How to solve path integrals in quantum mechanics

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A systematic classification of Feynman path integrals in quantum mechanics is presented and a table of solvable path integrals is given which reflects the progress made during the last 15 years, including, of course, the main contributions since the invention of the path integral by Feynman in 1942. An outline of the general theory is given which will serve as a quick reference for solving path integrals. Explicit formulas for the so-called basic path integrals are presented on which the general scheme to classify and calculate path integrals in quantum mechanics is based. © *1995 American Institute of Physics.*

I. INTRODUCTION

Path integrals had been introduced in physics for the first time by Feynman in his thesis.[1] By means of his path integral Feynman gave a new formulation of quantum mechanics "in which the central mathematical concept is the analog of the action in classical mechanics. It is therefore applicable to mechanical systems whose equations of motion cannot be put into Hamiltonian form. It is only required that some sort of least action principle be available.[1]" The idea to formulate quantum mechanics in terms of the Lagrangian instead of the Hamiltonian goes back to Dirac.[2] Feynman's first publication on path integrals appeared in 1948.[3] Historically of utmost importance was Feynman's generalization of the path integral to quantum electrodynamics[4] from which he derived for the first time the "Feynman rules" providing an extremely effective method for performing calculations in perturbation theory.

One of the great advantages of the path integral is that it gives a global (integral) solution of the quantum mechanical problem in question. This is in contrast to the standard approach to quantum mechanics based on the Schrödinger equation which gives a local (differential) formulation of the problem. Due to its global character, a proper definition of the path integral depends crucially on its regularization prescription and on the imposed boundary conditions. In the language of functional analysis, the question of the boundary conditions is closely related to the problem of finding the appropriate self-adjoint extension of a given Hamiltonian. The main point is that *the path integral contains all this information by its very construction!* Of course, in the evaluation of a particular path integral, one exploits information provided by functional analysis, the theory of special functions, and the theory of differential equations. Therefore the interplay between these fields and the theory of path integrals is very important to obtain useful results.

In this contribution we restrict ourselves to path integrals in quantum mechanics. Until fairly recently, only a few examples of exactly solvable path integrals were known; see the books by Feynman and Hibbs,[5] and by Schulman,[6] which give a good account of the state of art at the time of 1965 and 1981, respectively. However, the situation has drastically changed during the last 15 years, and it is no exaggeration to say that we are able to solve today essentially all path integrals in quantum mechanics which correspond to problems for which the corresponding Schrödinger equation can be solved exactly. (This, of course, excludes all classically chaotic systems.) It thus appears to us that the time has come to look for a systematic classification of path integrals in quantum mechanics. Our goal therefore is to evaluate as many path integrals as possible in order to build up quantum mechanics from the point of view of fluctuating paths. A comprehensive *Table of Feynman Path Integrals* will appear soon[7] and was already announced in Ref. 8. In this contribution we shall present the main ideas on how our classification scheme works and which

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classes of path integrals are exactly solvable. By our presentation the interested reader should be able to treat almost every path integral in quantum mechanics by a proper combination of the various methods. In addition our intention is to give a quick reference guide for solving them. For coherent state path integrals we have to refer to the literature.[9,10] In the following we are not able to give a complete list of references. A very extensive list of the literature on path integrals comprising more than 1500 papers will be given in our monography[11] which is in preparation.

II. GENERAL THEORY

A. Formulation of the path integral

Let us set up the definition of the Feynman path integral. We first consider the simple case of a classical Lagrangian $\mathscr{L}(\mathbf{x},\dot{\mathbf{x}})=(m/2)\dot{\mathbf{x}}^2-V(\mathbf{x})$ in D dimensions. Then the integral kernel $(\mathbf{x}',\mathbf{x}''\in\mathbb{R}^D,\ T=t''-t')$

$$K(\mathbf{x}'',\mathbf{x}';T)=\langle\mathbf{x}''|e^{-iH(t''-t')/\hbar}|\mathbf{x}'\rangle\Theta(t''-t') \tag{1}$$

of the time-evolution equation

$$\Psi(\mathbf{x}'',t'')=\int_{\mathbb{R}^D}K(\mathbf{x}'',\mathbf{x}';t'',t')\Psi(\mathbf{x}',t')d\mathbf{x}' \tag{2}$$

is represented in the form (Feynman path integral)

$$K(\mathbf{x}'',\mathbf{x}';T)=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int_{\mathbb{R}^D}d\mathbf{x}_k\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(\mathbf{x}_j-\mathbf{x}_{j-1})^2-\epsilon V(\mathbf{x}_j)\right]\right\} \tag{3a}$$

$$\equiv\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathscr{D}\mathbf{x}(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-V(\mathbf{x})\right]dt\right\}. \tag{3b}$$

Here we have used the abbreviations $\epsilon=(t''-t')/N\equiv T/N$, $\mathbf{x}_j=\mathbf{x}(t_j)$ $(t_j=t'+\epsilon j,\ j=0,\ldots,N)$, and we interpret the limit $N\to\infty$ as equivalent to $\epsilon\to 0$, T fixed.

The next step is to consider a generic classical Lagrangian of the form $\mathscr{L}(\mathbf{q},\dot{\mathbf{q}})=(m/2)g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b-V(\mathbf{q})$ in some D-dimensional Riemannian space $\mathbf{M}$ with line element $ds^2=g_{ab}(\mathbf{q})dq^a\,dq^b$. This case, as first systematically discussed by DeWitt,[12] requires a careful treatment. The Feynman path integral is most conveniently constructed by considering the Weyl-ordering prescription in the corresponding quantum Hamiltonian. The result then is (see, e.g., Refs. 13–17 and references therein)

$$K(\mathbf{q}'',\mathbf{q}';t'',t')=[g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int_{\mathbf{M}}d\mathbf{q}_k\cdot\prod_{l=1}^{N}\sqrt{g(\bar{\mathbf{q}}_l)}$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}g_{ab}(\bar{\mathbf{q}}_j)\Delta q_j^a\Delta q_j^b-\epsilon V(\bar{\mathbf{q}}_j)-\epsilon\Delta V(\bar{\mathbf{q}}_j)\right]\right\}, \tag{4a}$$

$$\equiv[g(\mathbf{q}')g(\mathbf{q}'')]^{-1/4}\int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathscr{D}_{MP}\mathbf{q}(t)\sqrt{g(\mathbf{q})}$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}g_{ab}(\mathbf{q})\dot{q}^a\dot{q}^b-V(\mathbf{q})-\Delta V(\mathbf{q})\right]dt\right\}. \tag{4b}$$

Here $\bar{\mathbf{q}}_j=\frac{1}{2}(\mathbf{q}_j+\mathbf{q}_{j-1})$ denotes the midpoint coordinate, $\Delta\mathbf{q}_j=(\mathbf{q}_j-\mathbf{q}_{j-1})$, and $\Delta V(\mathbf{q})$ is a well-defined "quantum potential" of order $\hbar^2$ having the form $[\Gamma_a=\partial_a \ln\sqrt{g},\ g(\mathbf{q})=\det(g_{ab})]$

$$\Delta V(\mathbf{q})=\frac{\hbar^2}{8m}[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_a)_{,b}+g^{ab}{}_{,ab}]. \tag{5}$$

The midpoint prescription (MP) together with the additional potential term ΔV is obtained in a completely natural way as an unavoidable consequence of the Weyl-ordering prescription in the corresponding quantum Hamiltonian

$$\begin{aligned}H&=-\frac{\hbar^2}{2m}\Delta_{LB}+V(\mathbf{q})=-\frac{\hbar^2}{2m}g^{-1/2}\partial_a g^{1/2}g^{ab}\partial_b+V(\mathbf{q})\\&=\frac{1}{8m}(g^{ab}p_ap_b+2p_ag^{ab}p_b+p_ap_bg^{ab})+V(\mathbf{q})+\Delta V(\mathbf{q}),\end{aligned} \tag{6}$$

where $p_a=-i\hbar(\partial_a+\frac{1}{2}\Gamma_a)$ denotes the momentum operator conjugate to the coordinate q_a in M. Of course, choosing another ordering prescription leads to a different lattice definition[18–20] in Eq. (4) and a different quantum potential ΔV. However, every consistent lattice definition of Eq. (4) can be transformed into another one by carefully expanding the relevant metric terms (integration measures and kinetic energy term).

In an alternative approach the metric tensor is assumed to be given as a product according to $g_{ab}=h_{ac}h_{cb}$.[20] Then we obtain for the Hamiltonian

$$H=-\frac{\hbar^2}{2m}\Delta_{LB}+V(\mathbf{q})=\frac{1}{2m}h^{ac}p_ap_bh^{cb}+\Delta V_{\rm PF}(\mathbf{q})+V(\mathbf{q}) \tag{7}$$

and for the path integral (PF–product-form) we obtain

$$K(\mathbf{q}'',\mathbf{q}';T)=\int_{\mathbf{q}(t')=\mathbf{q}'}^{\mathbf{q}(t'')=\mathbf{q}''}\mathcal{D}_{\rm PF}\mathbf{q}(t)\sqrt{g(\mathbf{q})}\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}h_{ac}(\mathbf{q})h_{cb}(\mathbf{q})\dot{q}^a\dot{q}^b-V(\mathbf{q})-\Delta V_{\rm PF}(\mathbf{q})\right]dt\right\} \tag{8a}$$

$$=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{k=1}^{N-1}\int d\mathbf{q}_k\sqrt{g(\mathbf{q}_k)}\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}h_{bc}(\mathbf{q}_j)h_{ac}(\mathbf{q}_{j-1})\Delta q_j^a\Delta q_j^b-\epsilon V(\mathbf{q}_j)-\epsilon\Delta V_{\rm PF}(\mathbf{q}_j)\right]\right\}. \tag{8b}$$

Here $\Delta V_{\rm PF}$ denotes the well-defined quantum potential

$$\Delta V_{\rm PF}(\mathbf{q})=\frac{\hbar^2}{8m}[g^{ab}\Gamma_a\Gamma_b+2(g^{ab}\Gamma_b)_{,b}+g^{ab}{}_{,ab}+2h^{ac}h^{bc}{}_{,ab}-h^{ac}{}_{,a}h^{bc}{}_{,b}-h^{ac}{}_{,b}h^{bc}{}_{,a}] \tag{9}$$

arising from the specific lattice formulation (8) of the path integral, respectively, the ordering prescription for position and momentum operators in the quantum Hamiltonian. We only use the lattice formulation (8) in this article unless otherwise (and explicitly) stated.

B. Transformation techniques

Indispensable tools in path integral techniques are transformation rules. In order to avoid cumbersome notation, we restrict ourselves to the one-dimensional case. For the general case we refer to Refs. 12–32 and references therein. We consider the path integral (3) and perform the coordinate transformation $x=F(q)$. Implementing this transformation, one has to keep all terms of $O(\epsilon)$ in Eq. (3). Expanding about midpoints, the result is

$$K(F(q''),F(q');T)=[F'(q'')F'(q')]^{-1/2}\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int dq_k\cdot\prod_{l=1}^{N}F'(\bar q_l)$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}F'^2(\bar q_j)(\Delta q_j)^2-\epsilon V(F(\bar q_j))-\frac{\epsilon\hbar^2}{8m}\frac{F''^2(\bar q_j)}{F'^4(\bar q_j)}\right]\right\}. \tag{10}$$

Note that the path integral (10) has the canonical form of the path integral (4). It is not difficult to incorporate the explicitly time-dependent coordinate transformation $x=F(q,t)$.[21,24] Then we obtain

$$K(F(q'',t''),F(q',t');t'',t')=A(q'',q';t'',t')\tilde K(q'',q';t'',t'), \tag{11}$$

with the prefactor $[F'(q,t)=\partial F(q,t)/\partial q,\ \dot F(q,t)=\partial F(q,t)/\partial t$, etc.]

$$A(q'',q';t'',t')=[F'(q'',t'')F'(q',t')]^{-1/2}$$
$$\times\exp\left\{\frac{im}{\hbar}\left[\int^{q''}F'(z,t'')\dot F(z,t'')dz-\int^{q'}F'(z,t')\dot F(z,t')dz\right]\right\}, \tag{12}$$

where the path integral representation for the kernel $\tilde K$ is given by $[\bar F_j=F(\bar q_j,\bar t_j),\ \bar t_j=\frac{1}{2}(t_j+t_{j-1})]$

$$\tilde K(q'',q';t'',t')=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int dq_k\cdot\prod_{l=1}^{N}\bar F'_l\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\bar F_j'^2(\Delta q_j)^2-\epsilon V(\bar F_j)\right.\right.$$
$$\left.\left.-\frac{\epsilon\hbar^2}{8m}\frac{\bar F_j''^2}{\bar F_j'^4}-\epsilon m\int^{\bar q_j}F'(z,t)\ddot F(z,t)dz\right]\right\}. \tag{13}$$

