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Conformal Haag-Kastler Nets, Pointlike Localized Fields and the Existence of Operator Product Expansions

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Abstract

Starting from a chiral conformal Haag-Kastler net on 2 dimensional Minkowski space we construct associated pointlike localized fields. This amounts to a proof of the existence of operator product expansions.

We derive the result in two ways. One is based on the geometrical identification of the modular structure, the other depends on a "conformal cluster theorem" of the conformal two-point-functions in algebraic quantum field theory.

The existence of the fields then implies important structural properties of the theory, as PCT-invariance, the Bisognano-Wichmann identification of modular operators, Haag duality and additivity.

1 Introduction

The formulation of quantum field theory in terms of Haag Kastler nets of local observable algebras ("local quantum physics" [Haag]) has turned out to be well suited for the investigation of general structures. Discussion of concrete models, however, is mostly done in terms of pointlike localized fields.

In order to be in a precise mathematical framework, these fields might be assumed to obey the Wightman axioms [StW]. Even then, the interrelation between both concepts is not yet completely understood (see [BaW, BoY] for the present stage).

In the Wightman framework, the postulated existence of operator product expansions [Wil] has turned out to be very fruitful, especially in 2d conformal field theory. The existence of a convergent expansion of the product of two fields on the vacuum could been derived from conformal covariance, but the existence of the associated local fields had to be postulated [Lüs, Mac, SSV].

In the Haag-Kastler framework, the existence of an operator product expansion might be formulated as the existence of sufficiently many Wightman fields such that their linear span applied to the vacuum is dense in the Hilbert space. Actually, we are able to derive a stronger result. We give an expansion of a local observable into fields with local coefficients and show that this expansion converges $*$-strongly on a dense domain in Hilbert space.

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Heuristically, Wightman fields are constructed out of Haag-Kastler nets by some scaling limit which, however, is difficult to formulate in an intrinsic way [Buc]. In a dilation invariant theory scaling is well defined, and in the presence of massless particles the construction of a pointlike field was performed in [BuF].

In this paper, we study the possibly simplest situation: Haag-Kastler nets in 2 dimensional Minkowski space with trivial translations in one light cone direction ("chirality") and covariant under the real Möbius group which acts on the other lightlike direction. We show that in the vacuum representation pointlike localized fields can be constructed. Their smeared linear combinations are affiliated to the original net and generate it. We do not know at the moment whether they satisfy all Wightman axioms, since we have not yet found an invariant domain of definition.

Our method consists of an explicit use of the representation theory of $SL(2,\mathbf{R})$ combined with recent results on the modular structure of conformally covariant nets, obtained by Fröhlich and Gabbiani [FröG] and by Brunetti, Guido and Longo [BGL] and based on Borchers' theorem [Bor1]. We review these results in section 3 and show that they directly imply additivity of the net.

Additionally, we give an alternative argument which does not use Borchers' theorem. It relies essentially on a conformal cluster theorem which we prove in section 3. The existence of conformal fields can then be derived directly, and the Bisognano-Wichmann property, Haag duality, PCT-covariance and additivity are consequences.

Part of this work is based on one of the authors' diploma thesis [Jör1]. The results have been announced in [Fre] and [Jör2].

2 Assumptions and Results

Let $\mathcal{A} = (\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a family of von Neumann algebras on some separable Hilbert space H. $\mathcal{K}_0$ denotes the set of nonempty bounded open intervals on $\mathbf{R}$. $\mathcal{A}$ is assumed to satisfy the following conditions.

i) Isotony:
$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \quad \text{for} \quad I_1 \subset I_2, \quad I_1, I_2 \in \mathcal{K}_0. \tag{1}$$

ii) Locality:
$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \quad \text{for} \quad I_1 \cap I_2 = \{\}, \quad I_1, I_2 \in \mathcal{K}_0 \tag{2}$$
$(\mathcal{A}(I_2)'$ is the commutant of $\mathcal{A}(I_2))$.

iii) There exists a strongly continuous unitary representation U of $G = SL(2,\mathbf{R})$ in H with $U(-1) = 1$ and
$$U(g)\,\mathcal{A}(I)\,U(g)^{-1} = \mathcal{A}(gI), \quad I, gI \in \mathcal{K}_0 \tag{3}$$
$(SL(2,\mathbf{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbf{R}\cup\{\infty\}$ by $x \mapsto \frac{ax+b}{cx+d}$ with the appropriate interpretation for $x, gx = \infty$).

iv) The conformal Hamiltonian $\mathbf{H}$, which generates the restriction of U to $SO(2)$, has non-negative spectrum.

v) There is a unique (up to a phase) U-invariant unit vector $\Omega \in H$.

vi) H is the smallest closed subspace containing the vacuum Ω which is invariant under $U(g)$, $g \in SL(2,\mathbf{R})$, and $A \in \mathcal{A}(I), I \in \mathcal{K}_0$ ("cyclicity"). [1]

In [FröG] and [BGL] it was proven that under these conditions there is always an antiunitary involution Θ (the PCT-operator) which acts on $\mathcal{A}$ by

$$\Theta\, \mathcal{A}(I)\, \Theta = \mathcal{A}(-I) \tag{4}$$

and on $U(SL(2,\mathbf{R}))$ by

$$\Theta\, U(g)\, \Theta = U(g_\vartheta)\,. \tag{5}$$

Here g_ϑ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ means $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

In section 3, we show as a further consequence of the results of [FröG, BGL] that the net is automatically additive (cf. section 3), i.e. if $I = \bigcup_\alpha I_\alpha$ with $I, I_\alpha \in \mathcal{K}_0$ then

$$\mathcal{A}(I) = \bigvee_\alpha \mathcal{A}(I_\alpha) \tag{6}$$

where $\bigvee$ denotes the generated von Neumann algebra.

