DESY 94-157

Metaplectic Spinor Fields and Global Anomalies

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Abstract

We investigate spinor fields on phase-spaces. Under local frame-rotations they transform according to the (infinite-dimensional, unitary) metaplectic representation of $Sp(2N)$ which plays a role analogous to the Lorentz group. We introduce a one-dimensional nonlinear sigma-model whose target space is the phase-space under consideration. The global anomalies of this model are analyzed, and it is shown that its fermionic partition function is anomalous exactly if the underlying phase-space is not a spin-manifold, i.e., if metaplectic spinor fields cannot be introduced consistently. The sigma-model is constructed by giving a path-integral representation to the Lie-transport of spinors along the hamiltonian flow.

1 Introduction

It is a well known fact that there exist space-times on which it is impossible to consistently introduce spinor fields[1]. It is quite remarkable that this phenomenon does not only occur on space-time manifolds, but also on the *phase-spaces* of certain physical systems. Originally the relevant spinor fields, known as metaplectic spinors[2], where studied in the context of geometric quantization[3] where they are related to semiclassical wave functions[4]. More recently metaplectic spinor fields also appeared in theories with Parisi-Sourlas supersymmetry[5], in the covariant quantization of the Green-Schwarz superstring[6], and in models of anyon superconductivity[7]. Metaplectic spinor fields are the spinors of the group $Sp(2N)$ which, for phase-spaces, plays the same role the Lorentz group plays for space-time. The purpose of the present paper is to relate the (im)possibility of defining metaplectic spinors on a given phase-space to the global anomalies of a special type of nonlinear sigma-model whose target space is the phase-space under consideration.

Before turning to this subject we have to recall some basic facts about spin structures on curved space-times[8]. Let us consider a n-dimensional Riemannian manifold $\mathcal{M}_n$[1]. We cover $\mathcal{M}_n$ by a set of open neighborhoods $\mathcal{O}_{(\alpha)}$ and introduce an n-bein field $e_{(\alpha)} \equiv \{e^a_{\mu(\alpha)}\}$ on each of them. In the overlap regions $\mathcal{O}_{(\alpha)} \cap \mathcal{O}_{(\beta)}$ the two n-beins are related by a local Lorentz transformation $\Lambda_{(\alpha\beta)} \in O(n)$

$$e_{(\alpha)}(x) \;=\; \Lambda_{(\alpha\beta)}(x)\; e_{(\beta)}(x) \tag{1.1}$$

The consistency of these relations requires that on triple overlap regions $\mathcal{O}_{(\alpha)} \cap \mathcal{O}_{(\beta)} \cap \mathcal{O}_{(\gamma)}$ the transition functions obey the cocycle condition

$$\Lambda_{(\alpha\beta)}(x)\; \Lambda_{(\beta\gamma)}(x)\; \Lambda_{(\gamma\alpha)}(x) \;=\; 1 \tag{1.2}$$

Spinor fields $\psi_{(\alpha)}$ on $\mathcal{O}_{(\alpha)}$ are assumed to transform on $\mathcal{O}_{(\alpha)} \cap \mathcal{O}_{(\beta)}$ according to

$$\psi_{(\alpha)}(x) \;=\; \tilde{\Lambda}_{(\alpha\beta)}(x)\psi_{(\beta)}(x) \tag{1.3}$$

where $\tilde{\Lambda}_{(\alpha\beta)} \in Spin(n)$ represents the Lorentz transformation $\Lambda_{(\alpha\beta)}$ on the spinors. Therefore we must have that on $\mathcal{O}_{(\alpha)} \cap \mathcal{O}_{(\beta)} \cap \mathcal{O}_{(\gamma)}$

$$\tilde{\Lambda}_{(\alpha\beta)}(x)\; \tilde{\Lambda}_{(\beta\gamma)}(x)\; \tilde{\Lambda}_{(\gamma\alpha)}(x) \;=\; 1 \tag{1.4}$$

As $Spin(n)$ is the double covering of $SO(n)$, there is a sign ambiguity in going from the vector to the spinor representation: both $+\tilde{\Lambda}$ and $-\tilde{\Lambda} \in Spin(n)$ correspond to the same

[1] We assume that $\mathcal{M}_n$ has Euclidean signature and that it is simply connected.

element $\Lambda \in O(n)$. As a consequence, (1.2) follows from (1.4) *but not vice versa*: for certain choices of the signs in the "lifting" $\Lambda_{(\alpha\beta)} \mapsto \pm\tilde{\Lambda}_{(\alpha\beta)}$ we might end up with a "-1" on the RHS of eq.(1.4). If we can choose the signs of the transition functions $\tilde{\Lambda}_{(\alpha\beta)}$ in such a way that (1.4) holds on any triple intersection region then $\mathcal{M}_n$ admits a spin structure. If this is impossible, no globally well defined spinor fields can exist on $\mathcal{M}_n$. Typical examples of manifolds which do not admit spin structures include the projective spaces $\mathbf{CP}^n$ for n even.

Whether or not $\mathcal{M}_n$ can support spinor fields is "measured" by the second Stiefel-Whitney class $w_2(\mathcal{M}_n)$ which takes values in the Čech cohomology group $H^2(\mathcal{M}_n,\mathbf{Z}_2)$. It is defined in terms of triple products of $\tilde{\Lambda}_{(\alpha\beta)}$'s similar to the one on the LHS of (1.4). It is a classic result that $\mathcal{M}_n$ is a spin manifold if and only if $w_2(\mathcal{M}_n)$ is trivial[2]. For our purposes it is advantageous to look at these topological obstructions from a slightly different point of view.

Consider a spin connection $\omega_\mu(x)$, i.e., a gauge field on $\mathcal{M}_n$ which assumes values in the Lie algebra of $O(n)$. Let γ be a loop on $\mathcal{M}_n$ parametrized by $x^\mu(\tau), \mu = 1\cdots n, \tau \in [0, 2\pi]$. Now we parallel-transport a vector $\psi^\mu(\tau)$ around this loop. After the circuit is completed, the original $\psi(0)$ has changed to $\psi(2\pi) = R\psi(0)$ where the $O(n)$-valued monodromy matrix R is given by

$$R \;=\; P\;\exp[\,\frac{1}{2}\int_0^{2\pi} d\tau\;\dot{x}^\mu \omega_{\mu\alpha\beta}\;\Sigma^{\alpha\beta}] \tag{1.5}$$

Here $\Sigma^{\alpha\beta}$ are the generators of $O(n)$, and P denotes the path-ordering. A possible inconsistency could arise as follows. Consider a loop in loop-space, i.e., a family of loops $\gamma_u, u \in [0,1]$ for which $\gamma_0 = \gamma_1$. The parametrizations $x^\mu_u(\tau)$ obey

$$x^\mu_u(\tau) \;=\; x^\mu_u(\tau + 2\pi),\ \forall u \in [0,1] \qquad \text{and} \qquad x^\mu_0(\tau) \;=\; x^\mu_1(\tau),\ \forall \tau \in [0, 2\pi] \tag{1.6}$$

so that $u = 0$ and $u = 1$ lead to the same loop $\gamma_0 = \gamma_1$. For each value of u we obtain a different monodromy matrix $R = R(u)$. Because $R(0) = R(1)$, the family of monodromy matrices $R(u), u \in [0,1]$, forms a loop on the group manifold of $O(n)$. This loop defines a certain element of the homotopy group $\Pi_1(O(n)) = \mathbf{Z}_2$. It can be shown [9] that $\mathcal{M}_n$ does *not* admit a spin structure if there exists a family γ_u such that the resulting monodromy matrices $R(u)$ give rise to a nontrivial element in $\Pi_1(O(n))$.

Witten [10] has studied the obstruction to defining spinors on $\mathcal{M}_n$ in terms of a one-dimensional nonlinear sigma-model whose target space is $\mathcal{M}_n$. It is defined by the

[2]Similarly, the first Stiefel-Whitney class $w_1(\mathcal{M}_n) \in H^1(\mathcal{M}_n,\mathbf{Z}_2)$ is an obstruction to the orientability of $\mathcal{M}_n$. Here we shall always assume that $\mathcal{M}_n$ is orientable.

$N = 1$ supersymmetric action

$$S \;=\; \int_0^{2\pi} d\tau\; [\frac{1}{2} g_{\mu\nu}(x(\tau))\dot{x}^\mu \dot{x}^\nu \;+\; \frac{1}{2}\psi^\mu g_{\mu\nu} Y^\nu{}_\rho \psi^\rho] \tag{1.7}$$

Here $g_{\mu\nu}$ is the metric on $\mathcal{M}_n$, and

$$Y^\mu{}_\nu \;\equiv\; i[\partial_t \delta^\mu{}_\nu + \dot{x}^\rho \omega_\rho{}^\mu{}_\nu] \tag{1.8}$$

The quantities $\psi^\mu(\tau)$ are real, anticommuting variables which transform as vectors under $O(n)$. If we integrate them out we obtain the fermionic partition function $\sqrt{\det Y}$ which is a functional of the bosonic "background field" $x^\mu(\tau)$. It turns out [10] that if one tries to define the theory (1.7) on a manifold which cannot support spinor fields, then the square root of $\det Y$ is not a globally well-defined functional of $x^\mu(\tau)$. This is a $\mathbf{Z}_2$-type global anomaly [11] related to the two possible signs of the square root: if we fix a definite sign for γ_0 and go through the family of loops γ_u we might arrive at the opposite sign for γ_1. This happens exactly if $R(u)$ is a nontrivial element of $\Pi_1(O(n))$. Thus we see that the anomalies of the *one-particle* theory (1.7) can be used in order to detect obstructions for spinor *fields* on the space-time $\mathcal{M}_n$.