It is obvious that the path integral representation (13) is not completely satisfactory. Whereas the transformed potential $V(F(q,t))$ may have a convenient form when expressed in the new coordinate q, the kinetic term $(m/2)F'^2\dot q^2$ is in general nasty. Here the so-called "time transformation" comes into play which leads in combination with the "space transformation" already carried out to general "space–time transformations" in path integrals. The time transformation is implemented (see, e.g., Refs. 13, 21, 22, 24, 26–32 and references therein) by introducing a new "pseudotime" s''. In order to do this, one first makes use of the operator identity

$$\frac{1}{H-E}=f_r(x,t)\frac{1}{f_l(x,t)(H-E)f_r(x,t)}f_l(x,t), \tag{14}$$

where H is the Hamiltonian corresponding to the path integral $K(t'',t')$, and $f_{l,r}(q,t)$ are functions of q and t, multiplying from the left or from the right, respectively, onto the operator $(H-E)$. Secondly, one introduces a new pseudotime s'' and assumes that the constraint

$$\int_0^{s''} ds\, f_l(F(q(s),s))f_r(F(q(s),s))=T=t''-t' \tag{15}$$

has for all admissible paths a unique solution $s''>0$ given by

$$s''=\int_{t'}^{t''}\frac{dt}{f_l(x,t)f_r(x,t)}=\int_{t'}^{t''}\frac{ds}{F'^2(q(s),s)}. \tag{16}$$

Here one has made the choice $f_l(F(q(s),s))=f_r(F(q(s),s))=F'(q(s),s)$ in order that in the final result the metric coefficient in the kinetic energy term is equal to 1. A convenient way to derive the corresponding transformation formulas uses the energy dependent Green's function $G(E)$ of the kernel $K(T)$ defined by

$$G(q'',q';E)=\langle q''|\frac{1}{H-E-i\epsilon}|q'\rangle=\frac{i}{\hbar}\int_0^\infty dT\, e^{i(E+i\epsilon)T/\hbar}K(q'',q';T). \tag{17}$$

For the path integral (3) one obtains the following transformation formula (here we consider the time-independent one-dimensional case only)

$$K(x'',x';T)=\int_{-\infty}^{\infty}\frac{dE}{2\pi i}\,e^{-iET/\hbar}G(q'',q';E), \tag{18}$$

$$G(q'',q';E)=\frac{i}{\hbar}\,[F'(q'')F'(q')]^{1/2}\int_0^\infty ds''\,\hat{K}(q'',q';s''), \tag{19}$$

where the transformed path integral $\hat{K}$ is given by

$$\begin{aligned}\hat{K}(q'',q';s'')=&\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int dq_k\\&\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(\Delta q_j)^2-\epsilon F'^2(\bar{q}_j)(V(F(\bar{q}_j))-E)-\epsilon\Delta V(\bar{q}_j)\right]\right\}\\&\equiv\int_{q(0)=q'}^{q(s'')=q''}\mathscr{D}q(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{q}^2-F'^2(q)(V(F(q))-E)-\Delta V(q)\right]ds\right\},\end{aligned} \tag{20}$$

with the quantum potential ΔV defined by

$$\Delta V(q)=\frac{\hbar^2}{8m}\left(3\,\frac{F''^2}{F'^2}-2\,\frac{F'''}{F'}\right). \tag{21}$$

A rigorous lattice derivation is far from trivial and has been discussed by many authors. Recent attempts to put it on a sound footing can be found in Refs. 25, 30–32. In terms of stochastic processes the time transformation is formulated as follows:

$$\int_{\mathscr{C}(\mathrm{R},x')} \mathscr{D}W[x]\,\delta(x(t)-x')\exp\left[-\frac{i}{\hbar}\int_{t'}^{t''} V(x(t))dt\right]$$
$$=\int_{\mathrm{R}} \frac{dE}{2\pi\hbar}\, e^{iET/\hbar}\int_0^\infty ds'' \int_{\mathscr{C}(\mathrm{R},q')} \mathscr{D}W[q]\,\delta(F(q(s))-x')$$
$$\times\exp\left\{-\frac{i}{\hbar}\int_0^{s''} [F'^2(q)(V(F(q))-E)+\Delta V(q)]ds\right\}. \tag{22}$$

Here $\mathscr{C}(\mathrm{R},x')$ denotes the set of paths in R which start at x' at t', the δ functions describe the boundary condition, and $\mathscr{D}W[x]$ is the stochastic measure for the Feynman process in real time, or the Wiener process in imaginary time after a Wick rotation.

Further refinements are possible and general formulae of practical interest and importance can be derived. Let us note that also an explicitly time-dependent "space–time transformation" can be formulated similarly, cf. Refs. 7, 11, 21, 24.

By the same technique the separation of variables in path integrals can also be stated, cf. Ref. 33. Let us consider a $D=d+d'$ dimensional system, where $\mathbf{x}$ denotes the d-dimensional coordinate and $\mathbf{z}$ the d'-dimensional coordinate. For simplicity we consider the special case where the metric tensor for the $\mathbf{x}$ coordinates is equal to $f^2(\mathbf{z})\mathbb{1}$, and the metric tensor for the $\mathbf{z}$ coordinate is diagonal and denoted by $\mathbf{g}=\mathbf{g}(\mathbf{z})$ with elements $g_i=g_{ii}(\mathbf{z})$, $i=1,\dots,d'$. Furthermore, we incorporate a potential of the special form $\hat{W}(\mathbf{x},\mathbf{z})=W(\mathbf{z})+V(\mathbf{x})/f^2(\mathbf{z})$ which also includes all quantum potentials arising from metric terms. Then $(g=\Pi\, g_i^2)$

$$\int_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''} \mathscr{D}\mathbf{z}(t) f^d(\mathbf{z})\sqrt{g}\int_{\mathbf{x}(t'')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathscr{D}\mathbf{x}(t)$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\left((\mathbf{g}\cdot\dot{\mathbf{z}})^2+f^2(\mathbf{z})\dot{\mathbf{x}}^2\right)-\left(\frac{V(\mathbf{x})}{f^2(\mathbf{z})}+W(\mathbf{z})\right)\right]dt\right\}$$
$$=[f(\mathbf{z}')f(\mathbf{z}'')]^{-d/2}\int dE_\lambda\, \Psi_\lambda^*(\mathbf{x}')\Psi_\lambda(\mathbf{x}'')\int_{\mathbf{z}(t')=\mathbf{z}'}^{\mathbf{z}(t'')=\mathbf{z}''} \mathscr{D}\mathbf{z}(t)\sqrt{g}$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\mathbf{g}\cdot\dot{\mathbf{z}})^2-W(\mathbf{z})-\frac{E_\lambda}{f^2(\mathbf{z})}\right]dt\right\}. \tag{23}$$

Here we assumed that the d-dimensional $\mathbf{x}$-path integration has the spectral representation

$$\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathscr{D}\mathbf{x}(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-V(\mathbf{x})\right]dt\right\}=\int dE_\lambda\, \Psi_\lambda^*(\mathbf{x}')\Psi_\lambda(\mathbf{x}'')e^{-iE_\lambda T/\hbar}. \tag{24}$$

C. Group path integration

We consider the Lagrangian $\mathscr{L}(\mathbf{x},\dot{\mathbf{x}})=(m/2)g_{ab}\dot{x}^a\dot{x}^b-V(\mathbf{x})$ $(\mathbf{x}\in\mathrm{R}^{p+q})$ as formulated, say, in a not necessarily positive definite space with signature

$$(g_{ab})=\mathrm{diag}(\underbrace{+1,\dots,+1}_{p\ \text{times}},\underbrace{-1,\dots,-1}_{q\ \text{times}}). \tag{25}$$

We want to evaluate the corresponding path integral in an abstract way by exploiting path integration over group manifolds. In order to illustrate the general method, let us assume that $\mathbf{x}\in\mathrm{R}^D$ and $V(\mathbf{x})=V(|\mathbf{x}|)$. We consider the short-time kernel of (3) $(\epsilon\to 0)$

$$K(\mathbf{x}_j,\mathbf{x}_{j-1};\epsilon)=\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{D/2}\exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\epsilon}(\mathbf{x}_j-\mathbf{x}_{j-1})^2-\epsilon V(|\mathbf{x}_j|)\right]\right\}. \tag{26}$$

Next we introduce D-dimensional polar coordinates with polar variable r and angular variables $\theta_1,\ldots,\theta_{D-2}$, ϕ.[13,34] Then we have $V(|\mathbf{x}_j|)=V(r_j)$ and $\mathbf{x}_{j-1}\cdot\mathbf{x}_j=r_{j-1}r_j\cos\Theta_{j-1,j}$, where $\Theta_{j-1,j}$ is the angle between the two vectors $\mathbf{x}_{j-1}$ and $\mathbf{x}_j$ which can be expressed by means of the addition theorem for polar coordinates in D dimensions in terms of the angular variables. We now seek an expansion of the function $e^{z\cos\Theta_{j-1,j}}$ in terms of the angular variables. This expansion is constructed in two steps. First we use the formula (Ref. 35, Chap. IX)

$$e^{z\cos\Theta}=\left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)\sum_{l=0}^{\infty}(l+\nu)I_{l+\nu}(z)C_l^{\nu}(\cos\Theta), \tag{27}$$

with $\nu=(D-2)/2$, where C_l^{ν} are Gegenbauer polynomials and I_μ a modified Bessel function. The addition theorem for the M linearly independent real surface (or hyperspherical) harmonics S_l^{μ} of degree l on the S^{D-1}-sphere has the form

$$\sum_{\mu=1}^{M}S_l^{\mu}(\mathbf{\Omega}_1)S_l^{\mu}(\mathbf{\Omega}_2)=\frac{1}{\Omega(D)}\frac{2l+D-2}{D-2}C_l^{(D-2)/2}(\cos\Theta_{1,2}). \tag{28}$$

Here $\mathbf{\Omega}=\mathbf{x}/r$ denotes a unit vector in R^D, $\Omega(D)=2\pi^{D/2}/\Gamma(D/2)$ the volume of the D-dimensional unit sphere, and $M=(2l+D-2)(l+D-3)!/l!(D-3)!$. The orthonormality relation is

$$\int d\mathbf{\Omega}\, S_l^{\mu}(\mathbf{\Omega})S_{l'}^{\mu'}(\mathbf{\Omega})=\delta_{ll'}\delta_{\mu\mu'}. \tag{29}$$

As a result we get the expansion formula

$$e^{z\cos\Theta_{1,2}}=2\pi\left(\frac{2\pi}{z}\right)^{(D-2)/2}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M}S_l^{\mu}(\mathbf{\Omega}_1)S_l^{\mu}(\mathbf{\Omega}_2)I_{l+(D-2)/2}(z). \tag{30}$$

Insertion into the path integral yields the "partial wave expansion"

$$\begin{aligned}K(\mathbf{x}'',\mathbf{x}';T)&=K(r'',r',\mathbf{\Omega}'',\mathbf{\Omega}';T)\\&=(r'r'')^{(2-D)/2}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M}S_l^{\mu}(\mathbf{\Omega}')S_l^{\mu}(\mathbf{\Omega}'')\lim_{N\to\infty}\left(\frac{m}{i\epsilon\hbar}\right)^{N}\prod_{k=1}^{N-1}\int_0^{\infty}r_k\,dr_k\\&\quad\times\prod_{j=1}^{N}\exp\left[\frac{im}{2\epsilon\hbar}(r_j^2+r_{j-1}^2)-\epsilon\frac{i}{\hbar}V(r_j)\right]I_{l+(D-2)/2}\left(\frac{m}{i\epsilon\hbar}r_jr_{j-1}\right)\end{aligned} \tag{31}$$

$$=(r'r'')^{(1-D)/2}\sum_{l=0}^{\infty}\sum_{\mu=1}^{M}S_l^{\mu}(\mathbf{\Omega}')S_l^{\mu}(\mathbf{\Omega}'')K_{l+(D-2)/2}(r'',r';T), \tag{32}$$

where the *radial path integral* is given by

$$K_{l+(D-2)/2}(r'',r';T)=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int_0^\infty dr_k\prod_{p=1}^{N}\mu_{l+(D-2)/2}[r_pr_{p-1}]$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(\Delta r_j)^2-\epsilon V(r_j)\right]\right\}\tag{33}$$
$$\equiv\int_{r(t')=r'}^{r(t'')=r''}\mathscr{D}r(t)\mu_{l+(D-2)/2}[r^2]\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot r^2-V(r)\right]dt\right\}.\tag{34}$$

The nontrivial functional weight is defined as

$$\mu_l[r_jr_{j-1}]=\sqrt{2\pi z_j}e^{-z_j}I_l(z_j),\tag{35}$$

with $z_j=mr_jr_{j-1}/i\epsilon\hbar$. Therefore we have achieved a twofold result. On the one hand we have expanded the exponential $e^{z\cos\Theta}$ in terms of the spherical harmonics, i.e., the matrix elements of the group SO(D), and on the other we have separated the D-dimensional path integral into an angular part and a radial path integral. The appearance of the modified Bessel function is interpreted as a nontrivial functional weight in the radial path integration of Refs. 13,34. Of course, the radial path integral cannot be further evaluated if the potential $V(r)$ is not specified. An example is given in Sec. III B.