Due to the positivity condition iv) the representation U is completely reducible into the elements of the "dicrete series" [Lang], and the irreducible components τ are up to equivalence uniquely characterized by the conformal dimension $n_\tau \in \mathbf{N}_0$ (n_τ is the lower bound of the spectrum of the conformal Hamiltonian H in the representation τ).

Associated with each irreducible subrepresentation τ of U we find for each $I \in \mathcal{K}_0$ a densely defined operator valued distribution φ_τ^I on the space $\mathcal{D}(I)$ of Schwartz functions with support in I such that the following statements hold for all $f \in \mathcal{D}(I)$.

i) The domain of definition of $\varphi_\tau^I(f)$ is given by $\mathcal{A}(I)'\,\Omega$.

ii)

$$\varphi_\tau^I(f)\,\Omega \in H_\tau\,. \tag{7}$$

iii)

$$U(g)\,\varphi_\tau^I(x)\,U(g)^{-1} = (cx+d)^{-2n_\tau}\varphi_\tau^{gI}(gx) \tag{8}$$

with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R}),\ I, gI \in \mathcal{K}_0$.

iv)

$$\varphi_\tau^I(f)^* \supset \varphi_{\bar\tau}^I(\bar f) \tag{9}$$

with $\bar\tau(\cdot) = \Theta\tau(\cdot)\Theta$ and $\bar f$ the complex conjugate of f; in particular, $\varphi_\tau^I(f)$ is closable.

v) The closure of $\varphi_\tau^I(f)$ is affiliated to $\mathcal{A}(I)$.

vi) $\mathcal{A}(I)$ is the smallest von Neumann algebra to which all operators $\varphi_\tau^I(f)$ are affiliated.

[1] This assumption is seemingly weaker than cyclicity of Ω w.r.t. the algebra of local observables on $\mathbf{R}$. The proof of the Reeh-Schlieder-theorem in section 3 is therefore nontrivial.

3 Generalities on Chiral Nets

It is convenient to extend the net to intervals I on the circle $S^1 = \mathbf{R} \cup \{\infty\}$ by setting

$$\mathcal{A}(I) = U(g)\,\mathcal{A}(g^{-1}I)\,U(g)^{-1}, \quad g^{-1}I \in \mathcal{K}_0,\ g \in SL(2,\mathbf{R}). \tag{10}$$

The covariance property guarantees that $\mathcal{A}(I)$ is well defined for all intervals I of the form $I = gI_0,\ I_0 \in \mathcal{K}_0,\ g \in SL(2,\mathbf{R})$, i.e. for all nonempty nondense open intervals on S^1 (we denote the set of these intervals by $\mathcal{K}$).

First we remark that Ω is cyclic and separating for all $\mathcal{A}(I),\ I \in \mathcal{K}$ (Reeh-Schlieder theorem [ReS, Bor2]; cf. [FröG] for a similar argument). Namely, let us look at $I = \mathbf{R}_+$ (without restriction of generality). $\mathbf{R}_+$ is mapped into itself by translations $x \mapsto x + a,\ a > 0$, and special conformal transformations $x \mapsto \frac{x}{-cx+1},\ c < 0$. Both 1-parameter groups have a positive generator under U, hence by the usual Reeh-Schlieder argument

$$H_0 := \overline{\mathcal{A}(\mathbf{R}_+)\,\Omega} = \overline{\bigcup_t U(g_t)\,\mathcal{A}(\mathbf{R}_+)\,\Omega} \tag{11}$$

for both 1-parameter groups (g_t). So H_0 is invariant under $\mathcal{A}(I)$ for all $I \in \mathcal{K}_0$ and under translations and special conformal transformations, thus under $U(SL(2,\mathbf{R}))$. By assumption vi) we conclude $H_0 = H$, i.e. Ω is cyclic for $\mathcal{A}(\mathbf{R}_+)$, and, by locality, separating for $\mathcal{A}(\mathbf{R}_-)$.

In the following paragraph, we review results of [FröG] and [BGL]. We start with the modular structure. The modular involution J_I and the modular operator Δ_I are obtained by polar decomposition of the closure S_I of the operator $A\Omega \mapsto A^*\Omega,\ A \in \mathcal{A}(I)$,

$$S_I = J_I\,\Delta_I^{\frac{1}{2}}, \quad I \in \mathcal{K}. \tag{12}$$

J_I implements an antiisomorphism between $\mathcal{A}(I)$ and $\mathcal{A}(I)'$ and $\Delta_I^{it},\ t \in \mathbf{R}$, automorphisms of $\mathcal{A}(I)$ [Tak]. Borchers has shown [Bor1] that every unitary 1-parameter group $(U(t))_{t\in R}$ with positive generator and Ω as a fixed point which induces endomorphisms of $\mathcal{A}(I)$ for $t > 0$ satisfies the commutation relations

$$\Delta_I^{it}\,U(a)\,\Delta_I^{-it} = U(e^{-2\pi t}a) \tag{13}$$

$$J_I\,U(a)\,J_I = U(-a). \tag{14}$$

Applying this to $I = \mathbf{R}_+$ and to the 1-parameter groups considered before we find that the operators

$$Z(s) = \Delta_{\mathbf{R}_+}^{is}\,U\begin{pmatrix} e^{\pi s} & 0 \\ 0 & e^{-\pi s} \end{pmatrix}, \quad s \in \mathbf{R}, \tag{15}$$

commute with $U(g)$ for all $g \in SL(2,\mathbf{R})$ (in particular, $s \mapsto Z(s)$ is a 1-parameter group). Moreover,

$$J_{\mathbf{R}_+}\,U(g)\,J_{\mathbf{R}_+} = U(g_\vartheta)\,. \tag{16}$$

With $\vartheta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,\mathbf{R})$ we have $\mathbf{R}_- = \vartheta\,\mathbf{R}_+$. Hence,

$$J_{\mathbf{R}_-} = U(\vartheta)\,J_{\mathbf{R}_+}\,U(\vartheta)^{-1}\,. \tag{17}$$

