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On the Local Equilibrium Condition

Hermann Heßling
Deutsches Elektronen–Synchrotron (DESY), Notkestraße 85, D–22603 Hamburg, Germany

A physical system is in local equilibrium if it cannot be distinguished from a global equilibrium by "infinitesimally localized measurements". This should be a natural characterization of local equilibrium, but the problem is to give a precise meaning to the qualitative phrase "infinitesimally localized measurements". A solution is suggested in form of a *Local Equilibrium Condition* (LEC), which can be applied to linear relativistic quantum field theories but not directly to selfinteracting quantum fields. The concept of *local temperature* resulting from LEC is compared to an old approach to local temperature based on the principle of maximal entropy. It is shown that the principle of maximal entropy does not always lead to physical states if it is applied to relativistic quantum field theories.

1. Introduction

An understanding of nature seems to be easier at very small length scales than at larger length scales. This is our every day experience here at DESY, where we analyze ZEUS data [1] to test predictions of the asymptotically free QCD.

The smaller the observables the less can be resolved: whatever the state of a physical system is, it cannot be distinguished from the vacuum state if the localization regions of the observables are shrinked to a point. This is the content of the Principle of Local Stability [7], [8]. What can be said about the state of a physical system if the localization region of a measurement is not completely shrinked to a point, but is "infinitesimally localized"? From general relativity we know that because of the Equivalence Principle gravitation is locally constant. In [10] a formulation of a Quantum Equivalence Principle (QEP) was suggested. According to QEP the states of all physical systems are locally constant. QEP was investigated in the Rindler spacetime [2] and it was shown that the Hawking–Bisognano–Wichman temperature [9], [1], [2] is a consequence of QEP [10].

In this paper we try to characterize local equilibrium states by formulating a condition for "infinitesimally localized measurements". This condition is presented in the next section. The next section starts with a collection of some more or less known facts and a presentation of our nomenclature. In the last section we compare our approach to local equilibrium to an approach based on the principle of maximal entropy.

2. Local Equilibrium Condition

In a quantum mechanical system of finite degrees of freedom the expectation value $<A>$ of any observable A in a state $<\cdot>$ can be characterized by a density matrix ρ

$$<A> \quad = \quad \frac{\mathrm{Tr}\,\rho A}{\mathrm{Tr}\,\rho}. \tag{1}$$

If one introduces the *modular Hamiltonian* $\tilde{H}$ by

$$e^{-\tilde{\beta}\tilde{H}} \quad = \quad \rho$$

where $\tilde{\beta}$ is a number introduced for later convenience, the *modular evolution*

$$\gamma_\tau(A) = e^{i\tilde{H}\tau} A e^{-i\tilde{H}\tau}$$

[1] For a description of the ZEUS detector see e.g. [13].
[2] The Rindler spacetime is a wedge in the Minkowski spacetime ($|t| < x^{(1)}$). It is a simple model of a black hole.

can be defined. Cyclicity of the trace gives the *KMS–condition* [11], [12]

$$<\gamma_\tau(A)B> \quad = \quad <B\gamma_{\tau+i\tilde{\beta}}(A)> . \qquad (2)$$

In quantum field theory the right hand side of (1) does not exist but the KMS–condition (2) can be used directly to characterize the state [6], [8]. For simplicity we concentrate on the Klein–Gordon field $\phi(x)$ in Minkowski spacetime.

If the system is in a global equilibrium state $<\cdot>_\beta$ with temperature $1/\beta$ the modular Hamiltionan is nothing but the Hamilton operator, $\tilde{H} = H$, which generates the time evolution α_t along an inertial time coordinate t in the Minkowski spacetime

$$\alpha_t\phi(0,\vec{x}) \; = \; e^{iHt}\phi(0,\vec{x})e^{-iHt} \quad = \quad \phi(t,\vec{x}).$$

The parameter $\tilde{\beta}$ represents the inverse temperature β of the system. If we replace the modular evolution γ_τ in (2) by the time evolution α_t and perform a Fourier transformation of the KMS–condition, the 2–point function $<\phi(x')\phi(x)>_\beta$ is representable in terms of a state independent commutator [10]

$$<\phi(x')\phi(x)>_\beta = \qquad (3)$$
$$\frac{i}{2\tilde{\beta}}\int \mathrm{d}\tau[\alpha_\tau\phi(x),\phi(x')]\coth\frac{\pi}{\tilde{\beta}}(\tau - i\epsilon)$$

