an extended (0,q)-supersymmetry transformation provided that

$$R_s = J_s \ , \tag{4.4}$$

which implies both (4.2) and (4.3) and so replaces them. The other commutators yield no further conditions. From (4.4) it is seen that $M = 2$ if $q = 2$ and $M = 3$ if $q = 4$. The $SO(N) \times SO(M)$ symmetry of the massless sigma-model may clearly be identified with a subgroup of the $SO(p) \times SO(q)$ automorphism group of the supersymmetry algebra without central charges.

We now turn to the $m \neq 0$ case. For reasons that will become apparent below we combine the rotation (4.1) with the extended (p,0) transformations to form the transformations

$$\begin{aligned} \delta_{(\eta,\xi)}\phi^i &= i\eta_-^r I_r{}^i{}_j D_+\phi^j \\ \Delta_{(\eta,\xi)}\psi_-^i &= \eta_-^r \hat{I}_r{}^i{}_j \mathcal{S}^j + \frac{im}{2}\eta_-^r (Z_r - v_r)^i + i\xi^s J_s{}^i{}_j \psi_-^j \ , \end{aligned} \tag{4.5}$$

which we shall call the *new* extended (p,0) transformations. The commutator on ϕ of the new extended (p,0) transformations among themselves does not produce any conditions not already found from the $m = 0$ case. Omitting terms proportional to field equations, the commutator on ψ_- is

$$\begin{aligned} [\delta_{(\eta,\xi)}, \delta_{(\eta',\xi')}]\psi_-^i &= -\Gamma^{(-)i}_{jk}\big\{[\delta_{(\eta,\xi)}, \delta_{(\eta',\xi')}]\phi^j\big\}\psi_-^k - 2i\eta_-'{}^r \eta_-^r \nabla^{(-)}_{\mp} \psi_-^i \\ &\quad - 2\xi'^s \xi^t \varepsilon_{stw} J_w{}^i{}_j \psi_-^j + \frac{m}{2}\big(\xi^s \eta_-'{}^r - \xi'^s \eta_-{}^r\big)\big(J_s(Z_r - v_r)\big)^i . \end{aligned} \tag{4.6}$$

The right hand side can be identified with known transformations provided that

$$\xi^r = D_+\eta_-^r \qquad r = 1, \ldots, \min(p-1, q-1) \ , \tag{4.7}$$

and

$$(v_r - Z_r)_i + (J_r)_{ij}(u - X)^j = 0 \ . \tag{4.8}$$

Indeed, using (4.7) the new extended (p,0) transformations (4.5) can be rewritten in terms of the single x-independent superfield η_-^r and, using (4.8), the on-shell commutator of these transformations is found to be

$$\begin{aligned} [\delta_\eta, \delta_{\eta'}]\psi_-^i &= -\Gamma^{(-)i}_{jk}\big\{[\delta_\eta, \delta_{\eta'}]\phi^j\big\}\psi_-^k \\ &\quad - 2i\big(\eta_-'{}^r \eta_-^r \nabla^{(-)}_{\mp}\psi_-^i - \frac{i}{2} D_+(\eta_-'{}^r \eta_-^r)\nabla^{(-)}_+ \psi_-^i\big) \\ &\quad - \frac{1}{2}\varepsilon_{rst}\big[m(D_+\eta_-'{}^r \eta_-^s + \eta_-'{}^r D_+\eta_-^s)(Z - v)_t^i + 4D_+\eta_-'{}^r D_+\eta_-^s J_t{}^i{}_j \psi_-^j\big] \ . \end{aligned} \tag{4.9}$$

The explicit connection term on the right hand side cancels the connection terms implicit in the covariant derivatives. One can then see that the commutator closes on (1,0) and new extended (p,0) transformations (up to field equations).

