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Effects of weak self-interactions in a relativistic plasma on cosmological perturbations

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Abstract

The exact solutions for linear cosmological perturbations which have been obtained for collisionless relativistic matter within thermal field theory are extended to a self-interacting case. The two-loop contributions of scalar $\lambda\phi^4$ theory to the thermal graviton self-energy are evaluated, which give the $O(\lambda)$ corrections in the perturbation equations. The changes are found to be perturbative on scales comparable to or larger than the Hubble horizon, but the determination of the large-time damping behavior of subhorizon perturbations requires a resummation of thermally induced masses.

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The theory of linear perturbations of otherwise homogeneous and isotropic cosmological models [1] plays a central rôle in the problem of large-scale structure formation in the early universe. The conventional approach to their study in cosmological models involving more complicated forms of matter than a perfect fluid [1] is based on the coupled Einstein-Boltzmann equations [2–6], which are usually solved numerically. In Ref. [7], a novel framework has been developed which employs perturbative thermal field theory [8] to derive self-consistent and automatically gauge-invariant perturbation equations. In this formalism, the connection of the perturbed metric and the perturbed energy-momentum tensor is provided by the thermal graviton self-energy. At one-loop order, the leading high-temperature contributions to the latter have been obtained in Ref. [9] (see also Ref. [10]) in momentum space, and conformal invariance allows one to transform these results directly to a curved space with vanishing Weyl tensor. The one-loop high-temperature contributions describe relativistic collisionless matter interacting with the metric perturbations, and the resulting perturbation equations have been shown [11] to be equivalent to a gauge-invariant reformulation of the Einstein-Vlasov equations [12]. Moreover, analytic solutions for scalar, vector, and tensor perturbations have been found for a spatially flat Friedmann-Robertson-Walker (or, Einstein-de Sitter) cosmological model in the general case of a two-component system containing also a perfect fluid part [7,11,13], as well as for higher-dimensional models [14].

In this Letter, we investigate the effects of weak self-interactions on the exact solutions that were found for relativistic collisionless matter. For simplicity we consider scalar particles with $\lambda\phi^4$ interaction Lagrangian. The leading self-interaction effects are described by two-loop corrections to the thermal graviton self-energy in the high-temperature limit. The corrections are such that the ensuing perturbation equations can still be solved exactly in terms of rapidly converging power series.

The background space-time of the Einstein-de Sitter cosmological model, which we shall consider, is given through the line element

$$ds^2 = S^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right) \ , \tag{1}$$

with τ being the conformal time which measures the size of the horizon in comoving coordinates, $R_H = \tau$. A thermal distribution of relativistic (i.e., effectively massless) scalar particles with $\lambda\phi^4$ self-interactions gives rise to a background energy-momentum tensor

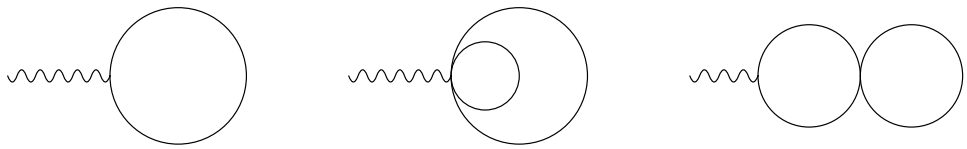

FIG. 1. One- and two-loop diagrams for the energy-momentum tensor. Wavy lines denote external gravitons and straight lines scalar particles.

$$\tilde{T}^{\mu}{}_{\nu} = u^{\mu}u_{\nu}(\tilde{E}+\tilde{P}) + \tilde{P}\delta^{\mu}_{\nu} \ , \qquad u_{\mu} = S\delta^{0}_{\mu} \ , \tag{2}$$

with $\tilde{E} = 3\tilde{P}$. Including order λ, $\tilde{T}^{\mu}{}_{\nu}$ is determined by the diagrams of Fig. 1, which at finite temperature T yield

$$\tilde{E}(S=1) = \left(\frac{\pi^2}{30} - \frac{\lambda}{16}\right) T^4 + O(\lambda^{3/2}) \ . \tag{3}$$

The scale dependence of the mean energy density is given by $\tilde{E}(S) = \tilde{E}(S=1)S^{-4}$, with $S \propto \tau$.