The purpose of the present paper is to investigate a similar one-particle theory which is sensitive to obstructions for *metaplectic* spinor fields. Our model describes a particle which propagates on a symplectic rather than a Riemannian manifold. This symplectic manifold, henceforth denoted as $\mathcal{M}_{2N}$, should be thought of as the phase-space of some hamiltonian system with N degrees of freedom. In order to motivate the specific form of our model, we introduce on $\mathcal{M}_{2N}$ a Hamiltonian H and investigate the behaviour of tensors and spinors under the classical hamiltonian flow [12] generated by H. In section 2 we set up a formalism which describes the "dragging" of geometrical objects along the hamiltonian flow in a path-integral language. We apply this construction to metaplectic spinors, whose most important properties are also briefly summarized in section 2. The action appearing in our path-integral defines a kind of non-linear sigma-model in one dimension and will play a role similar to (1.7). In section 3 we show that this sigma-model provides all the necessary ingredients needed in order to study metaplectic spin structures. Then, in sections 4 and 5, we analyze the anomalies of the model, and we describe how they are related to the topological obstructions which prevent us from defining metaplectic spinors on certain phase-spaces.

2 Lie-Transport via Path-Integrals

In this section we introduce the one-dimensional field theory model whose global anomalies will detect the obstructions to defining metaplectic spinor fields. Our presentation follows ref.[13] to which the reader is referred for further details.

We specify a system with N degrees of freedom by fixing a hamiltonian function H on a phase-space $\mathcal{M}_{2N}$ (with local coordinates ϕ^a) which is equipped with a symplectic two-form $\omega = \frac{1}{2}\omega_{ab}d\phi^a \wedge d\phi^b$. By assumption ω is closed and nondegenerate so that there exists an inverse matrix ω^{ab}: $\omega_{ac}\omega^{cb} = \delta^b_a$. Locally we can introduce canonical coordinates $\phi^a = (p^i, q^i)$, $i = 1 \cdots N, a = 1 \cdots 2N$. Then the only nonzero components of ω_{ab} are $\omega_{j,N+i} = -\omega_{N+i,j} = \delta_{ij}$. Given ω and H we can construct the hamiltonian vector field

$$h^a \;=\; \omega^{ab}\,\frac{\partial H}{\partial \phi^b} \;\equiv\; \omega^{ab}\partial_b H \tag{2.1}$$

so that Hamilton's equation reads $\dot{\phi}^a = h^a(\phi)$. We consider the vector field (2.1) as the generator of a symplectic diffeomorphism (canonical transformation) $\delta\phi^a = -h^a(\phi)$. Under this transformation any tensor or spinor field $\chi(\phi^a)$ changes by an amount $\delta\chi = l_h\chi$ where

$$l_h = h^a\partial_a - \partial_b h^a G^b{}_a \tag{2.2}$$

is the Lie-derivative. Here $G^b{}_a$ are the generators of $Sp(2N)$ in the representation of the field χ. It is useful to define

$$\begin{aligned} K_{ab}(\phi) &\equiv \partial_a\partial_b H(\phi) \\ \Sigma^{ab} &\equiv i(G^a{}_c\omega^{cb} + G^b{}_c\omega^{ca}) \end{aligned} \tag{2.3}$$

so that

$$l_h = h^a\partial_a + \frac{i}{2}K_{ab}\Sigma^{ab} \tag{2.4}$$

with $K_{ab} = K_{ba}$ and $\Sigma_{ab} = \Sigma_{ba}$. If $\chi(\phi^a)$ stands for a vector field $v^a(\phi)$, for example, the generators are given by

$$(\Sigma^{ab}_{\rm vec})^c{}_d = -i(\delta^a_d\omega^{bc} + \delta^b_d\omega^{ac}) \tag{2.5}$$

This leads to the familiar formula for the Lie-derivative of a vector field:

$$l_h v^a = h^b\partial_b v^a - \partial_b h^a v^b \tag{2.6}$$

It is easy to check that the matrices (2.5) are the generators of $Sp(2N)$ in its defining representation. Forming group elements $S^a{}_b$ with infinitesimal parameters $\kappa_{ab} = \kappa_{ba}$,

$$S^a{}_b = (1 - \frac{i}{2}\kappa_{cd}\Sigma^{cd}_{\rm vec})^a{}_b = \delta^a{}_b + \omega^{ac}\kappa_{cb}, \tag{2.7}$$

they preserve the symplectic two-form ω:

$$\omega_{ab} S^a{}_c S^b{}_d = \omega_{cd} \tag{2.8}$$

Conversely, any symplectic matrix infinitesimally close to the identity has the form (2.7)[4].

If some generic field $\chi_\alpha(\phi, t)$ transforms under $Sp(2N)$ in the matrix representation $(\Sigma^{ab})^\beta{}_\alpha$ then it evolves under the hamiltonian flow ("time evolution") according to

$$-\partial_t \chi_\alpha(\phi, t) = l_h \chi_\alpha(\phi, t) = \left[\delta^\beta{}_\alpha h^a \partial_a - \frac{i}{2} K_{ab}(\phi)(\Sigma^{ab})^\beta{}_\alpha\right] \chi_\beta(\phi, t) \tag{2.9}$$

This equation describes the Lie-transport of any geometric object χ_α along the flow lines of the vector field h^a. If χ is a scalar, for instance, the Σ-term on the RHS of eq.(2.9) is absent and we are left with Liouville's equation $-\partial_t \chi = h^a \partial_a \chi$ for density functions on phase-space.

Next we solve the Lie-transport equation (2.9) in terms of a path-integral. To this end we introduce a set of auxiliary variables λ_a, η^α and $\bar{\eta}_\alpha$ which, together with ϕ^a, are coordinates on a certain supermanifold [14], the extended phase-space $\mathcal{M}_{\rm ext}$. With respect to the symplectic structure on $\mathcal{M}_{\rm ext}$ [13], both (λ_a, ϕ^a) and $(\bar{\eta}_\alpha, \eta^\alpha)$ are conjugate pairs. The λ_a's are commuting quantities; under diffeomorphisms on $\mathcal{M}_{2N}$ they transform in the same way as the derivatives ∂_a. The ghosts η^α and $\bar{\eta}_\alpha$ are anticommuting; the $\bar{\eta}_\alpha$'s transform under $Sp(2N)$ in the same representation as χ_α and η^α in the representation dual to it. We "lift" the dynamics from the ordinary phase-space to the extended one by defining the following Hamiltonian on $\mathcal{M}_{\rm ext}$:

$$\widetilde{\mathcal{H}} = h^a(\phi)\lambda_a + \frac{1}{2}\bar{\eta}_\alpha K_{ab}(\phi)(\Sigma^{ab})^\alpha{}_\beta \eta^\beta \tag{2.10}$$

¿From the point of view of $\mathcal{M}_{\rm ext}$, ϕ^a and η^α are "position" variables, and λ_a and $\bar{\eta}_\alpha$ are the conjugate momenta. Therefore it is natural to consider the path-integral

$$\mathbf{K}(\phi_2, \eta_2, t_2 | \phi_1, \eta_1, t_1) \equiv \int \mathcal{D}\phi \ \mathcal{D}\lambda \ \mathcal{D}\eta \ \mathcal{D}\bar{\eta} \ \exp\left[i \int_{t_1}^{t_2} dt \{\lambda_a \dot{\phi}^a + i\bar{\eta}_\alpha \dot{\eta}^\alpha - \widetilde{\mathcal{H}}\}\right] \tag{2.11}$$

with the boundary conditions $\phi^a(t_{1,2}) = \phi^a_{1,2}$ and $\eta^\alpha(t_{1,2}) = \eta^\alpha_{1,2}$. There exists an equivalent operatorial formulation of the "quantum theory" defined by the functional integral (2.11). The relevant (graded) commutation relations can be read off from its action:

$$\begin{aligned} [\ \phi^a, \lambda_b] &= i\delta^a_b \qquad , \qquad [\phi^a, \phi^b] = 0 = [\lambda_a, \lambda_b] \\ [\ \bar{\eta}_\beta, \eta^\alpha] &= \ \delta^\alpha_\beta \qquad , \qquad [\bar{\eta}_\alpha, \bar{\eta}_\beta] = 0 = [\eta^\alpha, \eta^\beta] \end{aligned} \tag{2.12}$$

Contrary to quantum mechanics, all components of $\phi^a = (p^i, q^i)$ commute among themselves. In a Schrödinger-type representation of these commutator relations we represent ϕ^a and η^α as multiplicative operators and λ_a and $\bar{\eta}_\alpha$ as derivatives:

$$\lambda_a = -i\frac{\partial}{\partial \phi^a} \quad , \quad \bar{\eta}_\alpha = \frac{\partial}{\partial \eta^\alpha} \tag{2.13}$$

They act on "wave functions" of the form

$$\chi(\phi, \eta) = \sum_p \frac{1}{p!}\, \chi_{\alpha_1 \alpha_2 \cdots \alpha_p}(\phi)\, \eta^{\alpha_1} \eta^{\alpha_2} \cdots \eta^{\alpha_p} \tag{2.14}$$

The kernel $\mathbf{K}$ of eq.(2.11) provides the time-evolution of these wave-functions:

$$\chi(\phi, \eta, t) = \int d\phi_0\; d\eta_0\; \mathbf{K}(\phi, \eta, t | \phi_0, \eta_0, t_0)\; \chi(\phi_0, \eta_0, t_0) \tag{2.15}$$

