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Massless Particles, Electromagnetism, and Rieffel Induction

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Abstract

The connection between space-time covariant representations (obtained by inducing from the Lorentz group) and irreducible unitary representations (induced from Wigner's little group) of the Poincaré group is re-examined in the massless case. In the situation relevant to physics, it is found that these are related by Marsden-Weinstein reduction with respect to a gauge group. An analogous phenomenon is observed for classical massless relativistic particles. This symplectic reduction procedure can be ('second') quantized using a generalization of the Rieffel induction technique in operator algebra theory, which is carried through in detail for electromagnetism.

Starting from the so-called Fermi representation of the field algebra generated by the free abelian gauge field, we construct a new ('rigged') sesquilinear form on the representation space, which is positive semi-definite, and given in terms of a Gaussian weak distribution (promeasure) on the gauge group (taken to be a Hilbert Lie group). This eventually constructs the algebra of observables of quantum electromagnetism (directly in its vacuum representation) as a representation of the so-called algebra of weak observables induced by the trivial representation of the gauge group.

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1 Introduction

This paper is mainly concerned with the theory of the free electromagnetic field. Our reason for studying this system is that it provides the simplest physically relevant model on which to test certain new ideas to handle field theories with constraints. These ideas equally well apply to interacting gauge theories, and to some extent even to general relativity, so we hope that our formalism will turn out to be useful in the quantization theory of those theories, too.

The essential attribute of a gauge theory is that its equations of motion are simultaneously under- and overdetermined: the time-evolution of certain components of the gauge field is not specified at all, whereas there are constraints on the Cauchy data. For example, in classical electromagnetism (CEM) the Maxwell equations $\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0$ split up (on the choice of a Cauchy surface $x^0 = t = \text{const.}$) into the evolution equations $\Box \mathbf{A}^T = 0$ and the constraint $\dot{A}^L - A_0 = 0$ (Gauss' law); the evolution of A_0 and A^L is left undetermined (here $A_\mu = (A_0, \mathbf{A}^T + \nabla A^L)$, with $A^L = \Delta^{-1} \nabla \cdot \mathbf{A}$). It is a remarkable (yet little-known) feature of classical gauge theories that the double procedure of imposing the constraints and factoring out the remaining undetermined (and unphysical) fields is equivalent to a so-called Marsden-Weinstein reduction of the phase space of the unconstrained theory with respect to the gauge group [14, 5], obviating the need to explicitly perform the above split. Our main purpose is to exploit this feature in setting up a corresponding quantum theory.

We briefly recall this reduction procedure [32, 33, 1, 29]. Let a Lie group G act smoothly on a symplectic manifold S; G and S may be infinite-dimensional. It is required that the action is strongly Hamiltonian. This means firstly that the symplectic form ω is invariant under the group action, and secondly that for each $X \in \mathbf{g}$ (where $\mathbf{g}$ is the Lie algebra of G) one can find a function J_X satisfying $i_{\tilde{X}} \omega = dJ_X$ (where $\tilde{X}$ is the vector field on S defined by the infinitesimal action of X, that is, $(\tilde{X} f)(s) = d/dt f(\exp(tX)s)_{|t=0}$), and $\{J_X, J_Y\} = J_{[X,Y]}$, where $\{\ ,\ \}$ is the Poisson bracket on $C^\infty(S)$ derived from the symplectic form, and $[\ ,\ ]$ is the

Lie bracket on $\mathbf{g}$. In that case, one can define a so-called moment map $J : S \to \mathbf{g}^*$ (where $\mathbf{g}^*$ is the topological dual of $\mathbf{g}$) by $\langle J(s), X\rangle = J_X(s)$, which intertwines the given G-action on S and the co-adjoint action on $\mathbf{g}^*$. Then the Marsden-Weinstein reduced space S^0 is defined as $J^{-1}(0)/G$, which is a symplectic manifold provided certain technical conditions are satisfied.

In order to have an optimal setup for quantizing the theory, as well as to exploit the connection between electromagnetism and the representation theory of the Poincaré group P, we depart from the approach in [14, 5] in our choice of the classical phase space S. Namely, we would like the quantum theory to be meaningful in the context of algebraic quantum field theory [22, 23], and for this the time-evolution of all fields should be specified, and an action of P be defined on them. This necessitates a partial gauge fixing, which we impose by stipulating that the unphysical fields A_0 and $\mathbf{A}^L$ satisfy the same equation of motion as $\mathbf{A}^T$, so that $\Box A_\mu = 0$. The Gauss' law constraint may then be rewritten as $\partial^\mu A_\mu = 0$. This procedure is not quite the same as imposing the Lorentz gauge, for we treat the constraint $\partial^\mu A_\mu = 0$ as Gauss' law rather than as a gauge condition; it is not identically satisfied by the field A_μ. It may equivalently be arrived at by manipulating the Lagrangian, cf. [42, p. 143]).

We then take S to be the Hilbert space of those real weak solutions A_μ of the wave equation $\Box A_\mu = 0$ whose Cauchy data lie in L^2 in a suitable sense (cf. subsection 2.3). This space is equipped with a symplectic form defined by

$$\omega(B, C) = -\int d^3x\, B_\nu(x) \overset{\leftrightarrow}{\partial_0} C^\nu(x), \tag{1.1}$$

For the (residual) gauge group G we choose the Hilbert Lie group of scalar solutions λ (modulo constants) of the wave equation whose (exterior) derivative $d\lambda$ lies in S. This group acts on S in the usual way by gauge transformations, i.e., $\lambda \in G$ maps $A_\mu \in S$ to $A_\mu + \partial_\mu \lambda$. We will verify that all conditions for Marsden-Weinstein reduction are satisfied, so that the reduced space S^0 as defined above indeed coincides with the physical phase space of electromagnetism. The main point here is that the constraint space $J^{-1}(0) \subset S$ coincides with the space of fields satisfying $\partial^\mu A_\mu = 0$.

From the point of view of the Poincaré group, what happens here is the following. Though interpreted above as classical phase spaces of a field theory, S and S^0 are simultaneously Hilbert spaces of quantum states, which may be construed as the respective quantizations of certain one-particle phase spaces (the one corresponding to S^0 being a co-adjoint orbit of P). The natural action of P on S is non-unitary, reducible, and indecomposable if one regards S as a Hilbert space. But if one looks at S as a symplectic space, this action is symplectic, and intertwines with the action of the gauge group in such a way that it has a well-defined quotient action on S^0. This reduced action is exactly the irreducible unitary representation defined by massless particles of helicity ± 1. It is clear from the indecomposability of S under P that direct integral decompositions could not have achieved this reduction.

We now turn to the ('second') quantization of S (from an algebraic point of view, the object that is quantized is actually the Poisson algebra $C^\infty(S)$). Our main problem is finding a quantum analogue of the Marsden-Weinstein reduction procedure. The most convenient setting for doing this is provided by algebraic quantum field theory [22, 23], which has already been extensively used in the study of quantum electromagnetism (QEM) [9, 10, 16, 17, 19, 35]. As explained in the Introduction, this theory can only be used if the gauge is partially fixed. Hence we follow [9] in taking the field algebra $\mathfrak{F}$ of QEM to be the CCR algebra over the symplectic space S (cf. [8]).

The elimination of the unphysical degrees of freedom in $\mathfrak{F}$ was accomplished in [16, 17] at a purely algebraic level using the so-called T-procedure developed there. This technique is intended to provide a rigorous version of the Dirac method of dealing with quantum constrained systems [12, 42]. In this paper we propose to handle quantum electromagnetism using a completely different method, which has specifically been designed to quantize systems whose classical constraint structure is given by Marsden-Weinstein reduction. This method is based on the observation [26] that a generalization of Marsden-Weinstein reduction, in which the strongly Hamiltonian group action on S is replaced by a homomorphism of an arbitrary Poisson algebra into $C^\infty(S)$ [34, 46, 26], can be quantized by the operator-algebraic

technique of Rieffel induction [38, 13].

The aim of this induction technique is to find a representation π^0 of a (pre-) C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}^0$, given a representation π_0 of some other (pre-) C^*-algebra $\mathfrak{B}$ on a Hilbert space $\mathcal{H}_0$. To accomplish this, the original formulation started from a left $\mathfrak{A}$- and right $\mathfrak{B}$-module L. That is, L is merely a linear space, and a homomorphism of $\mathfrak{A}$ into the algebra $\mathcal{L}(L)$ of all linear maps on L is given, as well as an anti-homomorphism of $\mathfrak{B}$ into $\mathcal{L}(L)$. It is not required that the actions of $\mathfrak{A}$ and $\mathfrak{B}$ commute (although they do so in our applications). The key ingredient is then a so-called rigging map $\langle \cdot, \cdot \rangle_{\mathfrak{B}} : L \times L \to \mathfrak{B}$. The latter is a sesquilinear form (conventionally assumed linear in the second entry) taking values in $\mathfrak{B}$, with the additional properties holding for all $\Psi, \Phi \in L$:

$$\langle \Psi, \Phi \rangle_{\mathfrak{B}}^* = \langle \Phi, \Psi \rangle_{\mathfrak{B}}; \quad \langle \Psi, \Phi B \rangle_{\mathfrak{B}} = \langle \Psi, \Phi \rangle_{\mathfrak{B}} B$$

for all $B \in \mathfrak{B}$, and

$$\langle A\Psi, \Phi \rangle_{\mathfrak{B}} = \langle \Psi, A^* \Phi \rangle_{\mathfrak{B}}$$

for all $A \in \mathfrak{A}$. Positivity, in the sense that $\langle \Psi, \Psi \rangle_{\mathfrak{B}} \geq 0$ for all $\Psi \in L$, was required in [38], but dropped in [13] in favour of the weaker property $\pi_0(\langle \Psi, \Psi \rangle_{\mathfrak{B}}) \geq 0$, which suffices for the induction procedure, for it secures that $\mathcal{H}^0$ is a Hilbert space. Moreover, the bound

$$\pi_0(\langle A\Psi, A\Psi \rangle_{\mathfrak{B}}) \leq \| A \|^2 \, \pi_0(\langle \Psi, \Psi \rangle_{\mathfrak{B}})$$

is required to hold for all $A \in \mathfrak{A}$ and $\Psi \in L$; it guarantees that $\pi^0(\mathfrak{A})$ is a (pre) C^*-algebra. The induced space $\mathcal{H}^0$ is then constructed by first forming the algebraic tensor product $L \otimes \mathcal{H}_0$, and endowing it with a bilinear form $(\cdot, \cdot)_0$, defined by

$$(\Psi \otimes v, \Phi \otimes w)_0 = (\pi_0(\langle \Phi, \Psi \rangle_{\mathfrak{B}}) v, w),$$

where $(\cdot, \cdot)$ is the inner product in $\mathcal{H}_0$ (taken linear in the first entry, unlike the rigging map). This form is positive semi-definite. Secondly, one builds the quotient of $L \otimes \mathcal{H}_0$ by its subspace of vectors with vanishing $(\cdot, \cdot)_0$ norm, and completes the quotient (equipped with the form inherited from $(\cdot, \cdot)_0$) into a Hilbert space

$\mathcal{H}^0$. Denoting the image of an elementary vector $\Psi \otimes v \in L \otimes \mathcal{H}_0$ in $\mathcal{H}^0$ under the projection map onto the quotient by $\Psi \tilde{\otimes} v$, the representation $\pi^0(\mathfrak{A})$ is then defined on the subspace of $\mathcal{H}^0$ of finite linear combinations of such images by $\pi^0(A)(\Psi \tilde{\otimes}) v = (A\Psi)\tilde{\otimes} v$. By the above bound, this can be extended to both $\mathcal{H}^0$ and (if $\mathfrak{A}$ is not complete) to the completion of $\mathfrak{A}$ by continuity.

It was shown in [26] that the special case of quantum Marsden-Weinstein reduction with respect to a locally compact unimodular group G, acting continuously and properly on S, is covered as follows. We assume that the Poisson algebra $C^\infty(S)$ has been quantized into a C^*-algebra $\mathfrak{F}$, which is faithfully represented on a Hilbert space $\mathcal{H}$. This space should additionally carry a continuous unitary representation U of G (which plays the rôle of the quantization of the G-action on S). Then $\overline{\mathfrak{B}} = C^*(G)$ acts from the right (that is, in an anti-representation) on $\mathcal{H}$ by $\pi^-(f) = \int_G dx\, f(x) U(x^{-1})$, where dx is the Haar measure on G (this expression is defined on $C_c(G)$ and extended to $C^*(G)$ by continuity). We then try to identify a dense subspace $L \subset \mathcal{H}$, such that the integral $\int_G dx\, (U(x)\Psi, \Phi)$ is finite for all $\Psi, \Phi \in L$ (for G compact one may take $L = \mathcal{H}$; in general, many choices of L may exist, or none at all). If so, the rigging map defined by $\langle \Psi, \Phi \rangle_{\mathfrak{B}} : x \to (U(x)\Phi, \Psi)$ takes values in the pre-C^*-algebra $C(G) \cap L^1(G)$ (whose completion is $C^*(G)$). We now define the so-called weak algebra of observables $\mathfrak{F}^G$, which is the subalgebra of operators in $\mathfrak{F}$ which commute with all $U(x)$. If we take for $\mathfrak{A}$ a suitable dense subalgebra of $\mathfrak{F}^G$, then all properties required of the rigging map are satisfied.

The fact that our Marsden-Weinstein reduction is from the value $0 \in \mathbf{g}^*$ is then reflected by our taking the trivial representation of $C^*(G)$ (or G) on $\mathcal{H}_0 = \mathbb{C}$ to induce from; hence the rigged inner product is defined on L itself, and is given by

$$(\Psi, \Phi)_0 = \int_G dx\, (U(x)\Psi, \Phi). \tag{1.2}$$

(Thus for G compact, a case which even the naive Dirac formalism [12, 42] can handle, one simply has $(\Psi, \Phi)_0 = (P_0\Psi, P_0\Phi)$, where P_0 projects on the subspace of $\mathcal{H}$ carrying the trivial representation of G).

It should now be clear which problems we face if we wish to apply this scenario to QEM. The gauge group G is not locally compact (unless we equip it with the

discrete topology, which would be disastrous both for Marsden-Weinstein reduction, as this procedure is based on the use of Lie groups, as well as for Rieffel induction, for more subtle reasons to become clear in section 4), so it has no Haar measure. This means that the group algebra $C^*(G)$ is not defined, and also that (1.2) above makes no sense. (See [15] for a definition of a group algebra of groups which are a topological inductive limit of locally compact subgroups. It remains to be seen whether this is of any help for our problem.) Hence we have to rethink the Rieffel induction procedure, and in doing so it becomes clear that one neither needs the algebra $\mathfrak{B}$ nor the rigging map: the essential point is the rigged inner product $(\cdot,\cdot)_0$ on $L \otimes \mathcal{H}_0$ (which coincides with L in our case, where $\mathcal{H}_0 = \mathbb{C}$; in what follows we specialize to this case). It suffices to find a sesquilinear form on L with the following properties to hold for all $\Psi, \Phi \in L$ and all $A \in \mathfrak{A}$:

$$1.\ \ (\Psi,\Psi)_0 \geq 0; \tag{1.3}$$

$$2.\ \ (A\Psi,\Phi)_0 = (\Psi, A^*\Phi)_0; \tag{1.4}$$

$$3.\ \ (A\Psi, A\Psi)_0 \leq \| A \|^2\ (\Psi,\Psi)_0. \tag{1.5}$$

The second property implies that $\mathfrak{A}$ maps the null space of $(\cdot,\cdot)_0$ into itself. The induced space $\mathcal{H}^0$ and the induced representation $\pi^0(\mathfrak{A})$ are then defined as before, viz. $\pi^0(A)\tilde{\Psi} = \widetilde{A\Psi}$, where $\tilde{\Psi}$ is the image of Ψ in the quotient of L by the null space. Now (1.3) guarantees that $\mathcal{H}^0$ is a Hilbert space, and (1.4) implies that π^0 is a *-representation of $\mathfrak{A}$, which by (1.5) is continuous, and extendable to the completion $\overline{\mathfrak{A}}$.

Clearly, (1.3)-(1.5) are merely the conditions which a positive semi-definite sesquilinear form on a left $\mathfrak{A}$-module has to satisfy in order to produce a representation of (the completion of) $\mathfrak{A}$. As such, these conditions have nothing to do with Rieffel induction. The point is that Rieffel induction, or our slight bending of it, provides a systematic mechanism leading to such a form. Indeed, on the basis of the connection between symplectic reduction and Rieffel induction [26] it may be said that Marsden-Weinstein reduction itself provides this mechanism.

We will take $\mathcal{H} = \exp(S)$, the bosonic Fock space [8] (alternatively known as symmetric Hilbert space [20]) over S, which carries a natural representation of $\mathfrak{F}$ [9], and in addition a unitary representation U of the gauge group G. Inspired by (1.2), we attempt to construct the rigged inner product by

$$(\Psi, \Phi)_0 = \lim_n \int_{\mathcal{H}_n} \frac{d^n x}{\pi^{n/2}} (U(x)\Psi, \Phi). \tag{1.6}$$

Here $\{\mathcal{H}_n \subset G\}_n$ is an inductive family of Hilbert subspaces of G, which eventually exhaust it, and $d^n x$ is the Lebesgue measure on each $\mathcal{H}_n \simeq \mathbb{R}^n$. Remarkably, this limit indeed exists for a suitable choice of L, and can be written as a functional integral over G with respect to a Gaussian promeasure (cf. [11]; this is also called a weak distribution [40]). Moreover, the expression (1.6) allows us to prove properties 1-3 above quite easily (based on the corresponding proofs for the locally compact case [26]). The induced representation π^0 on $\mathcal{H}^0$ can be identified explicitly. For example, the Poincaré automorphisms on $\mathfrak{A}$ are implemented in π^0.