So, inserting $g = \vartheta$ in (16) we obtain that $J_{\mathbf{R}_-}$ coincides with $J_{\mathbf{R}_+}$. Thus, by locality and by the properties of modular involutions, we obtain

$$\mathcal{A}(\mathbf{R}_-) \subset \mathcal{A}(\mathbf{R}_+)' = J_{\mathbf{R}_+} \mathcal{A}(\mathbf{R}_+) J_{\mathbf{R}_+} \subset J_{\mathbf{R}_-} \mathcal{A}(\mathbf{R}_-)' J_{\mathbf{R}_-} = \mathcal{A}(\mathbf{R}_-). \tag{18}$$

This shows Haag duality for half lines $\mathcal{A}(\mathbf{R}_-) = \mathcal{A}(\mathbf{R}_+)'$ and, by conformal covariance, Haag duality for every $I \in \mathcal{K}$:

$$\mathcal{A}(I')' = \mathcal{A}(I). \tag{19}$$

Now we compute

$$\begin{aligned} Z(s) &= \Delta^{is}_{\mathbf{R}_+} U\begin{pmatrix} e^{\pi s} & 0 \\ 0 & e^{-\pi s}\end{pmatrix} \\ &= U\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \Delta^{is}_{\mathbf{R}_+} U\begin{pmatrix} e^{\pi s} & 0 \\ 0 & e^{-\pi s}\end{pmatrix} U\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \\ &= \Delta^{is}_{\mathbf{R}_-} U\begin{pmatrix} e^{-\pi s} & 0 \\ 0 & e^{\pi s}\end{pmatrix} \\ &= \Delta^{-is}_{\mathbf{R}_+} U\begin{pmatrix} e^{-\pi s} & 0 \\ 0 & e^{\pi s}\end{pmatrix} \\ &= Z(-s). \end{aligned} \tag{20}$$

Here we used essential duality and the fact that the modular operator of $\mathcal{A}(\mathbf{R}_+)'$ is $\Delta^{-1}_{\mathbf{R}_+}$. Since Z is a 1-parameter group it must be trivial. Therefore, we obtain

$$\Delta^{is}_{\mathbf{R}_+} = U\begin{pmatrix} e^{-\pi s} & 0 \\ 0 & e^{\pi s}\end{pmatrix}. \tag{21}$$

Moreover, we show that $J_{\mathbf{R}_+} = \Theta$. The commutation relations of $J_{\mathbf{R}_+}$ with $U(SL(2,\mathbf{R}))$ have already been determined; now let $I \in \mathcal{K}$ and $g \in SL(2,\mathbf{R}))$ with $gI = \mathbf{R}_+$. Thus,

$$\begin{aligned} J_{\mathbf{R}_+} \mathcal{A}(I) J_{\mathbf{R}_+} &= J_{\mathbf{R}_+} U(g) \mathcal{A}(\mathbf{R}_+) U(g)^{-1} J_{\mathbf{R}_+} \\ &= U(g_\vartheta) J_{\mathbf{R}_+} \mathcal{A}(\mathbf{R}_+) J_{\mathbf{R}_+} U(g_\vartheta)^{-1} = U(g_\vartheta) \mathcal{A}(\mathbf{R}_-) U(g_\vartheta)^{-1} \\ &= \mathcal{A}(-I). \end{aligned} \tag{22}$$

Note that $g_\vartheta \mathbf{R}_- = -I$ follows from $gI = \mathbf{R}_+$.

We now use the results of [FröG] and [BGL], presented above, to prove that the net is automatically additive. Let $I = \bigcup_\alpha I_\alpha$, $I, I_\alpha \in \mathcal{K}$, and let $I_0 \in \mathcal{K}$ such that $\bar{I}_0 \subset I$. Then there is a finite number of intervals $I_{\alpha_1}, \ldots I_{\alpha_n}$ which already cover I_0. According to the result above, $\Delta^{it}_{I_0}$ implements the 1-parameter subgroup $(g_t)_{t\in\mathbf{R}}$ of $SL(2,\mathbf{R})$ which has the boundary points of I_0 as fixed points. There is a sufficiently small interval $I_1 \in \mathcal{K}$, $I_1 \subset I_0$, such that for all $t \in \mathbf{R}$ the interval $g_t(I_1)$ is contained in one of the intervals I_{α_i}, $i = 1, \ldots n$. The algebra

$$\mathcal{A}_{I_1}(I_0) := \bigvee_{t\in\mathbf{R}} \alpha_{g_t}(\mathcal{A}(I_1)) \subset \mathcal{A}(I_0) \tag{23}$$

is invariant under the modular automorphism $\mathrm{Ad}\Delta^{it}_{I_0} = \alpha_{g_t}$ of $\mathcal{A}(I_0)$ and has Ω as a cyclic vector, hence coincides with $\mathcal{A}(I_0)$ [Tak]. Thus, $\mathcal{A}(I_0)$ is contained in $\bigvee_\alpha \mathcal{A}(I_\alpha)$ for all $\bar{I}_0 \subset I$. But a conformally covariant net is continuous from below (cf. e.g. [Jör1]),

$$\mathcal{A}(I) = \bigvee_{\bar{I}_0 \subset I} \mathcal{A}(I_0), \tag{24}$$

which implies additivity.