By the usual argument, eq.(2.15) is equivalent to the following Schrödinger-like equation:

$$i\partial_t \chi(\phi, \eta, t) = \widetilde{\mathcal{H}}(\phi, \lambda = -i\frac{\partial}{\partial \phi}, \eta, \bar{\eta} = \frac{\partial}{\partial \eta})\; \chi(\phi, \eta, t) \tag{2.16}$$

Using the explicit form of $\widetilde{\mathcal{H}}$, eq.(2.10), and the expansion (2.14) one can show that (2.16) amounts to the following equation of motion for the components $\chi_{\alpha_1 \cdots \alpha_p}(\phi, t)$:

$$\begin{aligned} -\partial_t \chi_{\alpha_1 \cdots \alpha_p}(\phi, t) &= l_h \chi_{\alpha_1 \cdots \alpha_p}(\phi, t) \\ &= h^a \partial_a \chi_{\alpha_1 \cdots \alpha_p} - \frac{i}{2}\, \chi_{\beta \alpha_2 \cdots \alpha_p}\, K_{ab} (\Sigma^{ab})^\beta{}_{\alpha_1} - \cdots \end{aligned} \tag{2.17}$$

To be precise, eq.(2.17) obtains only for a specific operator ordering, namely the "anti-Wick" ordering where all $\bar{\eta}$'s are put on the right of all η's:

$$:\eta^\alpha\, \bar{\eta}_\beta: = \eta^\alpha\, \bar{\eta}_\beta \quad , \quad :\bar{\eta}_\beta\, \eta^\alpha: = -\eta^\alpha\, \bar{\eta}_\beta, \qquad \text{etc.} \tag{2.18}$$

Therefore the Hamiltonian (2.10) is represented by the first order differential operator

$$\widetilde{\mathcal{H}} = -ih^a(\phi)\partial_a - \frac{1}{2} K_{ab}(\phi) (\Sigma^{ab})^\alpha{}_\beta\, \eta^\beta\, \frac{\partial}{\partial \eta^\alpha} \tag{2.19}$$

Comparing (2.17) to (2.9) we see that the path-integral does not only solve our original equation (corresponding to $p = 1$), but also time-evolves antisymmetric tensors $\chi_{\alpha_1 \cdots \alpha_p}(\phi)$ which transform under $Sp(2N)$ in the p-fold tensor product of the original representation. If this was the vector representation of $Sp(2N)$, then the wave functions (2.14) are (inhomogeneous) differential forms on $\mathcal{M}_{2N}$. In this case the ghosts η^a carry a vector index and provide a basis in the cotangent spaces $T_\phi \mathcal{M}_{2N}$. They can be identified with the differentials $d\phi^a$ [15]. Here instead we assume that the fermions $\eta^\alpha(t)$ transform in the

metaplectic representation. Before we come to the special properties of the path-integral in this representation we have to recall some properties of the metaplectic group $Mp(2N)$.

To the symplectic matrices $S \in Sp(2N)$ we can associate the metaplectic operators $M \in Mp(2N)$ which constitute a projective unitary representation of $Sp(2N)$. The relation between $Mp(2N)$ and $Sp(2N)$ is two-to-one, because both $+M(S)$ and $-M(S)$ represent the same symplectic matrix. (Therefore the notation $M = M(S)$ is slightly misleading in general. However, in our applications the correct sign will always be fixed by continuity arguments, see below.) We can construct the generators of the metaplectic group by imitating the procedure familiar from space-time spinors where representations of $Spin(n)$ are obtained from representations of the Clifford algebra $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$. In the case of $Mp(2N)$ we start from the corresponding metaplectic Clifford algebra [14, 5]:

$$\gamma^a\gamma^b - \gamma^b\gamma^a = 2i\omega^{ab} \tag{2.20}$$

Because this algebra involves a commutator rather than an anticommutator it has no finite-dimensional matrix representations. We look instead for representations in which the γ-matrices are operators on some infinite-dimensional Hilbert space $\mathcal{V}$. Let us fix a symplectic matrix $S^a{}_b$ and let us try to find an operator $M(S)$, acting on $\mathcal{V}$, such that

$$M(S)^{-1}\gamma^a M(S) = S^a{}_b\gamma^b \tag{2.21}$$

Now we assume that $S^a{}_b$ is close to the identity so that we may use the parametrization (2.7). If we make a similar ansatz for the metaplectic operator,

$$M(S) = 1 - \frac{i}{2}\kappa_{ab}\Sigma^{ab}_{\text{meta}} \quad , \tag{2.22}$$

then eq.(2.21) implies the following condition for the generators:

$$[\gamma^a, \Sigma^{bc}_{\text{meta}}] = i(\omega^{ab}\gamma^c + \omega^{ac}\gamma^b) \tag{2.23}$$

Its solution reads

$$\Sigma^{ab}_{\text{meta}} = \frac{1}{4}(\gamma^a\gamma^b + \gamma^b\gamma^a) \tag{2.24}$$

We see that once we have found a representation of the metaplectic γ-matrices, a corresponding representation of the group can be obtained in almost the same fashion as for $Spin(n)$. For this representation to be unitary we require the γ-matrices to be hermitian operators with respect to the inner product of the Hilbert space $\mathcal{V}$:

$$(\gamma^a)^\dagger = \gamma^a \quad , \quad (\Sigma^{ab})^\dagger = \Sigma^{ab} \quad , \quad M(S)^\dagger = M(S)^{-1} \tag{2.25}$$

The representation space $\mathcal{V}$ can be chosen in a variety of ways. If, for instance, $\mathcal{V}$ is taken to be the Fock space of N independent harmonic oscillators with commutation relations $[a_i, a_k^\dagger] = \delta_{ik}$, then the operators

$$\begin{aligned} \gamma^k &= a_k + a_k^\dagger \\ \gamma^{N+k} &= i(a_k - a_k^\dagger), \qquad k = 1 \cdots N \end{aligned} \tag{2.26}$$

are easily seen to obey eq.(2.20).

More generally, we can exploit the similarity between the metaplectic Clifford algebra and the Heisenberg algebra. We consider an auxiliary quantum system with position and momentum operators $\hat{x}^k$ and $\hat{\pi}^k$, respectively, satisfying $[\hat{x}^k, \hat{\pi}^j] = i\delta^{kj},\ k, j = 1 \cdots N$. If we combine $\hat{x}^k$ and $\hat{\pi}^k$ into $\hat{\varphi}^a \equiv (\hat{\pi}^k, \hat{x}^k)$, the canonical commutation relations can be written as

$$\hat{\varphi}^a\ \hat{\varphi}^b - \hat{\varphi}^b\ \hat{\varphi}^a = i\omega^{ab} \tag{2.27}$$

Thus, setting

$$\gamma^a = \sqrt{2}\ \hat{\varphi}^a \tag{2.28}$$

we obtain a representation of the metaplectic Clifford algebra on the quantum mechanical Hilbert space of the auxiliary system with operators $\hat{\varphi}^a \equiv (\hat{\pi}^k, \hat{x}^k)$. Furthermore, if we choose a basis in which $\hat{x}^k$ is diagonal we get

$$\begin{aligned} (\gamma^k)^x{}_y &= \sqrt{2}\ \langle x|\hat{x}^k|y\rangle = \sqrt{2}\ x^k \delta^N(x-y) \\ (\gamma^{N+k})^x{}_y &= \sqrt{2}\ \langle x|\hat{\pi}^k|y\rangle = -i\sqrt{2}\ \partial_k \delta^N(x-y) \end{aligned} \tag{2.29}$$

The generic index α stands for the eigenvalue of $\hat{x}^k$ now: $\alpha \to x \equiv (x^k) \in \mathbf{R}^N$. In this representation the metaplectic generators assume the form of Schrödinger operators with a quadratic potential:

$$(\frac{1}{2}\kappa_{ab}\Sigma^{ab}_{\mathrm{meta}})^x{}_y = [-\frac{1}{2}\kappa_{kj}\partial^k\partial^j - \frac{i}{2}\kappa_{N+k,j}(x^k\partial^j + \partial^j x^k) + \frac{1}{2}\kappa_{N+k,N+j}x^k x^j]\delta^N(x-y) \tag{2.30}$$

Let us return to the path-integral (2.11) now, where we take the generators Σ^{ab} in the metaplectic representation. The relevant Lagrangian reads then

$$\widetilde{\mathcal{L}} = \lambda_a(\dot{\phi}^a - h^a(\phi)) + i\bar{\eta}_x\Big[\delta^x{}_y\partial_t + \frac{i}{2}K_{ab}(\phi)(\Sigma^{ab}_{\mathrm{meta}})^x{}_y\Big]\eta^y \tag{2.31}$$

Here and in the following we use the summation (or rather integration) convention also for the "indices" $x, y, \cdots \in \mathbf{R}^{2N}$. In the Lagrangian (2.31) the auxiliary field λ_a acts as a Lagrange multiplier for Hamilton's equation so that the path-integral is strictly localized

on paths satisfying $\dot{\phi}^a = h^a(\phi)$. In the same way the antighost $\bar{\eta}$ enforces the equation of motion for η, which has a rather interesting interpretation:

$$i\partial_t \eta^x(t) = \frac{1}{2} K_{ab}(\phi(t))\ (\Sigma^{ab}_{\mathbf{meta}})^x{}_y\ \eta^y(t) \tag{2.32}$$

This is precisely the linearized Schrödinger equation for the wave function $\psi(x,t) \equiv \eta^x(t)$ which one obtains if one quantizes small fluctuations around a classical trajectory $\phi^a(t)$. From a geometrical point of view eq.(2.32) describes how the spinor η^x changes when it is dragged by the hamiltonian flow along the trajectory $\phi^a(t)$. This equation should be compared to its counterpart in the vector representation. If we use, instead of the metaplectic ones, the generators of eq.(2.5) in (2.31), then we get for the ghosts (which we denote as $c^a \equiv \eta^a$ in the vector representation):

$$\partial_t c^a = \partial_b h^a(\phi(t))\ c^b \tag{2.33}$$

This is exactly Jacobi's equation [15] which governs small *classical* fluctuations $\delta\phi^a(t) = c^a(t)$ around a given solution of $\dot{\phi}^a = h^a(\phi)$.