This procedure can now be put into a more general context.[36] We consider the generic Lagrangian $\mathscr{L}(\mathbf{x},\dot{\mathbf{x}})=(m/2)g_{ab}\dot x^a\dot x^b-V(\mathbf{x})$ $(\mathbf{x}\in\mathbb{R}^{p+q})$ and its corresponding short-time kernel $K(\mathbf{x}_j,\mathbf{x}_{j-1};\epsilon)$. The short-time kernel is evaluated by harmonic analysis with respect to the symmetry group of the Lagrangian. This is usually a Lie group. In order to do this one seeks for an expansion of $e^{z\mathbf{x}_{j-1}\cdot\mathbf{x}_j}$ in terms of representations of the group. This may be done in generalized polar coordinates involving generalized spherical harmonics. We assume that we can introduce a generalized polar variable τ and a set of generalized angular variables $\{\theta\}$ such that $x_\nu=\tau\hat e_\nu(\theta_1,\dots,\theta_{p+q-1})$ $(\nu=1,\dots,p+q)$, where the $\hat{\mathbf{e}}$'s are unit vectors in some suitably chosen (timelike, spacelike, or lightlike) set[36] with $V(\mathbf{x})=V(\tau)$, say. To perform the integration over the spherical harmonics the scalar product $\mathbf{x}_{j-1}\cdot\mathbf{x}_j$ must be rewritten in terms of a group element, say, a function $f(g_{j-1}^{-1}g_j)$, such that $e^{z\mathbf{x}_{j-1}\cdot\mathbf{x}_j}=e^{zf(g_{j-1}^{-1}g_j)}$. Since $g=g_{j-1}^{-1}g_j$ is a group element we set $F(g)=e^{zf(g)}$. The expansion then yields

$$F(g)=\int dE_\lambda\, d_\lambda\sum_m\hat F_m^\lambda(\tau)D_m^\lambda(g),\quad \hat F_m^\lambda=\int_G F(g)D_m^{\lambda*}(g^{-1})dg,\tag{36}$$

where dg is the invariant group (Haar) measure. $\int dE_\lambda$ stands for a Lebesgue–Stieltjes integral which includes discrete ($\int dE_\lambda\to\Sigma_\lambda$) as well as continuous representations. The summation index m may be a multi-index. d_λ denotes (in the compact case) the dimension of the representation; otherwise we take

$$d_\lambda\int_G D_m^\lambda(g)D_{m'}^{\lambda'*}(g)dg=\delta(\lambda,\lambda')\delta_{m,m'}\tag{37}$$

as a definition for d_λ. $\delta(\lambda,\lambda')$ can denote a Kronecker delta or a δ function, depending on whether the variable λ is a discrete or continuous parameter. For instance, in D-dimensional polar coordinates the functions D_{0m}^l $(l\in\mathbb{N}_0,m\in\mathbb{Z})$ are called associated spherical harmonics, and the D_{00}^l $(l\in\mathbb{N}_0)$ are the zonal harmonics. For the path integral we obtain

$$K(\mathbf{x}'',\mathbf{x}';T)=K(\tau'',\tau',\{\theta'',\theta'\};T)=\int dE_\lambda\, d_\lambda \sum_m D^\lambda_m(g'^{-1}g'')K_{\lambda,m}(\tau'',\tau';T), \tag{38}$$

$$K_{\lambda,m}(\tau'',\tau;T)=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int d\tau_k\prod_{l=1}^{N}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{(p-1/2)/2}\left(\frac{im}{2\pi\epsilon\hbar}\right)^{q/2}\hat{F}^\lambda_n(z_l)e^{-z_l}$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}(\Delta\tau_j)^2-\epsilon V(\tau_j)\right]\right\}, \tag{39}$$

with $z_l=m\tau_l\tau_{l-1}/i\epsilon\hbar$. We see that $(m/2\pi i\epsilon\hbar)^{(p-1/2)/2}(im/2\pi\epsilon\hbar)^{q/2}\hat{F}^\lambda_m[z]e^{-z}$ plays the role of a generalized functional weight in the path integral. The path integration over the group elements could be performed due to their orthonormality. Choosing a particular basis in the group it is then possible to expand the $D^\lambda_m(g'^{-1}g'')$ in terms of the wave functions $\Psi^\lambda_{mn}(\{\theta\})$, corresponding to the Casimir operator of the group. In Ref. 36 the authors concentrated on the cases where the harmonic analysis can be performed either with the $D^\lambda_m(g)$ as the characters of the group or the zonal spherical functions. However, the method is more general. In the case of SO(D) the Casimir operator is the Legendre operator and the wave functions are the hyperspherical harmonics $S_l(\mathbf{\Omega})$ which are products of Gegenbauer polynomials.

Using the technique of group path integration it is possible to choose different coordinate space representations to derive various path integral identities. Examples are the path integral identity for the Pöschl–Teller and modified Pöschl–Teller potentials.[36]

Another aspect of group path integration is the so-called interbasis expansion for problems which are separable in more than one coordinate system. In the case of potential problems, these potential are called superintegrable. This property is very closely connected with the necessary condition that such problems have several integrals of motion, and that the underlying dynamical symmetry group allows the representation of the problem in various coordinate space representations. Superintegrable systems can be found in Euclidean space, as well as in spaces of constant curvature. The basic formula is quite simple being

$$|\mathbf{k}\rangle=\int d\mathbf{l}\; C_{l,k}|\mathbf{l}\rangle, \tag{40}$$

where $|\mathbf{k}\rangle$ stands for a basis of eigenfunctions of the Hamiltonian in the coordinate space representation $\mathbf{k}$, and $\int d\mathbf{l}$ is the expansion with respect to the coordinate space representation $\mathbf{l}$ with coefficients $C_{\mathbf{l},\mathbf{k}}$ which can be discrete, continuous, or both. The main difficulty is, in case one has two coordinate space representations in the quantum numbers $\mathbf{k}$ and $\mathbf{l}$, respectively, to find the expansion coefficients $C_{\mathbf{l},\mathbf{k}}$. Well known are the expansions which involve Cartesian coordinates and polar coordinates. In the simple case of free quantum motion in Euclidean space, this means that exponentials representing plane waves are expanded in terms of Bessel functions and spherical waves (a discrete interbasis expansion), see Eq. (27).

This general method of changing a coordinate basis in quantum mechanics can now be used in the path integral. We assume that we can expand the short-time kernel, respectively, the exponential $e^{z\mathbf{x}_{j-1}\cdot\mathbf{x}_j}$, in terms of matrix elements of a group according to Eq. (36). Here a specific coordinate basis has been chosen. We then can change the coordinate basis by means of Eq. (40). Due to the unitarity of the expansion coefficients $C_{\mathbf{l},\mathbf{k}}$ the short-time kernel is expanded in the new coordinate basis, and the orthonormality of the basis allows to perform explicitly the path integral, exactly in the same way as in the original coordinate basis.

From the two (or more) different equivalent coordinate space representations, formulas and path integral identities can be derived. These identities actually correspond to integral and summation identities, respectively, between special functions. The case of the expansion from Cartesian coordinates to polar coordinates has been studied by Peak and Inomata[37] with their solution of the radial harmonic oscillator. The path integral solution of the radial harmonic oscillator in turn enables one to calculate numerous path integral problems related to the radial harmonic oscillator, actually problems which are of the so-called Berselian type, including the radial Coulomb problem.

Other expansions are not so well known. This is mostly due to the fact that either the involved coordinate systems are not familiar, or that the expansion coefficients have a complicated structure. Let us, for instance, consider the following two problems:

(i) *Free motion in* $\mathbb{R}^2$ *in elliptic coordinates*. One makes use of the expansion (Ref. 38, p. 185, $h=pd/2$)

$$\exp[ip(x\cos\alpha+y\sin\alpha)]=2\sum_{n=0}^{\infty} i^n ce_n(\alpha;h^2)M_n^{(1)}(\mu;h)ce_n(\nu;h^2)$$
$$+2\sum_{n=1}^{\infty} i^{-n} se_n(\alpha;h^2)M_{-n}^{(1)}(\mu;h)se_n(\nu;h^2), \tag{41}$$

where the se_n, ce_n, and $M_n^{(1)}$ are Matthieu functions and $\mu\geqslant 0$, $0\leqslant\nu\leqslant\pi$ the elliptic coordinates in $\mathbb{R}^2$ ($x=d\cosh\mu\cos\nu$, $y=d\sinh\mu\sin\nu$, and d is the interfocus distance). Performing the group path integration using Eq. (41) yields the path integral identity [$me_n(z)=\sqrt{2}ce_n(z)$, $me_{-n}(z)=-i\sqrt{2}se_n(z)$] (Refs. 39 and 40)

$$\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)d^2(\sinh^2\mu+\sin^2\nu)$$
$$\times\exp\left[\frac{im}{2\hbar}d^2\int_{t'}^{t''}(\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2)dt\right]$$
$$=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^\infty p\,dp\,e^{-i\hbar p^2T/2m}$$
$$\times M_n^{(1)*}\left(\mu';\frac{dp}{2}\right)M_n^{(1)}\left(\mu'';\frac{dp}{2}\right)me_n^*\left(\nu';\frac{d^2p^2}{4}\right)me_n\left(\nu'';\frac{d^2p^2}{4}\right). \tag{42}$$

(ii) *Free motion in* $\mathbb{R}^3$ *in prolate spheroidal coordinates*. One makes use of the expansion (Ref. 38, p. 315)

$$\exp[i\,pd(\sinh\mu\sin\nu\sin\theta\cos\phi+\cosh\mu\cos\nu\cos\theta)]$$
$$=\sum_{l=0}^{\infty}\sum_{h=-l}^{l}(2l+1)i^{l+2n}e^{in\phi}S_l^{n(1)}(\cosh\mu;pd)ps_l^n(\cos\nu;p^2d^2)ps_l^{-n}(\cos\theta;p^2d^2). \tag{43}$$

$\mu\geqslant 0$, $0\leqslant\nu\leqslant\pi$, $0\leqslant\phi<2\pi$ are prolate spheroidal coordinates, and $S_l^{n(1)}$, ps_l^n are spheroidal functions. d is the interfocus distance. Again performing the group path integration in terms of the spheroidal wave functions yields[39,40]

$$\int_{\mu(t')=\mu'}^{\mu(t'')=\mu''}\mathscr{D}\mu(t)\int_{\nu(t')=\nu'}^{\nu(t'')=\nu''}\mathscr{D}\nu(t)d^3(\sinh^2\mu+\sin^2\nu)\sinh\mu\,\sin\nu\int_{\phi(t')=\phi'}^{\phi(t'')=\phi''}\mathscr{D}\phi(t)$$
$$\times\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}d^2((\sinh^2\mu+\sin^2\nu)(\dot\mu^2+\dot\nu^2)\right.\right.$$
$$\left.\left.+\sinh^2\mu\,\sin^2\nu\dot\phi^2)+\frac{\hbar^2}{8md^2\sinh^2\mu\,\sin^2\nu}\right]dt\right\}$$
$$=\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\frac{2l+1}{2\pi^2}\frac{(l-n)!}{(l+n)!}e^{in(\phi''-\phi')}\int_0^\infty p^2\,dp\,e^{-i\hbar p^2T/2m}$$
$$\times ps_l^n(\cos\nu';p^2d^2)ps_l^n(\cos\nu'';p^2d^2)S_l^{n(1)*}(\cosh\mu',pd)S_l^{n(1)}(\cosh\mu'',pd). \qquad (44)$$

Further examples are the path integral representations (see, e.g., Refs. 39–41)

(iii) *for elliptic coordinates on the sphere* $S^{(2)}$,
(iv) *for elliptic cylindrical coordinates on the sphere* $S^{(3)}$,
(v) *for pseudoelliptic coordinate systems on the pseudosphere* $\Lambda^{(2)}$,
(vi) *for pseudoelliptic cylindrical coordinate systems on the pseudosphere* $\Lambda^{(3)}$,
(vii) *for the harmonic oscillator in* $\mathbb{R}^2$ *in elliptic coordinates, and*
(viii) *for the generalized Kepler–Coulomb problem in prolate spheroidal coordinates.*

D. Klein–Gordon particle

The path integral formulation of a Klein–Gordon particle was already presented by Feynman[42] in one of his classical articles. It goes as follows: One considers the Green function corresponding to the Klein–Gordon equation

$$(\Box+M^2)G(x'',x')=-\delta(x''-x'), \qquad (45)$$

where $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu=\partial_t^2-\Delta$ is the Klein–Gordon operator, and $\delta(x)$ the four-dimensional δ function. M is the mass of the particle. According to Ref. 42, 43 we can now write $G(x'',x')$ as

$$G(x'',x')=\frac{i}{2\hbar}\int_0^\infty d\tau\,e^{-iM^2\tau/2\hbar}K(x'',x';\tau), \qquad (46)$$

where $0<s<\tau$ is a new timelike variable ("fifth parameter"). The new propagator $K(x'',x';\tau)$ describes time evolution in τ from x' to x'' and has the following path integral representation:

$$K(x'',x';\tau)=\int_{x(0)=x'}^{x(\tau)=x''}\mathscr{D}x(s)\exp\left(\frac{i}{2\hbar}\int_0^\tau g_{\mu\nu}\dot x^\mu\dot x^\nu\,ds\right). \qquad (47)$$

This path integral satisfies the Schrödinger-like equation

$$i\,\frac{\partial K(x'',x';\tau)}{\partial\tau}=-\Box_{x''}K(x'',x';\tau) \qquad (48)$$

together with the initial condition $\lim_{\tau\to0}K(x'',x';\tau)=\delta(x''-x')$. Therefore, the propagator can be seen as a usual quantum mechanical path integral, defined on a four-dimensional manifold with metric $g_{\mu\nu}$. Potentials and magnetic fields can be incorporated in an obvious way.