Self-consistent perturbations of the metric $g_{\mu\nu} = \tilde{g}_{\mu\nu} + \delta g_{\mu\nu}$ are found by varying the Einstein equations,

$$\delta G^{\mu\nu} \equiv \delta(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = -8\pi G\delta T^{\mu\nu} \tag{4}$$

with

$$\delta T_{\mu\nu}(x) = \int d^4y \frac{\delta T_{\mu\nu}(x)}{\delta g^{\alpha\beta}(y)} \delta g^{\alpha\beta}(y). \tag{5}$$

Hence, $\delta T_{\mu\nu}$ is determined by the graviton self-energy

$$\Pi_{\mu\nu\alpha\beta} \equiv \frac{\delta^2\Gamma}{\delta g^{\mu\nu}\delta g^{\alpha\beta}} = \frac{1}{2}\frac{\delta(\sqrt{-g}T_{\mu\nu})}{\delta g^{\alpha\beta}}, \tag{6}$$

where Γ contains all contributions to the effective action besides the classical Einstein-Hilbert action.

Γ is a nonlocal functional of the metric. With a flat background metric, the graviton self-energy $\Pi_{\mu\nu\alpha\beta}$ is most easily evaluated through momentum-space Feynman rules. In the high-temperature limit, Γ is conformally invariant [7] which makes it possible to directly transform the flat-space results to conformally flat cosmological models such as the Einstein-de Sitter one of Eq. (1) according to

$$\Pi^{\mu\nu\rho\sigma}(x,x')\big|_{g=S^2\eta} = S^{-2}(\tau)\int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \ \bar{\Pi}^{\mu\nu\rho\sigma}(k)\Big|_{\eta} S^{-2}(\tau') \ . \tag{7}$$

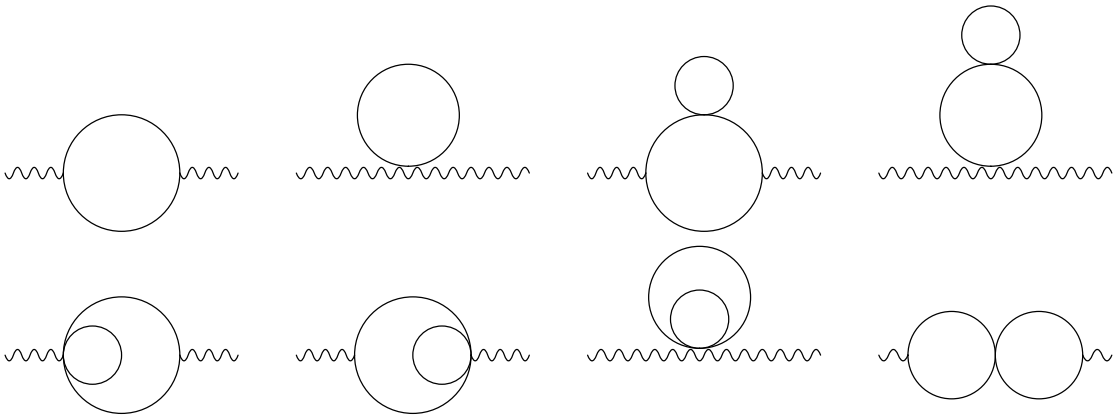

FIG. 2. The one- and two-loop diagrams for the graviton self-energy up to and including $O(\lambda)$. Only the last diagram contributes to the κ'-term in K_2.

The one-loop contribution to $\bar{\Pi}^{\mu\nu\rho\sigma}$ has been evaluated in Ref. [9] at temperature $T \gg k_0, k$, which is relevant for the case of cosmological perturbations [the momentum scale is set by the inverse Hubble radius $\sim (T/m_{\rm Planck})T$]. It describes collisionless matter, interacting solely with the metric perturbations. Since collisionless matter can sustain anisotropic pressure, this medium admits a rich spectrum of possible cosmological solutions. Those have been studied in great detail in Ref. [11] by two of the present authors.