The resulting structure may be compared with the Gupta-Bleuler (or BRST) formalism of QEM [41, 19]. There one has a representation of the field algebra $\mathfrak{F}$ on a space with indefinite metric, which contains a non-dense subspace on which the metric is positive semi-definite. Our form $(\cdot, \cdot)_0$, on the other hand, enjoys the latter property on a dense subspace L. The need to subsequently quotient out the null space arises in both formalisms. Hence in our formalism there is no need to first identify a 'physical' subspace of $\mathcal{H}$, and we have no negative norm states. The price to be paid for this is that $\mathcal{H}$ does not carry a unitary representation of the Poincaré group [9], and that the rigged inner product is only densely defined (in fact, as a quadratic form on $\mathcal{H}$ it is not even closable). One still has the Hilbert inner product $(\cdot, \cdot)$ on $\mathcal{H}$, but it is only used to construct the rigged inner product $(\cdot, \cdot)_0$, and plays no independent rôle whatsoever; the physically relevant Hilbert inner product is the one on the induced space $\mathcal{H}^0$, which comes from $(\cdot, \cdot)_0$ rather than $(\cdot, \cdot)$. The definition of the rigged inner product via an auxiliary Euclidean structure is responsible for the fact that the first step of our procedure is not Lorentz invariant; full Poincaré invariance is restored only on the induced space $\mathcal{H}^0$. It may be possible to construct L and $(\cdot, \cdot)_0$ in a different way, in which Poincaré invariance is manifest

at all stages.

More generally, the usual first step in the quantization of constrained systems of first finding a Hilbert space representation of the unconstrained systems should be replaced by the problem of finding a representation on a space L carrying a rigged inner product with the properties 1-3 listed above, where the weak algebra of observables $\overline{\mathfrak{A}}$ is the subalgebra of operators of the unconstrained system which commute with the constraints, or with the action of the gauge group. The actual algebra of observables of the physical system in question is then $\pi^0(\overline{\mathfrak{A}})$, which is isomorphic to the quotient of $\overline{\mathfrak{A}}$ by the kernel of π^0 (note that $\overline{\mathfrak{A}}$, unlike $\mathfrak{F}$, is not simple in general).

To close this Introduction, we briefly summarize the contents and the logic of this paper. As a preliminary, subsection 2.1 contains a brief review of the (non-unitary) 'covariant' (i.e., vector, tensor, etc.) representations of the Poincaré group P, and their connection with the irreducible unitary ('canonical') representations. This leads into subsection 2.3, which contains a theorem stating that (unitary) massless helicity 1 or 2 representations are obtained as Marsden-Weinstein reductions of canonical vector or tensor representations. An important ingredient of the proof is isolated in a finite-dimensional model in subsection 2.2. This result is central to the paper, for it provides the specific way of writing electromagnetism as a constrained system, that we are going to quantize systematically with Rieffel induction. In particular, the precise rôle of gauge transformations and gauge invariance in the classical theory is now formulated in such a way as to admit a quantization from first principles. As an aside, we show in subsection 2.4 how a similar theory of covariant versus canonical realizations of the Poisson algebra defined by P can be formulated at the classical one-particle level; here the rôle played by massless particles is seen to be very similar to the Hilbert space case.

As a preparation for our field-theoretic calculations, section 3 is devoted to a model with four degrees of freedom (mimicking the components of A_μ evaluated at a fixed point in momentum space). Performing Rieffel induction on this model already exhibits most of the combinatorial features of the full theory, while remaining

straightforward analytically. This model has the special feature that, on account of the Stone-von Neumann theorem, the unconstrained system admits a unique quantization (in the sense of an irreducible representation of the algebra generated by its degrees of freedom). In that case our construction becomes a strict algorithm - the only freedom left is that of choosing between various unitarily equivalent realizations. The key brickwork is the rigged inner product in subsection 3.2, which may be thought of as providing a suitable 'dual' harmonic decomposition of the delta-function on the dual of the gauge group as an integral over the gauge group. (Recall that in ordinary harmonic analysis one expresses the delta-function on the group, rather than its dual, as a Plancherel integral over the unitary dual of the group.) From this, the induced representation of the algebra of observables is derived in a straightforward fashion in subsection 3.3.

In section 4 we turn to full electromagnetism. Being infinite-dimensional, the unconstrained system now no longer has a unique quantization (that is, an irreducible representation of the field algebra), and in this sense our construction hinges on making a certain choice. We perform Rieffel induction on the Fermi representation of the field algebra of quantum electromagnetism, for this is the representation that most closely corresponds to the (unique) one used in the finite-dimensional model in section 3. Once this choice has been made, the rigged inner product (in subsection 4.2) is 'canonical'; it has the same interpretation as the one in the preceding section, with the new feature that our 'dual' harmonic analysis now involves a (rigorous) functional integral. We construct the corresponding algebra of observables in its vacuum representation in subsection 4.3.

We close with some loose remarks in section 5. The main part is subsection 5.3, where we comment on the subtle differences and analogies between our method and the T-procedure of Grundling and Hurst.

We wish to acknowledge our debt to the paper [9], which introduced the models we discuss in sections 3 and 4, and, using different techniques, analyzed many of their quantum-mechanical features. Helpful discussions with Hendrik Grundling took place at an early stage of this work, as well as after completion of the first

draft. Rainer Verch provided us with constructive criticism of the manuscript. Also, thanks to Bernard Kay for discussions on eq. (3.32), and pointing out ref. [2].

Our metric is $g = \mathrm{diag}\,(1, -1, -1, -1)$, Greek indices run from 0 to 3, and Latin ones from 1 to 3. If no confusion can arise, we omit the tilde or hat on Fourier-transformed functions. The Lie algebra of a Lie group G is denoted by $\mathbf{g}$.

2 Classical gauge theories and the representation theory of the Poincaré group

The aim of this section is to show that the passage from the representation of the Poincaré group P carried by the massless vector field A_μ to the unitary irreducible representation defined by massless particles of helicity ± 1 can be accomplished by Marsden-Weinstein reduction.

2.1 Review of covariant vs. canonical representations

References for this subsection are [43, 3, 4]. The Poincaré group is the regular semidirect product $P = L \ltimes N$, where L is Lorentz group and $N = \mathbb{R}^4$ (we are confident that the reader will not confuse this notation with the linear space L in Rieffel induction). The Mackey theory of induced representations of regular semidirect products applies to a general abelian factor N and locally compact group L, but since time inversion is represented by an anti-linear operator in physics, we can only use this theory for the proper orthochronous subgroup $P_+^\uparrow$ of P. Hence, in what follows L stands for $L_+^\uparrow$ (i.e., $\det \Lambda = 1$ and $\Lambda^0{}_0 \geq 0 \; \forall \Lambda \in L$).

Canonical and covariant representations are both examples of induced representations. That is, take a closed subgroup $H \subset P$ and a representation (not necessarily unitary) U_χ of H on a Hilbert space $\mathcal{H}_\chi$. Then choose a measurable section $s : P/H \to P$ of the canonical projection $pr_P : P \to P/H$ (i.e., $pr_P \circ s = id$). Assume that P/H has a P-invariant measure, w.r.t. which we define $L^2(P/H)$. The induced representation U^χ of P on the Hilbert space $\mathcal{H}^\chi = L^2(P/H) \otimes \mathcal{H}_\chi$ is then given by

$$(U^\chi(x)\psi)(q) = U_\chi(s(q)^{-1}\, x\, s(x^{-1}q))\psi(x^{-1}q). \tag{2.1}$$

This is unitary iff U_χ is unitary. The two relevant specializations of this scheme are:

1. Canonical representations. L acts on $\hat{N}$ (the dual group of N; for $N = \mathbb{R}^4$, $\hat{N} = \mathbb{R}^4$ and we take the Minkowski pairing between N and $\hat{N}$, i.e., $\langle p, a\rangle = \exp(ip_\mu a^\mu)$) by $\langle \Lambda p, a\rangle = \langle p, \Lambda^{-1} a\rangle$. Take $\tilde{p} \in \hat{N}$ fixed and let $L_{\tilde{p}}$ be the stability group of $\tilde{p}$. Then in the above scheme $H = L_{\tilde{p}} \ltimes N$, and $P/H \simeq L/L_{\tilde{p}}$. If we choose a section $\eta : L/L_{\tilde{p}} \to L$ of the canonical projection $pr_L \equiv pr : L \to L/L_{\tilde{p}}$, we have also obtained a section $s : L/L_{\tilde{p}} \to P$, given by $s(q) = (\eta(q), 0)$. A representation U_σ of $L_{\tilde{p}}$ on a Hilbert space $\mathcal{H}_\sigma$ defines a representation $U_{\tilde{p},\sigma}$ of H on $\mathcal{H}_{\tilde{p},\sigma} = \mathcal{H}_\sigma$ by $U_{\tilde{p},\sigma}(\Lambda, a) = \exp(i\tilde{p}a) U_\sigma(\Lambda)$, with $\tilde{p}a \equiv \tilde{p}_\mu a^\mu$. Thus $\mathcal{H}^{\tilde{p},\sigma} = L^2(L/L_{\tilde{p}}) \otimes \mathcal{H}_\sigma$. Up to unitary equivalence, the induced representation defined on this space only depends on the orbit of $\tilde{p}$, which for $\tilde{p}^2 \geq 0$ is of the form $O_m^\pm = \{p \in \mathbb{R}^4 | p^2 = m^2, \pm p^0 > 0\}$. Hence we relabel $\mathcal{H}^{\tilde{p},\sigma}$ as $\mathcal{H}^{m,\pm,\sigma}$; the induced representation follows from (2.1) as

$$(U^{m,\pm,\sigma}((\Lambda, a))\psi)(p) = e^{ipa} U_\sigma(\eta(p)^{-1}\, \Lambda\, \eta(\Lambda^{-1}p))\psi(\Lambda^{-1}p). \tag{2.2}$$

To emphasize that $L/L_{\tilde{p}}$ is identified with the $L-$orbit $O_{\tilde{p}}$ of $\tilde{p}$ in $\mathbb{R}^4$, we have denoted its points by p rather than q. It is well known that the unitarity and irreducibility of $U^{m,\pm,\sigma}$ is implied by the corresponding properties of U_σ.

2. Covariant representations. Here $H = L$, so that $P/H \simeq \mathbb{R}^4$. Now we induce from a finite-dimensional representation U_λ of L on $\mathcal{H}_\lambda$ (which is never unitary unless it is the trivial representation), and conveniently choose $s(x) = (\mathrm{id}, x)$. Hence on $\mathcal{H}^\lambda = L^2(\mathbb{R}^4) \otimes \mathcal{H}_\lambda$ the induced representation given by (2.1) becomes

$$(U^\lambda(\Lambda, a)\psi)(x) = U_\lambda(\Lambda)\psi(\Lambda^{-1}(x - a)), \tag{2.3}$$

or, after a Fourier transform $\hat{\psi}(p) = (2\pi)^{-1} \int d^4x\, \psi(x) \exp(ipx)$,

$$(U^\lambda(\Lambda, a)\hat{\psi})(p) = e^{ipa} U_\lambda(\Lambda)\hat{\psi}(\Lambda^{-1}p). \tag{2.4}$$

The first step in the reduction of this highly reducible representation is to decompose it as a direct integral over the orbits $O_{\pm m} = \{p \in \mathbb{R}^4 | p^2 = \pm m^2\}$ in $\hat{N}$. We take the $+$ sign only in what follows, and must then further decompose over the two orbits $O_m^\pm$. This leads to the Hilbert spaces $\hat{\mathcal{H}}^{m,\pm,\lambda} = L^2(\mathbb{R}^3, d'p) \otimes \mathcal{H}_\lambda$ with the invariant

measure $d'p = d^3p/(2\pi)^3 2\sqrt{\mathbf{p}^2+m^2}$. This space carries a representation $\hat{U}^{m,\pm,\lambda}$, which is the (improper) restriction of U^λ, and is therefore simply given by putting $p^0 = \pm\sqrt{\mathbf{p}^2+m^2}$ in (2.4).

The connection with the canonical representations discussed before follows by picking a point $\tilde{p} \in O_m^\pm \simeq L/L_{\tilde{p}}$, and a section η, as before, and defining the transformation V on $\hat{\mathcal{H}}^{m,\pm,\lambda}$ by

$$\psi(p) \equiv (V\hat{\psi})(p) = U_\lambda(\eta(p)^{-1})\hat{\psi}(p). \tag{2.5}$$

If we define $\mathcal{H}_\eta^{m,\pm,\lambda}$ to coincide with $\hat{\mathcal{H}}^{m,\pm,\lambda}$ as a vector space, but equipped with the modified inner product

$$(\psi,\varphi)_\eta = \int d'p (U_\lambda(\eta(p)^{-1})\psi(p), U_\lambda(\eta(p)^{-1})\varphi(p))_{\mathcal{H}_\lambda}, \tag{2.6}$$

and refer to $\hat{\mathcal{H}}^{m,\pm,\lambda}$ with its usual inner product as $\mathcal{H}^{m,\pm,\lambda}$, then $V : \mathcal{H}_\eta^{m,\pm,\lambda} \to \mathcal{H}^{m,\pm,\lambda}$ is evidently unitary. The representation $\hat{U}^{m,\pm,\lambda}$, but now defined on $\mathcal{H}_\eta^{m,\pm,\lambda}$ rather than $\hat{\mathcal{H}}^{m,\pm,\lambda}$, is relabeled as $U_\eta^{m,\pm,\lambda}$. The representation $U^{m,\pm,\lambda} = VU_\eta^{m,\pm,\lambda}V^{-1}$ is then given by

$$(U^{m,\pm,\lambda}((\Lambda,a))\psi)(p) = e^{ipa}U_\lambda(\eta(p)^{-1}\,\Lambda\,\eta(\Lambda^{-1}p))\psi(\Lambda^{-1}p). \tag{2.7}$$

That is, it coincides with (2.2) up to the fact that (2.7) contains $U_\lambda(L_{\tilde{p}})$ on $\mathcal{H}_\lambda$ (i.e., the restriction of U_λ from L to its subgroup $L_{\tilde{p}}$), whereas (2.2) has the representation $U_\sigma(L_{\tilde{p}})$ on $\mathcal{H}_\sigma$ (which we assume to be unitary and irreducible). Since $U_\lambda(L_{\tilde{p}})$ is generically reducible, so is $U^{m,\pm,\lambda}$. To recap: the pair $(\hat{U}^{m,\pm,\lambda}, \hat{\mathcal{H}}^{m,\pm,\lambda}) = (\hat{U}^{m,\pm,\lambda}, \mathcal{H}^{m,\pm,\lambda})$ is independent of η, but irrelevant. In $(U_\eta^{m,\pm,\lambda}, \mathcal{H}_\eta^{m,\pm,\lambda})$ only $\mathcal{H}_\eta^{m,\pm,\lambda}$ explicitly depends on η through its inner product. This pair is unitarily equivalent to $(U^{m,\pm,\lambda}, \mathcal{H}^{m,\pm,\lambda})$, in which only $U^{m,\pm,\lambda}$ depends on η.

For $m > 0$ the reduction of $U_\lambda(L_{\tilde{p}})$ is straightforward, because $L_{\tilde{p}} = SO(3)$ is compact, so that $U_\lambda(L_{\tilde{p}})$ is completely reducible, and the desired irreducible component can be projected out by further covariant subsidiary conditions. For $m = 0$, on the other hand, the potential problem arises that $L_{\tilde{p}} = E(2) = SO(2) \ltimes \mathbb{R}^2$, which is neither semi-simple nor compact, has indecomposable representations. However,

the representation $D_{(j_1,j_2)}$ of L, restricted to $E(2)$, contains a subrepresentation of helicity h iff $h = j_2 - j_1$. Hence the tensor $\tilde{F}_{\mu\nu}$, which carries the representation $D_{(1,0)} \oplus D_{(0,1)}$, is suitable to describe the helicity ± 1 representations of the physical photon; the free Maxwell equations merely project these representations out of the reducible $F_{\mu\nu}$ (note that the tilde on $F_{\mu\nu}$ is written to indicate that physically $\tilde{F}_{\mu\nu} = F_{\mu\nu}(\tilde{p})$, see below).

By the same token, the vector $\tilde{A}_\mu$ carrying the representation $D_{(\frac{1}{2},\frac{1}{2})}$ of L (which is defined by $D_{(\frac{1}{2},\frac{1}{2})}(\Lambda)\tilde{A}_\mu = \Lambda_\mu{}^\nu \tilde{A}_\nu$), does not contain these subrepresentations. The reason is well-known, but it is worthwhile to recall it, and reformulate the ensuing discussion in the language of symplectic reduction theory.