E. Dirac particle

We cite the path integral representation for the one-dimensional Dirac particle (Refs. 5, 44–46) ($p_x=-i\hbar\partial_x$)

$$\mathbf{K}(x'',x';T)=\langle x''|\exp\left[-\frac{i}{\hbar}T(\boldsymbol{\sigma}_{\mathbf{x}}\,p_x+mc^2\boldsymbol{\sigma}_{\mathbf{z}}+\mathbf{V}(x))\right]|x'\rangle$$

$$=\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}\nu(t)\exp\left(-\frac{i}{\hbar}\int_{t'}^{t''}\mathbf{V}(x)dt\right). \tag{49}$$

$\mathbf{V}$ may be a matrix-valued potential. The support property of the measure $\mathscr{D}\nu$ is defined in such a way that it selects paths of N steps each of length $c\epsilon$ ($\epsilon=T/N$ in the lattice representation) that start at x' in the direction α, and end at x'' in the direction β, where α and β take the values "right" and "left." The path integration then is a summation over all reversings of directions (Ref. 5). $\boldsymbol{\sigma}_{\mathbf{x}}$ and $\boldsymbol{\sigma}_{\mathbf{z}}$ are the Pauli matrices. Simple applications are the free particle,[5,46] and a point interaction.[47]

F. The fermionic path integral

The fermionic path integral in the coherent state representation is defined as follows[10] (for simplicity we restrict ourselves to a Fermi system with a single spin variable):

$$K(\bar\eta'',\eta';T)=\int_{\eta(t')=\eta'}^{\bar\eta(t'')=\bar\eta''}\mathscr{D}\bar\eta(t)\mathscr{D}\eta(t)\exp\left\{\bar\eta''\eta(t'')+\frac{i}{\hbar}\int_{t'}^{t''}[i\bar\eta(t)\dot\eta(t)-H(\bar\eta,\eta;t)]dt\right\}$$

$$=\lim_{N\to\infty}\prod_{k=1}^{N-1}\int d\bar\eta_{N-k}\,d\eta_{N-k}$$

$$\times\exp\left\{\bar\eta_N\eta_N+\frac{i}{\hbar}\sum_{j=1}^{N}[i\bar\eta_j(\eta_j-\eta_{j-1})-\epsilon H(\bar\eta_j,\eta_{j-1};t)]\right\}. \tag{50}$$

Here η_j and $\bar\eta_j$ denote Grassmann variables satisfying the anticommutation relations $\{\eta_k,\eta_l\}=\{\bar\eta_k,\bar\eta_l\}=\{\bar\eta_k,\eta_l\}=0$ for all l and k. The boundary conditions are imposed by requiring $\eta(t)$ to be fixed at $t=t'$, $\eta(t')=\eta'$, and $\bar\eta$ to be fixed at $t=t''$, $\bar\eta(t'')=\bar\eta''$. $H(\bar\eta,\eta;t)$ is obtained from a given Hamiltonian $H(a^\dagger,a;t)$ in "normal ordered form" by replacing the fermion creation and annihilation operators according to $a^\dagger\to\bar\eta$, $a\to\eta$.[9]

G. Perturbation expansions

The general method for the time-ordered perturbation expansion is quite simple. Let us assume that we are given a potential $W(x)=V(x)+\tilde V(x)$ in the path integral and suppose that W is so complicated that a direct path integration is not possible. However, the path integral $K^{(V)}$ corresponding to $V(x)$ is assumed to be known. We expand the integrand of the path integral containing $\tilde V(x)$ in a perturbation expansion about $V(x)$. The result has a simple interpretation on the lattice: the initial kernel corresponding to V propagates during the short-time interval ϵ unperturbed, then it interacts with $\tilde V$ in order to propagate again in another short-time interval ϵ unperturbed, and so on, up to the final state. One then obtains the following series expansion (Refs. 5, 47–56) ($x\in\mathbb{R}$):

$$K(x'',x';T)=K^{(V)}(x'',x';T)+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\left(\prod_{j=1}^{n}\int_{t'}^{t_{j+1}}dt_j\int_{-\infty}^{\infty}dx_j\right)$$
$$\times K^{(V)}(x_1,x';t_1-t')\tilde{V}(x_1)K^{(V)}(x_2,x_1;t_2-t_1)\cdots$$
$$\times\tilde{V}(x_{n-1})K^{(V)}(x_n,x_{n-1};t_n-t_{n-1})\tilde{V}(x_n)K^{(V)}(x'',x_n;t''-t_n). \tag{51}$$

Here we have ordered time as $t'=t_0<t_1<t_2<\cdots<t_{n+1}=t''$ and paid attention to the fact that $K(t_j-t_{j-1})$ denotes the retarded propagator and thus is different from zero only if $t_j\geqslant t_{j-1}$. Several problems in path integration which are definitely non-Gaussian, non-Besselian, or non-Legendrian can be addressed by a perturbation expansion approach. Let us mention the incorporation of point interactions[47,51–56] and boundary conditions at finite distances. Also $1/r$ (Refs. 48, 52) and $1/r^2$-potentials[52] can be treated.

A specific kind of a perturbation expansion was developed by Devreese *et al.*[48–50] by performing a Fourier transformation of the potential which enables one to make an exact path integration of the emerging quadratic Lagrangian problem. One obtains the infinite series $(c>1)$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\, e^{sT/\hbar}\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathscr{D}\mathbf{x}(t)\exp\left\{-\frac{1}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2+V(\mathbf{x})\right]dt\right\}$$
$$=\sum_{n=0}^{\infty}(-1)^n\int_{\mathrm{R}^D}\frac{d\mathbf{k}_0}{(2\pi\hbar)^D}\int_{\mathrm{R}^D}\frac{d\mathbf{k}_1}{(2\pi\hbar)^D}\hat{V}(\mathbf{k}_1)\cdots\int_{\mathrm{R}^D}\frac{d\mathbf{k}_n}{(2\pi\hbar)^D}\hat{V}(\mathbf{k}_n)$$
$$\times\frac{\exp((i/\hbar)\mathbf{x}'\cdot\Sigma_{j=1}^{n}\mathbf{k}_j-(i/\hbar)\mathbf{x}''\cdot\mathbf{k}_0)}{[s+(\mathbf{k}_0^2/2m)]\cdots[s+(\mathbf{k}_0+\cdots+\mathbf{k}_n)^2/2m]}$$
$$=\sum_{n=0}^{\infty}(-1)^n\int_{\mathrm{R}^D}\frac{d\mathbf{k}_0}{(2\pi\hbar)^D}\prod_{j=1}^{n}\int_{\mathrm{R}^D}\frac{d\mathbf{k}_j}{(2\pi\hbar)^D}\hat{V}(\mathbf{k}_j-\mathbf{k}_{j-1})\frac{\exp[(i/\hbar)(\mathbf{x}'\cdot\mathbf{k}_n-\mathbf{x}''\cdot\mathbf{k}_0)]}{(s+\mathbf{k}_0^2/2m)\cdots(s+\mathbf{k}_n^2/2m)}. \tag{52}$$

$\hat{V}(\mathbf{k})$ is the Fourier transform of the potential $V(\mathbf{x})$.

III. BASIC PATH INTEGRALS

In this section we present the path integrals which we consider as the basic path integrals.

A. Path integral for the harmonic oscillator and related path integrals

The first elementary example is the path integral for the harmonic oscillator. It has been first evaluated by Feynman.[3] We have the identity $(x\in\mathrm{R})$

$$\int_{x(t')=x'}^{x(t'')=x''} \mathscr{D}x(t) \exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(\dot{x}^2-\omega^2x^2)dt\right]$$
$$= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}} \exp\left\{\frac{im\omega}{2\hbar}\left[(x'^2+x''^2)\cot\omega T-\frac{2x'x''}{\sin\omega T}\right]\right\} \tag{53}$$

$$= \sum_{n\in\mathbb{N}_0} e^{-i\omega T(n+1/2)}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{1}{2^n n!}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x'\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}x''\right)$$
$$\times\exp\left(-\frac{m\omega}{2\hbar}(x'^2+x''^2)\right). \tag{54}$$

The expansion into wave functions has been achieved by means of the Mehler formula; $H_n(x)$ denotes the Hermite polynomials.

The path integral for general quadratic Lagrangians can also be stated exactly as ($\mathbf{x}\in\mathbb{R}^D$)

$$\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathscr{D}\mathbf{x}(t)\exp\left(\frac{i}{\hbar}\int_{t'}^{t''}\mathscr{L}(\mathbf{x},\dot{\mathbf{x}})dt\right)$$
$$=\left(\frac{1}{2\pi i\hbar}\right)^{D/2}\sqrt{\det\left(-\frac{\partial^2 S_{Cl}[\mathbf{x}'',\mathbf{x}']}{\partial x_a''\partial x_b'}\right)}\exp\left(\frac{i}{\hbar}S_{Cl}[\mathbf{x}'',\mathbf{x}']\right). \tag{55}$$

Here $\mathscr{L}(\mathbf{x},\dot{\mathbf{x}})$ denotes any classical Lagrangian at most quadratic in $\mathbf{x}$ and $\dot{\mathbf{x}}$ and $S_{Cl}[\mathbf{x}'',\mathbf{x}'] = \int_{t'}^{t''}\mathscr{L}(\mathbf{x}_{Cl},\dot{\mathbf{x}}_{Cl})dt$ the corresponding classical action evaluated along the classical solution $\mathbf{x}_{Cl}$ satisfying the boundary conditions $\mathbf{x}_{Cl}(t')=\mathbf{x}'$, $\mathbf{x}_{Cl}(t'')=\mathbf{x}''$ (here we assume that the classical dynamics allows only a single classical path). The determinant appearing in Eq. (55) is known as the van Vleck–Pauli–Morette determinant (see, e.g., Refs. 12, 57 and references therein). The explicit evaluation of $S_{Cl}[\mathbf{x}'',\mathbf{x}']$ may have any degree of complexity due to complicated classical solutions of the Euler–Lagrange equations as the classical equations of motion. The path integral (55) includes the following important cases:

1. *The linear potential*
2. *The (time-dependent forced) harmonic oscillator*
3. *A particle in a crossed time-dependent electric and magnetic field*
4. *Oscillators with magnetic fields*
5. *Oscillators with friction*
6. *Coupled oscillators*
7. *Penning trap potential*
8. *Oscillators with two-time actions*
9. *Second derivative Lagrangians, i.e., with a term $(\kappa/2)\ddot{x}^2$ which can describe stiffness of polymers or surface tensions in statistical mechanics*
10. *The periodic-orbit theory of Gutzwiller*[58,59]

Furthermore, the formula for the general quadratic Lagrangian (55) serves as a starting point for the semiclassical expansion, and the general moments formula in the path integral.[60–62] Let us just mention the semiclassical expansion formula as derived in Refs. 60, 61. It has the form

$$K(x'',x';T)=K_{\mathrm{WKB}}(x'',x';T)\Bigg(1+\sum_{j=1}^{\infty}\frac{1}{j!}\left(-\frac{i}{\hbar}\right)^j\sum_{n_1=3}^{\infty}\cdots\sum_{n_j=3}^{\infty}\int_0^T\frac{dt_1\cdots dt_j}{n_1!\cdots n_j!}$$

$$\times V^{(n_1)}(t_1)\cdots V^{(n_j)}(t_j)\int_{\mathbb{R}^{n_j}}\frac{d\mathbf{u}}{(2\pi i\hbar)^{n_j}\sqrt{\det\mathbf{W}\det\mathbf{S}}}x^{n_1}\cdots x^{n_j}$$

$$\times\exp\left\{\frac{i}{2\hbar}(\mathbf{u}-\mathbf{a})^{\dagger}[\mathbf{W}^{-1}+\mathbf{W}^{-1}\mathbf{S}\mathbf{C}^{-1}\tilde{\mathbf{C}}\mathbf{W}^{-1}](\mathbf{u}-\mathbf{a})\right\}\Bigg), \tag{56}$$

where $K_{\mathrm{WKB}}(T)$ is the semiclassical kernel as given, e.g., by Eq. (55), and $\mathscr{L}$ has been expanded up to second order in coordinates and velocities. The following abbreviations have been used:

$$a_i=\langle\mu_i,\bar{q}\rangle,\quad b_i=\langle\nu_i,\bar{p}\rangle, \tag{57}$$

$$\left.\begin{aligned}W_{ij}&=\int_0^T\int_0^T G_{ab}(t,t')d\mu_i(t)d\mu_j(t')\\ C_{ij}&=\int_0^T\int_0^T \bar{G}(t,t')d\mu_i(t)d\mu_j(t')\\ V_{ij}&=\int_0^T\int_0^T G_p(t,t')d\nu_i(t)d\nu_j(t')\end{aligned}\right\} \tag{58}$$

$$\mathbf{S}=\mathbf{V}-\tilde{\mathbf{C}}\mathbf{W}^{-1}\mathbf{C},\quad \tilde{\mathbf{C}}=\mathbf{C}^t. \tag{59}$$

Here $\bar{q}$ and $\bar{p}$ denote an average over the classical paths. μ, ν are integration measures, $\langle\cdot,\cdot\rangle$ is a scalar product with respect to these measures, and

$$\mathscr{G}(t,t')=\begin{pmatrix}G_{ab}(t,t') & \bar{G}(t,t')\\ \bar{G}(t',t) & G_p(t,t')\end{pmatrix}$$

is the Feynman Green function, i.e., the Green function of the small disturbance operator in phase space.