The commutator of the (0,1) transformations with the new extended (p,0) ones is

$$[\delta_\eta, \delta_\zeta]\phi^i = \frac{1}{2} D_+\zeta\eta_-^r (\hat{I}_r - I_r)^i{}_j \mathcal{S}^j - im\eta_-^r D_+\phi^k (\mathcal{L}_X I_r)^i{}_k \\ - im\eta_-^r D_+\zeta Z_r^i + iD_+\zeta D_+\eta_-^r J_r{}^i{}_j \psi_-^j \ . \tag{4.10}$$

For simplicity we shall again consider on-shell closure, which means that we may ignore the first term on the right hand side (we refer the reader to [1] for a more complete discussion of this point). The second term vanishes because X is holomorphic with respect to all complex structures. For *constant* parameters η_-^r the last term vanishes while the third term can interpreted as a central charge transformation. This is not possible when the parameters η_-^r are x-independent superfields rather than constants, and in this case the commutator produces a potentially-new symmetry for which the variation of ϕ is

$$\delta\phi^i = iD_+(\eta_-^r D_+\zeta) J_r{}^i{}_j \psi_-^j - im(\eta_-^r D_+\zeta) Z_r^i \ . \tag{4.11}$$

We can identify this transformation as that of an extended (0,q) supersymmetry transformations (2.8) provided that

$$Z_r = -Y_r \qquad r = 1, \ldots, \min(p-1, q-1) \ . \tag{4.12}$$

It turns out that the commutator on ψ_- closes in the same way without the need of any further conditions. Clearly, at most a diagonal $SO\big(\min(N,M)\big)$ subgroup of the $SO(N) \times SO(M)$ symmetry of the massless model can be realized in the massive case.

Using the condition $\mathcal{L}_{Y_s} I_r = 0$ required for closure of the supersymmetry algebra, the on-shell commutator on ϕ of the extended (0,q) transformations (2.8) with the new extended (p,0) ones is

$$[\delta_\eta, \delta_\kappa]\phi^i = iD_+\kappa_-^r D_+\eta_-^s \varepsilon_{rst} J_{tj}^i \psi_-^j - iD_+\kappa_-^r D_+\eta_-^r \psi_-^i \\ + imD_+\kappa_-^r \eta_-^s Z_{sr}^i \ . \tag{4.13}$$

For $p = q = 2$ the right hand side can be identified with known transformations

provided that

$$Z_{11} \equiv T = X \ . \tag{4.14}$$

Similarly, for $p = q = 4$ the right hand side can be identified with known transformations provided that

$$Z_{sr} = \delta_{sr} X - \sum_t \varepsilon_{srt} Y_t \ . \tag{4.15}$$

We shall not trouble the reader with the complications of the $p \neq q$ cases except to say that if we take $p < q$ then the results are essentially the same as those of the (p,p) model. No additional conditions arise from consideration of the commutator on ψ_-. Finally, the commutators of the new extended (p,0) transformations with the (1,0) supersymmetry transformations close without the need of any further conditions. We have still to consider whether the action is invariant under the new extended (p,0) transformations; a calculation shows that the action is invariant as a consequence of the conditions required for closure of the algebra.

5. Automorphism algebra for p=q

We have just established the conditions for invariance of the sigma model under additional bosonic symmetries which we have called 'automorphism' symmetries. It is clear from the way they were found that they are indeed automorphisms of the supersymmetry transformations in the sense that a commutator of an 'automorphism' transformation with a supersymmetry transformation yields a further supersymmetry transformation. The connection with the automorphism group of the Poisson bracket algebra of supersymmetry charges is less clear, however, particularly in view of the fact that the Killing vector matrix $Z_{I'I}$ is not necessarily proportional to the central charge matrix because the latter may contain topological charges. It is also not so clear why only an $SO(3)$ automorphism can be realized in the massive (4,4) models. These points will be adressed in the course of this and the following section. Here we shall show for the general $p = q$ sigma model

that the the group realized by the automorphism transformations indeed coincides with the automorphism group of the Poisson bracket algebra of the supersymmetry charges. We also explain why $SO(4)$ cannot be realized in (4,4) models.