Using the well known non–equal time commutator for a massless Klein–Gordon field

$$[\phi(x),\phi(x')] = \frac{1}{2\pi i}\mathrm{sign}(t-t')\delta((x-x')^2)$$

we obtain for the 2–point function

$$<\phi(x')\phi(x)>_\beta = \frac{1}{4\pi\beta\dot{\sigma}}\left(\coth\frac{\pi\tau_+}{\beta} - \coth\frac{\pi\tau_-}{\beta}\right) \qquad (4)$$

where

$$\tau_\pm = t'-t\pm\sqrt{(\vec{x}'-\vec{x})^2} - i\epsilon, \;\; \dot{\sigma} = 2\sqrt{(\vec{x}'-\vec{x})^2}$$

We understand local equilibrium as a state which cannot be distinguished from a global equilibrium state by "infinitesimally localized measurements". If the localization region of an observable A is made smaller and smaller the expectation value $<A>$ of the observable A in a local equilibrium state $<\cdot>$ should become more and more identical to the expectation value $<A>_{\beta_*}$ of A in an equilibrium state $<\cdot>_{\beta_*}$ at a certain temperature β_*. To describe the shrinking of the localization region of an observable A we use a one–parametric scaling procedure $\delta_\lambda A$.

For the n–point observable $A = \phi(x_1)\ldots\phi(x_n)$ the scaling procedure is defined as [7], [5], [10]

$$\delta_\lambda A \quad = \quad N(\lambda)^n\,\phi(\chi_\lambda x_1)\ldots\phi(\chi_\lambda x_n) \qquad (5)$$

where (with respect to inertial coordinates x^μ)

$$(\chi_\lambda x)^\mu = {x_*}^\mu + \lambda(x^\mu - {x_*}^\mu) \qquad (6)$$

is a 1-parametric scaling diffeomorphism with $\chi_1 x = x$ and $\chi_0 x = x_*$. In the limit $\lambda \to 0$ the localization points $x_1\ldots x_n$ of the n–point observable $\delta_\lambda A$ are scaled into the point x_*. The *scaling function* $N(\lambda)$ has to be adjusted in such a way that the *scaling limit* of the n–point function, $\lim_{\lambda\to 0}<\delta_\lambda A>$, is well defined. A suitable scaling function for the Klein–Gordon field is $N(\lambda) = \lambda$.

According to the Quantum Equivalence Principle (QEP) [10] the scaled observable $\delta_\lambda A$ has to fulfil two requirements for small values of the scaling parameter λ: the expectation value $<\delta_\lambda A>$ has to be locally constant around the scaling point x_* and its the scaling limit, $\lim_{\lambda\to 0}<\delta_\lambda A>$, has to be continuous in x_*. In linear quantum field theories the first requirement can be written as the extremum conditon

$$\lim_{\lambda\to 0}\frac{d}{d\lambda}<\delta_\lambda A> \quad = \quad 0. \qquad (7)$$

Therefore the first nontrivial information about the state of a linear quantum field is beyond the first order in λ.

We say that a state $<\cdot>$ fulfils the *Local Equilibrium Condition* in the point x_*, if it does not differ from a global equilibrium state $<\cdot>_{\beta_*}$ of temperature $1/\beta_*$ up to second order in the scaling parameter λ

$$\lim_{\lambda\to 0}\frac{d^2}{d\lambda^2}\Big(\; <\delta_\lambda A> - <\delta_\lambda A>_{\beta_*}\Big) \quad = \quad 0. \qquad (8)$$

For a massless Klein–Gordon field the equilibrium part of LEC can be calculated from (4)

$$\lim_{\lambda\to 0}\frac{d^2}{d\lambda^2}<\lambda^2\phi(\chi_\lambda x')\phi(\chi_\lambda x)>_{\beta_*} \quad = \quad \frac{1}{12\;{\beta_*}^2}. \qquad (9)$$

