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# A Real-Space Renormalization Group Transformation for Classical Systems.

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**Abstract**

We formulate and describe a Wilson real space renormalization group approach for classical field theories defined on a lattice. In particular, we investigate the classical $\lambda\phi^4$ and two dimensional SU(2) Yang-Mills theories.

The main idea is to separate the angle variables of the oscillators into the fast ones and the slow ones and to average over the fast modes. As a result, one finds an effective Hamiltonian for the remaining slow variables. In the case of classical $\lambda\phi^4$ theory, we obtain by constructing "blockspin" variables some conditions on those classical solutions for which the effective Hamiltonian can be compared with the starting Hamiltonian. In the case of a gauge system, we present the general idea for the definition of "blockspin" variables and for the averaging over the fast modes.

*Finally, we suggest a method to estimate the evolution of the coupling constant with the RGT.*

# Contents

# Chapter 1

# Introduction

In the investigation of nonabelian gauge fields, one is confronted with a system of infinitely many degrees of freedom, and the most used method in the investigation of the properties of the quantum field theory is the Perturbation Theory. It is, in general, a powerful tool to describe the quantum physic. However, this method can not describe all effects, for example, color confinement, which is simply a non-perturbative effect. Therefore non-perturbative methods are required to understand this phenomenon. Important approaches are the investigation of solutions of the equation of motion and the numerical studies based upon the lattice formulation.

The study of classical field solutions, as for example: monopole [1] and instanton [2] has helped in the understanding of the quantum physics. In order to find these solutions the special symmetry properties of the nonabelian gauge theory are used and not the general considerations on the integrability of the non-linear field equations. In any case, we still do not have a clear picture of the classical solutions in the nonabelian gauge theory. In order to understand the complicated structure of QCD, a simplified version of this theory, the SU(2) Yang-Mills for spatialy constant fields, has attracted considerable interest. One can show that the field theory is reduced to a nonlinear mechanical system with two degrees of freedom (sometimes called Yang-Mills Classical Mechanics YMCM). Its investigation was initiated by Matinyan et al. [3]. They found stochastic oscillations which can be evidence for nonintegrability of the classical Hamiltonian. Motivated by this unexpected result, the dynamics of this YMCM was analyzed by many other authors using different methods of modern nonlinear mechanics [4]. The YMCM, apart from being a rather simplified version of QCD served also as a toy model in the attempt of understanding the semiclassical limit of quantum mechanics whose classical counterpart is chaotic [5]. This simple model is described by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + x_1^2 x_2^2) \tag{1.1}$$

In ref. [6] it was demonstrated that islands of stability exist in some fraction of phase space. Later on, chaotic behaviour was numerically demonstrated in the classical SU(2) Yangs-Mills system in the case where the field configuration depends only on one spatial coordinate and time [7] and in the case of spherical symmetry of the YM-classical equations [8]. However, a general solution of the equations of motion in the continuum has not yet been attempted. The proof of the existence of a non-integrable subsystem of the YM-classical System is a serious argument in favor of the non-integrability of these equations in the general case. However,



at the time, there is no demonstration that this characteristic is a general property of the dynamics of the gauge theory. The study of the physical significance of chaotic dynamics in the quantization of a classical field theory is a very important but also very difficult work. A sufficient understanding of the role of chaotic behavior of the classical systems for the cases of hard chaos and mixed behavior, could bring light to future investigations in the quantum mechanic formulation in this direction.

As a first step towards a more general investigation, one use the lattice formulation of Yang-Mills theory. The lattice regularization provides a powerful tool in the context of a non-perturbative formulation, specially when qualitative aspects of the models are considered [9, 10]. Lattice calculations are performed in Euclidean space, and thus field theories defined on a Minkowski space have to be transformed into Euclidean theories by analytical continuation to imaginary time. This analytical continuation shows the equivalence of the field theory to a system of classical statistical mechanics [10, 11]. The equivalence is also manifest in the path integral representation. In the lattice regularization formulated by Wilson, a field theory is defined on a space-time lattice with spacing (lattice constant) $a$ in a finite volume, and with the fields (in the case of scalar field) being defined on each lattice point. An alternative approach, working in the temporal gauge was suggested by Kogut and Susskind [12], keeping time continuous and considering only space as discrete: In their formulation, which can be derived from Wilson's theory by using the transfer matrix [13] the theory is reduced to a nonlinear Hamiltonian system of a finite number of degrees of freedom. The investigation of the integrability of the SU(2) gauge theory defined in the semiclassical limit is not rigorously performed but only the Lyapunov characteristic exponents of SU(2) gauge theory are determined for the case of small lattice [14].

In order to study the correspondence between quantum theory and classical theory, one can introduce different way to describe this transition. The most direct route to this investigation is the Feynman path-integral formalism. Here, the transition amplitude is expressed directly as the sum (a functional integral) over all possible paths between the initial and final states, weighted by the exponential of the action of the path. The study of the semiclassical limit for a system with one degree of freedom is successful, but for that systems with two or more degrees of freedom, the problem becomes more complicated and a good understanding of the classical trajectories is necessary in order to work out the functional summation. If the system is integrable, the standard WKB-approximation can be applied, but, if it is not integrable, the situation changes completely. No method like WKB is possible to obtain the classical limit [16]. However, it was found evidence [15] that some special trajectories in the region of low energy for a gauge theory on the lattice give a good description of the semiclassical approach. Moreover, in the case of a classical system [47] quantitative evidence has been found which indicates that even in the chaotic region of the phase space, sufficient regularity is seen on a time scale to allow the use of the standard EBK-quantization (Einstein, Brillouin and Keller) techniques. These results are then used in a semiclassical quantization, allowing results which are seen to be in excellent agreement with the quantum results. Therefore, it is very important to find out, whether the classical system is chaotic or not and to have a good understanding of the solution of the equations of motion.

Lately several investigations, both analytical and numerical, have been carried out in order to understand the dynamical behavior of nonlinear Hamiltonian systems with a large number of degrees of freedom. In the absence of any nonlinearity, such a system is integrable and its

trajectories move on $n$-tori in the $2n$-dimensional $p$ $q$-phase space [18]. When the nonlinear interaction is turned on, many of these tori are destroyed but the majority is preserved [see KAM-theorem]. The preserved tori are those whose frequencies are sufficiently incommensurable. It has been shown [19] that the Canonical Perturbation Theory provides a meaningful prediction and describes the system for the case of small perturbation. As the strength of the interaction increases, more and more tori are destroyed, and the system eventually becomes chaotic. As to Hamiltonians with two degrees of freedom, the KAM-tori have two dimensions, and they separate different regions of the three-dimensional surface of constant energy. This fact allows the coexistence of disjoint regions characterized by a different behavior, i.e. regular and chaotic regions exist [20]. When the number of degrees of freedom exceeds three, the KAM tori no longer separate the phase space, and one can observe that in a weak chaotic region, a slow diffusion mechanism can cause a trajectory to fill all the accessible phase space. This phenomenon is called Arnold diffusion [21]. However, for systems with a finite number of degrees of freedom and a small perturbation of strength $\lambda < \lambda_c$ ($\lambda_c$ is some critical value) the KAM-theory provides a rigorous proof of the existence of a positive measure of invariant tori. From the analytical standpoint, it is not completely clear how to estimate the critical strength $\lambda_c$ and how to determine the dependence on the number of degrees of freedom. It has been shown by numerical simulations that $\lambda_c$ has a strong dependence upon the number of degrees of freedom, and it is model dependent.