The operators $\widehat{\varphi}^a \equiv (\widehat{\pi}^k, \widehat{x}^k)$ should not be confused with the ones used in canonical quantization, $\widehat{\phi}^a \equiv (\widehat{p}^k, \widehat{q}^k)$. The latter act on a single (global) Hilbert space, whereas the former act in the fibers a whole bundle of Hilbert spaces: at each point of $\mathcal{M}_{2N}$ we have a local Hilbert space on which $Mp(2N)$ is represented by unitary transformations. This "Hilbert bundle" is the spinorial analogue of the tangent bundle, where $Sp(2N)$ acts on the fibers in its vector representation.

This concludes our brief review of the metaplectic path-integral introduced in ref.[13], to which we refer the reader for further details. In the rest of this paper we shall analyze in detail the anomalies of this model.

3 Loops in Loop-Space

Before we return to the metaplectic path-integral we have to discuss the topological properties of $Mp(2N)$ which are responsible for the occurrence of anomalies.

Recall that $Mp(2N)$ is a double covering of $Sp(2N)$. There is a sign ambiguity in the map $S \mapsto M(S)$ so that the representation is a projective one:

$$M(S_1)M(S_2) = \pm M(S_1 S_2) \tag{3.1}$$

Next consider a smooth loop $\mathcal{C}$ on the group manifold of $Sp(2N)$. We parametrize it by a one-parameter family of matrices $S(t) \in Sp(2N), t \in [0, 2\pi]$ with $S(0) = S(2\pi) = 1$. Topologically speaking, $Sp(2N)$ is a product of the form [4]

$$Sp(2N) \;=\; \mathbf{R}^{N(N+1)} \;\times\; SU(N) \;\times\; U(1) \tag{3.2}$$

Because both $\mathbf{R}^{N(N+1)}$ and $SU(N)$ are simply connected, its first homotopy group is

$$\Pi_1(Sp(2N)) = \Pi_1(U(1)) = \mathbf{Z}.$$

In pictorial terms this means that the $Sp(2N)$-manifold contains one single "hole" which is caused by the $U(1)$-factor. As a consequence, loops on $Sp(2N)$ can be classified according to their winding number $W[\mathcal{C}]$ which counts the number of times $\mathcal{C}$ winds around this hole.

Now we look at the image of the curve $\mathcal{C}$ on the group manifold of $Mp(2N)$. We fix the initial conditions $S(0) = 1$ and $M(0) = 1$ and therefore, by continuity, obtain a locally single-valued map $S(t) \mapsto M(S(t))$. A closed curve in $Sp(2N)$ does not necessarily lead to a closed curve in $Mp(2N)$, however. After having completed a full loop in $Sp(2N)$ we might arrive at $M = -1$ in $Mp(2N)$. In fact, this happens always when the winding number is odd:

$$M(S(2\pi)) \;=\; e^{-i\pi\mu/2}\; M(S(0)) \;=\; (-1)^{W[\mathcal{C}]}\; M(S(0)) \tag{3.3}$$

Here $\mu = 2W[\mathcal{C}] \equiv 2W[S(t)]$ is the Maslov index of the curve $\mathcal{C}$. It plays an important role in semiclassical quantization, for instance. The proof of eq.(3.3), as well as an explicit algorithm for the calculation of $W[\mathcal{C}]$, can be found in refs.[4].

Now we consider a classical hamiltonian system which is constituted by a simply connected phase-space $\mathcal{M}_{2N}$, $\Pi_1(\mathcal{M}_{2N})= 0$, and a hamiltonian vector field $h^a = \omega^{ab}\partial_b H$. It gives rise to loops on $Sp(2N)$ in the following natural manner. Let us pick some closed path on phase-space, $\phi^a(t), t \in [0, 2\pi], \phi^a(0) = \phi^a(2\pi)$, which is not necessarily a solution of the equations of motion. This path defines a map from the circle S^1, parametrized by t, into the Lie algebra of $Sp(2N)$:

$$\mathcal{A}^a{}_b(t) \;=\; -\frac{i}{2}\, K_{cd}(\phi(t))(\Sigma^{cd}_{\rm vec})^a{}_b \;=\; \omega^{ac}K_{cb}(\phi(t)) = \;\partial_b h^a(\phi(t)) \tag{3.4}$$

The matrix $\mathcal{A}^a{}_b$ is indeed an element of $sp(2N)$ because it has the same sturcture as the term $\omega^{ac}\kappa_{cb}$ in eq.(2.7). The function $\mathcal{A}(t)$ should be thought of as a one-dimensional $Sp(2N)$-gauge field. In fact, under a canonical transformation $\phi \longrightarrow \phi'(\phi)$ it transforms as a Yang-Mills field:

$$\mathcal{A}' \;=\; U\mathcal{A}U^{-1} \;+\; (\partial_t U)U^{-1} \quad , \qquad U \in Sp(2N) \tag{3.5}$$

Here $U^a{}_b \equiv \partial\phi'^a/\partial\phi^b$ has to be evaluated on the trajectory $\phi^a(t)$.

The gauge field $\mathcal{A}$ can be used in order to "parallel-transport" a vector c^a along the curve $\phi^a(t)$. Writing

$$c^a(t) \;=\; S^a{}_b(t)\; c^b(0) \tag{3.6}$$

the matrix $S(t) \in Sp(2N)$ is given by the path-ordered exponential

$$S(t) \;=\; P\;\exp\int_0^t dt'\;\mathcal{A}(t') \tag{3.7}$$

Note that in general $S(2\pi) \neq 1$, i.e., eq.(3.7) does *not* define a loop on $Sp(2N)$ and has no winding number associated to it. If we take the time derivative of (3.6) we see that $c^a(t)$ obeys the same equation as the ghosts of the path-integral in the vector representation, namely eq.(2.33).

In an analogous fashion we introduce a gauge field which assumes values in the Lie-algebra of $Mp(2N)$:

$$\mathcal{M}^x{}_y(t) \;=\; -\frac{i}{2}\;K_{ab}(\phi(t))(\Sigma^{ab}_{\mathrm{meta}})^x{}_y \tag{3.8}$$

Its path-ordered exponential,

$$M(S(t)) \;=\; P\;\exp\int_0^t dt'\mathcal{M}(t') \quad , \tag{3.9}$$

transports spinors η^x along $\phi^a(t)$:

$$\eta^x(t) \;=\; M(S(t))^x{}_y\;\eta^y(0) \tag{3.10}$$

The spinor $\eta^x(t)$ obeys the equation of motion of the metaplectic ghosts, eq.(2.32). In any representation, this gives a natural interpretation to the fermionic variables η^α in the path-integral: they are "world-line" vectors, tensors, spinors, etc., which are parallel-transported along the trajectory $\phi^a(t)$ with respect to the connection $K_{ab}\Sigma^{ab}$ provided by the hamiltonian flow.

Up to now we considered a single loop $\phi^a(t)$ on $\mathcal{M}_{2N}$ which, via eq(3.7), gave rise to a certain (open) curve on $Sp(2N)$. In order to investigate possible obstructions for metaplectic spin structures, we introduce now a loop in the space of such loops. This is a one-parameter family of trajectories, $\phi^a_u(t), t \in [0,2\pi], u \in [0,1]$, with

$$\phi^a_u(t+2\pi) = \phi^a_u(t), \quad \forall u \in [0,1] \qquad \text{and} \qquad \phi^a_0(t) = \phi^a_1(t), \quad \forall t \in [0,2\pi] \tag{3.11}$$

Because $u = 0$ and $u = 1$ correspond to the same loop, we are dealing with a map from the torus $S^1 \times S^1$ into phase-space, where the first S^1 is parametrized by t and the second

one by u. Each loop $\phi^a_u(t)$ has its own gauge potential $\mathcal{A}_u(t)$ and its parallel-transport operator $S_u(t)$. Important information is contained in the resulting monodromy matrices

$$R(u) \;=\; S_u(2\pi) \;=\; P\;\exp\int_0^{2\pi} dt'\;\mathcal{A}_u(t') \tag{3.12}$$

Because $\phi^a_0(t) = \phi^a_1(t)$ implies $\mathcal{A}_0(t) = \mathcal{A}_1(t)$ and therefore $R(0) = R(1)$, the map $u \mapsto R(u)$ describes a *closed* curve on the $Sp(2N)$ group-manifold. The question of central importance is whether its image in $Mp(2N)$ is also closed, i.e., whether

$$M(R(u)) \;=\; M(S_u(2\pi)) \;=\; P\;\exp\int_0^{2\pi} dt\;\mathcal{M}_u(t) \tag{3.13}$$

returns for $u = 1$ to its original value $M(R(0))$. According to eq.(3.3) this is the case only if the winding number of the loop $R(u),\ W[R(u)]$, is even:

$$M(R(u=1)) \;=\; (-1)^{W[R(u)]}M(R(u=0)) \tag{3.14}$$

If there exists a family of loops on $\mathcal{M}_{2N}$, $\phi^a_u(t)$, for which the associated loop of monodromy matrices, $R(u)$, has an odd winding number, then there is no consistent way of defining metaplectic spinors on $\mathcal{M}_{2N}$. The reason is that the parameter values $u = 0$ and $u = 1$ yield the same curve in phase-space, but the resulting monodromy operators in the metaplectic representation are different: $M(R(1)) = -M(R(0))$. Therefore we cannot assign an unambiguous metaplectic holonomy operator to the loop $\phi^a_0(t) = \phi^a_1(t)$.