2.2 Marsden-Weinstein reduction for the frozen photon field

We conventionally choose the fixed point $\tilde{p} \in O_0^+$ to be $\tilde{p}^\mu = (1,0,0,1)$, and label the complex Fourier coefficients $A_\mu(\tilde{p})$ as $\tilde{A}_\mu \in \mathbb{C}^4$. The little group $L_{\tilde{p}} = E(2)$ (consisting of those Lorentz transformations which leave $\tilde{p}$ stable) is embedded in L by identifying the abelian generators $\hat{T}_1$, $\hat{T}_2$ of the former with $\hat{K}_1 - \hat{J}_2$, $\hat{K}_2 + \hat{J}_1$, respectively, and its generator of $SO(2)$ with $\hat{J}_3$; here $\hat{K}_i = \hat{M}_{0i}$ and $\hat{J}_i = \frac{1}{2}\epsilon_{ijk}\hat{M}_{jk}$, where $\hat{M}_{\mu\nu} = -\hat{M}_{\nu\mu}$ are the usual generators of the Lie algebra $\mathbf{l}$ of the Lorentz group. To describe how $\mathcal{H}_{(\frac{1}{2},\frac{1}{2})} = \mathbb{C}^4$ decomposes under $D_{(\frac{1}{2},\frac{1}{2})}(L \restriction E(2))$, we choose a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_+, \mathbf{u}_-\}$, which is related to the usual basis $\{\mathbf{e}_\mu\}$ of $\mathbb{C}^4$ by $\mathbf{u}_1 = \mathbf{e}_1$, $\mathbf{u}_2 = \mathbf{e}_2$, $\mathbf{u}_\pm = \frac{1}{2}(\mathbf{e}_0 \pm \mathbf{e}_3)$. Since $\tilde{p}_\mu = (1,0,0,-1)$, $T \equiv \mathbb{C}\,\mathbf{u}_-$ is invariant under $E(2)$, and so is $N \equiv \mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_-)$, but the latter does not decompose, since $\hat{T}_i \mathbf{u}_i = \mathbf{u}_-$ $(i = 1,2)$.

The desired representation $U_1 \oplus U_{-1}$ of $E(2)$ on $\mathbb{C}^2$, which characterizes photons, is obtained by quotienting N by T: the natural action of $E(2)$ on the quotient indeed coincides with $U_1 \oplus U_{-1}$ (i.e., $\mathbb{R}^2$ acts trivially and $SO(2)$ acts in its (complexified) defining representation). Hence one proceeds in two steps: first a constraint $\tilde{p}^\mu \tilde{A}_\mu = 0$ is imposed, and then one factorizes the solution space N of the constraint by the equivalence relation $\tilde{A}_\mu \sim \tilde{A}_\mu + \lambda \tilde{p}_\mu$, $\lambda \in \mathbb{C}$.

This procedure may be reformulated as a Marsden-Weinstein reduction. To treat

$\mathbb{C}^4$ as a classical phase space (S, ω) we equip it with the symplectic form ω defined by

$$\omega(\tilde{B}, \tilde{C}) = 2\,\mathrm{Im}\,(\tilde{B}, \tilde{C})_M \equiv 2\,\mathrm{Im}\,\tilde{B}^\mu \overline{\tilde{C}}_\mu. \tag{2.8}$$

Then the frozen gauge group $G = \mathbb{C}$ (regarded as additive group) acts on S in that $\lambda \in G$ maps $\tilde{A} \in S$ to $\tilde{A} + \lambda\tilde{p}$. This action is strongly Hamiltonian, with moment map $J : S \to \mathfrak{g}^* = \mathbb{C}$ given by $J(\tilde{A}) = i\tilde{p}^\mu \tilde{A}_\mu$. Hence the constraint set N coincides with $J^{-1}(0)$, and the reduced space $\mathbb{C}^2$ constructed above is nothing but the Marsden-Weinstein quotient $S^0 = J^{-1}(0)/G = N/T$.

We now compute the symplectic form ω^0 on S^0. Let $\tilde{A} \in N$, and $\tilde{B}$, $\tilde{C} \in T_{\tilde{A}}N \simeq N$. We denote the equivalence classes of these vectors in N/T by $[\cdot]$. By the theory of Marsden-Weinstein reduction, ω^0 is defined at $[\tilde{A}]$ by

$$\omega^0([\tilde{B}], [\tilde{C}]) = \omega(\tilde{B}, \tilde{C}). \tag{2.9}$$

In general, the right-hand side should be evaluated at any lift $\tilde{A}$ of $[\tilde{A}]$, but in the present case ω is the same at all points of S. In any case, by the G-invariance of ω, the r.h.s. is independent of the chosen representative $\tilde{\cdot}$ of the class $[\tilde{\cdot}]$. Hence if we define an inner product $(\cdot, \cdot)_{S^0}$ on S^0 by

$$([\tilde{B}], [\tilde{C}])_{S^0} = (\tilde{B}_T, \tilde{C}_T)_S, \tag{2.10}$$

where the inner product on the r.h.s. is the Euclidean one in $\mathbb{C}^4$, and $\tilde{B}_T = (0, \tilde{B}_1, \tilde{B}_2, 0)$ (guaranteeing independence of the choice of representatives), then

$$\omega^0([\tilde{B}], [\tilde{C}]) = -2\,\mathrm{Im}\,([\tilde{B}], [\tilde{C}])_{S^0}. \tag{2.11}$$

Moreover, the reduced representation of $E(2)$ on S^0 is unitary with respect to the inner product on S^0.

Anticipating the quantization of the model in section 3, one may introduce a classical 'field algebra' $\mathcal{F} = C^\infty(\mathbb{C}^4)$, whose algebraic structure is given by the Poisson bracket derived from ω and by pointwise multiplication. The algebra of weak observables $\mathcal{A}$ is then given by $\mathcal{F}^G$, which stands for the set of those elements f of $\mathcal{F}$ which are gauge-invariant (that is, satisfying $f(\tilde{A}_\mu + \lambda\tilde{p}_\mu) = f(\tilde{A}_\mu)$ for all

$\tilde{A} \in S$ and $\lambda \in G$). The elements of $\mathcal{A}$ have a well-defined action on S^0, which may be thought of as providing a classical induced representation of $\mathcal{A}$, cf. [26]. The quotient of $\mathcal{A}$ by the kernel of this representation is then the algebra of observables of the model.

We will now see how this procedure works for the entire field.

2.3 Marsden-Weinstein reduction for the photon field

In the notation we used for canonical representations, the one-photon Hilbert space is $\mathcal{H}_{\rm photon} = \mathcal{H}^{0,+,1} \oplus \mathcal{H}^{0,+,-1}$, where the labels ± 1 refer to the helicity ± 1 representations of $E(2)$. Our notation implies that $\mathcal{H}_{\rm photon}$ is to be seen as a P-module carrying the unitary representation

$$U_{\rm photon} = U^{0,+,1} \oplus U^{0,+,-1}, \tag{2.12}$$

cf. (2.2). The momentum space gauge field $A_\mu(\mathbf{p})$ is taken to transform under the covariant representation

$$U_{\rm gaugefield} = U^{0,+,(\frac{1}{2},\frac{1}{2})} \tag{2.13}$$

induced by $D_{(\frac{1}{2},\frac{1}{2})}$ of L, cf. (2.7) . Hence the gauge field Hilbert space is $\mathcal{H}_{\rm gaugefield} = \mathcal{H}^{0,+,(\frac{1}{2},\frac{1}{2})}$, identified with $L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^4$. The connection between (2.13) and (2.4) is somewhat subtle. The space of real weak solutions $A_\mu(x)$ of the wave equation $\Box A_\mu = 0$ with Fourier coefficients (in the sense that $A_\mu(x) = \int d'p\, A_\mu(\mathbf{p}) \exp(-ipx) +$ *c.c.*) in $L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^4$ forms a real Hilbert space, and defines a real representation of P through (2.3). This corresponds to a real subrepresentation of $U^{0,+,(\frac{1}{2},\frac{1}{2})} \oplus U^{0,-,(\frac{1}{2},\frac{1}{2})}$, which in quantum field theory is juggled so as to be replaced by the complex representation (2.13). Accordingly, we have taken the shortcut of starting from (2.13) directly.

Our problem is now to pass from $\mathcal{H}_{\rm gaugefield}$ to $\mathcal{H}_{\rm photon}$; as we have pointed out, this is not possible using a Hilbert space decomposition, but it can be achieved by Marsden-Weinstein reduction. For this purpose we equip $\mathcal{H}_{\rm gaugefield}$ with a symplectic form defined by

$$\omega(B, C) = 2\,\mathrm{Im} \int d'p B_\mu(\mathbf{p}) \overline{C^\mu(\mathbf{p})} \equiv 2\mathrm{Im}(B, C)_M, \tag{2.14}$$

cf. (2.8). In x-space, this form was already given in (1.1). We will henceforth refer to $\mathcal{H}_{\text{gaugefield}}$ with ω as S, to stress that it is now regarded as real symplectic (Hilbert) manifold.

It is easily shown that S is strongly symplectic (for one can modify the complex structure of S so that ω becomes twice the imaginary part of the inner product, cf. [9]).

We take the gauge group to be the Hilbert space $G = L^2(\mathbb{R}^3, d^3p\,|\mathbf{p}|/2(2\pi)^3)$, that is, we take non-constant weak solutions λ of the wave equation whose Fourier-transformed exterior derivative $d\lambda$ lies in S. The norm is accordingly

$$\| \lambda \|_{\frac{1}{2}} = \tfrac{1}{2}(d\lambda, d\lambda)_S = \left(\int d'p\,|\mathbf{p}|^2 |\lambda(\mathbf{p})|^2 \right)^{\frac{1}{2}}. \tag{2.15}$$

The abelian group structure is given by addition, and the gauge group acts on S by gauge transformations, i.e., in x-space $A \to A + d\lambda$ as usual, and hence in momentum space by $A_\mu(\mathbf{p}) \to A_\mu(\mathbf{p}) - ip_\mu\lambda(\mathbf{p})$ (with the standing notation $p_0 = |\mathbf{p}|$); for brevity we will write the latter action as $A \to A + d\lambda$ as well. The topology on G has been chosen precisely so that this action is smooth.

This action preserves the symplectic form and is even strongly Hamiltonian, with moment map $J : S \to \mathrm{Lie}(G)^*$ given by

$$J_X(A) \equiv \langle J(A), X \rangle = \omega(dX, A), \tag{2.16}$$

cf. (2.14). Thus, if we regard $\mathbf{g} = \mathrm{Lie}(G) \simeq G$ as a real Hilbert space, and identify $\mathbf{g}^*$ with $\mathbf{g} = G$ under the pairing $\langle \theta, X \rangle = 2\,\mathrm{Im}(X, \theta)$, then $J(A)$ is the element $\hat{A}$ of G whose value at $\mathbf{p}$ is given by $p^\mu A_\mu(\mathbf{p})/|\mathbf{p}|^2$; note that this function indeed lies in $L^2(\mathbb{R}^3, d^3p\,|\mathbf{p}|/2(2\pi)^3)$ if A is in $L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^4$. It easily follows that J is smooth, and because $J(A + d\lambda)(\mathbf{p}) = J(A)(\mathbf{p}) - ip^2\lambda(\mathbf{p})/|\mathbf{p}|^2 = J(A)(\mathbf{p})$ (for $p^2 = 0$), it is clearly equivariant (recall that the co-adjoint action of G on $\mathbf{g}^*$ is trivial, as G is abelian). We now show that the Marsden-Weinstein reduced space of S exists and carries the desired representation of P.

Theorem 1 *The Marsden-Weinstein reduced space $S^0 = J^{-1}(0)/G$ is a symplectic Hilbert manifold. In addition, it is a linear space, which inherits an inner product and a P-action (see (2.13)) from S. With respect to these inherited structures, it is*

unitarily equivalent to the Hilbert space $\mathcal{H}_{\rm photon}$ *carrying the representation* $U_{\rm photon}$ *(cf. (2.12)).*

Proof. We first show that S^0 is a symplectic manifold. Given that we already know that (S,ω) is strongly symplectic, that J is equivariant, and that S is a Hilbert manifold (so that, in particular, its model space is reflexive), this follows from the theory of (infinite-dimensional) Marsden-Weinstein reduction (cf. [1], and esp. [29, Ch. 6]) if: i) 0 is a regular value of J, and ii) G acts freely and properly on $J^{-1}(0)$.

To show i), we compute the derivative $J_*(A)$ of J at any A to be $J_*(A) : B \in T_AB \to \hat{B} \in T_{J(A)}\mathbf{g}^* \simeq \mathbf{g}^* \simeq G$ (cf. the preceding par.), which of course is independent of A. Thus an arbitary $\psi \in G$ is the image of $B_\mu(\mathbf{p}) = \frac{1}{2}p^\mu\psi(\mathbf{p})$ (note that $p^\mu p^\mu = 2|\mathbf{p}|^2$), so that $J_*(A)$ is surjective at any $A \in S$, hence certainly at $A \in J^{-1}(0)$.

As to ii), the freeness of the G-action is trivial, whereas the properness follows from the equivalent property that $A_n \to A$ and $\lambda_n \cdot A_n \to B$ in S together must imply that the sequence $\{\lambda_n\}$ in G has a convergent subsequence. Since the group action is given by $\lambda \cdot A = A + d\lambda$, this follows immediately from the topology we have put on G (in which $\lambda_n \to \lambda$ in G is the same as $d\lambda_n \to d\lambda$ in S).

The set $J^{-1}(0)$ consists of those A satisfying $p^\mu A_\mu(\mathbf{p}) = 0$ for almost all $\mathbf{p}$, and is closed: if $A_n \in J^{-1}(0)$ and $A_n \to A$ in S, then $A \in J^{-1}(0)$ because of (2.16) and Cauchy-Schwartz. We now fix a Lorentz frame, and define the projector P_T on S by

$$P_T A_0 = 0; \quad (P_T A_i)(\mathbf{p}) = A_i(\mathbf{p}) - p_i p_j A_j(\mathbf{p})/|\mathbf{p}|^2, \tag{2.17}$$

Thus $J^{-1}(0)$ has the (non Lorentz-invariant) orthogonal decomposition $J^{-1}(0) = P_T S \oplus dG$, where $d : G \to S$ is given by $(d\lambda)_\mu(\mathbf{p}) = -ip_\mu\lambda(\mathbf{p})$, cf. [9, p. 636]. Hence $J^{-1}(0)/G \simeq P_T S$, which shows that S^0 is a Hilbert space. (This is *not* an isomorphism as carrier spaces of the action of the Poincaré group P, which does not leave $P_T S$ stable).

Furthermore, P acts on S by symplectic transformations, maps $J^{-1}(0)$ into itself (as the condition defining this space is invariant), and also maps a vector of the type $d\lambda$ into another vector of this type (for $U_{\rm gaugefield} d\lambda = dU_{\rm gaugefield}\lambda$), so that

it preserves the G-foliation. The P-action on S therefore quotients to an action on S^0, and it follows from the isomorphism $S^0 \simeq P_T S$ as Hilbert spaces that this quotient action is the one on the one-photon space of QED in the Coulomb gauge. The theorem then follows from the well-known properties of the latter, or by explicit computation. ■

Hence in x-space $J^{-1}(0)$ consists of those fields which satisfy the Lorentz condition $\partial^\mu A_\mu(x) = 0$ (weak derivatives), which is the precise sense in which this formalism implements Gauss' law at this stage.

The symplectic form ω^0 on S^0 (determined by the Marsden-Weinstein reduction procedure) may be described as in subsection 2.2.: we now define an inner product on S^0 by

$$([B],[C])_{S^0} = (P_T B, P_T C)_S \tag{2.18}$$

in terms of the usual inner product on $S = L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^4$, and obtain

$$\omega^0([B],[C]) = -2\,\mathrm{Im}\,([B],[C])_{S^0}. \tag{2.19}$$

Consequently, the quotient representation U_{photon} of P is unitary.

As in the previous subsection, the Poisson algebra $\mathcal{A}$ of gauge-invariant smooth functions on S (which is a subalgebra of the classical field algebra $\mathcal{F}$ of all smooth functions on S) is represented on the reduced space S^0, and the quotient of $\mathcal{A}$ by the kernel of this representation is the algebra of observables of classical electromagnetism.

Finally, let us sketch the entirely analogous development for helicity ± 2, that is, linearized Einstein gravity. Here the starting point is the covariant representation $U^{0,+,(1,1)}$ of P, realized on the space of symmetric tensor fields $h_{\mu\nu}(x)$ on Minkowski space, which satisfy the wave equation $\Box h_{\mu\nu} = 0$, with Fourier coefficients in $L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^{10}$. The gauge group G now consists of those non-constant weak solutions $\xi_\mu(x)$ of the wave equation for which the quantity $(\delta\xi)_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - g_{\mu\nu}\partial\cdot\xi$ lies in S. G acts on S by $h \to h + \delta\xi$. With the symplectic structure on S given by

$$\omega(h,k) = \mathrm{Im} \int d'p\, (h_{\mu\nu}(\mathbf{p})\overline{k}^{\mu\nu}(\mathbf{p}) - \tfrac{1}{2} h^\mu_\mu(\mathbf{p})\overline{k}^\nu_\nu(\mathbf{p})),$$

the group action is strongly Hamiltonian, with moment map given by $J_\xi(h) = \omega(\delta\xi, h)$. Hence $J^{-1}(0) = \{h \in S | \partial^\mu h_{\mu\nu} = 0\}$, and $S^0 = J^{-1}(0)/G$, which is the space of physical degrees of freedom of gravitational waves, carries the representation $U^{0,+,2} \oplus U^{0,+,-2}$ of P.

2.4 Canonical and covariant Poisson actions of the Poincaré group

This subsection provides a conceptual link between the preceding theory, which describes classical massless field theory and quantum one-particle theory, and the theory of massless classical particles. It may be skipped without losing the main thread of the paper.

We show that the Hilbert space theory of the representations of P has an analogue at the level of symplectic manifolds, which play the rôle of phase spaces of classical relativistic particles. The main idea here is that the classical analogue of a unitary representation of P is a strongly Hamiltonian action of P on a symplectic manifold [1, 21, 45] (also cf. the Introduction above); the requirement of irreducibility of a Hilbert space representation is then replaced by the transitivity of the action. A well-known theorem of Kostant and Souriau (see [1, 21, 45]) then asserts that an irreducible symplectic manifold for P must be a co-adjoint orbit of P, or a covering space thereof; we will refer to such manifolds (along with the action of P on them) as canonical realizations.