Based on the solution of the harmonic oscillator and the quadratic Lagrangian, respectively, it is possible to derive expressions for the generating functional[63] in a perturbative approach which is also applicable in quantum field theory (Feynman graphs!). They are based on the moments formula for arbitrary functionals F of positions and momenta (the analog of Wick's theorem in quantum mechanics).[61] Some important moments formulas can also be found in Ref. 62.

Furthermore very satisfying expressions exist for the trace of the Euclidean time-evolution kernel, i.e., the partition function in terms of an effective potential[5,28,64]

$$\int_{\mathbb{R}} dx_0\, K(x_0,x_0;-iT)=\sum_n e^{-E_nT/\hbar}\equiv \oint \mathscr{D}x(t)\exp\left\{-\frac{1}{\hbar}\int_0^T\left[\frac{m}{2}\dot{x}^2+V(x)\right]dt\right\}$$
$$=\sqrt{\frac{m}{2\pi\hbar T}}\int_{-\infty}^{\infty} dx_0\, e^{-TW_1(x_0)/\hbar}. \tag{60}$$

The effective potential $W_1(x_0)$ is evaluated in the following way. One considers the smeared version of the potential $V(x)$ according to

$$V_{a^2(x_0)}(x_0)=\int_{-\infty}^{\infty}\frac{dx}{2\pi a^2}\, V(x)\exp\left(-\frac{(x-x_0)^2}{2a^2}\right), \tag{61}$$

$$a^2(x_0)=\frac{1}{T\Omega^2(x_0)}\left(\frac{T\Omega(x_0)}{2}\coth\frac{\Omega(x_0)T}{2}-1\right), \tag{62}$$

where $\Omega(x_0)$ is the frequency of a harmonic oscillator in the trial Lagrangian which emerges in a Fourier mode expansion of the partition function. Then one considers the quantity

$$\tilde{W}_1(x_0,a^2,\Omega)=V_{a^2(x_0)}(x_0)-\frac{1}{2}\,\Omega^2(x_0)a^2(x_0)+\frac{2}{\hbar T}\,\frac{\sinh\,(\Omega T/2)}{\Omega T} \tag{63}$$

and minimizes it such that the equations

$$a^2(x_0)=\frac{1}{T\Omega^2(x_0)}\left(\frac{\Omega T}{2}\coth\frac{\Omega T}{2}-1\right), \tag{64}$$

$$\Omega^2(x_0)=2\,\frac{\partial}{\partial a^2}\,V_{a^2}(x_0)=\frac{\partial^2}{\partial x_0^2}\,V_{a^2}(x_0) \tag{65}$$

are fulfilled. The emerging effective potential is denoted by $W_1(x_0)$ and inserted into the expression (60) of the partition function. The result is a generalization of Ref. 5. For details we refer to the literature.[5,28,64]

B. Path integral for the radial harmonic oscillator

In order to evaluate the path integral for the radial harmonic oscillator, one has to perform a separation of the angular variables, as discussed in Sec. II C. Here we are not going into the subtleties of the Besselian functional weight (35) connected with the Bessel functions which appear in the lattice approach,[13,25,34,37,65] which is actually necessary for a correct definition and explicit evaluation of the radial harmonic oscillator path integral. One obtains (modulo the above mentioned subtleties caused by the functional weight $\mu_\lambda[r^2]$) $(r',r''\geqslant 0)$

$$\int_{r(t')=r'}^{r(t'')=r''} \mathscr{D}r(t)\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\dot{r}^2-\hbar^2\frac{\lambda^2-1/4}{2mr^2}-\frac{m}{2}\omega^2r^2\right)dt\right]$$

$$\equiv\int_{r(t')=r'}^{r(t'')=r''} \mathscr{D}r(t)\mu_\lambda[r^2]\exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2r^2)dt\right]$$

$$:=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{N/2}\prod_{k=1}^{N-1}\int_0^\infty dr_k\prod_{l=1}^{N}\mu_\lambda[r_lr_{l-1}]\cdot\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\Delta^2r_j-\epsilon V(r_j)\right]\right\} \tag{66}$$

$$=\sqrt{r'r''}\frac{m\omega}{i\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2i\hbar}(r'^2+r''^2)\cot\omega T\right]I_\lambda\left(\frac{m\omega r'r''}{i\hbar\sin\omega T}\right) \tag{67}$$

$$=\frac{2m\omega}{\hbar}\sqrt{r'r''}\sum_{n\in\mathbb{N}_0}e^{-i\omega T(2n+\lambda+1)}\frac{n!}{\Gamma(n+\lambda+1)}\left(\frac{m\omega}{\hbar}r''r''\right)$$
$$\times\exp\left(-\frac{m\omega}{2\hbar}(r'^2+r''^2)\right)L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r'^2\right)L_n^{(\lambda)}\left(\frac{m\omega}{\hbar}r''^2\right), \tag{68}$$

where $I_\lambda(z)$ denotes the modified Bessel function, and $L_n^{(\lambda)}(z)$ a Laguerre polynomial. The expansion into the wave functions has been performed by means of the Hille–Hardy formula.

C. Path integral for the Pöschl–Teller potential

There are two further basic path integral solutions based on the SU(2) (Refs. 31, 36, 66, 67) and SU(1,1) (Ref. 36) group path integration, respectively. The first yields the following path integral identity for the Pöschl–Teller potential ($0<x<\pi/2$):

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-\frac{\hbar^2}{2m}\left(\frac{\alpha^2-1/4}{\sin^2x}+\frac{\beta^2-1/4}{\cos^2x}\right)\right]dt\right\}$$
$$=\frac{m}{2\hbar^2}\sqrt{\sin 2x'\sin 2x''}\frac{\Gamma(m_1-L_E)\Gamma(L_E+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$
$$\times\left(\frac{1-\cos 2x'}{2}\frac{1-\cos 2x''}{2}\right)^{(m_1-m_2)/2}\left(\frac{1+\cos 2x'}{2}\frac{1+\cos 2x'}{2}\right)^{(m_1+m_2)/2}$$
$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1-m_2+1;\frac{1-\cos 2x_<}{2}\right)$$
$$\times{}_2F_1\left(-L_E+m_1,L_E+m_1+1;m_1+m_2+1;\frac{1+\cos 2x_>}{2}\right) \tag{69}$$

$$=\sum_{n\in\mathbb{N}_0}\frac{\Phi_n^{(\alpha,\beta)}(x')\Phi_n^{(\alpha,\beta)}(x'')}{E_n-E}, \tag{70}$$

$$\Phi_n^{(\alpha,\beta)}(x)=\left[2(\alpha+\beta+2n+1)\,\frac{n!\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\right]^{1/2}\times(\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}P_n^{(\alpha,\beta)}(\cos 2x), \tag{71}$$

$$E_n=\frac{\hbar^2}{2m}\,(\alpha+\beta+2n+1)^2, \tag{72}$$

with $m_{1/2}=\frac{1}{2}(\beta\pm\alpha)$, $L_E=-\frac{1}{2}+\frac{1}{2}\sqrt{2mE}/\hbar$. Here $x_{(,)}$ denotes the larger, smaller of x', x'', respectively. ${}_2F_1(a,b;c;z)$ is the hypergeometric function, and $P_n^{(\alpha,\beta)}(z)$ are Jacobi polynomials. Here we have used the fact that it is possible to state closed expressions for the (energy dependent) Green's functions for the Pöschl–Teller and modified Pöschl–Teller potentials (see below), respectively, by summing up the spectral expansion.[68]

D. Path integral for the modified Pöschl–Teller potential

Similarly one can derive a path integral identity for the modified Pöschl–Teller potential. One gets ($m_{1,2}=\frac{1}{2}(\eta\pm\sqrt{-2mE}/\hbar)$, $L_\nu=\frac{1}{2}(-1+\nu)$, $r>0$)

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{r(t')=r'}^{r(t'')=r''}\mathscr{D}r(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\,\dot r^2-\frac{\hbar^2}{2m}\left(\frac{\eta^2-1/4}{\sinh^2 r}-\frac{\nu^2-1/4}{\cosh^2 r}\right)\right]dt\right\}$$

$$=\frac{m}{\hbar^2}\,\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}(\cosh r'\,\cosh r'')^{-(m_1-m_2)}(\tanh r'\,\tanh r'')^{m_1+m_2+1/2}$$

$$\times{}_2F_1\left(-L_\nu+m_1,L_\nu+m_1+1;m_1-m_2+1;\frac{1}{\cosh^2 r_<}\right)$$

$$\times{}_2F_1(-L_\nu+m_1,L_\nu+m_1+1;m_1+m_2+1;\tanh^2 r_>) \tag{73}$$

$$=\sum_{n=0}^{N_M}\frac{\Psi_n^{(k_1,k_2)*}(r')\Psi_n^{(k_1,k_2)}(r'')}{E_n-E}+\int_0^\infty dp\,\frac{\Psi_p^{(k_1,k_2)*}(r')\Psi_p^{(k_1,k_2)}(r'')}{\hbar^2p^2/2m-E}. \tag{74}$$

The bound states are explicitly given by

$$\Psi_n^{(k_1,k_2)}(r)=N_n^{(k_1,k_2)}(\sinh r)^{2k_2-1/2}(\cosh r)^{-2k_1+3/2}\times{}_2F_1(-n,-k_1+k_2-\kappa+1;2k_2;-\sinh^2 r) \tag{75a}$$

$$=\left[\frac{2n!(2\kappa-1)\Gamma(2k_1-n-1)}{\Gamma(2k_2+n)\Gamma(2k_1-2k_2-n)}\right]^{1/2}(\sinh r)^{2k_2-1/2}\times(\cosh r)^{2n-2k_1+3/2}\times P_n^{[2k_2-1,2(k_1-k_2-n)-1]}\left(\frac{1-\sinh^2 r}{\cosh^2 r}\right), \tag{75b}$$

$$N_n^{(k_1,k_2)}=\frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa-1)\Gamma(k_1+k_2-\kappa)\Gamma(k_1+k_2+\kappa-1)}{\Gamma(k_1-k_2+\kappa)\Gamma(k_1-k_2-\kappa+1)}\right]^{1/2}, \tag{75c}$$

$$E_n=-\frac{\hbar^2}{2m}\,(2\kappa-1)^2=-\frac{\hbar^2}{2m}\,[2(k_1-k_2-n)-1]^2, \tag{76}$$

with $k_1=\frac{1}{2}(1\pm\nu)$, $k_2=\frac{1}{2}(1\pm\eta)$, $n=0,1,\ldots,N_M<k_1-k_2-\frac{1}{2}$, $\kappa=k_1-k_2-n$. The continuum states are $[\kappa=\frac{1}{2}(1+ip),p\geqslant 0]$

$$\Psi_p^{(k_1,k_2)}(r)=N_p^{(k_1,k_2)}(\cosh r)^{2k_1-1/2}(\sinh r)^{2k_2-1/2}\times{}_2F_1(k_1+k_2-\kappa,k_1+k_2+\kappa-1;2k_2;-\sinh^2 r), \tag{77a}$$

$$N_p^{(k_1,k_2)}=\frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}[\Gamma(k_1+k_2-\kappa)\Gamma(-k_1+k_2+\kappa)\times\Gamma(k_1+k_2+\kappa-1)\Gamma(-k_1+k_2-\kappa+1)]^{1/2}. \tag{77b}$$

E. Path integration on the pseudo-Euclidean plane

In this section we summarize the various path integral representations on the pseudo-Euclidean plane. On the pseudo-Euclidean plane there are according to Kalnins and Miller[69–71] ten coordinate systems which separate the Schrödinger equation. A glance on the various representations shows that indeed various self-adjoint extensions (and new interbasis expansions) are required. However, because a thorough treatment of the path integration on the pseudo-Euclidean plane will be given elsewhere,[39] and in order not to make the article too lengthy, we restrict ourselves to the statement of the representations and some short comments. Details will be presented elsewhere.[39]

We shall use the following notation: $J_\nu(z)$ is Bessel function, $K_\nu(z)$ is the modified Bessel function, $E_\nu^{(0,1)}(z)$ are even and odd parabolic cylinder functions, $\mathrm{Ai}(z)$ denotes the Airy function, and $\mathrm{Me}_\nu(z)$ and $\mathrm{Me}_\nu^{(3)}(z)$ are Mathieu functions. In the parametric coordinate systems d is a positive parameter.