Finally, we derive a bound on conformal two-point-functions in algebraic quantum field theory. This bound specifies the decrease properties of conformal two-point-functions in the algebraic framework to be exactly those known from theories with pointlike localization.

Conformal cluster theorem: Let $(\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a conformally covariant local net on $\mathbf{R}$. Let $a, b, c, d \in \mathbf{R}$ and $a < b < c < d$. Let $A \in \mathcal{A}((a,b))$, $B \in \mathcal{A}((c,d))$, $n \in \mathbf{N}$ and $P_k A\Omega = P_k A^*\Omega = 0$, $k < n$. P_k here denotes the projection on the subrepresentation of G with conformal dimension k. We then have

$$|(\Omega, BA\Omega)| \leq \left(\frac{(b-a)(d-c)}{(c-a)(d-b)}\right)^n \|A\|\,\|B\|. \tag{25}$$

Proof: Choose $R > 0$. We consider the following 1-parameter subgroup of $G = SL(2,\mathbf{R})$

$$g_t : x \longmapsto \frac{x\cos\frac{t}{2} + R\sin\frac{t}{2}}{-\frac{x}{R}\sin\frac{t}{2} + \cos\frac{t}{2}}. \tag{26}$$

Its generator $\mathbf{H}_R$ is within each subrepresentation of G unitarily equivalent to the conformal Hamiltonian H. Therefore, the spectrum of $A\Omega$ and $A^*\Omega$ w.r.t. $\mathbf{H}_R$ is bounded below by n. Let $0 < t_0 < t_1 < 2\pi$ such that $g_{t_0}(b) = c$ and $g_{t_1}(a) = d$. We now define

$$F(z) = \begin{cases} (\Omega,\, B z^{-\mathbf{H}_R} A\Omega) & |z| > 1 \\ (\Omega,\, A z^{\mathbf{H}_R} B\Omega) & |z| < 1 \\ (\Omega,\, A\,\alpha_{g_t}(B)\,\Omega) & z = e^{it},\ t \notin [t_0, t_1], \end{cases} \tag{27}$$

a function analytic in its domain of definition, and then

$$G(z) = (z - z_0)^n (z^{-1} - z_0^{-1})^n F(z), \quad z_0 = e^{\frac{i}{2}(t_0+t_1)}. \tag{28}$$

At $z = 0$ and $z = \infty$ the function $G(\cdot)$ is bounded because of the bound on the spectrum of $\mathbf{H}_R$ and can therefore be analytically continued. As an analytic function it reaches its maximum at the boundary of its domain of definition, which is the interval $[e^{it_0}, e^{it_1}]$ on the unit circle:

$$\sup|G(z)| \leq \|A\|\,\|B\|\,|e^{it_0} - e^{\frac{i}{2}(t_0+t_1)}|^{2n} = \|A\|\,\|B\|\,(2\sin\frac{t_0 - t_1}{4})^{2n}. \tag{29}$$

This leads to

$$\begin{aligned} |(\Omega,\, BA\Omega)| &= |F(1)| = |G(1)|\,|1 - e^{\frac{i}{2}(t_0+t_1)}|^{-2n} = |G(1)|\,(2\sin\frac{t_0+t_1}{4})^{-2n} \\ &\leq \sup|G|\,(2\sin\frac{t_0+t_1}{4})^{-2n} \leq \|A\|\,\|B\| \left(\frac{\sin\frac{t_0-t_1}{4}}{\sin\frac{t_0+t_1}{4}}\right)^{2n}. \end{aligned} \tag{30}$$

Determining t_0 and t_1 we obtain

$$\lim_{R\to\infty} R\,t_0 = 2(c-b) \quad\text{and}\quad \lim_{R\to\infty} R\,t_1 = 2(d-a). \tag{31}$$

We now assume $a - b = c - d$ and find $\left(\frac{t_0-t_1}{t_0+t_1}\right)^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)} =: x$. Since the bound on $|(\Omega,\, BA\Omega)|$ can only depend on the conformal cross ratio x, we can drop the assumption and the theorem is proven.

The conformal cluster theorem now allows to derive further properties of conformal two-point-functions of local observables:

It has been shown (cf. e.g. [Jör1]) that a two-point-function $(\Omega,\, B\,U(x)\,A\Omega)$ of a chiral local net with translation covariance is of Lebesgue class L^p for any $p > 1$. The Fourier transform of this two-point-function is a measure concentrated on the positive half line. Therefore, it is - with the possible exception of a trivial delta function at zero - fully determined by the Fourier transform of the commutator function $(\Omega, [B, U(x)\,A\,U(x)^{-1}]\Omega)$. Since A and B are local observables, the commutator function has compact support and an analytic Fourier transform $G(p)$. The restriction $\Theta(p)\,G(p)$ of this analytic function to the positive half line is then the Fourier transform of $(\Omega,\, B\,U(x)\,A\Omega)$.

In the conformally covariant case with $P_k A\Omega = P_k A^*\Omega = 0$, $k < n$, the conformal cluster theorem implies that the two-point-function $(\Omega,\, B\,U(x)\,A\Omega)$ decreases as x^{-2n}. Therefore, its Fourier transform is $2n-2$ times continuously differentiable and can be written as $\Theta(p)\,p^{2n-1}\,H(p)$ with an appropriate analytic function $H(p)$.