To discuss the (2,2) models it is convenient to define

$$(Y_1, w^1) = (Y, w), \quad (Z_1, v^1) = (Z, v), \quad (Z_{11}, v^{11}) = (T, n) \ . \tag{5.1}$$

We showed in the previous section that these models are invariant under an $SO(2)$ symmetry provided that

$$(v - Z)_i + (J)_{ij}(u - X)^j = 0 \tag{5.2}$$

and

$$Z = -Y \qquad T = X \ . \tag{5.3}$$

These conditions are of course additional to those of (2.11). The combined set of equations implies that

$$v = -w \qquad n = u \tag{5.4}$$

and

$$\begin{aligned} (w - Y)_i + J^j{}_i(u - X)_j = 0 \\ (w + Y)_i - I^j{}_i(u + X)_j = 0 \ . \end{aligned} \tag{5.5}$$

The *independent* conditions are (5.3), (5.4), and (5.5). The significance of equations (5.3) and (5.4) is that they ensure that the central charge matrix is invariant under an $SO(2)$ subgroup of the $SO(2) \times SO(2)$ automorphism group of the supersymmetry algebra without central charges. Equation (5.3) is obviously necessary for this, but the necessity of (5.4) is perhaps less obvious. To see why it is necessary consider the Noether charges $Q_{I'I}$ given in (3.6): when $Z = -Y$ and $T = X$ we

find that

$$\begin{aligned} Q_{01} &= -Q_{10} + \text{ surface term} \\ Q_{00} &= Q_{11} + \text{ surface term} \ , \end{aligned} \tag{5.6}$$

where the surface terms can be interpreted as topological charges. These topological charges must also vanish if the central charge matrix is to be $SO(2)$ invariant, and the conditions of (5.4) ensure that this occurs, i.e. that the central charge matrix Q takes the form

$$Q = \begin{pmatrix} Q_X & -Q_Y \\ Q_Y & Q_X \end{pmatrix} \ , \tag{5.7}$$

where $Q_X \equiv Q_{00}$ and $Q_Y \equiv Q_{01}$.

We now turn to the (4,4) models. We have found that these models are invariant under an $SO(3)$ symmetry provided that the conditions

$$(v_r - Z_r)_i + (J_r)_{ij}(u - X)^j = 0 \ , \tag{5.8}$$

and

$$\begin{aligned} Z_r &= -Y_r \\ Z_{sr} &= \delta_{sr} X - \sum_t \varepsilon_{srt} Y_t \ , \end{aligned} \tag{5.9}$$

hold. Again, these are in addition to those of (2.11). The combined set of equations implies

$$v_r = -w_r \qquad v_{sr} = \delta_{sr} u - \sum_t \varepsilon_{srt} w_t \tag{5.10}$$

and

$$\begin{aligned} (w_r - Y_r)_i + J_r{}^j{}_i (u - X)_j &= 0 \\ (w_r + Y_r)_i - I_r{}^j{}_i (u + X)_j &= 0 \ . \end{aligned} \tag{5.11}$$

The *independent* equations are those of (5.9), (5.10), and (5.11). The equations

(5.9) and (5.10) ensure that the central charge matrix Q takes the form

$$Q = \begin{pmatrix} Q_X & -Q_1 & -Q_2 & -Q_3 \\ Q_1 & Q_X & -Q_3 & Q_2 \\ Q_2 & Q_3 & Q_X & -Q_1 \\ Q_3 & -Q_2 & Q_1 & Q_X \end{pmatrix} , \tag{5.12}$$

where $Q_X \equiv Q_{00}$ and $Q_r \equiv Q_{r0}$. This matrix is a sum of a multiple of the identity matrix and a self-dual matrix. This is the general form for a matrix that is invariant under an antiself-dual $SO(3)$ subgroup of the diagonal $SO(4)$.

The above analysis can also be carried out directly at the level of the Poisson brackets. The Noether charges associated with the automorphism symmetries are

$$A^r = \frac{1}{2}\int dx\big(- iI_{rij}\lambda^i_+\lambda^j_+ + iJ_{rij}\psi^i_-\psi^j_-\big). \tag{5.13}$$