In order to investigate the effects of weak self-interactions, one has to include higher loop corrections. With $\lambda\phi^4$ interactions, the order λ contributions are contained in the two-loop diagrams shown in Fig. 2. Evaluating again the high-temperature limit $T \gg k_0, k$ (full details will be presented elsewhere) gives a contribution to $\bar{\Pi}^{\mu\nu\rho\sigma}(k)$ that again satisfies the Ward identity corresponding to conformal invariance of Γ which is necessary for the simple transformation law of Eq. (7).

Since we did not have to single out a particular gauge, the perturbed Einstein equations (4) involve only gauge invariant combinations of the perturbed metric components and can be given entirely in terms of Bardeen's gauge invariant potentials [15]. Expanding the latter in Fourier modes (plane waves with respect to the conformally flat background of Eq. (1)) then leads to equations depending only on the conformal time τ, which can be combined with the wavelength of the metric perturbations $\ell \equiv 2\pi/k$ into the dimensionless variable

$$x \equiv k\tau = \frac{2\pi R_{\rm H}}{\ell} \; . \tag{8}$$

$x/\pi = 1$ defines the point in conformal time when half a wavelength fits inside the growing Hubble horizon.

Because of the nonlocality of the effective action Γ, the resulting equations are integro-differential equations rather than ordinary differential equations, involving a convolution integral of $\Pi^{\mu\nu\rho\sigma}(x, x')$ with the metric perturbations.

In the case of scalar metric perturbations, there are two independent equations for the two gauge invariant potentials Φ_H and Φ_A of Bardeen [15]. Using instead the combinations

$$\begin{aligned} \Phi &= \frac{1}{2}\Phi_H, \\ \Pi &= -(\Phi_H + \Phi_A), \end{aligned} \tag{9}$$

which are related to the gauge-invariant (cf. [15]) density contrast $\epsilon_{\rm m} = x^2\Phi/3$ and to the anisotropic pressure $\pi_{\rm anis} = x^2\Pi$, one equation is of the form

$$(x^2-3)\Phi(x) + 3x\Phi'(x) - 6\Pi(x) = -12\int dx' K(x-x')(\Phi+\Pi)'(x') \ , \tag{10}$$

while the second is an ordinary differential equation by virtue of the vanishing of the trace of the energy-momentum tensor in our ultrarelativistic setting,

$$\Phi'' + \frac{4}{x}\Phi' + \frac{1}{3}\Phi + \frac{2}{x}\Pi' - \frac{2}{3}\Pi = 0 \ . \tag{11}$$

The one-loop contribution to the kernel K reads [7]

$$K_1(x-x') = \frac{i}{2\pi}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{dk_0}{k} e^{-ik_0(\tau-\tau')}\frac{1}{2}\ln\frac{k_0+k}{k_0-k} = j_0(x-x')\theta(x-x'), \tag{12}$$

with $j_0(x) \equiv \sin(x)/x$. The step function gives an upper bound of x in the integral in Eq. (10); a lower bound at some initial x_0 can be adopted by adding an inhomogeneous term

$$\sum_{n=0}^{\infty} \gamma_n K^{(n)}(x-x_0)$$

to the covolution integral in Eq. (10), encoding the infinite number of initial conditions one has to impose by specifying the likewise infinite number of moments of the initial particle distribution [11].

The two-loop contributions have more complicated structure, but can still be evaluated exactly:

$$\begin{aligned} K_2(x-x') &= \frac{5i\lambda}{32\pi^3}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon} d\omega\, e^{-i\omega(x-x')}\left[\omega\ln^2\frac{\omega+1}{\omega-1} - \ln\frac{\omega+1}{\omega-1} - \frac{2\omega}{\omega^2-1}\right] \\ &= \frac{5\lambda}{8\pi^2}(2\kappa' - j_0 - \cos)(x-x')\theta(x-x') \ , \end{aligned} \tag{13}$$

where we have introduced

$$\kappa(x) \equiv \frac{1}{x}\left[\sin(x)\mathrm{Si}(2x) + \cos(x)\{\mathrm{Ci}(2x) - \gamma - \ln(2x)\}\right]$$
$$= 2\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+2)!} \sum_{j=0}^{m} \frac{1}{2j+1} \ , \tag{14}$$

with Si and Ci being the sine and cosine integral [16].