It is known that a generic Hamiltonian system is neither completely regular nor completely chaotic [20]. However, it is not a simple task to determine such features, particularly in high dimensional systems. Therefore, it is not obvious whether the fraction of a regular phase space tends to decrease or increase as $n$ increases. A central point in numerical and theoretical investigation of these features, is the choice of stochastic (chaotic) indicators. Several parameters have been introduced in the literature [20] for a numerical investigation: a measure of the rate of energy exchange among normal modes, maximal Lyapunov characteristic exponents, Kolmogorov-Sinai entropy, etc. Generally, one observes a transition from a regular motion at small nonlinearity to seemingly chaotic behavior at large nonlinearity. As the number of degrees of freedom increases at a fixed total energy, depending on the coupling, the system approaches either complete stochastic or complete regular motion. By studying the behavior of the average ratio the amplitud of the resonance to distance between resonance was shown [20], that an example of the former case is the Leonard-Jones gas, and of the latter is the limit of a continuous systems, such as a vibratory string.

Here we will investigate the possibility of describing the "chaotic" component of the Hamiltonian system of many degrees of freedom ($n$) at low energy and assuming that it is below the SST-energy (Strong-Stochasticity-Threshold) [23] by the studying the nonlinear coupling parameter $\lambda$ ($\lambda(n) < \lambda_c$), which measures the strength of the perturbation. The idea is to construct a series of canonical transformations to define a new effective Hamiltonian which describes a system of $n' < n$ degrees of freedom and which takes the same analytical form as the original one but with new values of the nonlinear coupling. This program can be performed in the context of canonical perturbation theory which involves averaging over a set of "fast" variables. For a system which is close to an integrable one and has an ordering of the frequencies, it is possible to introduce a division on the variables into fast and slow ones. In this case, one can introduce a canonical transformation which allows us to define a new Hamiltonian by averaging over the fast degrees of freedom. This new Hamiltonian describes the effective interaction of the remaining slow modes in the presence of the averaged fast variables which are adiabatic



invariants. In general, this Hamiltonian has a more general analytical form than the original one, but the investigation of the special case, when this reduced Hamiltonian is of the same form as the starting one, leads to restriction on the parameters of the systems and shall let us conclude about the behavior of the effective couplings. We want to find out whether $\lambda_0'$ the new strength of the nonlinear coupling, is smaller or bigger than $\lambda_0$. When $\lambda_0' > \lambda_0$, one can expect stronger coupling between the modes in the new Hamiltonian system, and the measure of the KAM-tori in the reduced phase space is decreased. The implication of this result is that as the number of modes increases, the regular motion envelops a bigger region of the phase space. On the contrary, $\lambda_0' < \lambda_0$ implies that for this specific nonlinear system, the region of regular solution in the phase space goes to zero, when $n \to \infty$. The construction of the different canonical transformations, the choice of variables which allows the separation into fast and slow modes, and the method of performing the averaging over the fast variables is completely analogous to the idea of a renormalization group transformation (RGT) for quantum field theory.

Renormalization group techniques have become an important tool to investigate systems with a very large number of degrees of freedom in a non-perturbative way, such as in statistical mechanics [24] and quantum field theories near a phase transition point [25]. Following Wilson's renormalization ideas, renormalization techniques in both momentum-space and position-space or real-space [26] have been developed. In the momentum-space approach [27], a renormalization group transformation of the parameter of the theory with cut-off $\Lambda$ is obtained by eliminating the high frequency components of the field. This can be done, at least in principle, by performing the functional integral in steps. In every step, one integrates out the high frequency parts within the momentum slice between the old and the new ultraviolet cut-off. The new cut-off is lower by some scale factor $L > 1$. All degrees of freedom with momenta below the new cutoff are kept fixed in such an integration step. In this way, one obtains a sequence of effective actions. They describe the theory for different but decreasing ultraviolet cut-offs, or, equivalently, at and above different but increasing length scales. After a RGT-step, the effective action has a new cut-off $\Lambda'$ given by $\Lambda' = \frac{\Lambda}{L}$. In a lattice formulation, the lattice spacing $a$ determines the U.V. cut-off, thus one concludes that the new action is a new theory defined on a lattice with lattice constant $La$. Finally, a subsequent rescaling of the lengths and of the field are introduced. A change of the ultraviolet cut-off ($\Lambda \to \Lambda' < \Lambda$) leads to a modification of the parameters (masses and coupling constants) of the initial Hamiltonian. Since near a phase transition point, the effective Hamiltonian has the same form as the original one, it is possible to repeat this transformation many times. This approach has provided a fundamental explanation for many of the qualitative features of critical phenomena [26].

In case of gauge theory the application of the RGT in momentum-space is more complicated. On the original lattice the momentum may only take the values $(-\frac{\pi}{a}, \frac{\pi}{a})$ and after a RGT-step one obtains an effective theory on the new lattice, where the momentum takes the values in the region $(-\frac{\pi}{La}, \frac{\pi}{La})$. Thus one would like to integrate out the high momentum variables of the original lattice which do not occur on the new lattice $\Lambda'_{La}$. But, unfortunately, due to the gauge invariance of the system, a simple decomposition of high and low frequency modes does not work and another RG-approach is necessary.

The other class of renormalization-schemes is the real-space renormalization (RSRG, or Block Spin Transformation.) [26]. A block spin transformation maps a lattice theory with lattice spacing $a$ into an effective theory on a "block-lattice" with lattice spacing $La$. In this formulation, one of the first steps for a block-spin transformation is the choice of the new

block variables defined on the new lattice. For the boson case, it is easy to define block spin variables [28], but it is not straightforward for gauge variables and fermions [29]. General arguments are required to support the choice of new variables. For the definition of gauge block-spin variables, it is the requirement that the gauge symmetry of the theory are preserved. As a result, different choices of block spin variables for the gauge field are available. Balaban [30] introduced a definition of blocked gauge fields in order to study, in a rigorous mathematical way, the non-abelian lattice gauge theory. Another definition of the Block-spin type "dielectric" was introduced by Mack [31].