In the following sections we shall demonstrate that the model (2.31) is sensitive to precisely this type of topolgical obstructions. It is afflicted by a global anomaly exactly if its target space $\mathcal{M}_{2N}$ is such that there are loops $R(u)$ with an odd winding number, i.e., if $\mathcal{M}_{2N}$ cannot be endowed with a metaplectic structure.

4 Partition Function and Fermion Determinant

Now we return to the Lagrangian (2.31) and study its partition function

$$\mathbf{Z}_{\rm meta}(T) \;=\; \int \mathcal{D}\phi\;\mathcal{D}\lambda\;\exp\left[i\;\int_0^T dt\;\lambda_a(\dot{\phi}^a - h^a(\phi))\right]\;Z[\phi(t)] \tag{4.1}$$

with the fermionic part

$$Z[\phi(t)] \;=\; \int \mathcal{D}\eta\;\mathcal{D}\bar{\eta}\;\exp\left[i\int_0^T dt\;\bar{\eta}_x\Big[i\delta^x{}_y\partial_t - \frac{1}{2}K_{ab}(\phi)(\Sigma^{ab}_{\rm meta})^x{}_y\Big]\eta^y\;\right] \tag{4.2}$$

We use periodic boundary conditions for ϕ^a, and both periodic (P) and antiperiodic (A) ones for η^x. In the latter case the path-integral (4.2) represents the true partition function,

in the former it yields the alternating sum over the sectors of different fermion number which enters the Witten index:

$$Z_A = \mathrm{Tr}\left[P \exp\left(-i\int_0^T dt\widetilde{\mathcal{H}}_F(t)\right)\right] \tag{4.3}$$

$$Z_P = \mathrm{Tr}\left[(-1)^F P \exp\left(-i\int_0^T dt\widetilde{\mathcal{H}}_F(t)\right)\right] \tag{4.4}$$

Here

$$\widetilde{\mathcal{H}}_F = \frac{1}{2} K_{ab}(\phi)(\Sigma^{ab}_{\mathrm{meta}})^x{}_y \, \eta^y \, \bar\eta_x \tag{4.5}$$

$$F = \eta^x \, \bar\eta_x \tag{4.6}$$

are the fermionic Hamiltonian and the fermion number operator, respectively. Formally the Gaussian integral (4.2) yields

$$\widetilde{Z}_{P,A}[\mathcal{A}] = \det\left[i\partial_t - \frac{1}{2}\mathcal{A}^a{}_b(t)\,\Sigma^b{}_a\right] \tag{4.7}$$

where $\Sigma^b{}_a \equiv \Sigma^{bc}_{\mathrm{meta}}\omega_{ca}$ and where the operator on the RHS of (4.7) acts on periodic or anti-periodic functions of t, respectively. Because of subtleties related to the discretization of the path-integral (4.2), the equality $\widetilde{Z}_{P,A} = Z_{P,A}$ holds only up to a normal-ordering term, to which we shall come back later on.

In the following we consider the determinant $\widetilde{Z}$ [3] as a functional of the gauge field $\mathcal{A}$. The crucial question is whether the determinant is invariant under gauge transformations. Classically the theory is canonically covariant which, in this context, means invariant under the Yang-Mills transformations (3.5). However, $\widetilde{Z}$ is the determinant of a one-dimensional Dirac operator and there might be anomalies which spoil its gauge invariance. It is known [16, 17] that in a one-dimensional "field" theory there are no *local* anomalies. Determinants such as (4.7) can always be regularized in such a way that the invariance under gauge transformations continuously connected to the identity is preserved. On the other hand, *global* anomalies can exist in one dimension, and they actually will occur in our model.

As there are no local anomalies, $\widetilde{Z}$ can depend only on gauge covariant quantities formed from $\mathcal{A}$. For a gauge field on S^1 the only such quantity is the monodromy matrix $R[\mathcal{A}] = P\exp\int_0^{2\pi} dt\mathcal{A}$, so that $\widetilde{Z} \equiv \widetilde{Z}[R]$. (We set $T = 2\pi$ from now on.) In order for the path-integral (4.1) to make sense, the fermion determinant must be a single-valued functional on the space of bosonic configurations $\phi^a(t)$. To check if this is the case, we look

[3] Occasionally we shall omit the subscripts P and A.

at the family of loops $\phi_u^a(t)$ and the associated loop of monodromy matrices $R[\mathcal{A}_u] \equiv R(u)$. In this manner we arrive at a family of determinants: $\tilde{Z}(u) = \tilde{Z}[R(u)]$. It might happen that a smooth interpolation from $u = 0$ to $u = 1$ does not lead us back to our starting point: $\tilde{Z}(1) \neq \tilde{Z}(0)$. In this case $\tilde{Z}$ is anomalous, it is not a single-valued functional of $\phi^a(t)$.

Let us evaluate the fermion determinant now. It depends on $\mathcal{A}$ only via $R[\mathcal{A}]$ which responds to a gauge transformation (3.5) as follows [16, 17]:

$$R[\mathcal{A}'] \;=\; U(2\pi)R[\mathcal{A}]U(0)^{-1} \tag{4.8}$$

This can be exploited in order to choose a convenient gauge for R. We recall [4] that any symplectic matrix can be written as the product of a positive-definite, symmetric matrix $\mathcal{T}$ and an orthogonal matrix $\mathcal{R}$ such that both $\mathcal{T}$ and $\mathcal{R}$ are also in $Sp(2N)$. The matrix $\mathcal{T}$ has a well-defined logarithm, i.e., there exists a matrix $\mathcal{G}$ such that $\mathcal{T} = \exp(\mathcal{G})$. Hence, for a given $\mathcal{A}$, we may write $R[\mathcal{A}] = \exp(\mathcal{G})\mathcal{R}$. The first factor of this product can be removed by the gauge transformation $U(t) = \exp(-\frac{t}{2\pi}\mathcal{G})$. Furthermore, by appropriately rotating the coordinate frame, the orthogonal matrix $\mathcal{R}$ can be made block-diagonal:

$$R[\mathcal{A}] \;=\; \mathrm{diag}\left[\begin{pmatrix} \cos\Theta_1 & -\sin\Theta_1 \\ \sin\Theta_1 & \cos\Theta_1 \end{pmatrix}, \cdots, \begin{pmatrix} \cos\Theta_N & -\sin\Theta_N \\ \sin\Theta_N & \cos\Theta_N \end{pmatrix} \right] \;\equiv\; R_D(\Theta_k) \tag{4.9}$$

Because all gauge fields with the same holonomy $R[\mathcal{A}]$ are equivalent, it is sufficient to calculate the determinant for the simplest possible $\mathcal{A}$ which reproduces (4.9), namely

$$\mathcal{A}_D(\Theta_k) \;=\; \mathrm{diag}\left[\frac{1}{2\pi}\begin{pmatrix} 0 & -\Theta_1 \\ \Theta_1 & 0 \end{pmatrix}, \cdots, \frac{1}{2\pi}\begin{pmatrix} 0 & -\Theta_N \\ \Theta_N & 0 \end{pmatrix} \right] \tag{4.10}$$

This gauge field is time independent, and it is easy to see that (4.9) is recovered from $R_D = \exp(2\pi\mathcal{A}_D)$.

As for the family of loops $\phi^a(t)$, we can transform $R[\mathcal{A}_u]$ and $\mathcal{A}_u$, for each fixed value of u, to the forms (4.9) and (4.10), respectively. The angles $\Theta_k = \Theta_k(u)$ depend on u then. Because $R(u)$ is periodic, the Θ_k's have to change between $u = 0$ and $u = 1$ by integer multiples of 2π:

$$\Theta_k(1) \;=\; \Theta_k(0) + 2\pi\mathcal{N}_k\ , \quad \mathcal{N}_k \in \mathbf{Z},\ k = 1, \cdots, N \tag{4.11}$$

We show in Appendix A that the winding number $W[R(u)]$ depends only on the integers $\mathcal{N}_k$. Hence every loop $R(u)$ is homotopic to a loop for which the Θ_k's depend on u linearly:

$$\Theta_k(u) \;=\; \Theta_k + 2\pi\mathcal{N}_k u \tag{4.12}$$

As a consequence, we may work with

$$\mathcal{A}_u = \mathcal{A}_D(\Theta_k + 2\pi\mathcal{N}_k u) \tag{4.13}$$

$$R(u) = R_D(\Theta_k + 2\pi\mathcal{N}_k u) \tag{4.14}$$

It is important to know the homotopy class of the loop $R(u)$ in $\Pi_1\left(Sp(2N)\right)$. In appendix A we find that its winding number is given by the sum of the $\mathcal{N}_k$'s:

$$W[R(u)] = \sum_{k=1}^{N} \mathcal{N}_k \tag{4.15}$$

The gauge transformation which led to the block-diagonal potential $\mathcal{A}_D$ has turned the original dynamical system into a set of non-interacting harmonic oscillators. In fact, the gauge potential (4.10) follows by eq.(3.4) from the quadratic Hamiltonian

$$H^{(2)} = \frac{1}{2}\sum_{k=1}^{N} \frac{\Theta_k}{2\pi}\left[(p^k)^2 + (q^k)^2\right] \tag{4.16}$$

where $(p^1, q^1; \ldots; p^N, q^N)$ are canonical coordinates for this auxiliary system. Thus the fermion determinant factorizes for the gauge field (4.13):

$$\widetilde{Z}_{P,A}[\mathcal{A}_u] = \prod_{k=1}^{N} \widetilde{Y}_{P,A}(\Theta_k + 2\pi\mathcal{N}_k \cdot u) \tag{4.17}$$