4 Construction of Local Fields

The idea for the definition of conformal fields is the following: Let A be a local observable,

$$A \in \bigcup_{I\in\mathcal{K}_0} \mathcal{A}(I), \tag{32}$$

and P_τ the projection onto an irreducible subrepresentation τ of U. The vector $P_\tau A\Omega$ may then be thought of as $\varphi_\tau(h)\,\Omega$ where φ_τ is a conformal field of dimension $n_\tau =: n$ and h is an appropriate function on $\mathbf{R}$. The relation between A and h, however, is unknown at the moment, up to the known transformation properties under G,

$$U(g)\,P_\tau A\Omega = \varphi_\tau(h_g^{(n)})\,\Omega \tag{33}$$

with $h_g^{(n)}(x) = (cx-a)^{2n-2}\,h(\frac{dx-b}{-cx+a})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. We may now scale the vector $P_\tau A\Omega$ by dilations $D(\lambda) = U\begin{pmatrix} \lambda^{\frac12} & 0 \\ 0 & \lambda^{-\frac12} \end{pmatrix}$ and find

$$D(\lambda)\,P_\tau A\Omega = \lambda^n\,\varphi_\tau(h_\lambda)\,\Omega \tag{34}$$

where $h_\lambda(x) = \lambda^{-1}\,h(\frac{x}{\lambda})$. Hence, we obtain formally for $\lambda \downarrow 0$

$$\lambda^{-n}\,D(\lambda)\,P_\tau A\Omega \longrightarrow \int dx\, h(x)\,\varphi_\tau(0)\,\Omega. \tag{35}$$

In order to obtain a Hilbert space vector in the limit, we smear over the group of translations $T(a) = U\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with some test function f and obtain formally

$$\lim_{\lambda\downarrow 0} \lambda^{-n} \int da\, f(a)\,T(a)\,D(\lambda)\,P_\tau A\Omega = \int dx\, h(x)\,\varphi_\tau(f)\,\Omega. \tag{36}$$

We now interpret the left hand side as a definition of a conformal field φ_τ on the vacuum, and try to obtain densely defined operators with the correct localization by defining

$$\varphi_\tau^I(f)\,A'\Omega = A'\varphi_\tau^I(f)\,\Omega, \quad f \in \mathcal{D}(I),\ A' \in \mathcal{A}(I)',\ I \in \mathcal{K}. \tag{37}$$

In the following, we want to make this formal construction meaningful. There are two problems to overcome.

The first one is the fact that the limit on the left hand side of (36) does not exist in general if $A\Omega$ is replaced by an arbitrary vector in H. This corresponds to the possibility that the function h on the right hand side might not be integrable. We will show that after smearing the operator A with a smooth function on G the limit is well defined. Such operators will be called regularized.

The second problem is to show that the smeared field operators $\varphi_\tau^I(f)$ are closable, in spite of the nonlocal nature of the projections P_τ. This problem is solved by the fact that the modular operators coincide with conformal transfomations [Bor1], as explained in section 3. An independent argument without recourse to Borchers' theorem is based on the conformal cluster theorem (see section 3) and will be outlined, together with its consequences, in section 5.

In order to investigate the limit in (36), we use the fact that $P_\tau H$ can be identified with $L^2(\mathbf{R}_+, p^{2n-1}\,dp)$, where G acts according to

$$\left(U_n\begin{pmatrix} a & b \\ c & d \end{pmatrix}\Phi\right)(p) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\pi}\int_{\mathbf{R}} dx \int_{\mathbf{R}_+} dq\, e^{-ip(x+i\varepsilon)+iqg^{-1}(x+i\varepsilon)}(a-c(x+i\varepsilon))^{2n-2}\Phi(q). \tag{38}$$

Now let $\Phi \in P_\tau H$ be smeared out with a test function on G such that Φ is C^∞, i.e. $g \mapsto U_n(g)\,\Phi$ is C^∞. We will show below that such $\Phi(\cdot)$ are continuous and bounded in p. Then straight forward calculation leads to

$$\left(\int da\, f(a)\,T(a)\,D(\lambda)\,\lambda^{-n}\,\Phi\right)(p) = \tilde f(p)\,\Phi(\lambda p) \tag{39}$$

and

$$\int dp\, p^{2n-1}\,|\tilde f(p)|^2\,|\Phi(\lambda p) - \Phi(0)|^2 \longrightarrow 0 \tag{40}$$

for $\lambda \downarrow 0$, showing the convergence of (36).

It remains to be shown that $\Phi(\cdot)$ is continuous and bounded. From the above assumption it follows that Φ is in the domain of definition of all powers of the conformal Hamiltonian $\mathbf{H}$. Hence, in an expansion of Φ into eigenvectors of $\mathbf{H}$,

$$\Phi = \sum_{k\ge n} c_k\Phi_k, \quad \mathbf{H}\Phi_k = k\Phi_k,\ \|\Phi_k\| = 1, \tag{41}$$

the sequence c_k is strongly decreasing. The normalized eigenfunctions are of the form

$$\Phi_k^{(n)}(p) = L_{n+k-1}^{2n-1}(2p)\,e^{-p} \tag{42}$$

with the normalized associated Laguerre polynomials L_{n+k-1}^{2n-1}. In the appendix we show

$$\sup_p|\Phi_k^{(n)}(p)| \le C\,k^n + D \tag{43}$$

with appropriate constants C and D. This directly implies continuity and boundedness of $\Phi(\cdot)$.

We thus obtained for each τ and each $\Phi \in P_\tau H \cap C^\infty$ with the complex number $\Phi(0) \neq 0$ a multiple of a unitary map $V_{\tau,\Phi}\ :\ L^2(\mathrm{R}_+, p^{2n-1}dp) \longrightarrow P_\tau H$ which is defined on the dense set $\{\tilde f|_{\mathbf{R}_+} \mid f \in \mathcal{D}(\mathbf{R})\}$ by

$$V_{\tau,\Phi}\ :\ \tilde f|_{\mathbf{R}_+} \longmapsto \Phi(0)\ |(\tilde f|_{\mathbf{R}_+})> := \lim_{\lambda\downarrow 0}\lambda^{-n}\int da\, f(a)\ T(a)\,D(\lambda)\,\Phi \tag{44}$$

and intertwines the irreducible representations of G.