It is this idea of performing RG-transformations that we want to apply to formulate a RGT for classical systems. As an application, we can use this formulation for a study of the regular solution of classical systems changing the number of degrees of freedom and the strength of the coupling parameter. We will consider a classical Hamiltonian system of $n$-weakly coupled harmonic oscillators, such as a classical field theory defined on a lattice in a finite volume. We shall assume that the parameters of the theory are given special values (quite similar to those which make the quantum analogous of this system critical.), and that the coupling is still small such that many KAM-tori are preserved (or slowly deformed). Then one may ask whether there exist special classical orbits among the preserved tori which show scaling behaviour when looked upon at different length scales. In order to get information on these trajectories to a very high order of perturbation theory, we will have to introduce and define a renormalization group transformation of the classical system. This can be done by applying the Canonical Perturbation Theory to the classical Hamiltonian [32] .

Bartels and Chang [33] have investigated this idea for a classical $\lambda\phi^4$ field theory on a lattice in a finite volume. In this paper, the averaging over a set of fast degrees of freedom provides a mechanism which is equivalent to the integration over the large momentum degrees of freedom in Wilson's RGT in momentum-space. The result of this averaging procedure which is formulated as a canonical transformation, is an effective Hamiltonian for the remaining slower variables. Following the idea of Wilson's RG-analysis in the vicinity of a critical point, one then might search for a situation where the new Hamiltonian is of the same form as the original one. It was shown that the details of the RGT for this classical system are analogous to the RGT in $\lambda\phi^4$ field theory, i.e. the appearance of diagrams with closed momentum loops and the concept of mass and wave function. Finally, the main results of their paper are the fixed point condition and the scaling law for the field amplitude, which define restrictions on those classical solutions for which this procedure works.

The main goal of this thesis is the formulation of a RG-transformation in the real-space for classical Hamiltonian systems. In the first part, we have reformulated the program of Bartels and Chang following the ideas of the RS-RG. Here we investigate the definition of a "block-spin" transformation for the scalar $\lambda\phi^4$-Hamiltonian, in particular, the definition of the "block-spin"-variables and the method of averaging the fast modes. We do this in order to find an effective Hamiltonian which is derived by identifying that these variables ("block-spin") as the slow modes of the system. In terms of these variables, one can introduce a set of canonical transformation which are discussed in chapter 3 and define completely our renormalization group formalism.

In the second part of this work, we turn to a pure gauge Hamiltonian system. Here we discuss first the general steps of constructing a sequence of canonical transformations which



enabled us, finally, to define a new effective Hamiltonian by averaging over the "fast" modes. This Hamiltonian can be interpreted as an effective Hamiltonian defined on a blocked-lattice. In this case, the formulation of a RG-transformation is more complicated since resonance between the frequencies of the modes are present. Then, the formulation of the RG is divided into three important parts. In the first part, one must define the classical gauge Hamiltonian and define the slow and the fast modes. In the second part, the averaging over the "fast" modes must be performed. Here we present a strategy to handle the resonance in order to define the effective Hamiltonian on the block lattice. In the third part, we present an approach to calculated the effective coupling constant using a numerical solution. The second part of this program can not be completely performed.

Let us discuss the different steps of this formulation. As starting point, let us given the definition of the classical Hamiltonian for a gauge theory on a finite lattice. In the lattice Hamiltonian formulation discussed in chapter 3, one deals with a conventional quantum mechanic system with the Hamiltonian Operator depending on the link variables on the spatial lattice and their conjugate momenta. In this case, the system is described by the time independent Schrödinger equation, and the wave function $\Psi$ depends on the set of all link variables. Since the theory is invariant under time independent gauge transformations, we have to add a subsidiary condition called Gauss' law. In the context of attempts to understand the complicated dynamics of QCD on the lattice, different approximations were introduced. A strong coupling [35] expansion has been used to prove some properties of the wave function and confinement in the region of large lattice coupling constant. On the other hand, the small coupling limit of gauge theories has been investigated using the semiclassical approximation [15]. The way in which the bare lattice coupling constant $g$, which becomes small in the continuum limit, enters in lattice Hamiltonian is analogous to the Planck constant in the Schrödinger equation. Thus, this suggests the use of the semiclassical approximation with $g$ as a small parameter.

Due to the above considerations, it is important to first investigate the SU(2) classical Hamiltonian system, which is derived in chapter 5 through the following correspondence

$$\hat{p} \to p \qquad \hat{q} \to q, \tag{1.2}$$

which can be considered as the opposite to the canonical quantization approach. On a periodic $d$-dimensional lattice with $n^d$- sites, the result is a Hamiltonian with $N = 3d(n-1)^d$-degrees of freedom described by

$$H = H_0 + \sum_{n>1} g^n H_n, \tag{1.3}$$

where

$$H_0 = \frac{1}{2}\left[\sum_{l,links} \vec{p}_l \vec{p}_l + \sum_{p,plaq.} \vec{u}_p \vec{u}_p\right] \tag{1.4}$$

and the nonlinear interaction terms $H_1, H_2$ are defined by equations ( 5.6 and 5.7). Due to the invariance of wave function under SU(2) gauge transformation, we find at each lattice point 3 constraints which are

$$\Omega^i(s) = \sum_{\mu}\{p^i(s,\mu) - p^i(s-\mu,\mu) + g(L^i(s,\mu) + L^i(s-\mu,\mu))\}. \tag{1.5}$$

A systematic study of constraint in SU(2) and SU(3) YMCM was performed [36], and demonstrated that the constraints are non-holonomic.

The key problem in the formulation of a RGT is the choice of a set of independent variables on the original lattice which completely parametrizes the theory and allows the transition to a new lattice, when some variables are eliminated by a gauge fixing and by an averaging process. In order to define a new set of variables we introduce a blocking with size 3 on the two dimensional lattice in analogous form to Quantum Field Theory. This allows us to separate the variables into "irrelevant gauge degrees of freedom" which can be eliminated through a canonical transformation (" gauge fixing"), and another set of variables which can be interpreted as" new links and loops variables" on the block-lattice. In this case, the sites on the block-lattice are defined as the block-center. Now, the elimination of these gauge degrees of freedom is restricted by the requirement for the reduced Hamiltonian to remain invariant under local gauge transformation on the new lattice (i.e. on the block-center). The definition of this gauge and the investigation of the structure of the reduced Hamiltonian represents the first part of this RGT-program. As an important result, we find that the frequencies of the slow and fast modes are separated by a "gap mass" which leads to the condition $w_{slow} << w_{fast}$. This result is obtained by introducing the normal modes, which reduce the integrable term of the Hamiltonian to a normal quadratic form and allows the definition of slow and fast variables.

In the second part of this program, we apply the method of Classical Perturbation Theory in order to obtain the effective Hamiltonian. In analogy to the scalar case, we average over the set of "fast modes". However, due to the gauge invariance, our Hamiltonian is degenerated. Thus, one must use the Degenerated Perturbation Theory. In this context, the procedure of averaging is the following:
a) New variables are introduced around the resonance. The result is that the integrable term is only depending on a set of fast degrees of freedom. Then the averaging is performed over these variables.
b) The averaged Hamiltonian is still dependent on the remaining set of fast variables which are, in general, no Action-Angle variables. The existence of an elliptic fixed point in the reduced phase space, let us introduce an expansion around it and define Action-Angle variables.
c) Now, the remaining Angle variables define at the elliptic point can be averaged. The final result is an effective Hamiltonian which describes the dynamics of "slow modes" in the presence of adiabatic invariants corresponding to the fast modes.
Finally, one can investigate the condition needed to reduce the effective Hamiltonian to the original form.