1. Canonical realizations. In the semidirect product $P = L \ltimes \mathbb{R}^4$, L will again stand for $L_+^\uparrow$ in what follows; the inclusion of time inversion and parity is complicated also in classical mechanics, where the former is represented as an anti-canonical transformation. See [45] for a careful treatment of the full Poincaré group in this context.

We wish to derive an expression analogous to (2.2) for the co-adjoint action of P on an orbit in the dual $\mathbf{p}^*$ of its Lie algebra $\mathbf{p}$. The co-adjoint action α_{co} of $(\Lambda, a) \in P$ on $(M, p) \in \mathbf{p}^* = \mathbf{l}^* \oplus \mathbb{R}^4$ is then given by [21, 31]

$$\alpha_{\mathrm{co}}(\Lambda, a)(M, p) = (\Lambda M + \vartheta_a(\Lambda p), \Lambda p), \tag{2.20}$$

where ΛM stands for the co-adjoint action of Λ on M, and $(\Lambda p)_\mu = \Lambda_\mu{}^\rho p_\rho$. The element $\vartheta_a(p) \in \mathfrak{l}^*$ is defined by $\langle \vartheta_a(p), X\rangle = \langle p, Xa\rangle$. It follows that the co-adjoint orbit $\mathcal{O}^P_{(M,p)}$ through (M,p) is characterized by the L-orbit of p in $\mathbb{R}^4$ and the L_p-orbit in $\mathfrak{l}_p^*$ through $\pi_p(M)$, where $\pi_p : \mathfrak{l}^* \to \mathfrak{l}_p^*$ is given by restriction to $\mathfrak{l}_p$. In what follows we assume that $p^2 \geq 0$ and $p^0 > 0$, so that we may take p to be our favourite point $\tilde{p} \in O_m \equiv O_m^+$ (the theory is identical for O_m^-). As before, we then have $L_{\tilde{p}} = SO(3)$ for $m > 0$ and $L_{\tilde{p}} = E(2)$ for $m = 0$. (The connection between these orbits and the canonical representations of P can be made explicit by either geometric quantization [36, 39] or by the reduction-induction theory of [26]).

Various descriptions of the orbits under this action exist [21, 45, 26], but here we find it convenient to exploit the fact (proved in [31] for general semidirect products) that the orbit $\mathcal{O}^P_{(\tilde{M},\tilde{p})}$ through $(\tilde{M},\tilde{p})$ is realized as the symplectic leaf of the Poisson manifold $(T^*L)/L_{\tilde{p}}$ (where $L_{\tilde{p}}$ acts on the cotangent bundle T^*P, equipped with its canonical Poisson structure, by pulling back its right-action on L) which corresponds to the co-adjoint orbit $\mathcal{O}^{L_{\tilde{p}}}_{\pi_{\tilde{p}}(\tilde{M})}$ in $\mathfrak{l}^*_{\tilde{p}}$. To avoid cumbersome notations, we will simply refer to the latter orbit in $\mathfrak{l}^*_{\tilde{p}}$ as $\mathcal{O}^{L_{\tilde{p}}}_\sigma$, where σ is some label characterizing the co-adjoint orbits of $L_{\tilde{p}}$. The leaf $L^{m,\sigma}$ in question can be written as

$$L^{m,\sigma} = L \times_{L_{\tilde{p}}} \pi_{\tilde{p}}^{-1}(\mathcal{O}^{L_{\tilde{p}}}_\sigma); \tag{2.21}$$

here (evidently) $\pi_{\tilde{p}}^{-1}(\mathcal{O}^{L_{\tilde{p}}}_\sigma) \subset \mathfrak{l}^*$, and $L_{\tilde{p}}$ acts on $T^*P \simeq L \times \mathfrak{l}^*$ by $R_h(\Lambda,\theta) = (\Lambda h^{-1}, \alpha_{\rm co}(h)\theta)$. The points of $L^{m,\sigma}$ are then equivalence casses $[\Lambda,\theta]$, where $(\Lambda,\theta) \sim R_h(\Lambda,\theta)$ for all $h \in L_{\tilde{p}}$. Note that $L_{\tilde{p}}$ indeed maps $\pi_{\tilde{p}}^{-1}(\mathcal{O}^{L_{\tilde{p}}}_\sigma)$ into itself under the co-adjoint action.

Assuming that $\pi_{\tilde{p}}(\tilde{M}) \in \mathcal{O}^{L_{\tilde{p}}}_\sigma$, we will denote $\mathcal{O}^P_{(\tilde{M},\tilde{p})}$ by $\mathcal{O}^{m,\sigma}$. The symplectomorphism $\rho : \mathcal{O}^{m,\sigma} \to L^{m,\sigma}$ is given by

$$\rho(M, \Lambda\tilde{p}) = [\Lambda, \Lambda^{-1}M]; \tag{2.22}$$

note that both sides are independent of the choice of Λ in the class $\{\Lambda h\}_{h\in L_{\tilde{p}}}$. The fact that ρ is indeed a symplectomorphism relative to the canonical ('Lie-Poisson-Kirillov-Kostant-Souriau-Arnold') symplectic form on $\mathcal{O}^{m,\sigma}$ [21, 1] and the one on $L^{m,\sigma}$ inherited from T^*L is proved in [31].

It follows from (2.20) and (2.22) that the action $\alpha_{\rm co}^{m,\sigma} = \rho \circ \alpha_{\rm co} \circ \rho^{-1}$ of P on $L^{m,\sigma}$ is given by

$$\alpha_{\rm co}^{m,\sigma}(\Lambda, a)[\tilde{\Lambda}, \theta] = [\Lambda\tilde{\Lambda}, \theta + \theta_a(\Lambda\tilde{\Lambda})], \tag{2.23}$$

where $\theta_a(\Lambda) = \vartheta_{\Lambda^{-1}a}(\tilde{p})$, i.e., $\langle \theta_a(\Lambda), X\rangle = \langle \tilde{p}, X\Lambda^{-1}a\rangle$. Note that $\pi_{\tilde{p}}(\theta_a(\Lambda)) = 0$, so that $\theta + \theta_a(\Lambda\tilde{\Lambda})$ in (2.23) indeed lies in $\pi_{\tilde{p}}^{-1}(\mathcal{O}_\sigma^{L_{\tilde{p}}})$ if θ does. Choosing a section $\eta : L/L_{\tilde{p}} \simeq O_m \to L$, as in subsection 2.1, leads to a local trivialization of $L^{m,\sigma}$, regarded as a bundle over O_m. In this trivialization P acts by

$$(\alpha_{\rm co}^{m,\sigma})^\eta(\Lambda, a)(p, \theta) = (\Lambda p, \alpha_{\rm co}(\eta(\Lambda p)^{-1}\Lambda\eta(p))\theta + \theta_a(\eta(\Lambda p))). \tag{2.24}$$

We will not use this expression in what follows, but it is the classical analogue of (2.2). The 'cocycle' $\eta(\Lambda p)^{-1}\Lambda\eta(p)$ takes values in $L_{\tilde{p}}$, so that the action is well-defined.

Since P acts transitively on $L^{m,\sigma}$, its symplectic form $\omega^{m,\sigma}$ is fully determined by specifying its value on the vector fields $\tilde{X}$ tangent to the flows $\alpha_{\rm co}^{m,\sigma}(\exp tX)$, $X \in \mathbf{p}$. Using (2.22) and the canonical symplectic structure on $\mathcal{O}^{m,\sigma}$ we obtain

$$\langle \omega^{m,\sigma}|\tilde{X}, \tilde{Y}\rangle([\Lambda, \theta]) = \langle \Lambda\theta + \Lambda\tilde{p}|[X, Y]\rangle. \tag{2.25}$$

An important special case is $\sigma = \{0\}$, in which case the well-known diffeomorphism $L \times_{L_{\tilde{p}}} \mathbf{l}_{\tilde{p}}^\perp \simeq T^*(L/L_{\tilde{p}})$ (where $\mathbf{l}_{\tilde{p}}^\perp \subset \mathbf{l}^*$ is the annihilator of $\mathbf{l}_{\tilde{p}}$) [25], plus a short computation showing that the canonical symplectic form on $T^*(L/L_{\tilde{p}})$ pulls back to the form (2.25) on $L^{m,0}$ under this diffeomorphism, implies that $L^{m,0} \simeq T^*O_m$ as symplectic manifolds.

Another pleasant special case pertains when $m > 0$ (and σ arbitrary; this label now specifies the radius of a two-sphere S^2 in $\mathbf{so(3)}^* = \mathbb{R}^3$). For in this case $\mathbf{l}$ has the reductive decomposition $\mathbf{l} = \mathbf{so(3)} \oplus \mathbf{m}$, where $\mathbf{so(3)}$ is generated by the rotation generators $\hat{J}_i$, and $\mathbf{m}$ is the span of the boost generators $\hat{K}_i$ $(i = 1, 2, 3)$. Reductive here means that $[\mathbf{so(3)}, \mathbf{m}] \subseteq \mathbf{m}$, and this property implies that the principal bundle $(L, L/L_{\tilde{p}}, L_{\tilde{p}})$ has an L-invariant connection, viz. the one defined by the reductive decomposition [25]. This is important, for any connection on this bundle may be used to define a projection $pr : L^{m,\sigma} \to T^*O_m$ [21], and therefore a splitting of $L^{m,\sigma}$

into orbital and spin degrees of freedom. The fact that the connection is L-invariant then means that this splitting is Poincaré covariant. In this case, one even has the global form $L^{m,\sigma} = T^*O_m \times S^2_\sigma$, which is well-known, except that in the literature [45, 21] the first factor T^*O_m is written as $T^*\mathbb{R}^3$, where $\mathbb{R}^3$ is x-space; that is, the rôle of base space and fibre is reversed. We leave it to the reader to apply the beautiful formalism of [21, 30] to the case at hand, and obtain the Poisson bracket on $L^{m,\sigma}$ as the sum of the canonical ones on T^*O_m and S^2, and an extra term depending on the curvature of the connection.

For our main concern is the massless case, which is dramatically different from the massive one. In this case there exists no reductive decomposition $\mathbf{l} = \mathbf{e}(\mathbf{2}) \oplus \mathbf{m}$, and the orbital degree of freedom does not factorize in a meaningful (i.e., covariant) manner. Recalling that $\mathbf{e}(\mathbf{2})$ is generated by $\hat{T}_1 = \hat{K}_1 - \hat{J}_2, \hat{T}_2 = \hat{K}_2 + \hat{J}_1, \hat{J}_3$, the most useful decomposition is to take $\mathbf{m}$ to be the linear span of $\hat{K}_1 + \hat{J}_2, \hat{K}_2 - \hat{J}_1, \hat{K}_3$ (which, incidentally, in contrast to the massive case, closes under Lie bracket, and generates the Lie algebra of Bianchi type V; in this classification the Lie algebra of $E(2)$ is of type VII_0, and that of $SO(3)$ of type IX).

In any case, the co-adjoint orbits of $E(2)$ which are of interest to us are the points $\mathcal{O}_{\pm 1}$, whose value on $\hat{T}_1$ and $\hat{T}_2$ vanishes, whereas the value on $\hat{J}_3$ is ± 1. These orbits are the classical analogues of the helicity ± 1 representations $U_{\pm 1}$ of $E(2)$, but note that the co-adjoint action of $E(2)$ on these orbits is (evidently) trivial, whereas the action of $U_{\pm 1}$ on $\mathbb{C}$ is not. Nonetheless, the corresponding orbits $\mathcal{O}^{0,\pm 1} \simeq L^{0,\pm 1}$ of P are not equivalent to $L^{0,0}$ as realizations of P. We shall refer to the values ± 1 as helicity.

2. Covariant realizations. The covariant symplectic realizations of P [21, sect. I.20] are initially defined on symplectic spaces of the type $T^*\mathbb{R}^4 \times \mathcal{O}^L_{a,s}$, where $\mathcal{O}^L_{a,s}$ is a co-adjoint orbit in $\mathbf{l}^*$ of the Lorentz group L, equipped with the canonical symplectic structure. We follow [21] in identifying both the Lie algebra $\mathbf{l}$ of L and its dual $\mathbf{l}^*$ with the space of antisymmetric 4×4 matrices $M_{\mu\nu}$; the pairing between $\mathbf{l}^*$ and $\mathbf{l}$ is given by contraction using the Minkowski metric g. Then $(\Lambda M)_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma M_{\rho\sigma}$.

Then such a co-adjoint orbit is characterized by the conditions [21, 45] $\frac{1}{2} M_{\mu\nu} M^{\mu\nu} =$

s^2 and $|\epsilon^{\mu\nu\rho\sigma}M_{\mu\nu}M_{\rho\sigma}| = 24a$. In physically relevant cases $a = 0$, and later we will only examine the orbit where $s = 1$, which leads to spin or helicity 1. In terms of the variables $K_i = M_{0i}$ and $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ this means that $\mathbf{J}\cdot\mathbf{K} = 0$ and $|\mathbf{J}|^2 - |\mathbf{K}|^2 = 1$.

We take $\omega = dx^\mu \wedge dp_\mu$ as the symplectic form on $T^*\mathbb{R}^4$ (where p_μ are the fiber co-ordinates on $T^*\mathbb{R}^4$); the total space has the product symplectic structure, and P acts on it by the product $\alpha^{a,s}$ of the pull-back of its defining action on $\mathbb{R}^4$ and the co-adjoint action of L on $\mathcal{O}^L_{a,s}$. This action is obviously strongly Hamiltonian, but it is 'reducible' because it is not transitive.

The first step in the 'reduction' of $T^*\mathbb{R}^4 \times \mathcal{O}^L$, is to impose the constraint $p^2 = m^2$ (and $p_0 > 0$), and quotient the constraint surface $\mathbb{R}^4 \times O_{\tilde{p}}$ by the null foliation of the induced presymplectic form. This is the same as performing a Marsden-Weinstein reduction w.r.t. the action of $\mathbb{R}$ on $T^*\mathbb{R}^4$, given by $(x^\mu, p_\mu) \to (x^\mu + p^\mu\tau, p_\mu)$. The reduced space may be identified with T^*O_m, which we realize as $L \times_{L_{\tilde{p}}} \mathbf{l}^\perp_{\tilde{p}}$, as before: a point $[x, \Lambda\tilde{p}]$ in the reduced space (brackets refering to the $\mathbb{R}$-foliation) may be identified with $[\Lambda, \theta_x(\Lambda)] \in L \times_{L_{\tilde{p}}} \mathbf{l}^\perp_{\tilde{p}}$ (brackets refering to $L_{\tilde{p}}$-equivalence classes). On account of the property $\theta_x(\Lambda h) = h^{-1}\theta_x(\Lambda)$ for all $h \in L_{\tilde{p}}$, this identification is well-defined.

The P-action $\alpha^{m,a,s}$ on $S^{m,a,s} = T^*O_m \times \mathcal{O}^L_{a,s}$ is given by the reduction of $\alpha^{a,s}$; we obtain

$$\alpha^{m,a,s}(\Lambda, a)([\tilde{\Lambda}, \theta], M) = ([\Lambda\tilde{\Lambda}, \theta + \theta_a(\Lambda\tilde{\Lambda})], \Lambda M). \tag{2.26}$$

Using the previous observation that $T^*O_m \simeq L^{m,0} \simeq \mathcal{O}^{m,0}$, we find that the symplectic structure $\omega^{m,a,s}$ on $S^{m,a,s}$ has a particularly simple form when evaluated on the vector fields generating the P-action. For $X, Y \in \mathbf{p}$ one has

$$\langle \omega^{m,a,s} | \tilde{X}, \tilde{Y} \rangle([\Lambda, \theta], M) = \langle \Lambda\theta + M + \Lambda\tilde{p} | [X, Y] \rangle. \tag{2.27}$$

Since P does not act transitively, this does not determine $\omega^{m,a,s}$, but the above expression is sufficient for our purposes.

We now specialize to the case $a = 0, s = 1$, and attempt to make P act transitively by imposing a further constraint (this was first done in [21, I.20] in the massive case, using a slightly different formalism). The appropriate constraint is $p^\mu M_{\mu\nu} = 0$,

which in the present setting comes from the function $\Phi : S^{m,0,1} \to \mathbb{R}^4$, defined by $\Phi([\Lambda,\theta],M) = M\Lambda\tilde{p}$ (which is indeed independent of the chosen representative of the class $[\Lambda,\theta]$). The constraint is then $\Phi = 0$; as we shall see, this stands for two independent conditions only. The solution set C_m is given by

$$C_m = \{([\Lambda,\theta],M) \in S^{m,0,1} | M \in \Lambda(\mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1})\}, \tag{2.28}$$

which is well-defined, for $L_{\tilde{p}}$ maps $\mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$ into itself, so that changing Λ by Λh, $h \in L_{\tilde{p}}$, alters nothing.