Using the conditions derived in section 4 one finds that the Poisson brackets of these charges with themselves and the other charges of the supersymmetry algebra are, for the $p = q = 2$ model,

$$\begin{aligned} &\{A, A\} = 0, \qquad \{A, S_+\} = S^1_+, \qquad \{A, S^1_+\} = -S_+, \\ &\{A, S_-\} = S^1_-, \qquad \{A, S^1_-\} = -S_-, \qquad \{A, H\} = 0, \\ &\{A, P\} = 0, \qquad \{A, Q\} = 0 \ , \end{aligned} \tag{5.14}$$

where $A \equiv A^1$ and $Q \equiv Q^{11}$, and, for the $p = q = 4$ model,

$$\begin{aligned} &\{A^r, A^s\} = -2\varepsilon_{rst}A^t, \qquad \{A^r, S_+\} = S^r_+ \\ &\{A^r, S^s_+\} = -\delta_{rs}S_+ - \varepsilon_{rst}S^t_+ \\ &\{A^r, S_-\} = S^r_-, \qquad \{A^r, S^s_-\} = -\delta_{rs}S_- - \varepsilon_{rst}S^t_- \\ &\{A^r, H\} = 0, \qquad \{A^r, P\} = 0, \qquad \{A^r, Q^{I'I}\} = 0 \ . \end{aligned} \tag{5.15}$$

As expected, in both cases the automorphism charges A^r transform the supersymmetry charges amongst themselves but leave the Hamiltonian, H, the momentum, P, and the central charges, $Q_{I'I}$, invariant.

Given the fact that an $SO(2)$ symmetry can be realized for certain (2,2) models, one might have expected to be able to realize an $SO(4)$ symmetry for some (4,4) models. It seems, however, that at most an $SO(3)$ subgroup of $SO(4)$ can be realized[$\star$. The relevant $SO(4)$ group for the massive models is the *diagonal* subgroup of the $SO(4)_L \times SO(4)_R$ automorphism group acting on the left (L) and right (R) handed supercharges of the massless sigma model. It can be shown that only one of the two $SO(3)$ subgroups of this $SO(4)$ can be realised by transformations on the fields of a massive (4,4) supersymmetric sigma model.

We shall first show that for the massless (4,4) supersymmetric sigma models, one can realise only an $SO(3)_L \times SO(3)_R$ subgroup of the $SO(4)_L \times SO(4)_R$ automorphism group of the supersymmetry algebra. For this, it is sufficient to examine the (4,0) sector of the algebra since the proof for the other sector is identical. We first observe that the diagonal $SO(4)_L$ acts on the supersymmetry charges S_+^I via its fundamental representation D;

$$D(T_{KL})^J{}_I = \delta_{KI}\delta_L{}^J - \delta_{LI}\delta_K{}^J \tag{5.16}$$

where $K, L, I, J = 0, \ldots, 3$ and $T_{KL} \equiv -T_{LK}$ is a basis in the Lie algebra of $SO(4)_L$. The self-dual and antiself-dual parts of this representation are

$$D^{(\pm)}_{KL} = D(T_{KL}) \pm \frac{1}{2}\varepsilon_{KL}{}^{MN} D(T_{MN}) \ . \tag{5.17}$$

They form four-dimensional representations of two commuting $SO(3)$ subgroups of $SO(4)_L$ as can be seen by definining

$$D^{(\pm)}_r = D^{(\pm)}_{0r} \tag{5.18}$$

$\star$ A related phenomenon occurs in the work of [7] on the coupling of $N = 4$ three-dimensional supersymmetric sigma-models to $N = 4$ supergravity; the $SO(4)$ invariance of the pure supergravity action is maintained in the matter-coupled action by virtue of the fact that the supergravity fields couple to *two* sigma-models, each of which contributes an $SO(3)$ factor.

and observing that

$$D_r^{(\pm)} D_s^{(\pm)} = -\delta_{rs} \pm \sum_t \varepsilon_{rst} D_t^{(\pm)} \tag{5.19}$$

and

$$[D_r^{(+)}, D_s^{(-)}] = 0 \ . \tag{5.20}$$

We shall denote by $SO(3)_L^+$ and $SO(3)_L^-$ the corresponding $SO(3)$ subgroups of $SO(4)_L$. The matrices $D^{(\pm)}$ form a four dimensional representation of $SO(3)_L^\pm$ and neither $SO(3)$ factor of the $SO(4)_L$ automorphism group leaves invariant any of the supercharges S_+^I. To see this, let us assume that there is one, say $S_+ \equiv S_+^0$, which is invariant under the transformation $D^{(\pm)} = \xi^r D_r^{(\pm)}$ for infinitesimal parameter ξ, i.e. $D^{(\pm)} S_+ = 0$. We then observe that $(D^{(\pm)})^2 = -(\xi)^2 \mathbf{1}$ and thus $(\xi)^2 S_+ = 0$ which implies that $\xi = 0$.