In this part we begin with the first step of the averaging (step a) over the "fast resonant modes". This calculation is performed to second and third order. In analogy to Quantum Field Theory, this corresponds to one loop calculation. As a result, we found the structure of the effective vertices: two, three, and the vacuum energy (zero slow external lines). Moreover, we obtained an expression for the derivation of the fixed points. In general, we have shown that all these effective expressions are depending on the transformation matrix used to define the normal modes represented in this case by a $10 \times 10$ matrix. An analytic representation of the total matrix was not possible to find. However, a numerical determination is possible to perform. Due to this problem, the study of the fixed points and the averaging over the remaining Angle variables can not be performed analytically.

In the third part, we discuss a numerical approach in order to determine the behavior of the coupling constant under the RGT. Here, we introduce a gauge invariant definition of the entropy (Kolmogorov entropy $h$) which is consider as a global indicator of the stability of the



system. If $h = 0$ then the system is integrable and in the other case, one finds chaotic motion in the phase space. Let us point out the main idea of this approach: a RGT is performed over a Hamiltonian (with a set of coupling constants) in order to find the "effective Hamiltonian" (renormalized) and then one calculates the entropy which is constant under the RGT. Now, one introduces an "approximate" Hamiltonian which is considered to be close to the renormalized Hamiltonian. This Hamiltonian is characterized by a set of coupling constants with another values and it allows to calculate the entropy. Now, one can require that the entropies take the same values which is only possible for a determined values of the coupling constant from the "approximate" Hamiltonian. Assuming that this Hamiltonian corresponds to the renormalized, one can estimated the behavior of the coupling constant under a blocking step. But this numerical calculation is left out in this work.

This thesis is organized a follows: In the next chapter, we introduced some background materials in the Classical Perturbation Theory. A discussion on resonance and their elimination is carried out. In chap. 3 the work of Bartels and Chang is recalled and the block spin transformation approach for the classical scalar field on the lattice is investigated.
The next chapter is devoted to review some topics on Hamiltonian formulation of gauge theory on the Lattice, and the block spin definition for gauge theories. In chap. 5 the definition of the classical Hamiltonian and the definition of the new set of variables is carried out. In the next chap., we dealt on the application of RGT on our Hamiltonian to different order in perturbation theory. Chapter VII contains a discussion of a method for a numerical solution. Chapter VIII contain a discussions of our results. Some details of our calculations are put in the appendices.

# Chapter 2

# Review of Classical Systems.

In this Chapter we review general ideas of integrability and perturbation theory approaches for Hamiltonian Systems. We begin with a brief description of canonical perturbation theory and adiabatic approximation to systems which are near-integrable, and for the case that the frequencies are incommensurable. The following section treats the case of commensurable frequencies. The study of perturbation theory to higher order is carried out in the Lie Transformation Formalism as described in the next subsection.

## 2.1 Perturbation Theory.

The question of the integrability of a Hamiltonian system is a difficult issue. It is found by the Hamilton-Jacobi equation technique [38], that a solution always seems possible, for systems of one degree of freedom. However, when we consider the problem of systems with $n$ degrees of freedom, it is not clear whether the Hamilton-equations are integrable or not. On the other hand, if the Hamilton-Jacobi equation is separable into $n$ independent equations, one for each degree of freedom, then we say that the Hamiltonian is integrable. The separation constants $F_i$ are known as isolating integrals. A Hamiltonian with N degrees of freedom is integrable if and only if $N$-independent isolating integrals exist. The $N$-independent integrals must be in involution, i.e. their Poisson-brackets with each other must vanish:

$$[F_i, F_j] = 0 \qquad i,j = 1...N. \tag{2.1}$$

This ensures that the F's form a complete set of new momenta in some transformed coordinates system.

The choice of the separation constant $F_i$ as the new momenta is arbitrary for a time independent Hamiltonian with $N$-degrees of freedom, which is integrable. One could just as well choose as the new momenta some quantities $J_i$, which are dependent functions of the $F_i$

$$J_i = J_i(F_j), \tag{2.2}$$

with the property that the new Hamiltonian is given by

$$\bar{H} = \bar{H}(\mathbf{J}) \tag{2.3}$$



where $\mathbf{J}$ is an N-dimensional vector, and the Hamiltonian equations may be trivially solved.

Furthermore, if the motion is periodic in each of the $q_k$, i.e., either $p_i$-$q_i$ are periodic functions of the time with the same period (libration), or $p_i$ is a periodic function of $q_i$ (rotation), then a set of action variables can be defined by:

$$I_i = \oint_{C_i} p_k dq_k, \tag{2.4}$$

where $C_k$ is the closed path corresponding to a complete libration motion.

We obtain the conjugate coordinate $\theta_i$ from the new generating function $\bar{S}(\mathbf{q},\mathbf{I})$, which is given by:

$$\theta_i = \frac{\delta S}{\delta I_i}, \tag{2.5}$$

where $\theta_i$ is the angle variable. The transformed Hamiltonian $\bar{H}(\mathbf{I})$ provides the canonical equation,

$$\dot{I}_k = \frac{\delta H}{\delta \theta_i} = 0 \tag{2.6}$$

$$\dot{\theta}_k = \frac{\delta H}{\delta I_i} = \omega_k(I_1,...,I_N), \tag{2.7}$$

where the $\omega_k$ is the frequency associated with each degree of freedom and the equations are trivially integrated to give

$$I_k = const. \tag{2.8}$$

$$\theta_k = \omega_k(I_1,...,I_N) + \delta_k. \tag{2.9}$$

If the frequencies are not in the ratio of a rational number, then the motion is called conditionally periodic. The action-angle variable formulation thus provides a convenient way to obtain the frequencies of oscillation without solving for the details of the motion. When describing the motion of near-integrable systems, one almost invariably makes a preparatory transformation to action-angle variables of the integrable part of the given system.

Normally,, for a system with more than one degree of freedom, it is not obvious that a given Hamiltonian will be separable and that the action variables can be defined. To integrate a Hamiltonian system of $N$ degrees of freedom it is necessary to find $N-1$ additional constants of motion to the constant energy. In general the isolating integrals can be associated with symmetries of the dynamical system. Uncovering an isolating integral is not a straightforward process at all when the symmetry is not obvious, such as the case of the Toda lattice [39]. There is no standard procedure presently known to determine all isolating integrals of a general Hamiltonian system, or even to find their total number. However, it is often by numerical experiment that the presence of a hidden isolating integral is suggested, and one is thus led to its uncovery.