For $m > 0$, the set $\mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$ consists of those $(\mathbf{J},\mathbf{K}) \in \mathbf{l}$ for which $\mathbf{K} = 0$ and $|\mathbf{J}| = 1$. Consequently, C_m is symplectic (in the traditional language of constraint theory [12, 42], the constraints are second-class), as is evident from the fact that we can find a symplectomorphism $\varphi : C_m \to L^{m,1}$, given by

$$\varphi([\Lambda,\theta],M) = [\Lambda, \theta + \Lambda^{-1}M]. \tag{2.29}$$

That this is a diffeomorphism is immediate from the definition $L^{m,1} = L \times_{L_{\tilde{p}}} \pi_{\tilde{p}}^{-1}(S^2_1)$ and the above description of $\mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$; that φ is symplectic is equally immediate from (2.25) and (2.27). Moreover, the appropriate P-actions are correclty intertwined by φ, in the sense that $\varphi \circ \alpha^{m,0,1} = \alpha^{m,1}_{\mathrm{co}} \circ \varphi$.

For $m = 0$ the set $\mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$ is given by $\{(\mathbf{J},\mathbf{K}) \in \mathbf{l} | K_1 = -J_2, K_2 = J_1, J_3 = \pm 1, K_3 = 0\}$, so that C_0 is the union of two components $C_0^{\pm}$, the $\pm$ corresponding to the sign of J_3. This time, however, the analogue of the map (2.29) is not a diffeomorphism. For $\mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$ is $\mathrm{span}(\hat{T}_1,\hat{T}_2) \pm \hat{J}_3$, whereas $\mathbf{l}^{\perp}_{\tilde{p}} = \mathrm{span}(\hat{T}_1,\hat{T}_2,K_3)$. This reflects the phenomenon that $\mathbf{l}^{\perp}_{\tilde{p}} \oplus \mathbf{l}_{\tilde{p}} \neq \mathbf{l}$ for $m = 0$ (i.e., $\mathbf{l}_{\tilde{p}} = \mathbf{e(2)}$), whereas equality does hold for $m > 0$ (where $\mathbf{l}_{\tilde{p}} = \mathbf{so(3)}$). Thus, for example, the points $([1,(T_1+\lambda_1,T_2+\lambda_2,K_3)],(T'_1-\lambda_1,T'_2-\lambda_2,J_3=\pm 1))$ have the same image under φ for any $\lambda_i \in \mathbb{R}$.

More systematically, the pull-back of the symplectic form $\omega^{0,0,1}$ to $C_0^{\pm}$ is not symplectic; $C_0^{\pm}$ are co-isotropically embedded in $S^{0,0,1}$, and the null directions of its presymplectic form may be found by computing the transformations generated by the constraint Φ, or by a local computation using (2.27) (it follows from the explicit

structure of $C_0^{\pm}$ that P acts transitively on each component). Looking at $\Lambda = 1$ to start with, the four conditions $\Phi = 0$ amount to $\Phi_1 = K_1 + J_2 = 0$, $\Phi_2 = K_2 - J_1 = 0$, and $\Phi_0 = \Phi_3 = K_3 = 0$. Now recall that the orbit $\mathcal{O}^L_{0,1}$ was specified by the conditions $\mathbf{J} \cdot \mathbf{K} = 0$ and $|\mathbf{J}|^2 - |\mathbf{K}|^2 = 1$, which on imposition of the first two constraints imply the third one, which is therefore superfluous. Thus we need to identify two independent constraints in Φ, and this may be done by choosing a global piecewisely smooth section $\eta : O_0 \to L$. The two independent constraints are then given by Φ_1^η and Φ_2^η, where $\Phi^\eta([\Lambda, \theta], M) = \eta(\Lambda\tilde{p})^{-1}\Phi([\Lambda, \theta], M)$. (If the constraints are written as $p^\mu M_{\mu\nu} = 0$, the need to specify a section η comes from the fact that one has to specify an isomorphism between $p^\perp/\mathbb{C}p$ and $\mathbb{R}^2$ at each p.)

The independent constraints generate an action β of $\mathbb{R}^2$ on $S^{0,0,1}$, which on C_0 is given by

$$\beta_\lambda([\Lambda, \theta], M) = ([\Lambda, \theta + \Lambda^{-1}\eta(\Lambda\tilde{p})\lambda], M - \eta(\Lambda\tilde{p})\lambda). \tag{2.30}$$

Here $\lambda \in \mathbb{R}^2$ is identified with $\lambda_1\hat{T}_1 + \lambda_2\hat{T}_2 \in \mathbf{e}(\mathbf{2}) \subset \mathbf{l}$, on which L acts by the co-adjoint representation, as before. We stress that the linearized form (2.30) only holds on C_0; the transformation of points off C_0 has a more complicated form, which is not needed in what follows. Since $\Lambda^{-1}\eta(\Lambda\tilde{p}) \in L_{\tilde{p}}$, and $L_{\tilde{p}}$ maps $\mathbf{l}^\perp_{\tilde{p}}$ into itself under the co-adjoint action, and since $M - \lambda \in \mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$ if $M \in \mathbf{l}_{\tilde{p}} \cap \mathcal{O}^L_{0,1}$, the transformation (2.30) indeed maps C_0 into itself, proving that C_0 is co-isotropically embedded in $S^{0,0,1}$, as claimed.

Although the action of $\mathbb{R}^2$ is not defined canonically (in the sense that the section η was needed to specify it), its orbits are independent of η, and define a foliation of C_0. A local computation verifies that this is precisely the null foliation $\mathcal{F}_0$ with respect to the induced presymplectic form. The leaf space $\mathcal{L} = C_0/\mathcal{F}_0$ of this foliation has two components $\mathcal{L}^\pm = C_0^\pm/\mathcal{F}_0$. We now recall the definition of the co-adjoint orbits $\mathcal{O}^{0,\pm 1} \simeq L^{0,\pm 1} = L \times_{E(2)} \pi_{\tilde{p}}^{-1}(\mathcal{O}_{\pm 1})$ of the Poincaré group, which correspond to helicity ± 1. If one inspects (2.29), in which φ is now regarded as as a map from $C_0^\pm$ to $L^{0,\pm 1}$, one immediately sees that $\varphi \circ \beta_\lambda = \varphi$ (with β_λ defined by (2.30)), and that two points only have the same image under φ if they are thus related. Hence φ quotients to map $\tilde{\varphi} : C_0^\pm/\mathcal{F}_0 \to L^{0,\pm 1}$. It follows from (2.25) and (2.27) that $\tilde{\varphi}$ is

a symplectomorphism.

We now show that $\tilde{\varphi}$ intertwines the action of P. Firstly, it is immediate from (2.26) and (2.28) that P maps C into itself. Furthermore, we see from (2.26) and (2.30) that

$$\pi^{0,0,1}(\Lambda,a)\circ\beta_\lambda([\tilde{\Lambda},\theta],M)=\beta_{h(\Lambda,\tilde{\Lambda})\lambda}\pi^{0,0,1}(\Lambda,a)([\tilde{\Lambda},\theta],M), \tag{2.31}$$

where $h(\Lambda,\tilde{\Lambda})=\eta(\Lambda\tilde{\Lambda}\tilde{p})^{-1}\Lambda\eta(\tilde{\Lambda}\tilde{p})$, which lies in $E(2)$. This relation implies that P acts on C in a leaf-preserving way, so that its action quotients to the leaf space; we call this action $\tilde{\alpha}^{0,0,1}$. Finally, (2.26) and (2.29) lead to the conclusion that $\alpha^{0,1}_{\rm co}\circ\tilde{\varphi}=\tilde{\varphi}\circ\tilde{\alpha}^{0,0,1}$.

To sum up, we have proved

Theorem 2 *Let $O_0=L/E(2)=\{p\in\mathbb{R}^4|p^2=0,p^0>0\}$, and let $\mathcal{O}^L_{0,1}$ the co-adjoint orbit of L in $\mathfrak{l}\simeq\mathfrak{l}^*$ characterized by the conditions $\mathbf{J}\cdot\mathbf{K}=0$ and $|\mathbf{J}|^2-|\mathbf{K}|^2=1$. Then the symplectic reduction of $T^*O_0\times\mathcal{O}^L_{0,1}$ (equipped with the product of the respective canonical symplectic structures) with respect to the constraint $p^\mu M_{\mu\nu}=0$ ($p\in O_0, M\in\mathcal{O}^L_{0,1}$) is symplectomorphic to $\mathcal{O}^{0,1}\cup\mathcal{O}^{0,-1}$, where $\mathcal{O}^{0,\pm1}$ is the co-adjoint orbit of P (equipped with the canonical symplectic structure) characterized by $m=0$ and helicity ±1. Moreover, the natural action of P on $T^*O_0\times\mathcal{O}^L_{0,1}$ survives this reduction, and is intertwined with the co-adjoint action on $\mathcal{O}^{0,1}\cup\mathcal{O}^{0,-1}$ by the appropriate symplectomorphism.*

The above procedure suggests that the reduced space is a Marsden-Weinstein quotient with respect to an action of $\mathbb{R}^2$. Unfortunately, the action in question was not canonically defined, and in the way it was defined is only piecewisely smooth. A more intrinsic way to proceed would be to consider the vector bundle $L\times_{E(2)}\mathbb{R}^2$, where $\mathbb{R}^2$ is regarded as the abelian subgroup of $E(2)$, as before; the action of $E(2)$ on $\mathbb{R}^2$ is given by the restriction of the co-adjoint representation (that is, $\mathbb{R}^2$ itself acts trivially and $SO(2)$ acts by rotations). This vector bundle is to be regarded as a Lie groupoid, which coincides with the associated Lie algebroid, and it acts on $T^*O_0\times\mathcal{O}^L_{0,1}$ in the sense of groupoid actions [27]. One may then apply the generalized symplectic reduction procedure with respect to symplectic groupoid actions [46, 26] to reobtain the above results.

In any case, the parallel between the way one passes from covariant to canonical massless Hilbert space representations, and symplectic realizations, respectively, is

striking. In both cases the essential point is the need to quotient out the directions (in Hilbert or symplectic space) in which the undesired abelian subgroup of $E(2)$ acts nontrivially.

3 Rieffel induction for the frozen photon field

In this section we freeze the electromagnetic potential A_μ at the value $\tilde{A}_\mu = A_\mu(\tilde{p})$, and quantize the model with 4 degrees of freedom of subsection 2.2.

3.1 Tuning up

For simplicity, we write ψ_μ for $\tilde{A}_\mu$. Recalling that $S = \mathbb{C}^4$ and that the symplectic form on S is given by (2.8), we define the 'field algebra' $\mathfrak{F}$ of the model to be $\mathfrak{F} = \mathcal{W}(S, \omega)$, the Weyl algebra of canonical commutation relations (CCR) defined by S and ω. This is generated by unitary elements $W(\psi)$ satisfying $W(\psi)W(\varphi) = \exp(-\frac{1}{2}i\omega(\psi, \varphi))W(\psi + \varphi)$ (cf. [8] or, in the present context, [9]). For heuristic considerations it is useful to introduce the (unbounded) bosonic creation- and annihilation operators with commutation relations $[a_\mu, a^*_\nu] = -g_{\mu\nu}$, in terms of which formally $W(\psi) = \exp(a(\psi) - a(\psi)^*)$ (where $a(\psi) = a_\mu \overline{\psi}^\mu$, hence $a(\psi)^* = a^*_\mu \psi^\mu$). In view of this, we will henceforth write ψ with an upper index.

The gauge transformation generated by $\lambda \in G = \mathbb{C}$ is given by the inner automorphism $\alpha_\lambda[A] = W(-\lambda\tilde{p})AW(-\lambda\tilde{p})^*$: for if we take $A = W(\psi)$ we find from the CCR that $\alpha_\lambda[\exp(a(\psi) - a(\psi)^*)] = \exp((a + \lambda\tilde{p})(\psi) - (a + \lambda\tilde{p})(\psi)^*)$. The algebra of weak observables $\overline{\mathfrak{A}}$ is by definition the gauge-invariant subalgebra $\mathfrak{F}^G$ of $\mathfrak{F}$; equivalently, it is the commutant $\mathcal{W}(T, \omega)'$ in $\mathfrak{F}$ of the abelian algebra $\mathcal{W}(T, \omega) = C^*(G_d)$ (here the commutant is defined as the collection of all elements of $\mathfrak{F}$ which commute with all elements in the given subalgebra $\mathcal{W}(T, \omega)$). Cf. subsect. 2.2 for the definition of N and T; G_d stands for the group G equipped with the discrete topology. The following equality [18] explicitly identifies the weak algebra of observables.

$$\mathcal{W}(T, \omega)' = \mathcal{W}(N, \omega). \tag{3.32}$$

This result is valid for arbitrary CCR algebras (of the minimal type defined in [28]).

The essential point is that N is the symplectic orthoplement $T^\perp$ of T. Here the symplectic orthoplement $M^\perp$ of a subspace $M \subset S$ is defined as

$$M^\perp = \{\varphi \in S | \omega(\varphi, \psi) = 0 \, \forall \psi \in M\}.$$

Eq. (3.32) is Theorem 4.2 in [18], whose proof unfortunately contains a gap. The proof given below was arrived at after a correspondence with H. Grundling (intending to correct the proof in [18]).

Proof of Eq. (3.32), after H. Grundling. Things are made transparent if we look at the CCR algebra $\mathcal{W}(S, \omega)$ as the C^*-completion of the twisted convolution algebra $C_c^\omega(S_d)$, where S_d stands for S equipped with the discrete topology; the twisted convolution product is defined on $C_c^\omega(S_d)$ (which is $C_c(S_d)$ as a space; the upper index ω denotes the twist in the convolution) by $(fg)(\psi) = \sum_{\phi \in S} f(\phi) g(\psi - \phi) \exp(-\frac{1}{2} i \omega(\phi, \psi))$, and extended to $C_\omega^*(S) \equiv \mathcal{W}(S, \omega)$ by continuity, cf., e.g., [28]. We then have the inequalities

$$\| f \|_\infty \leq \| f \|_2 \leq \| f \| \tag{3.33}$$

where the first norm is the supremum one, the second norm is in $L^2(S_d)$ (with respect to the counting measure on S_d), and the third is in $C_\omega^*(S_d) \equiv \mathcal{W}(S, \omega)$. The first inequality is obvious (given the discreteness of the underlying measure space), the second follows from the existence of the tracial state ω_0, defined by continuous extension of $\omega_0(f) = f(0)$ [28]; indeed, $\| f \|_2^2 = \omega_0(f^* f)$. It follows that $C_\omega^*(S_d)$ as a Banach space (with its C^*-norm) is continuously embedded in $C_0(S_d)$ (with sup norm), for any element of the former is the limit of a Cauchy sequence in $C_c(S_d)$; by (3.33) this sequence must also converge in the sup norm, so that its limit must lie in $C_0(S_d)$.

Now take an arbitrary $f \in C_\omega^*(S_d)$, and a Cauchy sequence f_n in $C_c(S_d)$ converging to f in $C_\omega^*(S_d)$. Then (recalling [28] that $W(\varphi) = \delta_\varphi$, the delta function with support at φ) it follows from the CCR that $[f_n, W(\varphi)]$ is the function $f_n^{(\varphi)} : \psi \to 2 i f_n(\psi - \varphi) \sin(-\frac{1}{2} \omega(\psi, \varphi))$. Now $\lim_n f_n^{(\varphi)}$ exists in $C_\omega^*(S_d)$, hence in $C_0(S_d)$. The function $\psi \to \sin(-\frac{1}{2} \omega(\psi, \varphi))$ lies in $C_b(S_d)$, which is the multiplier

algebra of $C_0(S_d)$. Hence $f_n^{(\varphi)} \to f^{(\varphi)}$ (defined like $f_n^{(\varphi)}$, with f_n replaced by f) in $C_0(S_d)$. By uniqueness of the limit, we infer $f_n^{(\varphi)} \to f^{(\varphi)}$ in $C^*_\omega(S_d)$. We conclude that $[f, W(\varphi)] = f^{(\varphi)}$.

Now f is in $\mathcal{W}(T,\omega)'$ iff $[f, W(\varphi)]$ vanishes for all $\varphi \in T$. The preceding paragraph then yields $\| f^{(\varphi)} \| = 0$, whereupon (3.33) implies that $f^{(\varphi)}$ identically vanishes for such φ. Therefore (evaluating $f^{(\varphi)}$ at $\psi = \psi' + \varphi$, and using $\omega(\varphi,\varphi) = 0$), f must vanish whenever its argument does not lie in $T^\perp = N$, and (3.32) follows. ■

(Note that Araki [2, Theorem 1(5)] gave an arduous proof of the corresponding von Neumann algebra result $\pi(\mathcal{W}(N,\omega))'' = \pi(\mathcal{W}(T,\omega))'$ for any regular representation π (where the commutants now have their usual meaning). Since $\pi(\mathcal{W}(T,\omega)') = \pi(\mathcal{W}(T,\omega))' \cap \pi(\mathcal{W}(S,\omega))$, whose bicommutant is $\pi(\mathcal{W}(T,\omega))'$, this result follows immediately from (3.32). Moreover, the regularity assumption may evidently be dropped.)

Thus $\overline{\mathfrak{A}} = \mathcal{W}(N,\omega)$, and note that $\mathcal{W}(T,\omega)$ is the centre of $\overline{\mathfrak{A}}$. The notation $\mathfrak{A}$ is reserved for the dense subalgebra of $\overline{\mathfrak{A}}$ consisting of finite linear combinations of elements $W(\psi)$, $\psi \in N$ (hence $\mathfrak{A} = \mathcal{W}_0(N)$).

Refering to the Introduction or to [26] for motivation, we now wish to quantize the reduction procedure of subsection 2.2 using Rieffel induction. As we have explained, in the special case at hand we first need a representation $\pi(\mathfrak{F})$ on a Hilbert space $\mathcal{H}$, which carries a unitary representation U of G as well. The latter point is actually taken care of by putting $U(\lambda) = \pi(W(-\lambda\tilde{p}))$.