We now turn to the definition of pointlike localized fields. Take a regularized local observable $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$, such that $g \mapsto \alpha_g(A)$ is $\mathcal{C}^\infty$ in the strong operator topology. Let τ be an irreducible subrepresentation of U. Then the vector $P_\tau A\Omega$ is $\mathcal{C}^\infty$. Hence, we may define densely defined operator valued distributions $\varphi^I_{\tau,A}$ on $\mathcal{D}(I)$, $I \in \mathcal{K}$ by

$$\varphi^I_{\tau,A}(f)\, B'\Omega = B'\, V_{\tau,P_\tau A\Omega}\, \hat{f}|_{\mathbf{R}_+}, \quad f \in \mathcal{D}(I),\ B' \in \mathcal{A}(I)'. \tag{45}$$

It is easy to see that the fields transform covariantly,

$$U(g)\, \varphi^I_{\tau,A}(f)\, U(g)^{-1} = \varphi^{gI}_{\tau,A}(f^{(n)}_g). \tag{46}$$

The main problem consists in proving closability of the operators $\varphi^I_{\tau,A}(f)$. This is equivalent to the existence of densely defined adjoint operators. We show that the natural candidates $\varphi^I_{\bar\tau,A^*}(\bar f)$ are indeed restrictions of the adjoint operators. This amounts to the relation

$$(\, B'\Omega\, ,\, \varphi^I_{\tau,A}(f)\, C'\Omega\,) = (\, \varphi^I_{\bar\tau,A^*}(\bar f)\, B'\Omega\, ,\, C'\Omega\,), \quad B', C' \in \mathcal{A}(I)'. \tag{47}$$

Since for all local observables A, $f \in \mathcal{D}(I)$ and sufficiently small $\lambda > 0$

$$\int_{\mathbf{R}} dx\, f(x)\, U(x)\, D(\lambda)\, A\, D(\lambda)^*\, U(x)^* \in \mathcal{A}(I), \tag{48}$$

it is sufficient to show that

$$(\, B'\Omega\, ,\, P_\tau A\Omega\,) = (\, P_{\bar\tau} A^*\Omega\, ,\, B'^*\Omega\,), \quad A \in \mathcal{A}(I),\ B' \in \mathcal{A}(I)'. \tag{49}$$

But this follows from the established relation between modular operators and conformal transformations,

$$\begin{aligned} (\, P_{\bar\tau} A^*\Omega\, ,\, B'^*\Omega\,) &= (\, P_{\bar\tau} J_I \Delta_I^{\frac12} A\Omega\, ,\, J_I \Delta_I^{-\frac12} B'\Omega\,) \\ &= (\, \Delta_I^{-\frac12} B'\Omega\, ,\, J_I P_{\bar\tau} J_I \Delta_I^{\frac12} A\Omega\,) \\ &= (\, B'\Omega\, ,\, P_\tau A\Omega\,). \end{aligned} \tag{50}$$

Moreover, $\varphi^I_{\tau,A}(f)\,\Omega \in D(\Delta_I^{\frac12})$. Hence, $\varphi^I_{\tau,A}(f)^{**}$ is affiliated to $\mathcal{A}(I)$ (cf. [DSW] and Prop. 2.5.9 in [BrR]).

It remains to be shown that for each τ there is a nonzero field $\varphi_{\tau,A}$ obtained by this construction. Let $g_y = \begin{pmatrix} y^{-1} & 0 \\ 1 & y \end{pmatrix} \in SL(2,\mathbf{R})$, $y \neq 0$. Using the representation of $P_\tau H$ as $L^2(\mathbf{R}_+, p^{2n-1}dp)$ we find

$$\Big(P_\tau\, \alpha_{g_y}(A)\,\Omega\Big)(0) = \int_0^\infty dp\, e^{ipy}\, p^{2n-1}\, (P_\tau A\Omega)(p). \tag{51}$$

The left hand side is in y the boundary value of an analytic function in the upper half plane. Therefore, it cannot vanish on an open set if $P_\tau A\Omega \neq 0$. Hence, an accidental vanishing of $P_\tau A\Omega(0)$ can be avoided by a small conformal transformation of A.

We conclude that the spaces $\varphi^I_\tau(f)\,\Omega$, τ irreducible and $f \in \mathcal{D}(I)$, are dense in H. But the algebra generated by polar and spectral decomposition of all $\varphi^I_\tau(f)^{**}$, τ irreducible and $f \in \mathcal{D}(I)$, is invariant under the modular automorphisms $\mathrm{Ad}\Delta_I^{it}$, hence coincides with $\mathcal{A}(I)$ [Tak].