Recall now from section 4, that a massless (4,4) supersymmetric sigma model is invariant under *independent* rotations of the fermions λ_+ and ψ_-. The corresponding Noether charges are

$$A_L^r = -\frac{i}{2} \int dx I_{rij} \lambda_+^i \lambda_+^j, \qquad A_R^s = \frac{i}{2} \int dx J_{sij} \psi_-^i \psi_-^j \ . \tag{5.21}$$

After some computation, it is straightforward to show that

$$\{A_L^r, S_+^I\} = D_r{}^{(-)I}{}_J S_+^J \ . \tag{5.22}$$

From this it is clear that one can realise the $SO(3)_L^-$ subgroup of $SO(4)_L$ with rotations on the fermions λ_+ induced by the complex structures I_r of the sigma model manifold.

To realise the $SO(3)_L^+$ subgroup of $SO(4)_L$, one may consider introducing some further rotations on the fermion fields λ_+. For this, one needs a set of $(1,1)$ tensors, F_r say, that differ from I_r, but invariance of the action and closure of the algebra will require that $F_r = I_r$, so no realisation of $SO(3)_L^+$ is possible by a rotation of the

fermion fields. Alternatively, one might try to realise the $SO_L^+(3)$ by rotating the bosons instead of the fermions. This would involve consideration of transformations of the form

$$\begin{aligned} \delta\phi^i &= \xi^r k_r \\ \delta\lambda_+^i &= \xi^r \partial_j k_r^i \lambda_+^j \ , \end{aligned} \tag{5.23}$$

where ξ^r are parameters and k_r are vector fields which must be Killing and leave invariant H in order to be invariances. In order to qualify as automorphism symmetries they would also have to rotate the complex structures I_r into themselves ($\mathcal{L}_{k_r} I_s = \sum_t \varepsilon_{rst} I_t$). But because the vector fields k_r are Killing they leave invariant the particular supersymmetry charge $S_+ \equiv S_+^0$ (given in section 3) and hence cannot realise the $SO(3)_L^+$ subgroup of the automorphism group $SO(4)_L$.

The above arguments can be repeated for $SO(4)_R$, to prove that only its subgroup $SO(3)_R^-$ can be realised by transformations on the fields. Therefore only the subgroup $SO(3)_L^- \times SO(3)_R^-$ of the automorphism group $SO(4)_L \times SO(4)_R$ of the supersymmetry algebra without central charges can be realised by symmetries in the massless sigma model.

In the massive ($m \neq 0$) case, closure of the algebra of transformations (given in section 4) requires that the parameters of the left and right rotations of the fermions λ_+ and ψ_- be the same and the resulting Noether charge is $A^r = A_L^r + A_R^r$. Combining this fact with the above discussion for the massless model we conclude that for the massive model only the diagonal subgroup $SO(3)^-$ of $SO(3)_L^- \times SO(3)_R^-$ can be realised.

To conclude, we remark that the $SO(3)$ group that leaves invariant the central charge matrix Q of (5.12), and hence the automorphism group of the (4,4) supersymmetry algebra with central charges, is the $SO(3)^-$. To prove this, we note that

$$Q = Q_X \, \mathbf{1} - \sum_r Q_r D_r^{(+)} \tag{5.24}$$

and then that (5.20) implies $[D_r^{(-)}, Q] = 0$.

6. Non-chiral models revisited

We shall now present some more detailed results for the special case of the non-chiral models with extended supersymmetry, i.e. the (2,2) and (4,4) models without torsion. For vanishing torsion it is possible to set $J_r = I_r$ and here we shall consider only this case.