We essentially follow [9] in taking the so-called Fermi representation π_F of $\mathfrak{F}$. This is defined on $\mathcal{H} = \exp(\mathbb{C}^4)$ (the symmetric Hilbert space, or bosonic Fock space, over $\mathbb{C}^4$ [20, 8]); on this space the usual creation- and annihilation operators are defined, which we denote by $\hat{a}$, $\hat{a}^*$; their commutation relations are $[\hat{a}_\mu, \hat{a}^*_\nu] = \delta_{\mu\nu}$. The heuristic idea of the Fermi representation is to represent a_i by $\hat{a}_i$, but a_0 by $\hat{a}^*_0$, so that the distinction between $-g_{\mu\nu}$ and $\delta_{\mu\nu}$ in the CCR is taken care of.

A rigorous definition is arrived at by starting from the dense subset E of $\mathcal{H}$, which consists of finite linear combinations of so-called exponential vectors [20]. These are

defined on any symmetric Hilbert space $\exp(\mathcal{K})$, and given for arbitrary $\psi \in \mathcal{K}$ by

$$e^{\psi} = \Omega + \psi + \frac{1}{\sqrt{2!}}\psi \otimes \psi + \ldots + \frac{1}{\sqrt{n!}}\psi \otimes \ldots \psi + \ldots, \tag{3.34}$$

where $\Omega = \exp(0)$ is the vacuum vector in $\exp(\mathcal{K})$, and $\psi \otimes \ldots \psi$ stands for the tensor product of n copies of ψ. We write this as $\exp(\psi) = \sum_{n=0}^{\infty}(n!)^{-\frac{1}{2}}\psi^{\otimes n}$. Note the square root (perhaps rendering the notation $\exp(\psi)$ somewhat inappropriate); it guarantees that $(\exp(\psi), \exp(\varphi)) = \exp(\psi, \varphi)$.

Back to $\mathcal{K} = \mathbb{C}^4$, we define π_F by linear extension of

$$\pi_F(W(\psi))e^{\varphi} = e^{-\frac{1}{2}(\psi,\psi)-(\varphi,\psi')}e^{\varphi+\psi'}, \tag{3.35}$$

where $\psi'^{\mu} = (\overline{\psi^0}, -\psi^i)$. To arrive at this expression, simply use the heuristic idea explained above, the BCH-formula, and the relations [20] $\exp(\hat{a}(\psi))\exp(\varphi) = \exp(\varphi, \psi)\exp(\varphi)$, $\exp(\hat{a}(\psi)^*)\exp(\varphi) = \exp(\varphi + \psi)$ (note that E is in the domain of $a(\psi)$ and $a(\psi)^*$, as well as of their exponentials [8]). It follows that π_F is indeed a representation, which may then be extended to all of $\mathcal{H}$ by continuity. (Our discussion differs from [9] in that we have not altered the conventional complex structure on $\mathbb{C}^4$, but rather interchanged the role of a_0 and a_0^*, and also in our explicit construction of π_F using exponential vectors.)

We note that $\pi_F(\mathfrak{A})$ (which may be identified with $\mathfrak{A}$, defined in the second par. of this subsection) leaves E stable, hence we may attempt to perform Rieffel induction on $L = E$. The representation of G on $\mathcal{H}$ is given by

$$U(\lambda) = \pi_F(W(-\lambda\tilde{p})), \tag{3.36}$$

and $\pi_F(\mathfrak{A}) = \pi_F(\mathfrak{F})^G$, i.e., the set of elements of $\mathfrak{F}$ which commute with all $U(\lambda)$.

If we would really need to apply Rieffel induction in its original formulation [38, 13], we would face a pesky dilemma at this point: if we choose $\mathfrak{B} = C(G) \cap L^1(G)$, then the rigging map $\langle\psi, \varphi\rangle_{\mathfrak{B}} : \lambda \to (U(\lambda)\varphi, \psi)$ indeed takes values in $\mathfrak{B}$, but unfortunately $\mathfrak{B}$ (acting on $\mathcal{H}$ through U, cf. the Introduction) does not leave E stable. On the other hand, if we change the topology on G and try $\mathfrak{B} = L^1(G_d)$, then $\mathfrak{B} \subset \mathfrak{A}$, so that E is stable under its action, but this time the rigging map does

not take values in $\mathfrak{B}$. Fortunately, all we need is formula (1.2) for the rigged inner product (in which G has its usual, Euclidean topology); this formula follows from the first choice of $\mathfrak{B}$ mentioned, and the difficulty with the stability of L can simply be ignored.

3.2 The rigged inner product

Thus we regard $L = E$ as a left-$\mathfrak{A}$ module via π_F, and induce from the trivial representation of G. The first step in the induction procedure is the introduction of the sesquilinear form $(\cdot,\cdot)_0$ on L. From (1.2), this is given by

$$(\Psi,\Phi)_0 = \int_{\mathbb{C}} \frac{d\lambda\, d\overline{\lambda}}{2\pi i}(U(\lambda)\Psi,\Phi). \tag{3.37}$$

Using (3.36) and (3.35), we find that for $\Psi,\Phi \in E$ this integral indeed converges, and on elementary vectors the result is

$$(e^{\psi}, e^{\varphi})_0 = \exp(-\psi^0\psi^3 - \overline{\varphi}^0\overline{\varphi}^3 + \psi^1\overline{\varphi}^1 + \psi^2\overline{\varphi}^2). \tag{3.38}$$

For later use, we note that we may rewrite this expression as

$$(e^{\psi}, e^{\varphi})_0 = (e^{\psi^\perp}, \Omega)_0\, (\Omega, e^{\varphi^\perp})_0\, e^{(\psi^T,\varphi^T)}, \tag{3.39}$$

where $\psi^\perp = (\psi^0, 0, 0, \psi^3)$ and $\psi^T = (0, \psi^1, \psi^2, 0)$; in fact $(\exp(\psi^\perp),\Omega)_0 = (\exp(\psi),\Omega)_0$.

Since G is amenable, it follows from [26, Proposition 2] that property 1 (1.3) is satisfied, whereas property 3 (1.5) follows from [26, Proposition 2]; note that these results were proved for $\mathfrak{B} = C_c(G)$ rather than $C(G) \cap L^1(G)$, but the proofs are not changed by this modification; moreover, the only input in the proofs is the fact that the rigging map takes values in $\mathfrak{B}$, so that the stability of L under $\mathfrak{B}$ (which, as we have discussed, is not satisfied in our application) was not used. Property 2 (1.4) is immediately verified, since each $A \in \mathfrak{A}$ commutes with $U(\lambda)$. Later on we will, in fact, deduce (1.3) and (1.5) directly.

We now examine some properties of $(\cdot,\cdot)_0$. Regarded as a quadratic form on $\mathcal{H}$ with domain E, it is positive (hence symmetric), but not closable. For if it were, there would exist a positive self-adjoint operator A such that $(\Psi,\Phi)_0 = (A\Psi,\Phi)$.

From (3.38) with $\varphi = 0$ we then find that $A\Omega = \sum_{n=0}^{\infty}(n!)^{-1}\sqrt{(2n)!}\,\mathbf{e}_0 \otimes_s \ldots \mathbf{e}_0 \otimes_s \mathbf{e}_3 \ldots \otimes_s \mathbf{e}_3$ (the n'th term contains n copies of $\mathbf{e}_0$ and n copies of $\mathbf{e}_3$, and $\otimes_s$ stands for the symmetrized tensor product, normalized such that $\psi \otimes_s \ldots \psi = \psi \otimes \ldots \psi$). But $(A\Omega, A\Omega) = \sum_n 1 = \infty$, so A does not exist.

However, the domain of $(\cdot,\cdot)_0$ can be extended to the linear hull of $E \cup F$, where F is the dense subspace in $\mathcal{H}$ consisting of finite linear combinations of vectors with a finite number of particles, i.e., of the type $\psi_1 \otimes_s \ldots \psi_n$, $n < \infty$. The extension is obtained from (3.39) and the formula

$$\psi_1 \otimes_s \ldots \psi_n = \frac{1}{\sqrt{n!}} \frac{d}{dt_1} \ldots \frac{d}{dt_n} e^{\sum_{I=1}^{n} t_I \psi_I}|_{t_1=\ldots t_n=0}. \tag{3.40}$$

The most convenient expression for it is obtained by remarking that any vector in F is a finite linear combination of vectors of the type $\psi_1^{\perp} \otimes_s \ldots \psi_l^{\perp} \otimes_s \psi_{l+1}^{T} \otimes_s \ldots \psi_n^{T}$, and for those we obtain

$$\begin{aligned}(\psi_1^{\perp} \otimes_s \ldots \psi_l^{\perp} \otimes_s \psi_{l+1}^{T} \otimes_s \ldots \psi_n^{T}, \varphi_1^{\perp} \otimes_s \ldots \varphi_{l'}^{\perp} \otimes_s \varphi_{l'+1}^{T} \otimes_s \ldots \varphi_{n'}^{T})_0 = \\ (\psi_1^{\perp} \otimes_s \ldots \psi_l^{\perp}, \Omega)_0\, (\Omega, \varphi_1^{\perp} \otimes_s \ldots \varphi_{l'}^{\perp})_0\, (\psi_{l+1}^{T} \otimes_s \ldots \psi_n^{T}, \varphi_{l'+1}^{T} \otimes_s \ldots \varphi_{n'}^{T}).\end{aligned} \tag{3.41}$$

This expression vanishes if the number of transverse components does not match (i.e., if $n - l \neq n' - l'$), or if l or l' are odd. For $l = 2m$ even, one has

$$(\psi_1^{\perp} \otimes_s \ldots \psi_{2m}^{\perp}, \Omega)_0 = \frac{(-1)^m}{m!\sqrt{(2m)!}} \sum_{P \in S_{2m}} \psi_{P(1)}^{0} \ldots \psi_{P(2m)}^{3}, \tag{3.42}$$

where m ψ's on the r.h.s. carry the upper index 0, and m ψ's carry 3 upstairs. The sum is over all permutations of $\{1, \ldots, 2m\}$. Equation (3.42) is still valid if each $\psi_i^{\perp}$ is replaced by ψ_i. The expression (3.41) may alternatively be derived directly from (3.37), with Ψ and Φ chosen in F. A general formula for the rigged inner product on F is given in [44].

The anatomy of $(\cdot,\cdot)_0$ is then clear from the observation that $\psi_1^{\perp} \otimes_s \ldots \psi_l^{\perp} \otimes_s \psi_{l+1}^{T} \otimes_s \ldots \psi_n^{T}$ equals $(\psi_1^{\perp} \otimes_s \ldots \psi_l^{\perp}, \Omega)_0\, \psi_{l+1}^{T} \otimes_s \ldots \psi_n^{T}$ plus a vector in the null space $\mathcal{N}$ of $(\cdot,\cdot)_0$. Hence the non-transverse components merely provide a numerical factor to the transverse ones, up to null vectors.

Finally, we remark that $(\cdot,\cdot)_0$ does not preserve the adjoint: for example, the property $(\pi_F(W(\psi))\Psi,\Phi)_0 = (\Psi,(\pi_F(W(-\psi))\Phi)_0$ only holds if $\psi \in N$.

3.3 The induced representation

Let us now return to Rieffel induction on the domain $L = E$. Recalling the discussion of the classical situation in subsection 2.2, our aim is to show that the induced space $\mathcal{H}^0$ (which is the completion of $L/\mathcal{N}$) is naturally isomorphic to the bosonic Fock space $\exp(S^0)$. Remember that $S^0 = J^{-1}(0)/G = N/T$; in this case this is isomorphic to $\mathbb{C}^2$ as a Hilbert space, the inner product being given by (cf. (2.10))

$$([\psi^T],[\varphi^T])_{S^0} = (\psi^T,\varphi^T)_S. \tag{3.43}$$

Here $S = \mathbb{C}^4$, and $[\psi^T]$ is the image of ψ under the double projection map, which first projects ψ onto its component in N, and then onto N/T. We now attempt to define a map $V : L \to \exp(S^0)$ by linear extension of

$$Ve^{\psi} = (e^{\psi},\Omega)_0\, e^{[\psi^T]} = e^{-\psi^0\psi^3}\, e^{[\psi^T]}, \tag{3.44}$$

There is a subtle point here: we could have identified $\mathbb{C}^2$ with $\mathrm{span}\{\mathbf{e}_1,\mathbf{e}_2\}$ rather than with N/T, and define V by writing ψ^T for $[\psi^T]$ in (3.44). In view of (3.43) this would equally well satisfy the crucial relation (3.46) below; the properties of the induced representation $\pi^0(\mathfrak{A})$ would then compel us to re-identify $\mathbb{C}^2$ as $S^0 = N/T$ anyway, as will become clear shortly.

Another subtlety is that the basis $\{\exp\psi\}$ is overcomplete, so that it is not obvious that the linear extension of (3.44) is well-defined. To see that it is, consider the map $\hat{V} : F \to \exp(S^0)$, defined by

$$\hat{V}\,\psi_1^{\perp}\otimes_s\ldots\psi_l^{\perp}\otimes_s\psi_{l+1}^T\otimes_s\ldots\psi_n^T = \left(\frac{(n-l)!l!}{n!}\right)^{\frac{1}{2}}(\psi_1^{\perp}\otimes_s\ldots\psi_l^{\perp},\Omega)_0\,[\psi_{l+1}^T]\otimes_s\ldots[\psi_n^T], \tag{3.45}$$

extended to F by linearity; this extension is manifestly well-defined. This map $\hat{V}$ is unbounded, but it can be extended to exponential vectors; a short calculation shows that this extension coincides with V; hence V may consistently be extended to E

by linearity, and may be thought of as being defined on the linear hull of $E \cup F$. The point is that it follows from (3.44) and (3.39) that

$$(V\Psi, V\Phi) = (\Psi, \Phi)_0, \tag{3.46}$$

where the inner product on the l.h.s. is the one in $\exp(S^0)$. This firstly verifies that $(\cdot,\cdot)_0$ is postive semi-definite on L, and secondly that its null space $\mathcal{N}$ coincides with $\ker V$. Therefore, V may be extended to $\mathcal{H}^0 = \overline{L/\mathcal{N}}$, and provides a unitary isomorphism $\tilde{V}$ between the induced space $\mathcal{H}^0$ and $\exp(S^0)$.

The induced representation π^0 is therefore characterized by

$$\pi^0(A) \circ V = V \circ \pi_F(A) \tag{3.47}$$

for all $A \in \mathfrak{A}$. Since $\pi_F(\mathfrak{A})$ is a representation and $\tilde{V}$ is unitary, this also shows, incidentally, that π^0 is a representation of $\mathfrak{A}$, so that the bound (1.5) is satisfied. The intertwining relation (3.47) allows us to describe π^0 explicitly. For it follows from (3.35), (3.44) that $\pi^0(W(\tau)) = 1$ for all $\tau \in T$; another way to say this is that for all $\Psi \in L$, one has $\pi_F(W(\tau))\Psi = \Psi + \nu$, where $\nu \in \mathcal{N}$. That is, gauge transformations leave vectors in L invariant up to null vectors. Subsequently, the property $W(\psi + \tau) = W(\psi)W(\tau)$ for all $\psi \in N$ and $\tau \in T$ implies that $\pi^0(W(\psi + \tau)) = \pi^0(W(\psi))$. Moreover, we see that $\pi^0(W(\psi^T))$ coincides with the Fock representative of $W([\psi^T])$ on $\exp(S^0)$.

We now define the CCR algebra $\mathcal{W}(S^0, \omega^0)$ over S^0 with the usual symplectic structure ω^0 (cf. (2.11)), and define π_Φ to be the Fock representation of $\mathcal{W}(S^0, \omega^0)$ on $\exp(S^0)$ [20, 8]. Then (recalling that $N = J^{-1}(0)$ and $S^0 = J^{-1}(0)/G$) the properties of π^0 mentioned imply that

$$\pi^0(\mathcal{W}(J^{-1}(0), \omega)) \simeq \pi_\Phi(\mathcal{W}(S^0, \omega^0)), \tag{3.48}$$

where we have extended the induced representation π^0 from $\mathfrak{A}$ to its completion $\mathcal{W}(N, \omega)$, which is allowed because the bound (1.5) is satisfied [38]. In particular, since π_Φ is faithful we find that $\mathcal{W}(J^{-1}(0), \omega)/\ker \pi^0 \simeq \mathcal{W}(S^0, \omega^0)$, where $\ker \pi^0$ is generated by elements in $\mathcal{W}(N, \omega)$ of the type $W(\tau) - 1$, $\tau \in T$. The algebra

$\mathcal{W}(S^0, \omega^0)$ is the algebra of observables of the model, and we see that it is obtained from the 'field algebra' $\mathfrak{F} = \mathcal{W}(S, \omega)$ in two steps: first the gauge-invariant subalgebra $\mathfrak{F}^G = \mathcal{W}(N, \omega)$ is selected, and then the remaining gauge transformations are eliminated by building a quotient.

Finally, consider the action of the group $E(2)$. As described in subsection 2.2, $E(2)$ acts on $S = \mathbb{C}^4$; this action is symplectic, and therefore leads to an automorphic action of $E(2)$ on the field algebra $\mathfrak{F}$, defined by $\alpha_x[W(\psi)] = W(x\psi)$ (also cf. [9]). Since N is invariant under $E(2)$, this automorphism group may be restricted to $\mathcal{W}(N, \omega)$, and also to the dense subalgebra $\mathfrak{A}$ of the latter.