Since

$$K_2(x) \propto x^4 + O(x^6),$$

$K = K_1 + K_2$ still satisfies

$$K(0_+) = 1, \qquad K(0_+) + 3K''(0_+) = 0 \ , \tag{15}$$

as did K_1. Thereby the conditions [11] on the initial data (summarized by the set of γ_n) following from demanding regularity at $x = x_0 = 0$ are left unchanged.

In addition to regular solutions, there exist also ones that are singular as $x \to 0$. In the collisionless case those behave as $\Phi, \Pi \sim x^{-5/2}\cos[\sqrt{27/20}\ln(x)]$, thus showing oscillations even on superhorizon scales [11,17,18]. This behavior is modified by the weak self-interactions, to wit,

$$\Phi, \Pi \sim x^{-5/2}\cos[\sqrt{\frac{27}{20} - \frac{2\lambda}{\pi^2}}\ln(x)] \ , \tag{16}$$

up to a constant shift of phase.

In Ref. [7], it was pointed out that the coupled system of Eqs. (10) and (11) can be solved exactly for $x_0 = 0$, yielding power series representations with infinite radius of convergence. This still holds true after adding the self-interaction term K_2. For the particular initial condition determined by $\gamma_n = 0$, $n \geq 1$, the first few terms in the regular solutions are given by

$$\Phi(x) = \gamma_0\Big(\frac{28 - 10\lambda/\pi^2}{5} - \frac{4644 - 3735\lambda/\pi^2 + 350(\lambda/\pi^2)^2}{315(54 - 5\lambda/\pi^2)}\, x^2 \pm \cdots\Big), \tag{17}$$
$$\Pi(x) = \gamma_0\Big(-\frac{4 - 5\lambda/\pi^2}{5} + \frac{2808 - 5445\lambda/\pi^2 + 700(\lambda/\pi^2)^2}{630(54 - 5\lambda/\pi^2)}\, x^2 \mp \cdots\Big).$$

With $\lambda = 1$, which corresponds to a prefactor ≈ 0.06 in the last line of Eq. (13), the density contrast $\epsilon_{\mathrm{m}} = x^2\Phi/3$ is plotted in Fig. 3 by the dashed line and compared with the collisionless case (full line). On superhorizon scales, $x \ll 1$, the solutions are dominated by the constant modes in Φ and Π. The amount of anisotropic pressure associated with a given density contrast is determined by Π/Φ, which is seen to be reduced with increasing λ, in agreement with intuitive expectation, since the collision-dominated case of a perfect fluid has $\Pi \equiv 0$.