The technique how to define path integrals in spaces with indefinite metric is described in Ref. 36. In the polar system (see II, below) a similar interbasis expansion as Eq. (27) has been used which is known from hyperbolic geometry. In the three parabolic systems, one has used the results from the inverted (repelling) harmonic oscillator and the linear potential, respectively. In the elliptic and hyperbolic systems (VI–IX see below) an interbasis expansion similar to Eq. (41) has been used which generalizes Eq. (41) to pseudo-Euclidean geometry. The last system (X below) is the most difficult one. Here it is necessary to consider the self-adjoint extension of the inverted (repelling) Liouville problem. We now have

I. Cartesian, $(v_0,v_1)=v\in\mathbb{R}^{(1,1)}$:

$$\int_{v(t')=v'}^{v(t'')=v''}\mathscr{D}v(t)\exp\left(\frac{im}{2\hbar}\int_{t'}^{t''}(\dot v_0^2-\dot v_1^2)dt\right)=\frac{m}{2\pi\hbar T}\exp\left(\frac{im}{2\hbar T}|v''-v'|^2\right) \tag{78}$$

$$=\int_{\mathbb{R}^{(1,1)}}\frac{dp}{4\pi^2}\exp\left[-\frac{i\hbar T}{2m}p^2+ip\cdot(v''-v')\right]. \tag{79}$$

II. Polar, $\rho>0$, $\tau\in\mathbb{R}$:

$$\int_{\rho(t')=\rho'}^{\rho(t'')=\rho''}\mathscr{D}\rho(t)\rho\int_{\tau(t')=\tau'}^{\tau(t'')=\tau''}\mathscr{D}\tau(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot\rho^2-\rho^2\dot\tau^2)+\frac{\hbar^2}{8m\rho^2}\right]dt\right\}$$
$$=\int_{\mathbb{R}}\frac{dk}{2\pi}e^{ik(\tau''-\tau')}\int_0^\infty\frac{p\,dp}{\pi^2}K_{ik}(-ip\rho'')K_{ik}(ip\rho')e^{-i\hbar p^2T/2m}. \tag{80}$$

III. Parabolic 1, $\xi,\ \eta\in\mathbb{R}$:

$$\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathscr{D}\xi(t)\int_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathscr{D}\eta(t)(\xi^2-\eta^2)\exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(\xi^2-\eta^2)(\dot\xi^2-\dot\eta^2)dt\right]$$

$$=\int_{\mathbb{R}}d\zeta\int_{\mathbb{R}}\frac{dp}{32\pi^4}\,e^{-i\hbar p^2T/2m}$$

$$\times\begin{pmatrix}\left|\Gamma\left(\frac{1}{4}+\frac{i\zeta}{2p}\right)\right|^2E^{(0)}_{-1/2+i\zeta/p}(e^{-i\pi/4}\sqrt{2p}\,\xi'')E^{(0)}_{-1/2+i\zeta/p}(e^{-i\pi/4}\sqrt{2p}\,\eta'')\\ \left|\Gamma\left(\frac{3}{4}-\frac{i\zeta}{2p}\right)\right|^2E^{(1)}_{-1/2+i\zeta/p}(e^{-i\pi/4}\sqrt{2p}\,\xi'')E^{(1)}_{-1/2+i\zeta/p}(e^{i\pi/4}\sqrt{2p}\,\eta'')\end{pmatrix}$$

$$\times\begin{pmatrix}\left|\Gamma\left(\frac{1}{4}+\frac{i\zeta}{2p}\right)\right|^2E^{(0)}_{-1/2-i\zeta/p}(e^{i\pi/4}\sqrt{2p}\,\xi')E^{(0)}_{-1/2-i\zeta/p}(e^{-i\pi/4}\sqrt{2p}\,\eta')\\ \left|\Gamma\left(\frac{3}{4}-\frac{i\zeta}{2p}\right)\right|^2E^{(1)}_{-1/2-i\zeta/p}(e^{-i\pi/4}\sqrt{2p}\,\xi')E^{(1)}_{-1/2-i\zeta/p}(e^{-i\pi/4}\sqrt{2p}\,\eta')\end{pmatrix}\tag{81}$$

IV. Parabolic 2, $\xi,\ \eta\in\mathbb{R}$:

$$\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathscr{D}\xi(t)\int_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathscr{D}\eta(t)(\eta^2-\xi^2)\exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(\eta^2-\xi^2)(\dot\xi^2-\dot\eta^2)dt\right]$$

$$=\int_{\mathbb{R}}d\zeta\int_{\mathbb{R}}\frac{dp}{32\pi^4}\,e^{-i\hbar p^2T/2m}\begin{pmatrix}\left|\Gamma\left(\frac{1}{4}+\frac{i\zeta}{2p}\right)\right|^2E^{(0)}_{-1/2+i\zeta/p}(\sqrt{2p}\,\xi'')E^{(0)}_{-1/2+i\zeta/p}(\sqrt{2p}\,\eta'')\\ \left|\Gamma\left(\frac{3}{4}+\frac{i\zeta}{2p}\right)\right|^2E^{(1)}_{-1/2+i\zeta/p}(\sqrt{2p}\,\xi'')E^{(1)}_{-1/2+i\zeta/p}(\sqrt{2p}\,\eta'')\end{pmatrix}$$

$$\times\begin{pmatrix}\left|\Gamma\left(\frac{1}{4}+\frac{i\zeta}{2p}\right)\right|^2E^{(0)}_{-1/2-i\zeta/p}(\sqrt{2p}\,\xi')E^{(0)}_{-1/2-i\zeta/p}(\sqrt{2p}\,\eta')\\ \left|\Gamma\left(\frac{3}{4}+\frac{i\zeta}{2p}\right)\right|^2E^{(1)}_{-1/2-i\zeta/p}(\sqrt{2p}\,\xi')E^{(1)}_{-1/2-i\zeta/p}(\sqrt{2p}\,\eta')\end{pmatrix}.\tag{82}$$

V. Parabolic 3, $\xi,\eta\in\mathbb{R}$:

$$\int_{\xi(t')=\xi'}^{\xi(t'')=\xi''}\mathscr{D}\xi(t)\int_{\eta(t')=\eta'}^{\eta(t'')=\eta''}\mathscr{D}\eta(t)(\xi-\eta)\exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(\xi-\eta)(\dot\xi^2-\dot\eta^2)dt\right]$$

$$=16\int_0^\infty\frac{dp}{p^{1/3}}\int_{\mathbb{R}}d\zeta e^{-i\hbar p^2T/2m}$$

$$\times\mathrm{Ai}\left[-\left(\xi'+\sqrt{2m}\,\frac{\zeta}{p^2}\right)p^{2/3}\right]\mathrm{Ai}\left[-\left(\xi''+\sqrt{2m}\,\frac{\zeta}{p^2}\right)p^{2/3}\right]$$

$$\times\mathrm{Ai}\left[-\left(\eta'+\sqrt{2m}\,\frac{\zeta}{p^2}\right)p^{2/3}\right]\mathrm{Ai}\left[-\left(\eta''+\sqrt{2m}\,\frac{\zeta}{p^2}\right)p^{2/3}\right].\tag{83}$$

VI. Elliptic 1, $a\in\mathbb{R}$, $b>0$:

$$\int_{a(t')=a'}^{a(t'')=a''}\mathscr{D}a(t)\int_{b(t')=b'}^{b(t'')=b''}\mathscr{D}b(t)d^2(\sinh^2 a-\sinh^2 b)$$
$$\times\exp\left[\frac{im}{2\hbar}d^2\int_{t'}^{t''}(\sinh^2 a-\sinh^2 b)(\dot a^2-\dot b^2)dt\right]$$
$$=\frac{1}{8\pi}\int_0^\infty p\,dp\int_{\mathbb{R}}dk\,e^{-\pi k}e^{-i\hbar p^2T/2m}\mathrm{Me}_{ik}\left(b'';\frac{p^2d^2}{4}\right)\mathrm{Me}_{ik}^*\left(b';\frac{p^2d^2}{4}\right)$$
$$\times M_{ik}^{(3)}\left(a'';\frac{pd}{2}\right)M_{ik}^{(3)*}\left(a';\frac{pd}{2}\right). \tag{84}$$

VII. Elliptic 2, $a\in\mathbb{R}$, $b>0$:

$$\int_{a(t')=a'}^{a(t'')=a''}\mathscr{D}a(t)\int_{b(t')=b'}^{b(t'')=b''}\mathscr{D}b(t)d^2(\sinh^2 a+\cosh^2 b)$$
$$\times\exp\left[\frac{im}{2\hbar}d^2\int_{t'}^{t''}(\sinh^2 a+\cosh^2 b)(\dot a^2-\dot b^2)dt\right]$$
$$=\frac{1}{8\pi}\int_0^\infty p\,dp\int_{\mathbb{R}}dk\,e^{-\pi k}e^{-i\hbar p^2T/2m}\mathrm{Me}_{ik}\left(b''-i\frac{\pi}{2};\frac{p^2d^2}{4}\right)$$
$$\times\mathrm{Me}_{ik}^*\left(b'-i\frac{\pi}{2};\frac{p^2d^2}{4}\right)M_{ik}^{(3)}\left(a'';\frac{pd}{2}\right)M_{ik}^{(3)*}\left(a';\frac{pd}{2}\right). \tag{85}$$

VIII. Hyperbolic 1, $y_1, y_2\in\mathbb{R}$:

$$\int_{y_1(t')=y_1'}^{y_1(t'')=y_1''}\mathscr{D}y_1(t)\int_{y_2(t')=y_2'}^{y_2(t'')=y_2''}\mathscr{D}y_2(t)\,\frac{d^2}{8}(\sinh y_1-\sinh y_2)$$
$$\times\exp\left[\frac{im}{16\hbar}d^2\int_{t'}^{t''}(\sinh y_1-\sinh y_2)(\dot y_1^2-\dot y_2^2)dt\right]$$
$$=\frac{1}{32\pi}\int_0^\infty p\,dp\int_{\mathbb{R}}dk\,e^{-\pi k}e^{-i\hbar p^2T/2m}\mathrm{Me}_{ik}\left(\frac{y_1''}{2}-i\frac{\pi}{4};i\frac{p^2d^2}{4}\right)$$
$$\times\mathrm{Me}_{ik}^*\left(\frac{y_1'}{2}-i\frac{\pi}{4};i\frac{p^2d^2}{4}\right)M_{ik}^{(3)}\left(\frac{y_2''}{2}-i\frac{\pi}{4};\sqrt{i}\frac{pd}{2}\right)M_{ik}^{(3)*}\left(\frac{y_2'}{2}-i\frac{\pi}{4};\sqrt{i}\frac{pd}{2}\right). \tag{86}$$

IX. Hyperbolic 2, $y_1, y_2\in\mathbb{R}$:

$$\int_{y_1(t')=y_1'}^{y_1(t'')=y_1''}\mathscr{D}y_1(t)\int_{y_1(t')=y_1'}^{y_1(t'')=y_1''}\mathscr{D}y_1(t)(e^{2y_1}+e^{2y_2})\exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(e^{2y_1}+e^{2y_2})(\dot y_1^2-\dot y_2^2)dt\right]$$
$$=\frac{2}{\pi^4}\int_0^\infty dk\,k\sinh\pi k\int_0^\infty dp\,pe^{-i\hbar p^2T/2m}K_{ik}(e^{y_1''}p)K_{ik}(e^{y_1'}p)K_{ik}(-ie^{y_2'}p)K_{ik}(ie^{y_2''}p). \tag{87}$$

X. Hyperbolic 3, $y_1, y_2 \in \mathbb{R}$:

$$\int_{y_1(t')=y_1'}^{y_1(t'')=y_1''} \mathscr{D}y_1(t) \int_{y_1(t')=y_1'}^{y_1(t'')=y_1''} \mathscr{D}y_1(t)(e^{2y_1}-e^{2y_2})\exp\left[\frac{im}{2\hbar}\int_{t'}^{t''}(e^{2y_1}-e^{2y_2})(\dot{y}_1^2-\dot{y}_2^2)dt\right]$$

$$=\frac{1}{\pi^2}\int_0^\infty p\,dp\,e^{-i\hbar p^2T/2m}\left\{\int_0^\infty \frac{k\,dk}{2\sinh\pi k}K_{ik}(ipe^{y_1'})K_{ik}(-ipe^{y_1''})\right.$$

$$\times[J_{ik}(pe^{y_2'})+J_{-ik}(pe^{y_2'})]^*[J_{ik}(pe^{y_2''})+J_{-ik}(pe^{y_2''})]$$

$$\left.+\sum_{n\in\mathbb{N}}4nJ_{2n}(pe^{y_2''})J_{2n}(pe^{y_2'})K_{2n}(-ipe^{y_1''})K_{2n}(ipe^{y_1'})\right\}. \tag{88}$$

F. General formulas

For the classification of solvable path integrals, one also requires a few additional formulas which generalize the usual problems in quantum mechanics in a specific way. Here one has, e.g.,

(i) Explicitly time-dependent problems according to, e.g., $V(x) \mapsto V(x/\zeta(t))/\zeta^2(t)$,

(ii) Incorporation of δ function perturbation according to $V(x) \mapsto V(x)-\gamma\delta(x-a)$ (one dimension),

(iii) Incorporation of δ' function perturbation according to $V(x) \mapsto V(x)-\beta\delta'(x-a)$ (one dimension),

(iv) Boundary problems with impenetrable walls (half space, infinite boxes) which can be derived from (ii) by considering the limit $\gamma\to\infty$ (Dirichlet boundary conditions),

(v) Boundary problems with impenetrable walls (half space, infinite boxes) which can be derived from (iii) by considering the limit $\beta\to\infty$ (Neumann boundary conditions),

(vi) Point interactions in two and three dimensions.