This automorphism group happens to be unitarily implementable in $\pi_F(\mathfrak{F})$, but since in the full electromagnetic theory the corresponding property (for the Lorentz group) fails to hold, we will not exploit it. Rather, the relevant fact is that the generating functional of the induced representation $\pi^0(\mathfrak{A})$ is invariant: for

$$(\pi^0(W(\psi))\Omega^0, \Omega^0) = (\pi_F((W(\psi))\Omega, \Omega)_0 = e^{\frac{1}{2}(\psi,\psi)_M}, \tag{3.49}$$

where $\Omega^0 = V\Omega$ is the vacuum vector in $\exp(S^0)$, and $(\psi, \psi)_M = \psi^\mu \overline{\psi}_\mu$, as before. This follows from (3.35) and (3.38) if one uses $\psi^0 = \psi^3$ for all $\psi \in N$. Therefore, we can define a unitary representation U^0 of $E(2)$ in the usual way [7], viz. U^0 is given by extending the equation $U^0(x)\pi^0(W(\psi))\Omega^0 = \pi^0(W(x\psi))\Omega^0$ for all $\psi \in N$ by linearity and continuity. It may be checked [44] that this representation coincides with the second quantization of the representation $U_1 \oplus U_{-1}$, cf. subsection 2.2.

We invite the reader to compare our discussion with the treatment of this model in [9]. There the representation $\pi^0(\mathcal{W}(N, \omega))$ is constructed in a completely different way, which is arguably a bit *ad hoc* compared with our induction procedure. Moreover, one finds in [9] a whole family π^ξ of representations of $\mathcal{W}(N, \omega)$, where $\xi \in \mathbb{C}$. In our formalism, representations unitarily equivalent to these may be obtained by treating ξ as a character of the gauge group $G = \mathbb{C}$, and inducing not from the trivial representation of G, but from the one defined by ξ, cf. [44] for details.

4 Rieffel induction in electromagnetism

4.1 Preamble

We now treat the entire field $A_\mu(x)$ in the manner of the preceding section. The setting is almost exactly as in section 3.1, with the following changes (cf. subsection 2.3, or [9], in which ref. S stands for our $P_T S$). The phase space S is now given by $L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^4$, with symplectic form (2.14). We continue to denote its elements by ψ, φ. The closed subspaces N and T of S are now defined by

$$N = \{A \in S | p_\mu \psi^\mu(\mathbf{p}) = 0 \,(\text{a.e.})\}; \quad T = \{d\lambda | \lambda \in G\}, \tag{4.1}$$

where the gauge group G has been defined prior to (2.15); recall our symbolic notation $d\lambda$ for the element of S defined by $(d\lambda)^\mu(\mathbf{p}) = -ip^\mu \lambda(\mathbf{p})$, $p^0 = |\mathbf{p}|$. We still have $T^\perp = N = N^{\perp\perp}$. Also, in the Marsden-Weinstein reduction of S, $J^{-1}(0)$ coincides with N, and $S^0 = J^{-1}(0)/G = N/T$, cf. subsection 2.3. The field algebra is $\mathfrak{F} = \mathcal{W}(S, \omega)$, and since (3.32) holds without modification, $\overline{\mathfrak{A}} = \mathcal{W}(N, \omega)$ is again the commutant in $\mathfrak{F}$ of $\mathcal{W}(T, \omega) = C^*(G_d)$, which in turn is contained in $\overline{\mathfrak{A}}$ as its center. Gauge transformations in $\mathfrak{F}$ are given by automorphisms $\alpha_\lambda[W(\psi)] = W(d\lambda)W(\psi)W(d\lambda)^*$, $\lambda \in G$, so that $\overline{\mathfrak{A}} = \mathfrak{F}^G$.

A new feature, indicating that we are now in the setting of quantum field theory, is that, as explained in [10] (following the discussion in [24] for scalar fields), $\mathfrak{F}$ has a subalgebra with the structure of a local net of C^*-algebras in the sense of Haag-Kastler (cf. [22, 23]). Namely, the local algebra $\mathfrak{F}(\mathcal{O})$ ($\mathcal{O} \subset \mathbb{R}^4$ open) is obtained as the C^*-closure of the subalgebra of $\mathfrak{F}$ generated by those $W(\psi)$ for which $\psi^\mu(\mathbf{p})$ possesses an extension $\tilde{\psi}^\mu(p)$ off the mass-shell $p^2 = 0$ whose Fourier transform lies in the Schwartz space $\mathcal{D}(\mathcal{O})$.

The Fermi representation of $\mathfrak{F}$ on $\mathcal{H} = \exp(S)$ is again given by (3.35). (Note that the smeared operators $\hat{a}(\psi)$, $\hat{a}(\psi)^*$ satisfy the commutation relations $[\hat{a}(\psi), \hat{a}(\varphi)^*] = (\varphi, \psi)$, and act as indicated below (3.35)). The representation U of the gauge group is now given by (cf. (3.36))

$$U(\lambda) = \pi_F(W(d\lambda)). \tag{4.2}$$

4.2 Functional integral representation of $(\cdot,\cdot)_0$

Our aim is to construct a representation π^0 of $\mathfrak{A}$ which is Rieffel-induced from the trivial representation π_0 of the gauge group G, and the Fermi representation $\pi_F(\mathfrak{F} \restriction \mathfrak{A})$ on $\exp(S)$, restricted to the dense subspace $L = E$ of exponential vectors (cf. (3.34)). Thus we would now like to construct the rigged inner product $(\cdot,\cdot)_0$ by a formula analogous to (3.37), replacing the integral over the frozen gauge group $\mathbb{C}$ by one over the infinite-dimensional Hilbert Lie group G. We start by inspecting the equation

$$(U(\lambda)e^{\psi}, e^{\varphi}) = \exp(-\parallel \lambda \parallel^2_{\frac{1}{2}}) \times \\ \exp\Big((\psi,\varphi) + i[(p^0\psi^0,\overline{\lambda}) + (\mathbf{p}\cdot\psi,\lambda) + (\overline{\lambda},p^0\varphi^0) + (\lambda,\mathbf{p}\cdot\varphi)]\Big)\,, \qquad (4.3)$$

which follows from (4.2) and (3.35); recall (2.15). All the inner products occurring in the exponentials on the r.h.s. are in $L^2(\mathbb{R}^3, d'p)$, and we use the notations $p^0\psi^0$ and $\mathbf{p}\cdot\psi$ to denote the functions defined by $(p^0\psi^0)(\mathbf{p}) = |\mathbf{p}|\psi^0(\mathbf{p})$, and $(\mathbf{p}\cdot\psi)(\mathbf{p}) = p^i\psi^i(\mathbf{p})$, respectively. These inner products are all finite, as can be seen from the precise definition of the Hilbert space G. In the following measure-theoretic considerations, G is regarded as a real Hilbert space.

Inspired by (4.3), we consider the standard Gaussian weak distribution μ_* [40] (alternatively called a promeasure or cylindrical measure [11]) on G. This is defined by first considering an arbitrary n-dimensional Hilbert subspace $\mathcal{H}_n \subset G$ ($n < \infty$), which has a measure μ_n defined on an arbitrary Borel set $A \in \mathcal{H}_n$ by $\mu_n(A) = \pi^{-n/2}\int_A d^n\lambda \exp(-\parallel \lambda \parallel^2_{\frac{1}{2}})$. The orthogonal projection onto $\mathcal{H}_n$ is denoted by P_n. The measure of the cylinder set $P_n^{-1}(A)$ is then given by $\mu_*(P_n^{-1}(A)) = \mu_n(A)$. If we now choose an inductive family $\{\mathcal{H}_n\}_n$ of such $\mathcal{H}_n$ which eventually exhausts G (that is, $\mathcal{H}_n \subset \mathcal{H}_{n+1}$, and the closure of $\cup_n \mathcal{H}_n$ is G; we say that G is the inductive limit of the family; this does not mean that the topology on G is the corresponding inductive limit topology), then the promeasure μ_* is eventually defined on all cylinder sets in G by the equation above. Conversely, each finite-dimensional Hilbert subspace $\mathcal{K}$ (not necessarily contained in the inductive family) is then equipped with a Gaussian measure $\mu_\mathcal{K}$, defined as $\mu_\mathcal{K}(A) = \mu_*(P_\mathcal{K}^{-1}(A))$, where $P_\mathcal{K} : G \to \mathcal{K}$ is the orthogonal

projector onto $\mathcal{K}$.

The covariance of this promeasure is the unit operator, which is not nuclear (given that G is infinite-dimensional), and therefore μ_* cannot be extended to a Borel measure on G [40]. However, 'tame' functions (also called cylinder functions) can be integrated with respect to μ_*; these are functions on G of the type $f(\lambda) = f_{\mathcal{K}}(P_{\mathcal{K}}\lambda)$, where $f_{\mathcal{K}}$ is a Borel function on a finite-dimensional Hilbert subspace $\mathcal{K}$ of G. Note that $f(P_{\mathcal{K}}\lambda) = f(\lambda)$. The 'integral' $\int_G d\mu_* f$ is then by definition equal to the Lebesgue integral $\int_{\mathcal{K}} d\mu_{\mathcal{K}} f_{\mathcal{K}}$. The usual theorems and rules of Lebesgue integration theory often apply if only such tame functions are involved [40].

With this preparation, we can define and compute the rigged inner product on $L = E$.

Proposition 1 *Choose an inductive family $\{\mathcal{H}_n\}_n$ of finite-dimensional Hilbert subspaces of G, such that G is the inductive limit of this family. Let μ_* be the standard Gaussian weak distribution (promeasure) on G. Then*

$$(\Psi, \Phi)_0 = \lim_n \int_{\mathcal{H}_n} \frac{d^n\lambda}{\pi^{n/2}} (U(\lambda)\Psi, \Phi) \tag{4.4}$$

exists for all Ψ, $\Phi \in E \subset \exp(S)$, and on elementary vectors equals

$$\begin{aligned}(e^{\psi}, e^{\varphi})_0 = e^{(\psi,\varphi)} &\int_G d\mu_*(\lambda) \\ &\exp i[(p^0\psi^0, \overline{\lambda}) + (\mathbf{p}\cdot\psi, \lambda) + (\overline{\lambda}, p^0\varphi^0) + (\lambda, \mathbf{p}\cdot\varphi)].\end{aligned} \tag{4.5}$$

Proof. Since $\Psi \in E$, we can write $\Psi = \sum_{I=1}^n c_I \exp(\psi_I)$, $n < \infty$, $c_I \in \mathbb{C}$, $\psi_I \in S$; an analogous expansion holds for Φ. By (4.3), $(U(\lambda)\Psi, \Phi) = \exp(- \parallel \lambda \parallel^2_{\frac{1}{2}}) f(\lambda)$, where f is a tame function on G, for it is a finite sum of terms of the form $g(\lambda) = \exp((\mu_1, \lambda) + \ldots (\mu_l, \lambda))$, where the μ_I are of the form $(\pm i)\, p^0\psi^0_I$, $(\pm i)\, \mathbf{p}\cdot\psi_I$, etc. (recall that G is here regarded as a real Hilbert space, so there is no overall factor i). Hence in the above definition of tame functions, $\mathcal{K}$ is the Hilbert space spanned by $\mu_1, \ldots, \mu_l$.

By definition of μ_* and the projector P_n (explained above), one has

$$\pi^{-n/2} \int_{\mathcal{H}_n} d^n\lambda\, (U(\lambda)\Psi, \Phi) = \int_G d\mu_*(\lambda)\, f(P_n\lambda).$$

Now $f(P_n\lambda)$ is a finite sum of terms of the form $g_n(\lambda) = \exp((P_n\mu_1,\lambda)+\ldots(P_n\mu_l,\lambda))$, which are evidently still tame, and $\int_G d\mu_*\, g_n$ can be explicitly evaluated, cf. [40]. Since $P_n \to 1$ weakly by construction of the family $\{\mathcal{H}_n\}_n$, it is immediately verified that $\lim_n \int_G d\mu_*\, g_n = \int_G d\mu_*\, g$. This implies the proposition. ■

Note that the Lebesgue dominated convergence theorem could not be used in this proof, although $\lim_n f(P_n\lambda) = f(\lambda)$ pointwise, and each $f \circ P_n$ as well as the limit function f are tame and bounded by a μ_*-integrable function. The reason is that in order to apply this theorem, the limit function and (eventually) all functions in the sequence should depend on a given set of vectors μ_I, whereas in the above case these vectors depend on n.

From (4.5) onwards, we can sail through. The Gaussian functional integral can be computed by naive methods, for the integrand is tame, with the result

$$(e^{\psi}, e^{\varphi})_0 = e^{-(\psi^0,\overline{\psi}^L)-(\overline{\varphi}^0,\varphi^L)+(\psi^T,\varphi^T)}, \tag{4.6}$$

where $\psi^T = P_T\psi$, with $P_T : S \to S$ the usual projector onto the transverse (physical) degrees of freedom (see (2.17)), and $\psi^L = \mathbf{p}\cdot\psi/|\mathbf{p}|$. The first two inner products in the exponential at the r.h.s. are in $L^2(\mathbb{R}^3, d'p)$, and the third one is in $L^2(\mathbb{R}^3, d'p)\otimes\mathbb{C}^4$. Comparing this result with (3.38), we see that ψ^L and ψ^T play the rôle of ψ^3 and (ψ^1,ψ^2) in the frozen model, respectively; the status of ψ^0 is unchanged. The symbol $\psi^\perp \in \mathbb{C}^4$ used in section 3 now stands for $(\psi^0, (p^ip^j/|\mathbf{p}|^2)\psi^j) \in L^2(\mathbb{R}^3, d'p)\otimes\mathbb{C}^4$. The entire discussion in subsections 3.2 and 3.3 may then be taken over with the obvious notational modifications.

4.3 The induced representation for electromagnetism

Firstly, (3.41) is still valid as it stands, but (3.42) is replaced by

$$(\psi_1^\perp \otimes_s \ldots \psi_{2m}^\perp, \Omega)_0 = \frac{(-1)^m}{m!\sqrt{(2m)!}} \sum_{P\in S_{2m}} (\psi^0_{P(1)}, \overline{\psi_{P(2)}}^L)\ldots(\psi^0_{P(2m-1)}, \overline{\psi_{P(2m)}}^L), \tag{4.7}$$

where the inner products are in $L^2(\mathbb{R}^3, d'p)$. With N and T as defined in (4.1), and $S^0 = N/T$ equipped with the inner product (3.43), we define the map $V : L = E \to \exp(S^0)$ by the analogue of (3.44):

$$V e^{\psi} = (e^{\psi}, \Omega)_0\, e^{[\psi^T]} = e^{-(\psi^0,\overline{\psi}^L)}\, e^{[\psi^T]}. \tag{4.8}$$

(Alternatively, we could start from (3.45).) From (4.6) and (3.43) we infer that (3.46) is satisfied on L, eventually leading to the identification of the induced space $\mathcal{H}^0$ with $\exp(S^0)$. The desired properties (1.3)-(1.5) are verified in the same way as before, i.e., (1.3) follows from (3.46), (1.4) is immediate from the definition of $\mathfrak{A}$, and (1.5) follows from (3.47), or from (3.48). Alternatively, they can be proved directly from (4.4) and the results in [26] for amenable locally compact groups, for each approximant in (4.4) is an integral over such a group (namely $\mathbb{R}^n$), and the inequalities (1.3) and (1.5) are obviously stable under taking the limit in n.

As a first application of (4.8), we consider the Hamiltonian H_F on $\mathcal{H}$. It is defined as the operator which implements the time-evolution on $\mathfrak{F}$ in the Fermi representation π_F, in the sense that $\pi_F(\alpha_t[A]) = \exp(itH_F)\pi_F(A)\exp(-itH_F)$. The time-evolution on $\mathfrak{F}$ is given by the one-parameter automorphism α_t, defined by $\alpha_t[W(\psi)] = W(\exp(-itH^{(1)})\psi)$. The operator $H^{(1)}$ appearing in this definition is the one-particle Hamiltonian $H^{(1)}$ on S, given by $(H^{(1)}\psi^\mu)(\mathbf{p}) = p^0\psi^\mu(\mathbf{p})$ on the obvious domain $D(H^{(1)})$ (note that $H^{(1)}$ is already in diagonal form). However, the peculiar factor $-g_{\mu\nu}$ (rather than the normal $\delta_{\mu\nu}$) in the CCR defining $\mathfrak{F}$ leads to the perhaps somewhat surprising fact that $H_F = d\Gamma(\tilde{H}^{(1)})$, where Γ is the second quantization operation [37], and $\tilde{H}^{(1)}$ is defined (and self-adjoint) on $D(H^{(1)}) \subset S$ by $(\tilde{H}^{(1)}\psi^\mu(\mathbf{p}) = (-p^0\psi^0(\mathbf{p}), p^0\psi^i(\mathbf{p}))$. In other words, H_F is the closure of the operator (defined on those finite-particle states in $F \subset \mathcal{H}$ whose components lie in $D(H^{(1)})$)

$$H_F^{\text{core}} = -g^{\mu\nu}\int d'p\,|\mathbf{p}|\hat{a}_\mu(\mathbf{p})^*\hat{a}_\nu(\mathbf{p}), \tag{4.9}$$

where $\hat{a}_\mu(\mathbf{p})^*$, $\hat{a}_\nu(\mathbf{p})$ are the usual quadratic forms associated to the creation- and annihilation operators on $\mathcal{H}$ (cf. [37, X.7]).