The existence of sufficiently many fields such that their linear span applied to the vacuum is dense in the Hilbert space is a formulation of the existence of an operator product expansion in the Haag-Kastler framework. Actually, we derive a stronger result with local coefficients and covariance w.r.t. the modular $*$-operation S:

Theorem: Let $I \in \mathcal{K}$ and $A \in \mathcal{A}(I)$. Then for each irreducible τ the two simultaneous conditions

$$P_\tau A\Omega = \varphi_{\tau,A}(f_{\tau,A})\,\Omega \quad \text{and} \quad P_\tau A^*\Omega = \varphi_{\tau,A}(f_{\tau,A})^*\,\Omega \tag{52}$$

together fully determine the testfunction $f_{\tau,A}$:

$$\begin{aligned} f_{\tau,A}(x) &= -i^{2n-1} \int_{-\infty}^{x} dy_1 \int_{-\infty}^{y_1} dy_2 \cdots \int_{\infty}^{y_{2n-2}} dy_{2n-1}\, (\, \Omega, [\, \varphi_\tau(y_{2n-1})^*, A\,]\,\Omega\,) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp\, e^{ipx}\, \frac{\int_{-\infty}^{\infty} dy\, e^{-ipy}\, (\, \Omega, [\, \varphi_\tau(y)^*, A\,]\,\Omega\,)}{p^{2n-1}}. \end{aligned} \tag{53}$$

In particular,

$$\mathrm{supp} f_{\tau,A} \subset I. \tag{54}$$

Therefore, we obtain a local expansion

$$A = \sum_\tau \varphi^I_\tau(f_{\tau,A}) \tag{55}$$

which converges on $\mathcal{A}(I)'\Omega$ $*$-strongly (cf. the definition in [BrR]).

Proof: The formula for $f_{\tau,A}$ in (53) follows from straight forward calculation since the two conditions determine the positive and negative energy content of $f_{\tau,A}$ respectively. The conformal cluster theorem applied to the commutator function $(\,\Omega, [\,\varphi_\tau(x)^*, A\,]\,\Omega\,)$ specifies the Fourier transform $G(p)$ of the commutator function to be of the form $p^{2n-1}\,H(p)$, with an appropriate analytic function $H(p)$. Therefore, using the Paley-Wiener theorem ([Tre], theorem 29.2) we see that the support of $f_{\tau,A}(x) = \check{H}(x)$ is included in the support of the commutator function. Hence, it is included in I.

The local expansion (55) then follows directly from $\mathrm{supp} f_{\tau,A} \subset I$ and the definition of the field operators.

5 A proof independent of Borchers' theorem

In this section, we sketch an alternative proof of the results on the existence of conformal fields and on the structure of conformal Haag-Kastler nets both in this work and in [FröG, BGL] without making use of Borchers' theorem [Borl].

In this proof, the basic input will be the conformal cluster theorem (cf. section 3). This new result gives a bound for conformal two-point-functions of local observables. We need the conformal cluster theorem as a substitute for Borchers' theorem in the proof of the closability of the field operators. Deviating from the formulation in chapter 4, we now use P_n instead of P_τ in the definition of the conformal fields. In analogy to (45) and with a regularized local observable A, we obtain conformal fields

$$\varphi^I_{n,A}(f)\, B'\Omega = B'\, V_{n,P_n A\Omega}\, f, \quad f \in \mathcal{D}(I),\ B' \in \mathcal{A}(I)'. \tag{56}$$

The next theorem proves the closability of the field operators.

Theorem: Let $n \in \mathrm{N}$, $I \in \mathcal{K}$, $f \in \mathcal{D}(I)$, $B', C' \in \mathcal{A}(I)'$ and let A be a regularized local observable. We then have

$$(B'\Omega, \varphi^I_{n,A}(f)\, C'\Omega) \;=\; (\varphi^I_{n,A^*}(\bar f)\, B'\Omega, C'\Omega) \tag{57}$$
$$\varphi^I_{n,A}(f)^\dagger \;:=\; \varphi^I_{n,A}(f)^*\,|_{\mathcal{A}(I)'\Omega} \;=\; \varphi^I_{n,A^*}(\bar f). \tag{58}$$

$\varphi^I_{n,A}(f)$ is closable because $\varphi^I_{n,A}(f)^*$ has a dense domain.

Proof: The Casimir operator C_G belonging to the Lie group $SL(2,\mathrm{R})/\mathrm{Z}_2$ and its representation $U(\cdot)$ is known to have the following spectral decomposition [Lang]:

$$C_G = \sum_{i=1}^{\infty} i(i-1)\, P_i\,. \tag{59}$$

Let B_1, B_2 be regularized observables localized in disjoint intervals. As C_G is a second order differential operator in G and since B_1, B_2 commute, we obtain

$$(B_1\,\Omega, C_G\, B_2\,\Omega) = (C_G\, B_2^*\,\Omega, B_1^*\,\Omega)\,. \tag{60}$$

A priori, this is not known to be true for the individual projections P_n, which are not local.

Some algebra leads to

$$\begin{aligned}
&(B'\Omega, \varphi^I_{n,A}(f)\, C'\Omega)\\
&= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(C'^* B'\Omega, U(x)\, D(\lambda)\,\lambda^{-n} P_n A\Omega)\\
&= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(C'^* B'\Omega, U(x)\, D(\lambda)\,\lambda^{-n}\left(\prod_{i=1}^{n-1}\frac{C_G - i(i-1)}{n(n-1)-i(i-1)}\right) P_n A\Omega),\\
&= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(C'^* B'\Omega, U(x)\, D(\lambda)\,\lambda^{-n}\left(\prod_{i=1}^{n-1}\frac{C_G - i(i-1)}{n(n-1)-i(i-1)}\right)(1-\sum_{i=1}^{n-1} P_i)\, A\Omega),
\end{aligned}$$

(Because of equation (59) the polynomial in C_G has the property to act as the identity operator on P_n and as the zero operator on all $P_i, i < n$. By the conformal cluster theorem the contribution of conformal energies $\geq n+1$ vanishes in the limit $\lambda \to 0$.)