The existence of the $N$-integrals $F_i$ means that the phase-space trajectories will be confined to some $N$-dimensional manifold $M$ in the $2N$-dimensional phase-space. One can show that the Manifold has the topology of the $N$-dimensional Torus. The existence of the Tori in phase-space then provide the means of defining action variables in an invariant way. The $N$-torus is

a natural $N$-independent periodic object. One can define $N$-topologically independent closed paths $C_k$ on the Torus, where none of the paths can be deformed continuously into each other. The set of action variables can thus be defined as

$$I_k = \frac{1}{2\pi} \oint_{C_k} \sum_{i=1}^{N} p_i dq_i. \tag{2.10}$$

It is important to note that if the system is completely integrable the transformation action-angle is global, that is the whole phase-space is filled with tori, and a given trajectory will lie, permanently, on one torus or another. A given set of initial conditions $(\mathbf{q(0)}, \mathbf{p(0)})$ fixes particular values of the integrals, that is, $F_i = F_i(\mathbf{q(0)}, \mathbf{p(0)})$. This set of $F_i$ determines on which torus the trajectory lies, and the values of the angle variables $\theta_i$ at a given time determine the position of the trajectory on that torus.

One can note that, for systems with one-degree of freedom, the energy shell and the tori are the same one-dimensional manifold. This means that these systems are ergodic (in statistical mechanics the term ergodic means that any system trajectory uniformly explores the energy shell). The two dimensional tori are embedded in the three dimensional energy shell for $N = 2$. This means that they divide the energy shell into two parts, an inside and an outside. Thus, if there was somehow a "gap" between the tori (which occurs for non-integrable systems), then the trajectory in that gap can not escape from it, while for $N \geq 3$ trajectories in "gaps" between higher dimensional tori are able to escape to the other regions of the energy shell. This phenomenon is known as the Arnold diffusion [21].

### 2.1.1 Canonical Perturbation Theory

We turn now to a description of those generic Hamiltonian systems that we can treat as perturbation of integrable systems. They are referred to as near-integrable systems. The distinguishing feature of near integrable systems is the simultaneous presence of regular trajectories and a region of stochasticity intimately mixed together. The stochastic trajectories are self-generated as a consequence of the motion induced by Hamilton's equations, which are deterministic and do not contain additional "ad hoc" stochastic forces. Most multidimensional systems are not integrable. However, for systems that do not differ much from integrable ones, one can attempt to obtain a solution to a desired degree of accuracy by expanding the generating function in powers of a small parameter and then solving the equation of motion successively for each power. Another approach can be found in ref. [40].

We now consider an autonomous periodic near-integrable Hamiltonian system with $N$ degrees of freedom which is of the form:

$$H(\mathbf{p}, \mathbf{q}) = \mathbf{H_0}(\mathbf{p}, \mathbf{q}) + \epsilon \mathbf{H_1}(\mathbf{p}, \mathbf{q}), \tag{2.11}$$

where $H_0$ is an integrable Hamiltonian for $\epsilon = 0$, $H_1$ is a non integrable perturbation and $\epsilon$ is the small perturbative parameter.

Canonical Perturbation Theory exploits the special properties of the action-angle variables. The starting point is to express the system as a function of the action-angle variables $(I_i, \theta_i)$ of



the zero order system, that is,

$$H(\mathbf{p}, \mathbf{q}) = \mathbf{H_0}(\mathbf{I}) + \epsilon \mathbf{H_1}(\mathbf{I}, \theta), \tag{2.12}$$

where $H_0$ is a function of the action variables alone and $H_1$ is a periodic function in the phase variables $\theta_i$.

The generic character of the motion for such a system is now fairly well understood. The trajectories lie in the $2N - 1$ dimensional surface $H = const$ of the $2N$-dimensional phase space. According to the KAM-Theorem a finite fraction of the phase space trajectories is regular, that is, it is associated with the first integrals of the motion. The remaining fraction exhibit stochastic (chaotic) behavior.

The basic idea of Canonical Perturbation Theory is to find a new set of action-angle variables $(\bar{\mathbf{I}}, \bar{\theta})$, such that there is a canonical transformation to a new Hamiltonian which is a function only of the $\bar{\mathbf{I}}$ up to a determined order in the perturbation parameter $\epsilon$.

The Canonical Perturbation Theory is formulated in the following way. $H_1$ is considered to be a multiple periodic function of the angles

$$H_1(\mathbf{I}, \theta) = \sum_{\mathbf{m}} \mathbf{H_{1m}}(\mathbf{I}) \mathbf{e}^{\mathbf{i m}\theta}, \tag{2.13}$$

where

$$\mathbf{m}\theta = \mathbf{m_1}\theta_1 + ... + \mathbf{m_n}\theta_\mathbf{n}, \tag{2.14}$$

with the $m_i$ being integers, and the sum being $n$-fold over the $m_i$. We again seek a transformation to new vectors $\bar{\mathbf{I}}, \bar{\theta}$, for which the new Hamiltonian $\bar{H}_1$ is a function of $\bar{\mathbf{I}}$ alone.
We introduce the near-identity generating function whose first-order term is also a multiple periodic function in $\bar{\theta}$

$$S(\bar{\mathbf{I}}, \theta) = \bar{\mathbf{I}}\theta + \epsilon \mathbf{S_1}(\bar{\mathbf{I}}, \theta) \tag{2.15}$$

where

$$S_1(\bar{\mathbf{I}}, \theta) = \sum_{\mathbf{m}} \mathbf{S_{1m}}(\bar{\mathbf{I}}) \mathbf{e}^{\mathbf{i m}\theta}. \tag{2.16}$$

The old action variables and the new angles are found from the canonical transformation equation:

$$I_k = \frac{\delta S}{\delta \bar{\theta}_k}, \tag{2.17}$$

$$\bar{\theta}_k = \frac{\delta S}{\delta I_k} \tag{2.18}$$

respectively, as a power series in $\epsilon$,

$$I_k = \bar{I}_k + \epsilon \frac{\delta S}{\delta \bar{\theta}_k}, \tag{2.19}$$

$$\bar{\theta}_k = \theta + \epsilon \frac{\delta S}{\delta I_k}, \tag{2.20}$$

whereby $\bar{H}(\bar{\mathbf{I}}, \bar{\theta}) = \mathbf{H}(\mathbf{I}(\bar{\mathbf{I}}, \bar{\theta}), \theta(\bar{\mathbf{I}}, \bar{\theta}))$

Expanding the right hand side of this equation in a power series in $\epsilon$, we have the zero order,

$$\bar{H}_0 = H_0(\bar{\mathbf{I}}), \tag{2.21}$$

and a first order

$$\bar{H}_1 = \omega_{0,k}(\mathbf{I})\frac{\delta S_1}{\delta\theta_k} + H_1(\bar{\mathbf{I}},\bar{\theta}), \tag{2.22}$$

where $\omega_{0,k}(\mathbf{I}) = \frac{\delta H_0}{\delta I_k}$ is the frequency for the unperturbed motion.