Clearly, the spectrum of H_F is $\mathbb{R}$, but if we inspect (4.8) or (3.45), and recall that the null space $\mathcal{N}$ of $(\cdot,\cdot)_0$ is $\ker V$, we infer that any vector in L equals a vector with only transverse components plus a null vector; hence

$$(H_F\Psi, \Psi)_0 \geq 0 \quad \forall\Psi \in L, \tag{4.10}$$

for we see from (4.9) that H_F has positive expectation value in transverse states.

This 'rigged' positivity property eventually implies that the Hamiltonian in the induced representation π^0 has positive spectrum.

We now return to the construction of the induced space. In complete analogy with the frozen case, the crucial fact is that the gauge transformations (4.2) do not affect the rigged inner product $(\cdot,\cdot)_0$, in the sense that $U(\lambda)\Psi$ equals Ψ plus a null vector for all $\Psi \in L$. We can reformulate this property in a way that clarifies the relation between our formalism and the 'Fermi method' of quantizing the electromagnetic field (cf.[9]). To do so, first note that $\pi_F(\mathfrak{F})$ is a regular representation, so that the field potentials A^F_μ exist as operator-valued distributions on $\mathcal{H}$. They are related to $\mathfrak{F}$ by means of [9] $\pi_F(W(D*f)) = \exp(iA^F(f))$, where $f \in \mathcal{S}(\mathbb{R}^4)\otimes\mathbb{R}^4$, $A^F(f) = \int d^4x\, A^F_\mu(x) f^\mu(x)$, and D is the usual Pauli-Jordan commutator function. Choosing $f^\mu = \partial^\mu\lambda$, the above property is equivalent to

$$(\partial^\mu A^F_\mu(x)\Psi, \Phi)_0 = 0 \tag{4.11}$$

as a distribution, for all Ψ, Φ initially in $L = E$, and, as explained earlier, extendable to the linear hull of $E \cup F$.

It should be clear that the discussion following (3.47) can be copied here (with $\tau = d\lambda$); in particular, $\pi^0(W(d\lambda)) = 1$ for all $\lambda \in G$. With ω^0 now defined by (2.19), the identification (3.48) is valid as it stands in the present setting. As we have seen in subsection 2.3, S^0 is the space of physical degrees of freedom of electromagnetism. This is reinforced by showing that it carries the correct representation of the Poincaré group P, cf. (2.12).

Since S carries a representation $U_{\text{gaugefield}}$ of P which leaves the symplectic form ω invariant (cf. subsection 2.3), we can define P as an automorphism group on $\mathfrak{F}$ by means of $\alpha_x[W(\psi)] = W(x\psi)$, in somewhat symbolic but obvious notation. This automorphism group cannot be unitarily implemented in π_F [9] (though its subgroup $SO(3) \ltimes \mathbb{R}^4$ can). However, (3.49) is valid also in the present case, so that the state on $\overline{\mathfrak{A}} = \mathcal{W}(N,\omega)$ defined by Ω^0 and π^0 is Poincaré-invariant. Hence we obtain a representation U^0 of P on $\mathcal{H}^0$ by the procedure explained after (3.49), and it is easily checked that U^0 is the second quantization of U_{photon}, the photon representation which already emerged from Marsden-Weinstein reduction in subsection 2.3.

We sum up in a theorem. The definitions of the symplectic space (S, ω) (where $S = L^2(\mathbb{R}^3, d'p) \otimes \mathbb{C}^4$ as a Hilbert space), the gauge group G, and the Marsden-Weinstein quotient $S^0 = J^{-1}(0)/G$ (with the symplectic form ω^0 on S^0) are given in subsection 2.3; note that $S^0 = N/T$, with $N = J^{-1}(0)$ and T defined in (4.1).

Theorem 3 *Take the field algebra $\mathfrak{F}$ of quantum electromagnetism to be the CCR-algebra $\mathcal{W}(S, \omega)$, on which G acts by inner automorphisms. Let the 'algebra of weak observables' $\mathfrak{F}^G = \mathcal{W}(J^{-1}(0), \omega)$ be its gauge-invariant subalgebra. With $\mathfrak{A}$ the dense subalgebra of $\mathfrak{F}^G$ spanned by the Weyl operators $W(\psi)$, $\psi \in J^{-1}(0)$, we obtain a left-$\mathfrak{A}$ and right-G module L as follows: L is the subspace of $\exp(S)$ spanned by exponential vectors, the left-action of $\mathfrak{A}$ is given by the restriction of the Fermi representation $\pi_F(\mathfrak{F})$ (cf. (3.35)) to $\mathfrak{A}$, and the action U of G is given by $U(\lambda) = \pi_F(W(d\lambda))$ (see (4.2)).*

Using a generalized notion of Rieffel induction (in which the rigging map on L is replaced by the direct definition of a rigged inner product (4.4) on it), these data can be used to construct a representation $\pi^0(\mathfrak{F}^G)$ induced from the trivial representation π_0 of G. Then $\pi^0(\mathcal{W}(S, \omega)^G) \simeq \mathcal{W}(S^0, \omega^0)$ as C^*-algebras, with respect to the local structure, and with respect to the automorphic action of the Poincaré group P. Moreover, the induction procedure produces $\mathcal{W}(S^0, \omega^0)$, which is the algebra of observables of quantum electromagnetism, in its vacuum representation on $\mathcal{H}^0 = \exp(S^0)$, in which the action of P can be unitarily implemented. This realizes the induced space $\mathcal{H}^0$ as a Fock space of physical photons.

5 Discussion

In this final section we briefly discuss certain isolated aspects of our formalism. Our coverage is mostly incomplete and tentative, and is intended to inspire further work in this direction.

5.1 The use of quantum fields

Our construction was based on choosing the domain L, on which the rigged inner product (4.4) is defined, to be $L = E \subset \mathcal{H} = \exp(S)$, the space of exponential vectors (coherent states for physicists). This has the disadvantage that $E \cap F = \mathbb{C}\,\Omega$, where F is the finite-particle subspace of $\mathcal{H}$. In other words, E does not contain states with a finite number of particles (apart from the vacuum). Yet all computations of scattering amplitudes in physics start from states in F, so if we wish to entertain the hope that our method may be of some practical use, we should incorporate

such states. On the one hand, this is straightforward, for we have seen that the rigged inner product (regarded as a quadratic form on $\mathcal{H}$), albeit unclosable, can be extended to the linear hull of $E \cup F$. The problem lies in our use of the Weyl algebras $\mathfrak{F}$ and $\overline{\mathfrak{A}} = \mathfrak{F}^G$. In order to carry through the induction procedure, it is necessary that $\mathfrak{F}^G$ contains a dense subalgebra $\mathfrak{A}$ which leaves L stable. With $L = E$ we could take $\mathfrak{A}$ as specified in the theorem above, but for $L = F$ no such subalgebra exists.

The simplest way out of this dilemma is to work with unbounded operators, using the Borchers-Uhlmann-Maurin formulation of algebraic quantum field theory (cf. [23], and refs. therein). The Borchers algebra $\mathfrak{F}_u$ appropriate to quantum electromagnetism was defined and analyzed in [6]. Leaving a rigorous study to the future, one may expect that an unbounded analogue of the Fermi representation π_F on $\exp(S)$ can be defined, so that $\pi_F(f_1 \otimes \dots f_n) = A^F(f_1) \dots A^F(f_n)$; we already encountered the fields A^F_μ in (4.11) and surrounding text. As already discussed in [26], one can carry through Rieffel induction for Op^*-algebras $\mathfrak{A}_u$ of unbounded operators (cf. [23]) as long as L is a common invariant domain of $\mathfrak{A}_u$, and the crucial property (1.4) is satisfied. This is indeed the case if $L = F$, and $\mathfrak{A}_u \subset \mathfrak{F}_u$ (the subalgebra in which test functions lie in $N \cap \mathcal{S}(\mathbb{R}^4) \otimes \mathbb{R}^4$) acts on F through π_F.

The idea is then to compute physical amplitudes, which in theory can be expressed as (squared) matrix elements in $\mathcal{H}^0$, in $\mathcal{H}$, using the rigged inner product rather than the original one on F. The simplest object is the propagator $(TA^F_\mu(x)\Omega, A^F_\nu(y)\Omega)_0$ (where T is the time-ordering instruction), which is found to coincide with the one in Coulomb-gauge QED. (Due to the fact that the A^F_μ are not symmetric with respect to $(\cdot,\cdot)_0$, the above expression differs from the two-point function $(TA^F_\mu(x)A^F_\nu(y)\Omega, \Omega)_0$.) The main difficulty in setting up a perturbation theory for interacting models along these lines is that Wick's theorem does not hold, for the reason that $(A\Omega, \Omega)_0$ does not necessarily vanish for normal-ordered expressions A in the creation- and annihilation operators. See [44] for details, and some further steps.

5.2 Use of the temporal gauge

Our choice of (S, ω) for electromagnetism (cf. (2.14)) was motivated by the connection between canonical and covariant representations of the Poincaré group, and led to a covariant formalism in all stages of the classical theory, and most stages (field algebra, algebra of weak observables, induced representation, algebra of observables and its vacuum representation) of the quantum theory. Alternatively, a non-covariant approach based on the partial gauge-fixing $A_0 = 0$ is possible, and suitable for certain applications (e.g, thermal field theory, topological effects in non-abelian gauge theories, Hamiltonian approach to anomalies, ...).

In that case, we take $S = T^*Q$ with its canonical symplectic structure ω, where $Q = L^2_{\mathbb{R}}(\mathbb{R}^3) \otimes \mathbb{R}^3$ is the configuration space of the spatial field $\mathbf{A}$. We use canonical co-ordinates (A_i, E_j) on S. The gauge group G is the subspace of $\mathcal{S}'(\mathbb{R}^3)$ of distributions λ whose weak exterior derivative $d\lambda$ lies in S, modulo constants; in contrast to our previous formulation, this is a *real* Hilbert space without a complex structure. G acts on S by $\mathbf{A} \to \mathbf{A} + d\lambda$; this action is strongly Hamiltonian, with moment map $J : S \to \mathbf{g}^* \simeq G$ given by $J(\mathbf{A}, \mathbf{E}) = \Delta^{-1} \nabla \cdot \mathbf{E}$. Hence $J^{-1}(0)$ consists of those points in S where Gauss' law $\nabla \cdot \mathbf{E} = 0$ is satisfied, and the Marsden-Weinstein reduced space $S^0 = J^{-1}(0)/G$ consists of solutions of Gauss' law modulo gauge transformations (it is symplectomorphic to the space called S^0 in subsection 2.3.).

To quantize, we define the field algebra $\mathfrak{F} = \mathcal{W}(S, \omega)$, and represent it in the Fock representation π_Φ on the symmetric Hilbert space $\mathcal{H} = \exp(S)$ (identifying S with $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$). The gauge group acts on $\mathcal{H}$ through the unitary representation U, defined by $U(\lambda) = \pi_\Phi(W(0, d\lambda))$. We wish to produce an induced representation of $\mathfrak{A}$, which is the appropriate dense subalgebra of $\mathcal{W}(J^{-1}(0), \omega)$. Thus we choose $L = E \subset \mathcal{H}$, which is a left-$\mathfrak{A}$ and right-G module, and define a rigged inner product on L much as in Proposition 1, cf. (4.4). The result is

$$(e^{\psi}, e^{\varphi})_0 = e^{(\psi^L, \overline{\psi}^L) + (\overline{\varphi}^L, \varphi^L) + (\psi^T, \varphi^T)}, \tag{5.1}$$

where $\psi^T = P_T \psi$, with P_T defined in (2.17), and $\psi^L = P_L \psi$, with $P_L = 1 - P_T$. All inner products are in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^3$. Comparing this expression with (4.6), we

infer that the rôle of the inner product $(\psi^0, \overline{\psi}^L)$ is now played by $(\psi^L, \overline{\psi}^L)$. The anatomy of $(\cdot,\cdot)_0$ is still that all vectors in L can be decomposed as the sum of a purely transverse part and a null vector.

The construction of the induced representation in subsection 4.3 may then be adapted in an obvious way. The situation is actually simpler here, for the subtle distinction between $P_T S$ and N/T is irrelevant now. Thus $S^0 = P_T S$, so that the induced space $\mathcal{H}^0$ may be identified with $\exp(P_T S)$, and we may write ψ^T rather than $[\psi^T]$ in the present version of (4.8). One eventually recovers the results stated in Theorem 3 (replacing the claims about the Poincaré group by those about its subgroup $SO(3) \ltimes \mathbb{R}^4$).

5.3 Regularity

An alternative C^*-algebraic procedure to handle constrained systems was developed in [16, 17]. Restricting our discussion to the application of this so-called T-procedure to systems with first-class contraints, and grossly simplifying the story, this procedure is intended to resolve a spectral difficulty faced by the traditional Dirac method [12, 42]. Namely, given a representation π of the algebra of operators $\mathfrak{F}$ of the unconstrained system on a Hilbert space $\mathcal{H}$, the prescription of Dirac is to look for gauge-invariant vectors in $\mathcal{H}$, and obtain a representation of the algebra $\mathfrak{F}^G$ of gauge-invariant operators on the 'physical' subspace $\mathcal{H}_{\rm phys}$ of $\mathcal{H}$ spanned by such vectors by simply restricting $\pi_F(\mathfrak{F} \upharpoonright \mathfrak{F}^G)$ to $\mathcal{H}_{\rm phys}$. This prescription only works if 0 is in the discrete spectrum of all the constraints (with common corresponding eigenvectors), which is rarely the case.

For example, in the context of electromagnetism, we take $\mathfrak{F}$ and $\mathfrak{F}^G$ as in the main text, and $\pi = \pi_F$; the physical states Ψ should then satisfy $\pi_F(W(d\lambda))\Psi = \Psi$ for all $\lambda \in G$. This is impossible, and $\mathcal{H}_{\rm phys}$ as defined by Dirac is empty in this case. This example was generalized to a theorem in [18], which, applied to electromagnetism, states that $\mathcal{H}_{\rm phys}$ is empty whenever $\pi(\mathfrak{F})$ is a regular representation.

In due fairness to physicists using the Dirac procedure (e.g., in quantum gravity), it should be pointed out that in practice the equation $C\Psi = 0$ (where C is a

constraint) is solved not as an eigenvalue problem in Hilbert space, but as a partial differential equation for which nothing is said in advance about the space of solutions. The drawback of such a procedure is obviously that an inner product on the space of solutions has to be found from scratch.

In any case, the T-procedure starts from a representation-independent definition of physical states as those states ω on $\mathfrak{F}$ for which $\omega(W(d\lambda)) = 1$ for all $\lambda \in G$. It follows that such 'Dirac' states must be non-regular on $\mathcal{W}(S,\omega)$, and this has inspired an approach to gauge theories based on the use of representations in which gauge-variant fields do not exist [35]. In the T-procedure, the algebra of observables $\mathfrak{A}_{\rm obs}$, which we obtained as $\pi^0(\mathfrak{F}^G)$, is constructed in a representation-independent way. Subsequently, physical states which (after restriction and quotienting) are regular on $\mathfrak{A}_{\rm obs}$ may be obtained from those Dirac states which are regular on $\mathcal{W}(N,\omega)$.

We now briefly summarize how our approach manages to avoid the use of non-regular states. The traditional Dirac condition $\pi_F(W(d\lambda))\Psi = \Psi$ is replaced by $\pi_F(W(d\lambda))\Psi = \Psi + \nu$, where ν has to lie in the null space $\mathcal{N}$ of the rigged inner product $(\cdot,\cdot)_0$. As we have seen, this condition is actually satisfied by *all* vectors Ψ in the dense subspace $L \subset \mathcal{H}$ on which $(\cdot,\cdot)_0$ is defined. The price one pays is that the functional ψ on $\mathcal{W}(S,\omega)$, defined by $\psi(A) = (\pi_F(A)\Psi,\Psi)_0$ (where $\Psi \in L$), only defines a state on $\mathcal{W}(N,\omega)$. The reason for this is that the positivity property $(\pi_F(A^*A)\Psi,\Psi)_0 \geq 0$ fails for general $A \in \mathcal{W}(S,\omega)$: the rigged inner product only preserves hermiticity for $A \in \mathcal{W}(N,\omega)$.

For the purpose of comparison with the T-procedure, our formalism may be seen as a method of constructing states on $\mathcal{W}(N,\omega)$, hence eventually on the algebra of observables, from states in a regular representation of the entire field algebra. The need to consider non-regular states on $\mathfrak{F}$ does not arise at all. Clearly, the fact that our weakened version of the Dirac condition is identically satisfied on L means that the non-physical state vectors in the Dirac method (as well as in the T-procedure) play a different rôle in our approach. We do not need to exclude such state vectors by hand: their non-invariance perishes automatically when the induced space $\mathcal{H}^0$ is constructed.

These points are particularly clearly illustrated when G is compact, in which case $(\Psi, \Phi)_0 = (P_0\Psi, P_0\Phi)$, where P_0 projects on the subspace of $\mathcal{H}$ carrying the trivial representation of G (cf. the Introduction). Then, according to both the T-procedure and the Dirac method, the physical states in $\mathcal{H}$ are only those that lie in $P_0\mathcal{H}$. On the other hand, all vectors in $\mathcal{H}$ satisfy the condition $U(x)\Psi = \Psi + \nu$, $\nu \in \mathcal{N}$ for all $x \in G$; here $\mathcal{N} = (P_0\mathcal{H})^\perp$. The analogue of $\mathcal{W}(N, \omega)$ is the algebra of G-invariant bounded operators on $\mathcal{H}$ (the 'field' algebra now consisting of all bounded operators). The induced space is $\mathcal{H}^0 = \mathcal{H}/\mathcal{N} = P_0\mathcal{H}$, and we see clearly how the non-invariant part of each non-Dirac state disappears.

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