Let us apply LEC to Hadamard states. Hadamard states are definable in linear quantum field theories and are quasifree states[3] with a specific singularity structure: the symmetric part of the 2–point–function is identical with Hadamard's fundamental solution of the wave equation [4]

$$<\{\phi(x'),\phi(x)\}> \quad = \quad \frac{u}{\sigma} + v\ln\sigma + w \qquad (10)$$

where σ is the square of the geodesic distance between x' and x. The functions u, v, w are regular in x and x'. The information about the state is contained in w; u and v are state independent and are uniquely fixed by the geometry of the spacetime. It is conjectered that the 2–point function of any physical state of the Klein–Gordon field can locally be approximated by a Hadamard state [8]. Assuming the validity of this conjecture [4] the equilibrium state can be approximated by a state of the form

$$<\{\phi(x'),\phi(x)\}>_{\beta_*} \quad = \quad \frac{u}{\sigma} + v\ln\sigma + w_{\beta_*}. \qquad (11)$$

It follows that LEC reduces to

$$w(x_*,x_*) = w_{\beta_*}(x_*,x_*) \qquad (12)$$

since the state independent singular parts $u/\sigma, v\ln\sigma$ cancel because of the difference in (8) and since the state dependent parts $w(\chi_\lambda x', \chi_\lambda x)$ and $w_{\beta_*}(\chi_\lambda x', \chi_\lambda x)$ are regular in the limit $\lambda \to 0$. This means that LEC does not depend on the scaling function χ_λ in the sense that Eqn (6) can be replaced by any one–parametric scaling diffeomorphism χ_λ which has x_* as a fixpoint. We are therefore allowed to call $1/\beta_*$ a *local temperature*.

Combining (9) and (12) we see that all Hadamard states of the massless Klein–Gordon field with a non–negative $w(x_*,x_*)$ have the local temperature

$$1/\beta_* = \sqrt{12\ w(x_*,x_*)}$$

in the scaling point x_*. Hadamard states with a negative $w(x_*,x_*)$ have an imaginary local temperature and therefore cannot be local equilibrium states.

[3] A state is called quasifree if its truncated n–point–functions vanish for $n \neq 2$.

[4] For a massless Klein–Gordon field the following conclusions can directly be proven without this conjecture by using (4).

How can one describe local equilibrium states in curved spacetimes? In curved spacetimes the existence of global equilibrium states can no longer be expected. Therefore LEC cannot be applied directly. Nevertheless let us insist on our "global first – local next" point of view: before one can define local equilibrium one has to know what equilibrium in a *finite* region of spacetime is. If a physical system is influenced by a rapidly changing gravitational force, it is intuitively clear that the system has to react, i.e. it has to change its state. But stationarity is an important characteristic quality of equilibrium. We therefore exclude non–stationary gravitational fields from the finite spacetime region $\mathcal{O}$ where we would like to investigate local equilibrium states. This can be done by assuming that in the region $\mathcal{O}$ there is a timelike Killing field $\partial/\partial t$. Quantitatively we characterize equilibrium states in the region $\mathcal{O}$ by carrying over the concept of *local KMS–states* from the Minkowski spacetime [3]. A local KMS–state $<\cdot>_{\beta,\mathcal{O}}$ in the region $\mathcal{O}$ of temperature $1/\beta$ has a 2–point function $<\phi(x')\phi(t,\vec{x})>_{\beta,\mathcal{O}}$ which is analytical in the open interval $0 < \mathrm{Im}t < \beta$, continuous in the closed interval $0 \leq \mathrm{Im}t \leq \beta$ and fulfils the KMS–conditon

$$<\phi(x)\phi(x')>_{\beta,\mathcal{O}} \quad = \quad <\phi(x')\phi(t+i\beta,\vec{x})>_{\beta,\mathcal{O}}$$

for all points $x = (t,\vec{x}), x'$ in $\mathcal{O}$. (The step from the 2–point function to arbitrary observables is straightforward and therefore not writen down.) To formulate LEC in the spacetime region $\mathcal{O}$, one has to replace the reference state $<\cdot>_{\beta_*}$ in (8) by the local KMS–state $<\cdot>_{\beta_*,\mathcal{O}}$.