We first consider the (2,2) models; recall that $(Y_1, w^1) = (Y, w)$, $(Z_1, v^1) = (Z, v)$, and $(Z_{11}, v^{11}) = (T, n)$. Since $H = 0$ we have $u^{I'I} = da^{I'I}$, i.e.

$$u = da \qquad v = db \qquad w = dc \qquad n = de \tag{6.1}$$

for locally-defined functions a, b, c and e. From (2.11) we now deduce that

$$\begin{aligned} (Y - Z)_i &= I^k{}_i \partial_k (a + e) \\ (X + T)_i &= I^k{}_i \partial_k (b - c) \\ (X - T)_i &= I^k{}_i (Z + Y)_k \\ \partial_i (a - e) &= I^k{}_i \partial_k (c + b) \ . \end{aligned} \tag{6.2}$$

In addition we know that X, Y, Z and T are Killing vector fields that are holomorphic ($\mathcal{L}_X I = \mathcal{L}_Y I = \mathcal{L}_Z I = \mathcal{L}_T I = 0$) with respect to a covariantly constant ($\nabla I = 0$) complex structure. Given certain conditions on the global structure[$\star$ of the target manifold $\mathcal{M}$, any such holomorphic Killing vector field, k, can be expressed in terms of an associated real Killing potential U as $k_i = I^j{}_i \partial_j U$. Thus, from the first two equations of (6.2) we may identify $(a + e)$ and $(b - c)$ as the Killing potentials of $(Y - Z)$ and $(X + T)$, respectively. Similarly, the other two independent linear combinations may also be written as

$$(Y + Z)_i = I^k{}_i \partial_k \alpha \qquad (X - T)_i = I^k{}_i \partial_k \beta \ , \tag{6.3}$$

where the scalars α and β are the Killing potentials. It follows directly from (6.2)

$\star$ A sufficient condition is that $\mathcal{M}$ be compact and simply connected.

and (6.3) that

$$(Y+Z)_i = -\partial_i \beta \qquad (X-T)_i = \partial_i \alpha \ . \tag{6.4}$$

A solution to (6.3) and (6.4) is[†

$$Z = -Y \qquad T = X \tag{6.5}$$

with α and β constant. If $\mathcal{M}$ is either irreducible or compact and simply connected then this solution is unique. Either of these conditions is sufficient to prove uniqueness although neither is necessary. It can also be shown that the form of the scalar potential is unchanged if the general solution is used when the solution (6.5) fails to be unique. For simplicity, we shall assume here that $\mathcal{M}$ is such that (6.5) is the only solution of (6.3) and (6.4). From the first two equations of (6.2) we then find that

$$X_i = I^k{}_i \partial_k \left(\frac{c-b}{2} \right) \qquad Y_i = I^k{}_i \partial_k \left(\frac{a+e}{2} \right) \ , \tag{6.6}$$

from which we may identify the Killing potentials of the two independent Killing vector fields. Let γ be the Killing potential of Y; then

$$a + e = 2\gamma + \text{ constant} \ . \tag{6.7}$$

The last of eqs. (6.2) implies that $\frac{1}{2}(a-e)$ is the real part of a holomorphic function, i.e.

$$a - e = 2(h + \bar{h}) \ , \tag{6.8}$$

where h is holomorphic. Eliminating e from (6.7) and (6.8) we find that

$$a = \gamma + (h + \bar{h}) + \text{ constant} \tag{6.9}$$

† A sign error in a similar analysis in section (7) of [1] led to the incorrect equation $T = -X$.

and hence that

$$\begin{aligned} |da|^2 &= g^{ij}\left[\partial_i\gamma + \partial_i(h+\bar{h})\right]\left[\partial_j\gamma + \partial_j(h+\bar{h})\right] \\ &= |Y|^2 + |d(h+\bar{h})|^2 + 2g^{ij}\partial_i\gamma\partial_j(h+\bar{h}) \ . \end{aligned} \tag{6.10}$$

Now,

$$\begin{aligned} g^{ij}\partial_i\gamma\partial_j(h+\bar{h}) &= g^{ij}\partial_i\gamma\partial_j(a-e) \\ &= I^{ki}\partial_i\gamma\partial_k(b+c) \\ &= Y^k\partial_k(b+c) \\ &= \text{constant} \ , \end{aligned} \tag{6.11}$$

since $d(b+c)$ is Y-invariant. Thus

$$\begin{aligned} V &= \frac{1}{4}m^2\left(|X|^2 + |da|^2\right) \\ &= \frac{1}{4}m^2\left(|X|^2 + |Y|^2 + |d(h+\bar{h})|^2\right) \ , \end{aligned} \tag{6.12}$$

in agreement with eq. 50 of [3].