The determinant

$$\widetilde{Y}_{P,A}(\Theta) \equiv \det\left[i\partial_t - \widehat{h}\right] \tag{4.18}$$

refers to a single oscillator Hamiltonian:

$$\widehat{h} = \frac{1}{2}\left(\frac{\Theta}{2\pi}\right)\left[-\frac{d^2}{dx^2} + x^2\right] \tag{4.19}$$

In deriving these equations we used that $\partial_a\partial_b H\Sigma^{ab}_{\mathbf{meta}} = \partial_a\partial_b H\widehat{\varphi}^a\widehat{\varphi}^b$ in the representation (2.28). The eigenvalues of $\widehat{h}$ are $\frac{\Theta}{2\pi}(n+\frac{1}{2}), n = 0, 1, \cdots$, and those of $i\partial_t$ consist of all integers and half-integers, respectively, depending on whether we use periodic or anti-periodic boundary conditions. In the former case we have for the square of the determinant:

$$\begin{aligned}\widetilde{Y}^2_P(\Theta) &= \det\left[-\partial_t^2 - \widehat{h}^2\right] \\ &= \prod_{n=0}^{\infty}\prod_{m=-\infty}^{\infty}\left[m^2 - \left(\frac{\Theta}{2\pi}\right)^2\left(n+\frac{1}{2}\right)^2\right]\end{aligned} \tag{4.20}$$

This product is not well-defined yet and must be regularized. If we divide it by the divergent, but Θ-independent constant $\prod_n\prod_m m^4$ and use the identity

$$\sin(\pi x) = (\pi x)\prod_{m=1}^{\infty}[1 - \frac{x^2}{m^2}] \tag{4.21}$$

we arrive at

$$\widetilde{Y}_P(\Theta) \;=\; \prod_{n=0}^{\infty} \sin\left[\frac{\Theta}{2}\left(n+\frac{1}{2}\right)\right] \tag{4.22}$$

The analogous result for anti-periodic boundary conditions reads:

$$\widetilde{Y}_A(\Theta) \;=\; \prod_{n=0}^{\infty} \cos\left[\frac{\Theta}{2}\left(n+\frac{1}{2}\right)\right] \tag{4.23}$$

Eq.(4.22) is very similar to the corresponding result for spinors on space-time to which it should be compared[10]. The main difference is that we obtain an infinite rather than a finite product of sine-functions. This makes the analysis slightly more complicated. Ultimately we have to check how $\widetilde{Z}_{P,A}[\mathcal{A}_u]$ behaves when we smoothly interpolate from $u = 0$ to $u = 1$, i.e., when we increase the argument of $\widetilde{Y}_{P,A}(\Theta)$ from Θ to $\Theta + 2\pi\mathcal{N}$. However, if we naively insert this substitution in (4.22) or (4.23) we obtain an ill-defined expression for the change of $\widetilde{Y}_{P,A}(\Theta)$. In order to understand this point better, we reconsider the partition function from a hamiltonian point of view in the next section.

5 The Anomaly

Let us evaluate the partition functions by an explicit summation over a complete set of states now. For the diagonal gauge potential $\mathcal{A}_D$ the traces (4.3) and (4.4) factorize,

$$Z_{P,A}[\mathcal{A}_u] \;=\; \prod_{k=1}^{N} Y_{P,A}(\Theta_k + 2\pi\mathcal{N}_k \cdot u) \tag{5.1}$$

where, for periodic boundary conditions, say,

$$Y_P(\Theta) \;= \mathrm{Tr}\left[\; (-1)^F \; P \;\; \exp\left(-2\pi i \; \widetilde{\mathcal{H}}_F\right) \;\right] \tag{5.2}$$

The operator $\widetilde{\mathcal{H}}_F$ has been defined in eq.(4.5) originally. For the quadratic hamiltonian (4.16) or (4.19), respectively, it reduces to a second-quantized oscillator Hamiltonian:

$$\widetilde{\mathcal{H}}_F \;=\; -:\bar{\eta}_x \; \widehat{h}^x{}_y \; \eta^y: \equiv \; -\frac{1}{2}(\frac{\Theta}{2\pi})\int d^N x \;\; :\bar{\eta}\left[-\partial_x^2 + x^2\right]\eta: \tag{5.3}$$

If we denote the orthonormal eigenfunctions of $\widehat{h}$ by ψ_n^x and expand the spinor and its dual in terms of fermionic creation and annihilation operators,

$$\eta^x \;=\; \sum_{n=0}^{\infty} b_n \; \psi_n^x \qquad , \qquad \bar{\eta}_x \;=\; \sum_{n=0}^{\infty} b_n^\dagger \; (\psi_n^x)^* \tag{5.4}$$

then the commutation relations (2.12) imply $b_n b_m^\dagger + b_m^\dagger b_n = \delta_{nm}$. For the fermion number operator and the Hamiltonian we obtain, respectively:

$$F = \sum_{n=0}^{\infty} (1 - b_n^\dagger b_n) \tag{5.5}$$

$$\widetilde{\mathcal{H}}_F = \sum_{n=0}^{\infty} \frac{\Theta}{2\pi} \left(n + \frac{1}{2}\right) \left(1 - b_n^\dagger b_n\right) \tag{5.6}$$

Obviously we are dealing with infinitely many Grassmann oscillators, one for each excitation level of the (bosonic) oscillator Hamiltonian $\widehat{h}$. Hence, for a given value of n, the eigenvalues of $b_n^\dagger b_n$ are 0 and 1. We construct a Fock space in the usual way by acting with the $b_n^\dagger$'s on the vacuum defined by $b_n|0\rangle = 0$. The slightly unusual point is that, by eq.(5.5), the state $b_n^\dagger|0\rangle$ has fermion number $F = 0$, and $|0\rangle$ has $F = 1$ (if we ignore the oscillators with $n' \neq n$ for a moment). This is a consequence of the anti-Wick ordering and the Schrödinger representation [18] which was necessary to make sure that the Schrödinger equation (2.16) is equivalent to the Lie-transport equation (2.17).

Summing over the eigenvalues $b_n^\dagger b_n = 0, 1$ for $n = 0, 1, 2, \cdots$ yields

$$Y_P(\Theta) = \prod_{n=0}^{\infty} \left[1 - \exp\left(-i\Theta(n + \frac{1}{2})\right)\right] \tag{5.7}$$

$$Y_A(\Theta) = \prod_{n=0}^{\infty} \left[1 + \exp\left(-i\Theta(n + \frac{1}{2})\right)\right] \tag{5.8}$$

Naively we would have expected that eqs.(5.7) and (5.8) coincide with the results (4.22) and (4.23) obtained by writing the determinant (4.18) as a product of eigenvalues. Instead we find that

$$Y_{P,A}(\Theta) = \exp\left[-i\frac{\Theta}{2}\sum_{n=0}^{\infty}(n + \frac{1}{2})\right] \; \widetilde{Y}_{P,A}(\Theta) \tag{5.9}$$

Clearly the sum in the exponential is only formal and has to be regularized. Its origin is as follows. Strictly speaking, the functional integral (4.2) equals the partition functions Z_A and Z_P of eqs.(4.3) and (4.4) only if the Hamiltonian $\widetilde{\mathcal{H}}_F$ is Weyl-ordered [19]. This means that under the path-integral the c-number $\bar{\eta}\eta$, say, corresponds to the operator $(\bar{\eta}\eta - \eta\bar{\eta})/2$. However, here we have to use the anti-Wick ordering, where $\bar{\eta}\eta$ amounts to the operator $-\eta\bar{\eta}$. The statistical sums Z_A and Z_P with the anti-Wick ordered Hamiltonian $\widetilde{\mathcal{H}}_F$ are indeed the appropriate partition functions. Therefore $Y_{P,A}(\Theta)$ of (5.7) and (5.8), rather than $\widetilde{Y}_{P,A}(\Theta)$, is the correct result. The point is that our evaluation of the determinants $\widetilde{Y}_{P,A}$ applies only to the path-integral discretized according to the mid-point rule which corresponds to Weyl ordering[19]. The correct discretization is such that the path-integral equals exactly $Z_{P,A}$. It involves an "effective" Lagrangian which differs from

the naive one by a constant term containing the sum appearing in (5.9). If we reorder the Hamiltonian we find

$$2\pi\widetilde{\mathcal{H}}_F^{\rm Weyl} \;=\; 2\pi\widetilde{\mathcal{H}}_F \;-\; \frac{1}{2}\Theta\sum_{n=0}^{\infty}\,(n+\frac{1}{2}) \tag{5.10}$$

The second term on the RHS of this equation is exactly what is needed in order to establish eq.(5.9).