Since we are seeking a new Hamiltonian that is only a function of the new action variables, we must choose $S_1$ from the above equation in order to eliminate the angle dependent part of $H_1$.
Introducing the average part of $H_1$,

$$< H >_\theta = \frac{1}{(2\pi)^n}\int d\theta H_1\left(\bar{\mathbf{I}},\bar{\theta}\right), \tag{2.23}$$

and the oscillating part

$$\tilde{H}_1 = H_1 - < H_1 >, \tag{2.24}$$

we have to first order that the transformed Hamiltonian takes the form:

$$\bar{H} = H_0\left(\bar{\mathbf{I}}\right) + < H_1\left(\bar{\mathbf{I}},\bar{\theta}\right) >_{\bar{\theta}}, \tag{2.25}$$

and we use the condition:

$$\omega_{0,k}(\mathbf{I})\frac{\delta S_1}{\delta\theta_k} = -\tilde{H}_1\left(\bar{\mathbf{I}},\bar{\theta}.\right) \tag{2.26}$$

Finally, we solve for $S_1$ by integrating the Fourier series for $H_1$, term by term, to obtain.

$$S(\bar{\mathbf{I}},\theta) = \bar{\mathbf{I}}\theta + \epsilon\sum_m i\frac{H_{1m}(\mathbf{I})}{\mathbf{m}\omega_0}e^{i\mathbf{m}\theta}. \tag{2.27}$$

This perturbation series does not in a strict sense converge, and the perturbation theory becomes meaningless. This problem comes from the small divisors, and was clearly recognized by Poincaré. Since for any $\bar{\mathbf{I}}$ and $\mathbf{m}$ can be found that $\mathbf{m}\omega(\mathbf{I})_0$ is arbitrary close to zero, i.e. the fundamental frequencies $\omega_0$ are commensurable then the sum in the series is divergent, but it can be shown (KAM-Theorem) that under certain assumptions for the unperturbed frequencies, these must be "very irrationals", the series converges in certain region of the phase space for some solution for the motion taken over finite but long periods of time (KAM-Series).

**KAM-Theorem**: this theorem guarantees that, under certain assumptions in the case of a perturbation $\epsilon H_1$ with small enough $\epsilon$, the iterative series for the generating function S converges and thus the invariant tori are not destroyed.
The KAM-theorem is formulated by assuming that the Hamiltonian is analytic in a complex domain of the phase space and that the unperturbed motion is non-degenerated.
The conditions which must be fulfilled in order for invariant tori to survive a perturbation are:
1) Linear independence of the frequencies

$$\mathbf{m}\omega_0 \neq \mathbf{0} \tag{2.28}$$



over a certain region of $\bar{\mathbf{I}}$, where this equation is satisfied.
2) Existence of a sufficiently large number of continuous derivatives of $H$.
3) A "sufficiently large distance" from the resonance

$$|\mathbf{m}\omega_0| \geq \gamma|\mathbf{m}|^{-\tau}, \tag{2.29}$$

where $\tau$ is dependent on the number of degrees of freedom, the smoothness of $H_1$, and $\gamma$ is dependent on $\epsilon$.
Since 3) cannot be fulfilled if $\gamma(\epsilon)$ is too large and $\gamma$ increases with $\epsilon$, the smallness of the perturbation is a condition for the existence of KAM-tori.
This theorem was proven by Arnold (1963) [41] for analytic Hamiltonian following the conjectures of Kolmogorov (1954) [42] and Morse (1962) [43].

To be rigorous, this theorem was originally demonstrated for non-degenerate Hamiltonians, i.e, with

$$det\left[\frac{\delta^2 H_0}{\delta I_j \delta I_i}\right] \neq 0, \tag{2.30}$$

but after Birkhoff's work on normal forms [44] the condition of non-degeneracy can be relaxed and the domain of validity of the theorem is extended. Therefore, the special case that $H_0$ describes a system of $N$-independent harmonic oscillators is included in the KAM-theorem [45].

### 2.1.2 Canonical Adiabatic Perturbation.

It is known that for an oscillator with a slowly varying aperiodic parameter, an expansion can be constructed to obtain an adiabatic constant of motion. The expansion parameter $\epsilon$ is the ratio of the fast oscillation period to the scale for the slowly varying parameter. This same general procedure can be used, for multiple periodic systems, to obtain series that do not explicitly contain small denominators.

Kruskal [46] showed that the adiabatic invariant can be generated to all orders in the expansion parameter, and that the resulting series are asymptotic. We can recall an important property of these series: an adiabatic invariant constructed from an asymptotic series is, in the limit $\epsilon \to 0$, an approximation of order $e^{-b/\epsilon}$ of the exact motion.
We now describe the difference in the ordering of an expansion of a near-integrable Hamiltonian for small perturbations and for slow (or adiabatic) ones. For a small perturbation, the Hamiltonian has the general form:

$$H(\mathbf{p},\mathbf{q}) = \mathbf{H_0}(\mathbf{I}) + \epsilon\mathbf{H_1}(\mathbf{I},\theta), \tag{2.31}$$

where $H_0$ describes a completely integrable motion, and $\epsilon$ is the small parameter characterizing the magnitude of the non-integrable part of $H$. For a small perturbation, the derivatives of $H_0$ and $H_1$ are assumed to be of the same order as the $H_0$ and $H_1$ themselves.

For a slow perturbation, the terms produced by differentiation are assumed to be smaller by order of $\epsilon$ than the term from which they are derived, e.g. for slow time variation

$$\frac{\delta H_0}{\delta t} \sim \epsilon H_0. \tag{2.32}$$

In general, one is interested in systems for which the variation in all but one of the degree of freedom, and in the time, is slow. Accordingly, we write the Hamiltonian in the form

$$H = H_0(J, \epsilon y, \epsilon t) + \epsilon H_1(J, \theta, \epsilon y, \epsilon t), \tag{2.33}$$

where $J$ and $\theta$ are the action-angle variables for the unperturbed ($\epsilon = 0$) motion in the single fast degree of freedom, and the $\bar{y} = (\bar{\mathbf{q}}, \bar{\mathbf{p}})$ are the slow canonical variables, not necessarily in action-angle form, of the remaining degrees of freedom. The small parameter (see Lichtenberg [20]) $\epsilon$ in equation ( 2.32) will automatically keep track of the ordering when we differentiate $H$ to construct the perturbation series and may be set to unity at the end of the calculation.