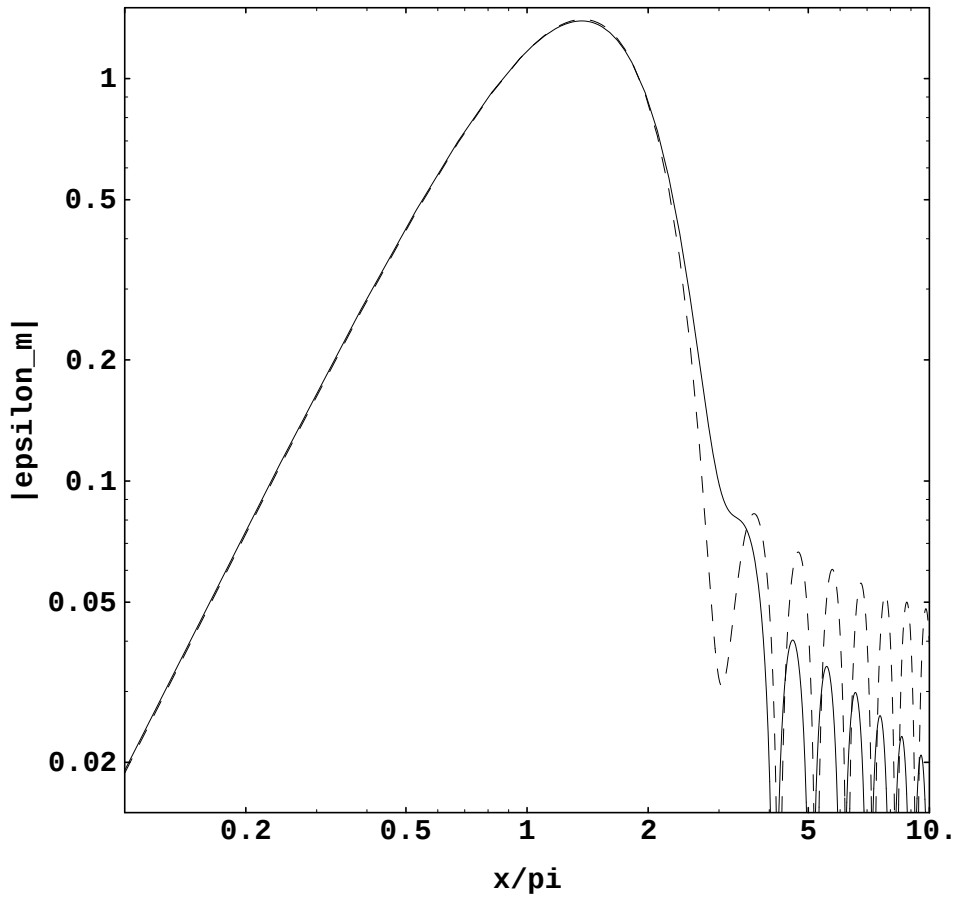

FIG. 3. The density contrast $|\epsilon_{\rm m}(x)|$ (in arbitrary normalization). The full line shows the collisionless situation, the dashed line includes self-interactions with $\lambda = 1$.

For large $x \gg 1$, the asymptotic behavior of the density contrast $\epsilon_{\rm m}(x)$ is proportional to the one of $K(x)$. In the collisionless case $K(x) = K_1(x) \sim \sin(x)/x$, whereas in the other extreme case of a perfect fluid $\epsilon_{\rm m} \sim \sin(x/\sqrt{3})$. With weak self-interactions, the asymptotic behavior is indeed modified towards weaker damping (see Fig. 3 and 4), which ceases completely for sufficiently large x where $K(x) \sim \lambda \cos(x)$. However, there the correction K_2 is overtaking the lowest order term K_1, which signals a possible breakdown of perturbation theory.

Inspecting the analytic structure that gives rise to the different asymptotic behavior of K_2, one finds that the discontinuity in the integrand of Eq. (13) is singular on the light-cone $\omega = k_0/k = \pm 1$. There is a logarithmically singular term which is responsible for $\kappa'(x) \sim \sin(x)\ln(x)/x$, but there are even poles at $\omega = \pm 1$, which contribute the term involving the $\cos(x)$. However, the scalar particles, whose rest masses are negligible at sufficiently high temperature, acquire thermal masses $m = \sqrt{\lambda}T$. A resummation of those will remove the singularities at $\omega = \pm 1$, which are associated with the masslessness of the

scalar particles, and thus will change the asymptotic behavior.

Without actually doing the higher order calculations, one can anticipate the result of such a resummation of thermal masses: Since in the resummed result there can no longer be poles at $\omega = \pm 1$, the corresponding term $\propto \cos(x)$ in K_2 has to be absorbed in a resummed K_1. Indeed, changing

$$K_1(x) \propto j_0(x) \to j_0((1 - \frac{5\lambda}{8\pi^2})x)$$

reduces

$$K_2(x) \to \frac{5\lambda}{4\pi^2}(\kappa' - j_0)(x)\theta(x)$$

up to higher orders in λ. However, the latter of the relations in Eq. (15) is now violated at order λ^2, which drastically restricts the solvability of the perturbation equations, so further modifications are needed. A natural solution is to change the phase velocity in K_2 by the same amount as in K_1 and to adjust the prefactor in K_2 appropriately. This leads to the remarkably simple result

$$K^{\rm resummed}(x) = \left(j_0 + \frac{1-v^2}{v^2}(\kappa' - j_0)\right)(vx)\,\theta(x) \tag{18}$$

with $v = 1 - 5\lambda/(8\pi^2)$. Up to and including order λ, this is the same as $K_1 + K_2$. But now the asymptotic behavior of $\epsilon_{\rm m}(x)$ for large x, which is linked to $K(x)$, is $\sim \sin(vx)\ln(x)/x$. Thus there is still strong damping of subhorizon-scale oscillations, which is only logarithmically weaker than in the collisionless case. Moreover, the phase velocity v is now smaller than 1, as one may expect from the comparison with the strongly collisional case of a perfect radiation fluid, where $v = 1/\sqrt{3}$. In Fig. 4 the effects of the above resummation are shown for $x/\pi \gtrsim 1$.