Let us now return to our original question of what happens to $Z_{P,A}[\mathcal{A}_u]$ when we interpolate between $u=0$ and $u=1$. For the individual factors $Y_{P,A}(\Theta)$ of the product (5.1) this means that their argument is increased (or decreased) by the amount $2\pi\mathcal{N}$ where $\mathcal{N}$ is some integer. We can read off from eqs.(5.7) and (5.8) that, if $\mathcal{N}$ is *even*, both Y_P and Y_A return to their original values:

$$Y_{P,A}(\Theta+2\pi\mathcal{N}) \;=\; Y_{P,A}(\Theta)\;, \qquad \mathcal{N}\ \text{even} \tag{5.11}$$

However, if $\mathcal{N}$ is *odd*, then Y_P and Y_A get interchanged during the excursion from $u=0$ to $u=1$:

$$\begin{aligned} Y_P(\Theta+2\pi\mathcal{N}) &= Y_A(\Theta) \\ Y_A(\Theta+2\pi\mathcal{N}) &= Y_P(\Theta)\;, \qquad \mathcal{N}\ \text{odd} \end{aligned} \tag{5.12}$$

The transition from Y_P to Y_A and vice versa signals a global anomaly: the partition function is not invariant under certain "large" gauge transformations. For definiteness, let us fix periodic boundary conditions at $u=0$. For a generic family of potentials $\mathcal{A}_u$ some of the integers $\mathcal{N}_k$ will be odd. Therefore, even if we start at $u=0$ from a product of Y_P's only, during the interpolation to $u=1$ we will produce some Y_A's:

$$\begin{aligned} Z_P[\mathcal{A}_0] &= Y_P(\Theta_1)Y_P(\Theta_2)\cdots Y_P(\Theta_N) \;\longrightarrow \\ Z_P[\mathcal{A}_1] &= Y_P(\Theta_1)Y_P(\Theta_2)\cdots Y_A(\Theta_i)\cdots Y_A(\Theta_j)\cdots Y_P(\Theta_N) \end{aligned} \tag{5.13}$$

What are the conditions for the flips $Y_P \to Y_A$ to occur? Let us recall from section 3 that if $\mathcal{M}_{2N}$ cannot be given a spin structure then there exist gauge fields $\mathcal{A}_u$ for which $R(u)$ has an odd winding number. According to eq.(4.15), $W[R(u)]$ is the sum of the the $\mathcal{N}_k$'s. Hence we have in this case:

$$(-1)^{W[R(u)]} \;=\; \prod_{k=1}^{N}(-1)^{\mathcal{N}_k} \;=\; -1 \tag{5.14}$$

This relation implies that some of the $\mathcal{N}_k$'s must be odd. We conclude that on those manifolds on which metaplectic spinors cannot be defined, the fermionic partition function

is necessarily anomalous, i.e., the functional $Z_{P,A}[\phi^a(t)]$ is not single-valued. Conversely, if there exists a family $\phi^a_u(t)$ for which the resulting gauge fields $\mathcal{A}_u$ cause an *odd* number of flips $Y_P \to Y_A$ then it follows that $\mathcal{M}_{2N}$ cannot support spinor fields. The restriction to an odd number of flips is necessary because an even number of odd $\mathcal{N}_k$'s adds up to a $W[R(u)]$ which is even. However, as the homotopy classes of $\Pi_1(Sp(2N))$ are labelled by a single integer only, we can always continuously deform $\mathcal{A}_u$ in such a way that only one $\mathcal{N}_k$, $\mathcal{N}_1$ say, is different from zero. Then $(-1)^W = (-1)^{\mathcal{N}_1}$, and the single-valuedness of the partition function is necessary and sufficient for the existence of a spin structure. This is the connection we wanted to establish. We emphasize that the question of whether or not there is an anomaly is unrelated to the form of the Hamiltonian H from which we started: the behavior of the fermionic partition functions, eqs.(5.11) and (5.12), depends only on the topology of the gauge fields, but not on H. Therefore we are testing a property of the manifold $\mathcal{M}_{2N}$ itself rather than of the dynamics taking place on it.

In order to complete the discussion we shall now give a slightly more rigorous derivation of the anomaly. The above argument was based upon formal manipulations with ill-defined infinite products. We can justify these steps by regularizing the determinant (4.20) using the zeta-function method. Because it can be applied to elliptic operators only, we "Wick-rotate" the angle $\Theta, \Theta \to i\Theta$, so that $\widehat{h}^2 \to -\widehat{h}^2$, which renders the resulting operator $-\partial_t^2 + \widehat{h}^2$ positive-definite. The regularized determinant

$$\widetilde{Y}^2_{P,A}(\Theta) \;=\; \det\left[-\partial_t^2 + \widehat{h}^2\right]_{\rm reg} \;=\; \exp\left[-\frac{1}{2}\,\zeta'_{P,A}(0)\right] \tag{5.15}$$

is defined in terms of the analytic continuation $s \to 0$ of the zeta-function

$$\zeta_{P,A}(s) \;=\; \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[(m+\frac{\varepsilon}{2})^2 + (\frac{\Theta}{2\pi})^2(n+\frac{1}{2})^2\right]^{-s} \tag{5.16}$$

where $\varepsilon = 0(1)$ for $\zeta_P(\zeta_A)$. The factor of $1/2$ in eq.(5.15) is necessary because in (5.16) we have included the negative integers in the sum over n. It is interesting that this zeta-function coincides exactly with the one for a free scalar field on a 2-dimensional torus with periodic/anti-periodic and anti-periodic/anti-periodic boundary conditions, respectively. It is needed for the computation of loop-amplitudes in string theory, for instance. If we define $q \equiv \exp(2\pi i\tau)$ and $\tau \equiv -\Theta/2\pi$ with Θ purely imaginary now, the result can be written as [20]

$$\widetilde{Y}_P(\Theta) \;=\; \left[\frac{\vartheta_{0,1}(0,\tau)}{\eta(\tau)}\right]^{1/2} \quad , \quad \widetilde{Y}_A(\Theta) \;=\; \left[\frac{\vartheta_{0,0}(0,\tau)}{\eta(\tau)}\right]^{1/2} \tag{5.17}$$

Here $\eta(\tau) = q^{1/24}\prod_{n=0}^{\infty}(1-q^n)$ is the Dedekind eta-function, and $\vartheta_{a,b}(z,\tau)$ denotes the

theta-function of the torus for the spin structure[4] (a,b) where $a,b = 0,1 \bmod 2$. In this notation one has $a,b = 0(1) \bmod 2$ for anti-periodic (periodic) boundary conditions around the cycles of the torus.[5] The ill-defined sum in eq.(5.9) we have to regularize as well. Using zeta-function regularization again, we replace it by an appropriate analytic continuation of the Hurwitz zeta-function $\zeta(z,w)$:

$$\sum_{n=0}^{\infty}(n+\frac{1}{2}) \quad \longrightarrow \quad \lim_{z\to -1}\sum_{n=0}^{\infty}(n+\frac{1}{2})^{-z} = \zeta(-1,\frac{1}{2}) = \frac{1}{24} \tag{5.18}$$

Thus $Y_{P,A} = q^{1/24}\widetilde{Y}_{P,A}$ and the final answer is given by

$$Y_P(\Theta) = \left[\frac{q^{1/24}}{\eta(\tau)}\,\vartheta_{0,1}(0,\tau)\right]^{1/2} \tag{5.19}$$

$$Y_A(\Theta) = \left[\frac{q^{1/24}}{\eta(\tau)}\,\vartheta_{0,0}(0,\tau)\right]^{1/2} \tag{5.20}$$

Letting $\Theta \to \Theta - 2\pi$, say, implies for τ the the modular transformation $\tau \to \tau + 1$. If we use one of the fundamental properties of the theta-functions,

$$\vartheta_{ab}(z,\tau+1) = \exp(i\pi a/4)\;\vartheta_{a,b+a+1}(z,\tau) \tag{5.21}$$

and the fact that $q^{1/24}/\eta(\tau)$ is invariant, we see that Y_P and Y_A are indeed interchanged if we vary Θ by an odd multiple of 2π. This confirms the relations (5.11) and (5.12).

6 Discussion and Conclusion

In this paper we started from the Lie-transport equation for metaplectic spinor fields on some phase-space $\mathcal{M}_{2N}$. We solved this equation in terms of a path-integral. The action appearing in this path-integral defines a one-dimensional non-linear sigma-model with target space $\mathcal{M}_{2N}$. Its global anomalies turned out to be related to certain topological properties of the phase-space. We found that, for any choice of the hamiltonian vector field, the fermionic partition function is anomalous precisely if there are obstructions which forbid globally well-defined metaplectic spinor fields on $\mathcal{M}_{2N}$.

To some extent, our approach is similar to Witten's discussion of space-time spinors [10]. However, there are also essential differences: the world-line fermions $\psi^{\mu}(\tau)$ in the

[4] It should not be confused with the spin structures on $\mathcal{M}_{2N}$.