$$\begin{aligned}
&= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(C'^* B'\Omega, U(x)\, D(\lambda)\,\lambda^{-n}\left(\prod_{i=1}^{n-1}\frac{C_G - i(i-1)}{n(n-1)-i(i-1)}\right) A\Omega)\\
&= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(U(x)\, D(\lambda)\,\lambda^{-n}\left(\prod_{i=1}^{n-1}\frac{C_G - i(i-1)}{n(n-1)-i(i-1)}\right) A^*\Omega, B'^* C'\Omega)\\
&= (\varphi^I_{n,A^*}(\bar f)\, B'^*\Omega, C'\Omega)\,.
\end{aligned} \tag{61}$$

Since in this approach arbitrary multiplicities of irreducible representations in $P_n H$ might appear, we have to ensure the existence of a sufficient number of orthogonal fields with conformal dimension $n \in \mathrm{N}$. Because of the cyclicity of the vacuum vector Ω w.r.t. the set of regularized local observables, appropriate operators $A_i, i \in M \subset \mathrm{N}$, can be found to construct a dense set of vectors $\{\varphi^I_{n,A_i}(f)\,\Omega \,|\, I \in \mathcal{K}_0, f \in \mathcal{D}(I), (A_i)_{i\in M} \text{ appropriate}\}$ in $P_n H$. Since the conformal two-point-function is determined up to a complex constant, we know

$$(\,\varphi_{n,A_i}(x)\,\Omega,\; \varphi_{n,A_j}(y)\,) = c_{ij}\,(y - x + i\varepsilon)^{-2n} \tag{62}$$

with suitable $c_{ij} \in \mathrm{C}$. According to Schmidt's orthogonalization system the matrix $(c_{ij})_{i,j}$ can then be transformed into a diagonal matrix $(\tilde c_{ij})_{i,j}$. Applying this system to the observables $A_i, i \in M$, we obtain new regularized local observables $\tilde A_i, i \in M$, giving rise to a set of conformal fields $(\varphi^I_{n,\tilde A_i}(\cdot))_{i\in M}$ orthogonal on the vacuum vector.

All other properties of the field operators can be proven the same way as in chapter 4.

Once having shown the existence of conformal fields, one can use them to derive important structures of the original conformal Haag-Kastler net, again and in contrast to [FröG, BGL] without making use of Borchers' theorem. The mere existence of conformal fields will be enough to establish the Bisognano-Wichmann identification of modular structures, Haag duality, the possibility to reconstruct the algebras from the fields, PCT-covariance and additivity.

The Bisognano-Wichmann result can be derived in the same way as in [BS-M]. The proof of Haag duality cannot yet use PCT-covariance, it has to rely on the identification of $J_{\mathbf{R}_+}$ and $J_{\mathbf{R}_-}$ (cf. [Bor1]). Haag duality indicates maximality and helps to prove that the original algebras can be reconstructed from the field operators. Now, this result naturally implies PCT-covariance of the net of algebras. Finally, additivity follows easily from the fact that we did not use products of fields in the reconstruction of the algebras.

More details of the argumentation can be found in [Jör1, Jör2].

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APPENDIX

A bound for $\sup_p|\Phi_k^{(n)}(p)|$

The eigenfunction of the conformal Hamiltonian for the eigenvalue $k \geq n$ in x-space is

$$f_k^{(n)}(x) = (1+ix)^{n-k-1}(1-ix)^{n+k-1}. \tag{63}$$

Its Fourier transform may be computed by the theorem of residues and turns out to be

$$\begin{aligned}
\tilde f_k^{(n)}(p) &= \int_{\mathbf{R}} dx\, e^{ipx}\, f_k^{(n)}(x)\\
&= N_k^{(n)}\, L^{2n-1}_{n+k-1}(2p)\, e^{-p}, \quad p > 0
\end{aligned} \tag{64}$$

with the normalized associated Laguerre polynomials L^{2n-1}_{n+k-1} and constants

$$N_k^{(n)} = 2^{n-1}\left(\frac{(k+n-1)!}{(k-n)!}\right)^{\frac12}(-1)^{k+n+1}. \tag{65}$$

For a bound on its modulus we choose as integration path a circle with center iR and radius $R \geq 1$. We obtain

$$\tilde f_k^{(n)}(p) = \int_0^{2\pi} d\varphi\; iRe^{i\varphi}\,(1 - R + iRe^{i\varphi})^{n-k-1}\,(1 + R - iRe^{i\varphi})^{n+k-1}\, e^{-pR+ipRe^{i\varphi}} \tag{66}$$

and find

$$|\tilde f_k^{(n)}| \;\leq\; 2\pi R\, \sup_\varphi\left((1+2R(R-1)(1+\sin\varphi))\,(1+2R(R+1)(1+\sin\varphi))^{\frac{n-1}{2}}\right).$$

$$\cdot \sup_\varphi \left(\frac{1+2R(R+1)(1+\sin\varphi)}{1+2R(R-1)(1+\sin\varphi)} \right)^{\frac{k}{2}} \tag{67}$$

$$\leq \; 2\pi R\,(1+4R^2)^{n-1} \left(\frac{R+1}{R-1} \right)^{\frac{k}{2}} . \tag{68}$$

We insert $R = k+1$, and find

$$|\tilde{f}_k^{(n)}(p)| \leq 2\pi\,(k+1)\,(5+8k+4k^2)^{n-1}\,(1+\frac{2}{k})^{\frac{k}{2}} \leq \frac{\pi}{2} e\,(5+4k)^{2n-1} . \tag{69}$$

As a bound on the normalized eigenfunctions we obtain

$$|\Phi_k^{(n)}(p)| = \frac{|\tilde{f}_k^{(n)}(p)|}{N_k^{(n)}} \leq \frac{C\,k^{2n}+D}{k^n} \leq C\,k^n + D \tag{70}$$

with appropriate constants C and D.

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