Similar issues arise also in the case of vector (rotational) as well as tensor perturbations (primordial gravitational waves). For small x, weak self-interactions affect mainly the singular solutions, and the same shift of frequency of the superhorizon oscillation occurs as in Eq. (16); for large x, vector and tensor perturbations differ from the scalar case in that their asymptotic behavior is not changed as much. Subhorizon-scale vector perturbations still have vorticity decaying $\sim \sin(x)/x$, and resummation changes only the phase velocity; tensor perturbations are stable against the kind of resummation performed above, and have the same asymptotic behavior as the collisionless ones, which indeed is already the same as in the perfect fluid case. All these findings as well as a generalisation to more-component systems will be presented in detail in a forthcoming publication.

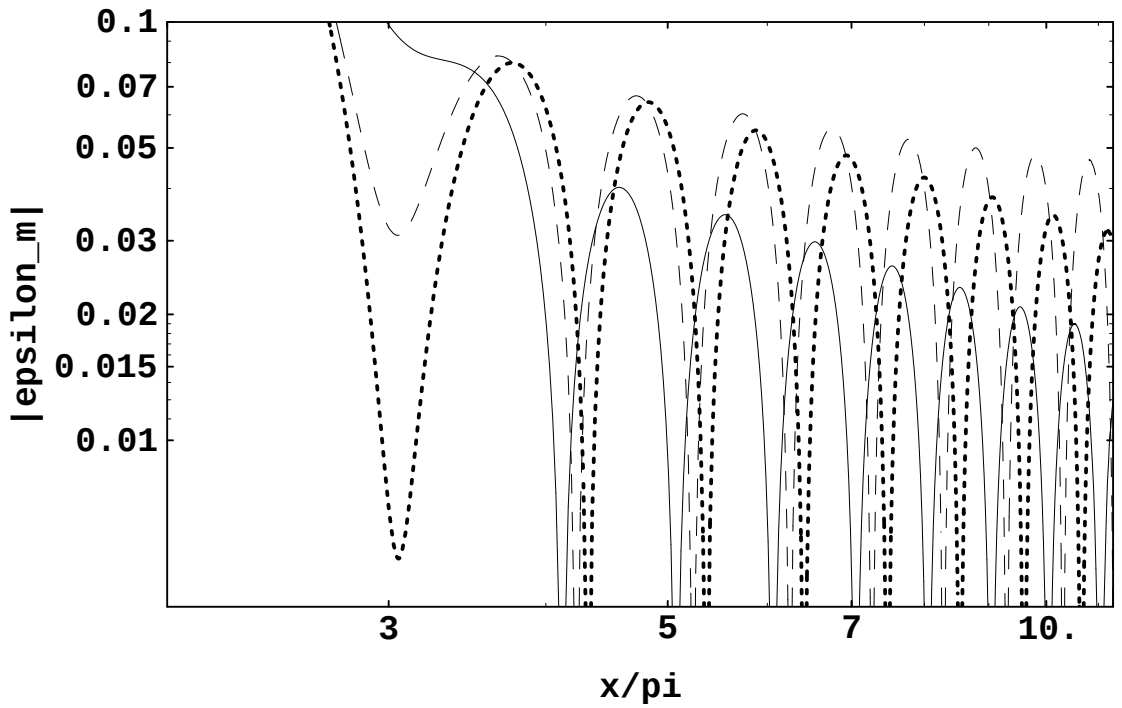

FIG. 4. In addition to the lines in Fig. 3, the dotted line shows the solution with the resummed kernel ($\lambda = 1$).

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