Observe that the restriction (6.5) on the Killing vector fields is such that the the vector-valued matrix $Z_{I'I}$ takes the $SO(2)$-invariant form

$$Z_{I'I} = \mu\delta_{I'I} + \nu\varepsilon_{I'I} \ . \tag{6.13}$$

As explained in section 4, this is necessary but not sufficient for the $SO(2)$ invariance of the action. For this one also needs $c = -b$ and $e = a$, in which case

$$X_i = -I^k{}_i\partial_k c \ , \qquad Y_i = I^k{}_i\partial_k a \ ,$$

and $h = 0$. Thus, $SO(2)$ invariance requires the superpotential, h, to vanish.

We turn now to the (4,4) models. Since $H = 0$ we have

$$u = da \qquad v_r = db_r \qquad w_r = dc_r \qquad v_{sr} = de_{sr} \ . \tag{6.14}$$

Defining

$$T_r \equiv Z_{rr} \qquad\qquad (r = 1, 2, 3) \ , \tag{6.15}$$

we deduce from (2.11) that

$$\begin{aligned}
(Y_r - Z_r)_i &= I_r{}^k{}_i \partial_k (a + e_{rr}) \qquad (r = 1, 2, 3) \\
(X + T_r)_i &= I_r{}^k{}_i \partial_k (b_r - c_r) \qquad (r = 1, 2, 3) \\
(X - T_r)_i &= I_r{}^k{}_i (Z_r + Y_r)_k \qquad (r = 1, 2, 3) \\
\partial_i (a - e_{rr}) &= I_r{}^k{}_i \partial_k (c_r + b_r) \qquad (r = 1, 2, 3) \ ,
\end{aligned} \tag{6.16}$$

which is the (4,4) analogue of (6.2). Now, however, (2.11) implies the further conditions

$$\begin{aligned}
(Z_{[rs]})_i &= -\frac{1}{2} (I_s I_r)^k{}_i \partial_k (e_{rr} + e_{ss}), \qquad (r \neq s), \\
\partial_i e_{[rs]} &= -\frac{1}{2} (I_s I_r)^k{}_i (T_r + T_s)_k, \qquad (r \neq s), \\
(Z_{(rs)})_i &= -\frac{1}{2} (I_s I_r)^k{}_i (T_s - T_r), \qquad (r \neq s), \\
\partial_i e_{(rs)} &= -\frac{1}{2} (I_s I_r)^k{}_i \partial_k (e_{rr} - e_{ss}) \qquad (r \neq s).
\end{aligned} \tag{6.17}$$

The third condition in (6.17) is similar to the third equation in (6.2), and applying the same arguments as in that case we can deduce that

$$Z_{(rs)} = 0, \qquad T_s - T_r = 0, \qquad (r \neq s) \ . \tag{6.18}$$

A consequence of this is that

$$e_{(rs)} = 0, \qquad e_{rr} - e_{ss} = 0, \qquad (r \neq s) \ , \tag{6.19}$$

(up to constants), and the fourth equation in (6.17) is automatically satisfied. Thus the three functions $\{e_{rr}; r = 1, 2, 3\}$ are actually the same function, and similarly

for the three Killing vectors T_r, so it is convenient to define

$$e_{rr} = e, \qquad T_r = T \qquad (r = 1, 2, 3) \ . \tag{6.20}$$

We can also write

$$\begin{aligned} Z_{[rs]} &= \sum_t \varepsilon_{rst}\, W_t \\ de_{[rs]} &= \sum_t \varepsilon_{rst}\, df_t \ . \end{aligned} \tag{6.21}$$