In order to calculate the effect of the perturbation $\epsilon H_1$, we proceed as in the above section. We try to find a transformation from $(J, \theta, \mathbf{y})$ to $(\bar{J}, \bar{\theta}, \bar{\mathbf{y}})$ such that the new Hamiltonian

$$\bar{H} = \bar{H}_0 + \epsilon \bar{H}_1 \tag{2.34}$$

is independent from the fast angle-variable $\bar{\theta}$. Introducing the near-identity generating function

$$S = \bar{\mathbf{J}}\theta + \bar{\mathbf{p}}\mathbf{q} + \epsilon \mathbf{S}_1\left(\bar{\mathbf{J}}, \bar{\theta}, \mathbf{y}\right), \tag{2.35}$$

we have, to first order, the transformation:

$$J_k = \bar{J}_k + \epsilon \frac{\delta S}{\delta \theta_k} \tag{2.36}$$

$$\theta_k = \theta_k + \epsilon \frac{\delta S}{\delta \bar{J}_k} \tag{2.37}$$

$$\mathbf{p} = \bar{p} + \epsilon \frac{\delta S}{\delta \mathbf{q}} \tag{2.38}$$

$$\mathbf{q} = \bar{q} - \epsilon \frac{\delta S}{\delta \mathbf{p}}. \tag{2.39}$$

Inserting these expressions in $H_0$, and expanding to first order in $\epsilon$ we obtain:

$$\bar{H}_0 = H_0(I) + \omega(I)\frac{\delta S_1}{\delta \theta}, \tag{2.40}$$

where $\omega = \frac{\delta H_0}{\delta I}$ is the fast frequency. Note that the other terms

$$\frac{\delta H_0}{\delta \bar{q}_i}\frac{\delta S_1}{\delta \bar{p}_i} \quad , \quad \frac{\delta H_0}{\delta \bar{p}_i}\frac{\delta S_1}{\delta \bar{q}_i} \tag{2.41}$$

are of second order in $\epsilon$ and have been dropped.

Now, since we require $\bar{H}$ to be independent from the fast angle, we choose $S_1$ to eliminate the oscillating part of the $H_1$, holding the slow variable fixed. One can define an average over the fast variable alone in an equivalent form to eq.(2.25). The reduced Hamiltonian is given by

$$\bar{H} = H_0(I) + < H_1(I, \bar{\mathbf{p}}, \bar{\mathbf{q}}) >_\theta \tag{2.42}$$

and the condition

$$\omega(I)\frac{\delta S_1}{\delta \theta} = -\tilde{H}_1\left(\mathbf{y}, \bar{\mathbf{I}}, \bar{\theta}\right), \tag{2.43}$$



which is easily integrable. This series is asymptotically convergent. Resonant denominators never appear and their effect is continually pushed into higher order as the expansion continues.

The adiabatic expansion given above can be carried to higher order. The description and expression for the adiabatic invariant will be presented in the following section using the Lie formalism.

The importance of this method lies in the construction of a hierarchy of invariants. The construction of an adiabatic invariant effectively reduces the system from $n$ to $n-1$ degrees of freedom within the limit of the adiabatic approximation. This follows because the new Hamiltonian is independent of $\theta$ (up to a determined order), hence $J$ is a constant. If one of the remaining degrees of freedom undergoes an oscillation that is fast compared to the other degrees of freedom, then we can introduce a second small parameter $\epsilon_2$, transformed to the action-angle variable of the fast motion for the unperturbed ($\epsilon_2 = 0$) system and find a second adiabatic invariant. This process may be continued to obtain a hierarchy of adiabatic invariants, until the system is reduced to one degree of freedom that can be integrated to obtain the final invariant or to a known system with $n_k$ degrees of freedom having characteristic properties. Then for practical applications, the hierarchy of adiabatic invariants may represent the actual motion to a very good approximation. This idea has been shown to be useful in the context of classical and quantum mechanic [47]. As was discussed, the adiabatic approximation is based on the supposition of widely separated time scales of the modes, i.e. for the usually case of two degrees of freedom, we have that $w_1 << w_2$ or $T_1 >> T_2$. Therefore, one can expecte that the perturbed series is valided in a period of time of the order $0 < t < T_1$. Moreover, an important result has been the realization that the adiabatic approximation ofter works even when a large time scale separation does not exist.

In the above discussion one can find adiabatic invariants through the hierarchy-approach, where the Hamiltonian is reduced to an effective System with $n-1$ degrees of freedom. We want now to consider the case of a Hamiltonian with $n$-degrees of freedom (n-harmonic oscillators), where a group of them is fast compared to the rest of the degrees of freedom.

Let us consider for this discussion a system of harmonic oscillators. In this case the frequencies, $w_k\,(k = 1...n)$, are constant and we introduce the condition that exist $w^*$, such that it divides the frequencies into two groups,

$$w_f = \{w'_k | w'_k > w^*,\ k' = 1...n_f\} \tag{2.44}$$

$$w_s = \{w'_k | w'_k < w^*,\ k' = 1...n_s = n - n_f\}; \tag{2.45}$$

and satisfy

$$\frac{w_s^{max}}{w_f^{min}} < \epsilon, \tag{2.46}$$

where $\epsilon$ is a small parameter. One can then introduce a partial average over the fast oscillators when these conditions are satisfied. This approach enables us to eliminate the fast degrees of freedom of the system, and defines a reduced Hamiltonian with $n - n_f$ degrees of freedom. One can find $n_f$ adiabatic invariant through this formulation, but the convergence of the series is affected by the problem of a small divisor in the case that some resonance conditions between the fast modus are satisfied.

In order to obtain the expression of first order, let us present the following notations; $(p, q)$ are the slow variables with $p, q \in R^{n-n_f}$; and $(P, Q)$ are the fast variables, with $P, Q \in R^{n_f}$ and

the corresponding frequencies satisfy the condition ( 2.46). Following the adiabatic perturbation method, we introduce a generating function $S$ which eliminates the fast angles associated to $P, Q$-variables. The result is

$$H = H_0(I'_f, p', q') + \epsilon < H_1 >_{\theta_f}, \tag{2.47}$$

and the $S_1$ is given by the condition:

$$\sum_{k'=1...n_f} w'_k \frac{\delta S_1}{\delta \theta_{k'}}. = -\tilde{H}_1.$$

The solution in the case that $\tilde{H}_1$ admits a Fourier decomposition is,

$$S_1 = \sum_{k'=1...n_f} i \frac{H_{1m}(I'_f)}{w_{k'} m_{k'}} e^{i m_{k'} \theta_{k'}}. \tag{2.49}$$

The existence of a resonance condition between the fast oscillators destroyed the convergence of the series, as can be seen in the last equation.

## 2.2 Perturbation Theory with Symmetry and Resonances

### 2.2.1 Introduction.

We learn from the previous section on perturbation theory that one can construct an expansion to obtain adiabatic-invariants of the motion. The expansion parameter is a small $\epsilon$-parameter and the series is only asymptotic convergent. The destruction of these adiabatic invariants due to resonance is related to the problem of small denominators appearing in the first order classical perturbation theory. There have been many methods devised to correct the perturbation techniques so that terms with small denominators do not occur to destroy the convergence of the series. One of such methods which we have used in this investigation is the degenerate perturbation theory or the method of secular perturbation [20, 37]. In this case, the resonance can be eliminated from the unperturbed Hamiltonian by a canonical transformation to a frame of reference that rotates with the resonance frequency. The new coordinates then measure the slow oscillation of the variables about their values at resonance, which is an elliptic fixed point of the new phase space. This technique has been applied to the nonlinear theory of accelerator dynamics [48] and to electron cyclotron resonance in a mirror field [49]. In general, these systems are described by a Hamiltonian with two or three degrees of freedom. We will discuss here the generalization to Hamiltonian with many degrees of freedom (multidimensional-systems), which are invariant under a symmetry transformation.