In [10] it was shown that the derivative condition (7) has to be modified if one wants to formulate QEP for asymptotically free quantum field theories like QCD, since the running coupling constant does not smoothly become zero in the short distance limit $\lambda \to 0$, but logarithmically. Because of the same reason the second derivative condition (8) needs a modificaton if one wants to characterize local equilibrium states for selfinteracting quantum fields.

3. LEC versus the principle of maximal entropy

Global equilibrium states in nonrelativistic quantum theories can be obtained by the extremalization of a certain functional: the entropy. The entropy of a state $<\cdot>= \mathrm{Tr}\hat{\rho}(\cdot)$ is defined as

$$S = -\mathrm{Tr}\hat{\rho}\ln\hat{\rho}$$

where $\hat{\rho} = \rho/\mathrm{Tr}\rho$ is the normalized density matrix. Consider the set of states with a given expectation value for the Hamilton operator

$$E = <H> \tag{13}$$

A global equilibrium state is characterizeable as the state of maximal entropy within this set. From the variational equation for the normalized density matrix, $\delta(S + c(1- <1>) + \beta(E- <H>)) = 0$, where c and β are Lagrange multipliers for the normalization condition $\mathrm{Tr}\hat{\rho} = 1$ and the constraint (13) respectively, one gets

$$\rho = e^{-\beta H}$$

It turns out that β is the inverse temperature of the system.

To determine the state of a system with a nonuniform temperature it was suggested [14] that the principle of maximal entropy has to be applied to a local form of the constraint (13)

$$\epsilon(\vec{x}) = <\mathcal{H}(\vec{x})> \tag{14}$$

where $\mathcal{H}(\vec{x})$ is the Hamiltion density at time zero. This leads to a continuum of Lagrange multipliers $\beta(\vec{x})$ and the density matrix

$$\rho = e^{-\int \mathrm{d}^3x\beta(\vec{x})\mathcal{H}(\vec{x})}. \tag{15}$$

$1/\beta(\vec{x})$ is interpreted as the *local temperature* of the system [14]. For the current status of this approach we refer to [15] and references therein.

What is the relation between the local temperature $1/\beta_*$ introduced in the last section and the local temperature $1/\beta(\vec{x})$ obtained with the principle of maximal entropy?

Let us choose a $\beta(\vec{x})$ which is linear in $x^{(1)}$

$$\beta(\vec{x}) = \tilde{\beta}x^{(1)}$$

Then the modular Hamiltonian of the density matrix (15)

$$\tilde{H} = \int \mathrm{d}^3x\ x^{(1)}\mathcal{H}(\vec{x}) \tag{16}$$

becomes the generator of a boost transformation in $x^{(1)}$–direction and for the modular evolution of the Klein–Gordon field one finds

$$\gamma_\tau\phi(0,\vec{x}) = \phi(x^{(1)}\sinh\tau, x^{(1)}\cosh\tau, x^{(2)}, x^{(3)})$$

This system was studied in [10] and it was shown that only one value of the parameter $\tilde{\beta}$ is allowed by QEP, namely $\tilde{\beta} = 2\pi$. We conclude that the principle of maximal entropy does not always lead to physical states. Consequently the interpretation of $1/\beta(\vec{x})$ as a local temperature does not make sense in general.

For the allowed parameter $\tilde{\beta} = 2\pi$ the 2–point function of the Klein–Gordon field in state given by (16) is just the 2–point function of the vacuum state in the Minkowski spacetime [1], [2], i.e. the local temperature in the sense of LEC is identically zero

$$1/\beta_* = 0$$

On the other hand the local temperature from the principle of maximal entropy

$$\frac{1}{\beta(\vec{x})} = \frac{1}{2\pi x^{(1)}}$$

is the Unruh temperature of an uniformly accelerated detector, where the Unruh temperature is measured with respect to the local proper time in the detector [8].

The two concepts of local temperature refer to different time concepts. The local temperature concept from the principle of maximal entropy refers to a time defined by a certain subclass of the modular evolutions. (It would be interesting to know under which conditions this approach makes sense physically.) In the Minkowski spacetime the local temperature concept of LEC is related to a time given by the clock of an experiment at rest.

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