The residual information contained in eqs. (6.17) can now be expressed in terms of the functions e and f_r and the Killing vector fields T and W_r as

$$\begin{aligned} T_i &= I_r{}^k{}_i \partial_k f_r \qquad (r = 1, 2, 3) \\ (W_r)_i &= -I_r{}^k{}_i \partial_k e \qquad (r = 1, 2, 3) \ , \end{aligned} \tag{6.22}$$

while using (6.21) to simplify (6.16) we deduce that

$$\begin{aligned} (Y_r - Z_r)_i &= I_r{}^k{}_i \partial_k (a + e) \qquad (r = 1, 2, 3) \\ (X + T)_i &= I_r{}^k{}_i \partial_k (b_r - c_r) \qquad (r = 1, 2, 3) \\ (X - T) &= I_r{}^k{}_i (Z_r + Y_r)_k \qquad (r = 1, 2, 3) \\ \partial_i (a - e) &= I_r{}^k{}_i \partial_k (c_r + b_r) \qquad (r = 1, 2, 3) \ . \end{aligned} \tag{6.23}$$

For a given value of r the equations (6.23) are precisely those of the (2,2) case, (6.2). With the same assumptions as before about the global structure of $\mathcal{M}$ we deduce that

$$T = X \qquad\qquad Z_r = -Y_r \qquad (r = 1, 2, 3) \ . \tag{6.24}$$

The remaining three equations of (6.23) combined with those of (6.22) then reduce

to

$$\begin{aligned} X_i &= \frac{1}{2} I_r{}^k{}_i \partial_k (b_r - c_r) \qquad (r = 1,2,3) \\ (Y_r)_i &= \frac{1}{2} I_r{}^k{}_i \partial_k (a + e) \\ (W_r)_i &= -I_r{}^k{}_i \partial_k e \\ \partial_i (a - e) &= I_r{}^k{}_i \partial_k (c_r + b_r) \qquad (r = 1,2,3) \end{aligned} \tag{6.25}$$

which imply that

$$(Y_r + W_r)_i = -\frac{1}{2} \partial_i (c_r + b_r) \ . \tag{6.26}$$

Hence the argument that previously led to the conclusion that $Z_r = -Y_r$ now leads to

$$W_r = -Y_r \qquad c_r = -b_r \ , \tag{6.27}$$

and using this to simplify (6.25) yields

$$\begin{aligned} X_i &= I_r{}^k{}_i \partial_k b_r \qquad (r = 1,2,3) \\ (Y_r)_i &= I_r{}^k{}_i \partial_k a \ . \end{aligned} \tag{6.28}$$

The first of these equations allows us to identify the functions b_r as the three Killing potentials of the triholomorphic Killing vector X. The second shows that the Killing vectors Y_r are holomorphic with respect to I_r with Killing potential a but we also know [1] that they must be triholomorphic, i.e. there exist functions m_{rs} such that

$$(Y_r)_i = I_s{}^k{}_i \partial_k m_{rs} \qquad (r,s = 1,2,3) \ . \tag{6.29}$$

Combining this with the equation for Y_r in (6.28) yields

$$\partial_i a = (I_s I_r)^k{}_i \partial_k m_{rs} \qquad (r,s = 1,2,3) \tag{6.30}$$

and this implies that

$$\partial_i a = I_r{}^k{}_i \partial_k m_r \ , \tag{6.31}$$

where m_t is defined by $dm_{[rs]} = \sum_t \varepsilon_{rst}\, dm_t$. This is the condition that a be a triholomorphic function. Substituting this result into the second of eqs. (6.28) we

see that $Y_r = -dm_r$ and then, by the previous argument for target spaces of the assumed global structure, $Y_r = 0$. Thus we have now shown that

$$da = 0 \tag{6.32}$$

and hence that the potential V is simply the length of the triholomorphic Killing vector field X.

From the results of section 3, we now see that the (4,4) models without torsion are $SO(3)$ invariant. The matrix of Killing vector fields is actually $SO(4)$ invariant but, as explained in section 3, this does not necessarily imply that the central charge matrix is $SO(4)$ invariant because of the possibility of topological charges. These topological charges vanish identically (i.e. for *all* sigma model field configurations) if and only the functions b_r are all constant. But in this case $X = 0$ and so the scalar potential V vanishes. Thus, the central charge matrix for massive (4,4) models without torsion is actually only $SO(3)$ invariant and this corresponds precisely to the group realized by the 'automorphism' transformations of the fields.

Acknowledgements: Discussions with C.M. Hull and H. Nicolai are gratefully acknowledged. G.P. was funded by a grant from the European Union.

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