(i) For the first class of problems, there is a general solution provided $\zeta(t)$ has a specific form. For $\zeta(t)=(at^2+2bt+c)^{1/2}$ one finds the general formula[21]

$$\int_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathscr{D}\mathbf{x}(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-\frac{1}{\zeta^2(t)}V\left(\frac{\mathbf{x}}{\zeta(t)}\right)\right]dt\right\}$$

$$=(\zeta''\zeta')^{-D/2}\exp\left[\frac{im}{2\hbar}\left(\mathbf{x}''^2\frac{\dot{\zeta}''}{\zeta''}-\mathbf{x}'^2\frac{\dot{\zeta}'}{\zeta'}\right)\right]K_{\omega',V}\left(\frac{\mathbf{x}''}{\zeta''},\frac{\mathbf{x}'}{\zeta'};\int_{t'}^{t''}\frac{dt}{\zeta^2(t)}\right), \tag{89}$$

with $\zeta'=\zeta(t')$, $\zeta''=\zeta(t'')$, etc. Here $\omega'^2=ac-b^2$, and $K_{\omega',V}$ denotes the path integral

$$K_{\omega',V}(\mathbf{z}'',\mathbf{z}';s'')=\int_{\mathbf{z}(0)=\mathbf{z}'}^{\mathbf{z}(s'')=\mathbf{z}''}\mathscr{D}\mathbf{z}(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{\mathbf{z}}^2-\frac{m}{2}\omega'^2\mathbf{z}^2-V(\mathbf{z})\right]ds\right\}. \tag{90}$$

Another class of time-dependent problems has a time dependence according to $V(x) \mapsto V(x-f(t))$. Here one gets (Ref. 72) $(q'=x'-f', f'=f(t')$, etc.)

$$\int_{x(t')=x'}^{x(t'')=x''} \mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x-f(t))\right]dt\right\}$$
$$=\exp\left\{\frac{im}{\hbar}\left[\dot{f}''(x''-f'')-\dot{f}'(x'-f')+\frac{1}{2}\int_{t'}^{t''}\dot{f}^2(t)dt\right]\right\}$$
$$\times\int_{q(t')=q'}^{q(t'')=q''} \mathscr{D}q(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2-V(q)-m\ddot{f}(t)q\right]dt\right\}. \tag{91}$$

Equations (89),(91) are special cases of Eq. (11) [note that $\dot{F}'(q,t)=0$ in Eq. (91) and therefore an additional term in the prefactor $A(t'',t')$ appears].

(ii) In the second class of general formulas we consider the incorporation of point interactions,[73] i.e., a δ function as an additional potential located at $x=a$ with strength γ. This kind of problems can be solved by an exact summation of a perturbation expansion according to Sec. II G. However, in general a closed formula can only be stated for the corresponding Green's function. One obtains[53]

$$\frac{i}{\hbar}\int_0^\infty dT e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''} \mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)+\gamma\delta(x-a)\right]dt\right\}$$
$$=G^{(V)}(x'',x';E)+\frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{1/\gamma-G^{(V)}(a,a;E)}. \tag{92}$$

$G^{(V)}(E)$ is the Green's function for the unperturbed problem (γ=0). Possible bound states are determined by the poles of $G(E)$, i.e., by the equation $G^{(V)}(a,a,E_n)=1/\gamma$. By repeating the procedure it is possible to incorporate an arbitrary number of δ function interactions.

(iii) The third class incorporates δ' function perturbation. This is achieved by considering the path integral formulation of the one-dimensional Dirac particle[5] together with a point interaction.[47] Taking the nonrelativistic limit one obtains for a δ' function perturbation in the path integral representation

$$\frac{i}{\hbar}\int_0^\infty dT e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''} \mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)+\beta\delta'(x-a)\right]dt\right\}$$
$$=G^{(V)}(x'',x';E)-\frac{G^{(V)}_{,x'}(x'',a;E)G^{(V)}_{,x''}(a,x';E)}{\hat{G}^{(V)}_{,x'x''}(a,a;E)+1/\beta}, \tag{93}$$

$$\hat{G}^{(V)}_{,xy}(a,a;E)=\left(\frac{\partial^2}{\partial x\partial y}G^{(V)}(x,y;E)-\frac{2m}{\hbar^2}\delta(x-y)\right)\Bigg|_{x=y=a}. \tag{94}$$

Note that in the path integral (93) the formal expression "$G^{(V)}_{,xy}(a,a;E)$" is automatically regularized by the removal of an "ultraviolet divergence." This regularization prescription is not put in "by hand" but is *a result*. This example shows in a nice way how boundary conditions are contained in the proper definition of a path integral and can be extracted by careful analysis. By repeating the procedure it is possible to incorporate an arbitrary number of δ' function interactions.

(iv) The fourth class of general formulas is obtained if we consider in Eq. (92) the limit $\gamma\to-\infty$. This has the consequence that an impenetrable wall appears at $x=a$. The result then is for the motion in the half space $x>a$ (Refs. 54, 55)

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}_{(x>a)}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=G^{(V)}(x'',x';E)-\frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)}. \tag{95}$$

Possible bound states are determined by the poles of $G(E)$, i.e., by the equation $G^{(V)}(a,a,E_n)=0$. Furthermore, for the motion inside a box with boundaries at $x=a$ and $x=b$ one obtains ($a<x<b$, Dirichlet–Dirichlet boundary conditions)

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}^{(DD)}_{(a<x<b)}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=\frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}(x'',b;E) & G^{(V)}(x'',a;E)\\ G^{(V)}(b,x';E) & G^{(V)}(b,b;E) & G^{(V)}(b,a;E)\\ G^{(V)}(a,x';E) & G^{(V)}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}{\begin{vmatrix} G^{(V)}(b,b;E) & G^{(V)}(b,a;E)\\ G^{(V)}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}. \tag{96}$$

(v) In an obvious way, as in the previous case, we can also obtain a path integral representation in a half space with Neumann boundary conditions at $x=a$ by letting $\beta\to-\infty$ in Eq. (93)[47]

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}^{(N)}_{(x>a)}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=G^{(V)}(x'',x';E)-\frac{G^{(V)}_{,x'}(x'',a;E)G^{(V)}_{,x''}(a,x';E)}{\hat{G}^{(V)}_{,x'x''}(a,a;E)}. \tag{97}$$

The same procedure as for the motion in a box $a<x<b$ with Dirichlet boundary conditions can be applied for Neumann boundary conditions at both boundaries

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}^{(NN)}_{(a<x<b)}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=\frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}_{,x'}(x'',a;E)\\ G^{(V)}_{,x''}(b,x';E) & \hat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x'x''}(b,a;E)\\ G^{(V)}_{,x''}(a,x';E) & G^{(V)}_{,x'x''}(a,b;E) & \hat{G}^{(V)}_{,x'x''}(a,a;E)\end{vmatrix}}{\begin{vmatrix} \hat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x'x''}(b,a;E)\\ G^{(V)}_{,x'x''}(a,b;E) & \hat{G}^{(V)}_{,x'x''}(a,a;E)\end{vmatrix}}. \tag{98}$$

Similar results can be obtained for Dirichlet boundary conditions at one boundary, and Neumann boundary conditions at the other. Radial boxes and rings can be taken into account as well, and potentials which depend on the absolute value $|x|$ by combining the results for Dirichlet and Neumann boundary conditions, i.e.,[47]

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(|x|)\right]dt\right\}$$

$$=G^{(V)}(x'',x';E)-\frac{G^{(V)}(x'',0;E)G^{(V)}(0,x';E)}{2G^{(V)}(0,0;E)}-\frac{G^{(V)}_{,x'}(x'',0;E)G^{(V)}_{,x''}(0,x';E)}{2\hat{G}^{(V)}_{,x'x''}(0,0;E)}. \tag{99}$$

(vi) It is also possible to incorporate a two- and three-dimensional δ function perturbation located at $\mathbf{x}=\mathbf{a}$ in the path integral[56,73]

$$\frac{i}{\hbar}\int_0^\infty dT\, e^{iET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathscr{D}_{\Gamma^{(V)}_{\alpha,a}}\mathbf{x}(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2-V(\mathbf{x})\right]dt\right\}$$

$$=G^{(V)}(\mathbf{x}'',\mathbf{x}';E)+(\Gamma^{(V)}_{\gamma,a}(E))^{-1}G^{(V)}(\mathbf{x}'',\mathbf{a};E)G^{(V)}(\mathbf{a},\mathbf{x}';E), \tag{100}$$

$$V(\mathbf{x})=\frac{\hbar^2}{2m}\frac{\lambda(\lambda-1)}{|\mathbf{x}|^2}+\frac{\eta}{|\mathbf{x}|}, \tag{101}$$

$$\Gamma^{(V)}_{\alpha,\mathbf{a}}(E)=\alpha g_{0,\lambda}-g_{1,\lambda}, \tag{102}$$

($\frac{1}{2}\leq\lambda<\frac{3}{2},\eta\in\mathbb{R}$). For the notation: $g(r)$ is a solution of the corresponding unperturbed problem, and α is the (regularized) coupling. The regularizing functions $g_{0,\lambda}g_{1,\lambda}$ are defined by

$$g_{0,\lambda}=\lim_{r\to0^+}\frac{g(r)}{G^{(0)}_\lambda(r)},\quad g_{1,\lambda}=\lim_{r\to0^+}\frac{g(r)-g_{0,\lambda}G^B_\lambda(r)}{r^\lambda}, \tag{103}$$

where $G^B_\lambda(r)$ denotes the asymptotic expansion of the irregular solution $G_\lambda(r)$ of the unperturbed problem. Generally one has $G^B_\lambda(r)=G^{(0)}_\lambda(r)+$additional terms, where $G^{(0)}_\lambda$ denotes the free particle case. Two special cases of $G^B_\lambda(r)$ can be stated for $\lambda=\frac{1}{2}$ and 1, i.e., for the Schrödinger operator in two and three dimensions, respectively, which will be sufficient for our purposes. Then

$$G^B_{1/2}(r)=G^{(0)}_{1/2}(r)=-\frac{m}{\pi\hbar^2}\sqrt{r}\ln r, \tag{104}$$

$$G^B_1(r)=\frac{m}{2\pi\hbar^2}\left(1-\frac{m\eta}{\hbar^2}r-\frac{2m\eta}{\hbar^2}r\ln r\right). \tag{105}$$

By these means, the incorporation of a point interaction in two and three dimensions located at $\mathbf{x}=\mathbf{a}$ in the path integral is then defined by Eq. (100). By repeating the procedure it is possible to incorporate an arbitrary number of two- and three-dimensional δ function interactions.

IV. A TABLE OF EXACTLY SOLVABLE FEYNMAN PATH INTEGRALS

We are now in the position to present a systematic classification and a list of exactly solvable Feynman path integrals. Of course, due to lack of space, an actual table cannot be presented in this article. We therefore list the name of the potential, i.e., the name of the quantum mechanical problem, and the basic path integrals to which the path integrals in question can be reduced and the method by which it can be solved.

In our table we order the quantum mechanical problems according to their underlying basic path integral. This classification is for one-dimensional potentials closely related to the classifica-

TABLE I. Application of potential problems.

Quadratic Lagrangian	Radial harmonic oscillator	Pöschl–Teller potential	Modified Pöschl–Teller potential
Infinite square well	Liouville potential	Scarf potentials	Reflectionless potential
Linear potential	Morse potential	Symmetric top	Rosen–Morse potential
Repelling oscillator	Uniform magnetic field	Magnetic top	Wood–Saxon potential
Forced oscillator	Motion in a section	Higgs oscillator on spheres	Hultén potential
Saddle point potential	Calogero model		Manning–Rosen potential
Uniform magnetic field	Aharonov–Bohm problems		Hyperbolic Scarf potential
Driven coupled oscillators	Coulomb potential		Hyperbolic barrier potential
Two-time action (Polaron)	Smorodinsky–Winternitz potentials		Hyperbolic spaces of rank one
Second derivative Lagrangians	Coulomb-like potentials in polar and parabolic coordinates		Kepler problem on (pseudo-) spheres
Semiclassical expansion	Nonrelativistic monopoles		Natanzon potentials
Generating functional	Kaluza–Klein monopole		Hyperbolic strip
Momenta formula	Poincaré plane +magnetic field +potentials		Higgs oscillator on pseudospheres
Effective potentials	Dirac Coulomb problem		Hermitean spaces
Anharmonic oscillator	Anyons		

tion scheme based on Schrödinger's factorization method as reviewed by Infeld and Hull,[74] respectively, to the related classification scheme of Gendenshteĭn[75] based on supersymmetric quantum mechanics.[76]

Path integrals on homogeneous spaces have been discussed in Ref. 36 (spheres and pseudospheres, bispherical coordinates). A comprehensive treatment of the various coordinate space representations can be found for two- and three-dimensional Euclidean space, and the two- and three-dimensional (pseudo-) sphere in Ref. 40.