[5] The result (5.17) could be expressed in terms of the Dedekind function alone, but the present form is more illuminating for our purposes.

sigma-model (1.7) transform as space-time *vectors* whereas our $\eta^x(t)$'s are spinors. So we might ask if there is also an anomaly in the path-integral for the Lie-transport in the vector representation. The answer to this question is no. In this case the relevant fermion determinant is

$$\det[\partial_t \delta^a{}_b - \mathcal{A}^a{}_b] \;=\; \int \mathcal{D}\bar{c}_a(t)\mathcal{D}c^a(t)\;\exp\left[-\int_0^{2\pi} dt\;\bar{c}_a(\partial_t\delta^a{}_b - \mathcal{A}^a{}_b)c^b\right] \tag{6.1}$$

It can be regularized in a way which preserves the invariance under both small and large gauge transformations[17]. Even though the determinant itself is single-valued, its square root is suffering from a sign ambiguity. It changes its sign if the gauge field executes a loop $\mathcal{A}_u$ with an odd winding number $W[R(u)]$:

$$\sqrt{\det(\partial_t - \mathcal{A}_1)} \;=\; (-1)^{W[R(u)]}\,\sqrt{\det(\partial_t - \mathcal{A}_0)} \tag{6.2}$$

This relation can be seen a consequence of the Atiyah-Patodi-Singer index theorem applied to an one-dimensional Dirac operator, see ref.[17] for a detailed discussion. This global $\mathbf{Z}_2$-anomaly is very similar to the one of a Weyl fermion coupled to a $SU(2)$ gauge field in four dimensions.[6] The partition function of a Dirac fermion is well-defined, but its square root, the partiton function for the Weyl fermion, is not single valued: going through a topologically nontrivial loop of background gauge fields we might pick up a minus-sign[11]. There is an obvious analogy with the present case. The path-integral for the Lie-transport in the vector representation involves two indepent integration variables, $\bar{c}_a$ and c^a. This pair is the analogue of a Dirac fermion: it has the single-valued partition function (6.1). The analogue of the Weyl fermion obtains by identifying $\bar{c}_a \equiv \omega_{ab}c^b$ and integrating over $c^a(t)$ only. The result is the square root of (6.1). In the context of our Lie-transport equations such "Weyl fermions" do not appear, but (with bosonic statistics) they play a central role in the semiclassical approximation of conventional quantum mechanical path-integrals. Assume the integral $Z = \int \mathcal{D}\phi^a \exp\{iS[\phi^a]\}$ is dominated by a classical path $\phi^a_{\rm cl}(t)$ so that we may expand $\phi^a(t) = \phi^a_{\rm cl}(t) + \varphi^a(t)$ and keep terms up to second order in φ^a only. Then the contribution of the fluctuations is precisely the (inverse) square root of (6.1):

$$\det[\partial_t \delta^a{}_b - \mathcal{A}^a{}_b]^{-1/2} \;=\; \int \mathcal{D}\varphi^a(t)\;\exp\left[-\int_0^{2\pi} dt\;\varphi^a\omega_{ab}[\partial_t\delta^a{}_b - \mathcal{A}^a{}_b]\varphi^b\right] \tag{6.3}$$

Because of (6.2) this functional of $\mathcal{A}$ is not single-valued. With periodic boundary conditions for φ, eq.(6.3) provides a path-integral representation of $\mathrm{Tr}[M(S(2\pi))]$ with

[6] Also the parity-violating anomalies in odd-dimensional Yang-Mills theories are of a similar nature [21, 22].

$M(S(t))$ defined in (3.9). By this identification, eqs.(6.2) and (3.14) are equivalent: $\mathrm{Tr}[M(R(u))] = (-1)^{W[R(u)]}\mathrm{Tr}[M(R(0))]$. These remarks show that if $\mathcal{M}_{2N}$ cannot carry metaplectic spinor fields, no semiclassical approximation is possible on this phase-space.

One might wonder about the precise relationship between the path-integral studied in this paper and the semiclassical one in eq.(6.3). Both of them are sensitive to the obstructions forbidding metaplectic spin structures. The difference is that the path-integral in (6.3) time-evolves objects such as η^x which carry only one "index" x. In our language they have fermion number $F = 1$. The metaplectic path-integral is a kind of many-particle generalization: it evolves multispinors $\chi_{x_1 x_2 \cdots x_p}(\phi)\eta^{x_1}\eta^{x_2}\cdots\eta^{x_p}$ which carry any number of x-type indices and have fermion number $F = p$. This suggests that the anomaly should be visible only in the sector with an odd fermion number F. This is indeed the case. If we return to the evaluation of the statistical sum (5.2) in the eigenvalue basis of $b_n^\dagger b_n = 0, 1$ and combine all contributions with fermion number f into the function Y_f, we obtain

$$Y_P(\Theta) \;=\; \sum_{f=0}^{\infty} (-1)^f \; Y_f(\Theta) \quad , \quad Y_A(\Theta) \;=\; \sum_{f=0}^{\infty} \; Y_f(\Theta) \tag{6.4}$$

Here

$$Y_f(\Theta) \;=\; \exp\left(-\frac{i}{2}\Theta f\right) \sum_{n=0}^{\infty} \; p(n,f)\exp(-in\Theta) \tag{6.5}$$

where $p(n,f)$ denotes the number of possibilities of dividing n into f distinct, non-negative integers. We see that

$$Y_f(\Theta + 2\pi\mathcal{N}) \;=\; (-1)^{f\mathcal{N}} \; Y_f(\Theta) \tag{6.6}$$

This shows that, for f even, Y_f is always invariant and that the anomaly is due to the sectors with odd fermion number.

Acknowledgement: I would like to thank E.Gozzi for many stimulating discussions.

Appendix

In this appendix we calculate the winding number of the loop of monodromy matrices $R(u)$ and prove eq.(4.15), which is instrumental in establishing the anomaly.

Let us recall[4] that every loop $S(t)$ on the $Sp(2N)$ group manifold is classified by a winding number $W[S(t)]$ which tells us to which homotopy class in $\Pi_1(Sp(2N)) = \mathbf{Z}$ this loop belongs. The basic properties of the functional $W[S(t)]$ are (S_0 is a constant symplectic matrix):

$$W[S_0 S(t)] \;=\; W[S(t)S_0] \;=\; W[S(t)] \tag{A.1}$$

$$W[S(t)^{-1}] \;=\; -W[S(t)] \tag{A.2}$$

$$W[S_1(t)S_2(t)] \;=\; W[S_1(t)] \;+\; W[S_2(t)] \tag{A.3}$$

Littlejohn and Robbins[4] have proposed a convenient algorithm for the explicit determination of $W[S(t)]$ for a given loop. It works as follows. First one fixes a canonical basis $(p^1, p^2, \cdots, p^N, q^1, q^2, \cdots, q^N)$ and decomposes the $2N \times 2N$ matrix $S(t)$ in $N \times N$ blocks according to

$$S(t) \;=\; \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} \tag{A.4}$$

Then one computes, as a function of t, the complex number $\det[A(t) - iB(t)]$ and counts how often this number winds around the origin of the complex plane while S executes its loop on $Sp(2N)$. It turns out that this number is exactly equal to $W[S(t)]$. We can summarize this in the formula

$$W[S(t)] \;=\; \oint \frac{dt}{2\pi i}\, \frac{d}{dt} \ln\, \det[A(t) - iB(t)] \tag{A.5}$$

Let us apply this algorithm to the loop $R(u)$ with u playing the role of t now. In eq.(4.14) we found

$$R(u) \;=\; R_D(\Theta_k + 2\pi \mathcal{N}_k u) \tag{A.6}$$

with the block-diagonal matrix $R_D(\Theta_k)$ which was defined in (4.9). It is convenient to rewrite $R(u)$ as a product of simpler matrices,

$$R(u) \;=\; \prod_{k=1}^{N} \bar{R}_k(u) \tag{A.7}$$

where $\bar{R}_k$ has nontrivial entries at the k-th position only:

$$\bar{R}_k(u) \;=\; \mathrm{diag}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} c_k & -s_k \\ s_k & c_k \end{pmatrix}, \cdots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \tag{A.8}$$

Here

$$\begin{aligned} c_k &= \cos(\Theta_k + 2\pi\mathcal{N}_k u) \\ s_k &= \sin(\Theta_k + 2\pi\mathcal{N}_k u) \end{aligned} \tag{A.9}$$

It is sufficient to calculate the winding number of the matrices $\bar{R}_k(u)$ because the property (A.3) implies that

$$W[R(u)] \;=\; \sum_{k=1}^{N} W[\bar{R}_k(u)] \tag{A.10}$$

It is not yet possible to apply the formula (A.5) to the loop (A.8) because different labellings of the canonical coordinates, i.e., different forms of ω_{ab}, are used here: eq.(A.5) refers to $(p^1, p^2, \cdots, p^N, q^1, q^2, \cdots, q^N)$, but (A.8) uses $(p^1, q^1; \cdots; p^N, q^N)$. If we reorder the rows and columns of (A.8) appropriately and then compare with the decomposition (A.4), we obtain (for k fixed):

$$\begin{aligned} A(u) &= \mathrm{diag}(1, 1, \cdots,\; c_k, 1, \cdots, 1) \\ B(u) &= \mathrm{diag}(0, 0, \cdots, -s_k, 0, \cdots, 0) \end{aligned} \tag{A.11}$$

It becomes obvious now that when u runs from 0 to 1, the phase factor

$$\det[A(u) - iB(u)] \;=\; \exp[i(\Theta_k + 2\pi\mathcal{N}_k u)] \tag{A.12}$$

executes exactly $\mathcal{N}_k$ revolutions around the origin. Therefore $W[\bar{R}_k(u)] = \mathcal{N}_k$, and the final results reads

$$W[R(u)] \;=\; \sum_{k=1}^{N} \mathcal{N}_k \tag{A.13}$$

This is precisely eq.(4.15) which we wanted to prove. We remark that it was not essential for the proof that Θ_k increases *linearly* from Θ_k to $\Theta_k + 2\pi\mathcal{N}_k$. Any interpolating function $\Theta_k(u)$ obeying the same boundary conditions would have led to the same winding number.

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