Our classification is according to the following scheme:

(i) The general Lagrangian which is at most quadratic in x and $\dot{x}$ (the harmonic oscillator being the simplest and best known example),
(ii) The radial harmonic oscillator,
(iii) The Pöschl–Teller potential,
(iv) The modified Pöschl–Teller potential,
(v) Path integrals on homogeneous spaces (group path integration, interbasis expansions),
(vi) Explicitly time-dependent problems,
(vii) Path integrals with point interactions, respectively, boundary conditions,
(viii) Path integrals with infinite boundaries at finite distances (half spaces, infinite walls, boxes, and rings),
(ix) Step potentials.

In the two tables we try to summarize our knowledge on how to solve path integrals in quantum mechanics. In Table I we display the various possibilities how the fundamental path integral solutions, i.e., the harmonic oscillator, the general quadratic Lagrangian, the radial harmonic oscillator, the Pöschl–Teller and the modified Pöschl–Teller potential, respectively, can be used to solve by path integration a wide range of other potential problems, including potential problems in spaces of constant curvature (i.e., Euclidean space, sphere, and pseudosphere).

TABLE II. Group path integration and perturbation expansion.

Group path integration	Perturbation expansions
Euclidean space	δ functions
Pseudo-Euclidean space	δ' functions
Spheres	Point interaction for Dirac particle
Single-sheeted pseudospheres	Dirichlet boundary conditions
Double-sheeted pseudospheres	Neumann boundary conditions
Bispherical coordinates	Boxes and radial rings
Pseudobispherical coordinates	Absolute value potentials
Klein–Gordon propagator	Point interactions in $\mathbb{R}^2,\mathbb{R}^3$
	Step potentials

Also some miscellaneous results are listed. All these problems can be called either Gaussian, Besselian, or Legendrian, respectively.

In Table II we list the kind of problems which are either related to path integration on group spaces, including their spectral expansion in more than one coordinate system, and path integral problems which are definitely non-Gaussian, Besselian or Legendrian at all. These problems can only be addressed by a perturbative approach, i.e., the exact summation of a perturbation expansion.

Of course, in the case of general quantum mechanical problems, more than just one of the basic path integral solutions is required. However, such problems can be conveniently put into a hierarchy according to which of the basic path integral is the most important one for its solution. For instance, in the path integral solution for the ring potential (an axially symmetric Coulomb-like potential), this hierarchy puts the radial harmonic oscillator path integral solution first, because it requires a space–time transformation to transform the Coulomb terms into a radial oscillator.

It is obvious that all potential problems can be generalized to more complicated problems, i.e., one can add an additional explicit time dependence, implement a δ function perturbation, and consider problems in half spaces and infinite boxes, cf. Eqs. (89)–(96), respectively. The construction of examples is left to the reader.

V. DISCUSSION AND OUTLOOK

In this contribution we have sketched our approach "How to Solve Path Integrals in Quantum Mechanics." We do not claim completeness; however, we have done our best to gather as much information as possible. In our presentation we did not give any proofs of the formulas. This will be postponed to our book.[7] Our intention was not to give a rigorous mathematical discussion of the existence of the path integral (cf., e.g., Refs. 57, 77–80) and the various transformation techniques (cf., e.g., Refs. 25, 30–32). Since Feynman's beautiful article[3] and his classic book written with Hibbs,[5] several textbooks and reports on path integration have been published.[5,6,28,62,63,79–85] Now the time seems to be ripe for a comprehensive summary and critical review including a systematic classification and extensive bibliography which we are going to complete soon.[7,11]

Summarizing, we can cover by the mentioned methods the so-called standard path integrals which are based on Gaussian, Besselian, and Legendrian path integrals. The solutions of these path integrals actually represent the matrix elements of group representations in a particular coordinate space representation. In this sense these three kinds of path integrals are but a special kind of the general path integration on group manifolds. One has to keep in mind that the chosen coordinate space representation is but one possibility. Other coordinate space representations will give other standard path integrals and corresponding path integral identities. As examples for the latter we have shown how to calculate the free motion in $\mathbb{R}^2$ and $\mathbb{R}^3$ in elliptic and spheroidal coordinates. As will be shown in the near future[39,41] we are able by this method to perform path integration for numerous problems (free motion in spaces of constant curvature, potential problems) in parametric coordinate systems.

The very formulation of the path integral already contains a specific kind of boundary condition. For the free motion or the standard potential problems these boundary conditions are usually chosen in such a way that the vanishing of the wave functions at $\pm\infty$ is required. Boundary conditions at finite distances must be incorporated more explicitly in the path integral. We have seen that Dirichlet and Neumann boundary conditions in one dimension can be incorporated into the path integral by point interactions, where one makes the strength of the point interaction infinitely repulsive. Actually point interactions represent boundary conditions for the wave function such that an infinite strength is but a special case. It is obvious that by this method we also can treat boundary conditions in D dimensions. They are incorporated by a δ function according to $-\gamma\delta(\mathbf{a}\cdot\mathbf{x})$, where $\mathbf{a}\cdot\mathbf{x}$ is a $(D-1)$-dimensional hyperplane. Making the strength of the hyperplane interaction infinitely repulsive gives Dirichlet boundary conditions along the hyperplane. Of course, the procedure is similar for Neumann boundary conditions.

The situation is more delicate for two- and three-dimensional point interactions. The peculiar feature of the self-adjoint extension of the corresponding Schrödinger operator defines a path integral representation with a special kind of boundary condition at the interaction point. Here Aharonov–Bohm effects (with magnetic moments) and point-particle interactions can be modeled.[73]

Mixed boundary conditions require the same kind of regularization procedure as Neumann boundary conditions. The latter are obtained by considering a δ' interaction in the path integral (as pointed out in Ref. 73, the notion of δ' must not be taken too literally, it only serves to describe a specific kind of boundary conditions at the location of the interaction). The δ' interaction in turn is derived by a regularization procedure which makes use of the path integral representation of the one-dimensional Dirac particle subject to a point interaction, i.e., a usual one-dimensional δ function. Here a four parameter family of self-adjoint extensions must be taken into account which covers a wide range of boundary conditions in the (then nonrelativistic, i.e., Schrödinger) limit.

We have also listed some formulas for explicitly time-dependent problems. The special feature of the time dependence is such that it is possible to remove it. However, additional (and sometimes imaginary) potential and measure terms, respectively, appeared. The imaginary potential can be on the one hand understood as a source or a sink for the probability, because the transformation of a time-independent Hamiltonian to a time-dependent one, say, has the consequence that the new Hamiltonian does not conserve energy; this is now exactly balanced by the imaginary potential in order to guarantee energy conservation of the entire (time-independent) system. On the other hand, this term can be interpreted as a "path dependent measure."

Notwithstanding the fact that a considerable progress has been achieved in recent years towards a deeper and more comprehensive understanding of path integration, many questions remain to be answered.

The familiar coordinate systems such as Cartesian, polar, or parabolic coordinates cover only a very limited range of possible studies in general coordinate systems and related questions, as separation of variables and the study of finitely integrable systems. The known techniques cover path integrals in coordinate systems which are parameter free. What is desirable is a treatment of *generic coordinate systems*, by which we mean coordinate systems depending on certain parameters in such a way that all standard coordinate systems can be obtained by degenerations of the generic ones. The elliptic and prolate spheroidal coordinates are simple examples for such systems. Such considerations are by no means just idle doings. The study of simple systems provides tools for the investigation of more complicated ones.

Let us shortly discuss the problem of the physical significance of considering the separation of variables in parametric coordinate systems. The free motion in a given space is, of course, the most symmetric one, and the search for the number of coordinate systems which allow the separation of the Hamiltonian is equivalent to the investigation how many inequivalent sets of observables can be found. The incorporation of potentials usually removes at least some of the symmetry properties of the space. Well-known examples are spherical systems, and they are most conveniently studied in spherical coordinates. For instance, the Coulomb potential is separable in four coordinate systems, namely, in conical, spherical parabolic, and prolate spheroidal II coordinates (for a comprehensive review with the focus on path integration cf. Ref. 86).

The separation of a quantum mechanical potential problem in more than one coordinate system has the consequence that there are additional integrals of motion and that the spectrum is degenerate. In the case of the isotropic harmonic oscillator one has in addition to the conservation of energy and the conservation of angular momentum, the conservation of the quadrupole moment; in the case of the Coulomb problem one has in addition to the conservation of energy and the angular momentum, the conservation of the Runge–Lenz vector. In total these additional conserved quantities add up to five integrals of motion (in classical mechanics) and observables (in quantum mechanics), respectively. It is even possible to introduce extra terms in the pure oscillator and Coulomb potential in such a way that one still has all these (slightly modified) integrals of motion. As it turns out, the so-constructed modified harmonic oscillator and Coulomb problem belong to a larger class of potentials which are called superintegrable.

Disturbing the spherical symmetry usually spoils the superintegrability. The first step consists of deforming the ring-shaped feature of the (superintegrable) modified oscillator and Coulomb potential. Here one gets in the former a ring-shaped oscillator and in the latter the Hartmann potential.

Disturbing the system further, one may end up with the situation that there is only one coordinate system left which allows separation of variables. A constant electric field (Stark effect) allows only the separation in parabolic coordinates. In this case it is interesting to remark that in the momentum representation of the hydrogen atom the bound state spectrum is described by the free motion on the sphere $S^{(3)}$, to be more precise, the dynamical group O(4) describes the discrete spectrum, and the Lorentz group O(3,1) the continuous spectrum. There are six coordinate systems on $S^{(3)}$ which separate the corresponding Laplacian. The solution in spherical and cylindrical coordinates correspond to the spherical and parabolic solution in the coordinate space representation. The elliptic cylindrical system on $S^{(3)}$ is of interest because it enables one to formulate a complete classification for the energy levels of the quadratic Zeeman effect.

The separation in parabolic coordinates is also possible in the case of a perturbation of the pure Coulomb field with a potential $\propto z/r$ which, however, still allows an exact solution.[87] The two-center Coulomb problem turns out to be separable only in spheroidal coordinates.

Another possibility to disturb the spherical symmetry is to remove the invariance under rotations with respect to a given axis (e.g., about a uniform magnetic field). Usually this invariance is used to illustrate the azimuthal quantum number m, i.e., the eigenvalue of the L_z operator. The physical meaning of this quantum number then is that there exists a preferred axis in space. This symmetry can be broken if one considers a Hamiltonian of a nucleus with an electric quadrupole moment Q and spin J in a spatially varying electric field. Here spheroconical coordinates are most convenient. Also the problem of the asymmetric top, a symmetric oblate top, or the case of tensorlike potentials can be treated best in spheroconical coordinates.

In order that a potential problem is separable in ellipsoidal coordinates, it is required that the shape of the potential resembles the shape of an ellipsoid. Of course, the anisotropic harmonic oscillator belongs to this class. Introducing quartic and sextic interaction terms then eventually allows only separation of variables in ellipsoidal coordinates. Another example is the Neumann model, i.e., a particle moving on a sphere subject to anisotropic harmonic forces.

A detailed study of the coordinate space representations of SU(1,1) is worthwhile. The group manifold of SU(1,1) (and its path integral representation) can be described as the free quantum motion on the hyperboloid

$$x_1^2+x_2^2-x_3^2-x_4^2=1, \tag{106}$$

where only the upper sheet is taken into account. As shown by Kalnins and Miller[69] the Laplace operator on this hyperboloid allows separation of variables in 74 coordinate systems! The spectrum of SU(1,1) contains an infinite discrete and a continuous spectrum. For instance, in the study of the two-sheeted pseudosphere only the continuous part is needed, whereas in the single-sheeted case both contributions are necessary. Furthermore, the coordinate space representations of SU(1,1) contain information about singular potentials as $V(r)=-\alpha/r^2(\alpha<1/4)$. Here a self-adjoint extension is required. Path integrals on homogeneous spaces with indefinite metric are plagued with such problems. If it is possible to extract the necessary information from a particular coordinate space representation, it is possible to set up a path integral formulation by means of path integration on groups, and one can derive new path integral identities.

It is our hope that a compilation of our present knowledge will help to spread the results achieved into the physical and mathematical community, making them available for critical consideration and further progress, with the ultimate goal of a comprehensive and complete path integral description of quantum mechanics and quantum field theory, including quantum gravity and cosmology.

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