The existence of resonances is closely linked to the convergence of the asymptotic series in multisystems. For periodic solutions represented as motion on an $n$-dimensional Torus with frequencies that are sufficiently incommensurable and for nonlinear coupling that is sufficiently small, the effect of the perturbation is only a slight deformation of the toroidal surface. However for frequencies that are commensurable, the perturbed Tori are deformed greatly and the particle orbits do not remain close to the unperturbed torus.



The procedure for determining the invariants of an $n$-dimensional system is rather involved, and we have outlined here the general steps as they will be used in the following sections.

1.) The Hamiltonian is divided into two parts. A zero order part which can be transformed to action-angle variables and is integrable; and a second part which is not integrable. In general this perturbation term is of the first order in some small parameter $\epsilon$.

2.) a: If one of the frequencies of the unperturbed system is much bigger than the other, the method of averaging is employed to obtain the first order invariants of the fast oscillators, including the perturbation. This method can be used to obtain invariants to higher order.
b: If two frequencies have a low order of commensurability, then a canonical transformation is made to a rotating frame in which there is only a single frequency for the unperturbed motion (intrinsic degeneracy) or two widely spaced frequencies (accidental degeneracy), and the averaging can be performed as in a). If after the transformation a resonance persists between the slow variables and the fast variables, then the additional resonance must be removed. In general, this last resonance is only accidental.

The averaged Hamiltonian has a dependence on the slow transformed variables after the series of canonical transformations, but these variables in the rotating frame are no longer in action-angle form and the averaged Hamiltonian defines a new Hamiltonian system with $n-1$ degrees of freedom. Now the Hamilton-Jacobi equation must be solved to re-introduce action-angle variables. The motion given by the reduced Hamiltonian is effectively the motion in a single degree of freedom for the case $n = 2$, and therefore integrable. The solution for this system and the definition of action-angle variables are not trivial problems for the case $n > 2$. On the other hand, one approximated solution can be found in the following steps:
i) the Hamiltonian is averaged over the fast variables,
ii) the averaged Hamiltonian is expanded around an elliptic point and action-angle variables are obtained from the lowest order of the expansion.
iii) Perturbation Theory is used to obtain adiabatic-invariants to higher order for the averaged Hamiltonian, and
iv) since the second order term in $\epsilon$ is angle dependent, a Perturbation Theory for higher orders can be applied. The process in 2a-b above is then repeated to obtain the new invariant of the motion.

The general theory to remove the degeneracy is discussed in the following subsections. In the last part, we consider an example of a nonlinear coupled oscillator with resonance which is a consequence of the symmetry of the system.

### 2.2.2 General Theory

We consider in this case an unperturbed Hamiltonian system for $n$-oscillators which is completely integrable. We note that in such a system, the frequencies are not dependent on the amplitude, i.e. they are constant.

Action-angle variables $(\vec{I}, \vec{\varphi})$ where $\vec{I} = (I_1, ..., I_n) \in B \subset R^n$, $\vec{\varphi} = (\varphi_1, ..., \varphi_n)\ mod\ 2\pi \in R^n$,

are introduced in the domain $B$ of $R^n$ where the variables $I_i$ are defined. Here $I_i$ are the first integrals of the motion where the equations $I_i = const.$ singles out an invariant $n$-dimensional torus and $\varphi_i$ are coordinates (phases) on this torus. $\vec{w} = (w_1, ..., w_n)$ are the constant frequencies of the unperturbed motion. The Hamiltonian $H_0$ of the unperturbed system depends only on the action variables:

$$H_0 = H_0(\vec{I}). \tag{2.50}$$

We introduce a small Hamiltonian perturbation, which is assumed to be periodic in $\vec{\varphi}$ with period $2\pi$, such that it can be expanded as

$$H_1 = \sum_{\vec{l}} H_{\vec{l}}(\vec{I}) e^{i(\vec{\varphi} \cdot \vec{l})}. \tag{2.51}$$

We will discuss the case that in the equation of the unperturbed motion, the frequencies are identically commensurable or resonant, i.e. that one or $r$ relations of the form:

$$\vec{k}\vec{w} = 0 \qquad \text{where} \qquad \vec{k} = (k_1, ..., k_n) \in Z^n/\{0\} \qquad \vec{k}\vec{w} = \sum_i k_i w_i \tag{2.52}$$

hold. The trajectories of the unperturbed motion on the torus $T^n$, then fill up the tori with dimensions smaller than $n$ and the averaging over the entire $T^n$ can not yield a correct description of the motion.

Given one or $r$ resonance relations (or simply resonances), i.d. $(\vec{k}\vec{w}) = 0$, one can prove that there exists a change of variables from $(I, \varphi)$ to $(J, \psi)$ so that in the new variables the Hamiltonian admits the following form.

$$H = H_0(J_1, \ldots, J_{n-r}) + H_1(\vec{J}, \vec{\psi}) \qquad r < n, \tag{2.53}$$

where

$$\vec{\psi} = (\gamma_1, ...., \gamma_r, \chi_1, ...., \chi_{n-r}) \qquad \text{and} \qquad \vec{J} \in R^n. \tag{2.54}$$

In this case the phases $\gamma_k$ $(k = 1...r)$ are the slow variables compared to $\chi_l$ $(l = 1...(n-r))$.

$$\begin{aligned} \dot{\gamma_k} &= \frac{\partial H}{\partial J_k} = O(\epsilon), \\ \dot{\chi_l} &= \frac{\partial H}{\partial J_l} = O(1). \end{aligned} \tag{2.55}$$

To give an approximate description of the evolution, in agreement with the averaging principle, one has to average the perturbed motion over the fast phases $\chi_l$. Now, we can state the following theorem (see ref. [19]: Arnold "Dynamical System III, page 170):

**Theorem**: In a Hamiltonian system with n degrees of freedom and $k' = (n - r) < n$ frequencies, the variables canonically conjugated to the fast variables are first integrals of the averaged system.

This theorem states that we can obtain, through averaging, a reduced Hamiltonian system with $r$-degrees of freedom for the slow variables (phases) and their canonically conjugated variables,

$$H = H_0(J'_1, ..., J'_{n-r}) + H_1(\vec{J'}, \gamma'_1, ..., \gamma'_r). \tag{2.56}$$



This procedure of eliminating fast variables in the presence of resonances is called partial average.

In the case of $n$ degrees of freedom and $r < n$ resonance conditions, a reduced Hamiltonian with $r$ degrees of freedom is yielded. The solution of this new system is not trivial. However,a considerable amount of information on the motion in the original system can be obtained through the investigation of the singular point or equilibrium points of the reduced Hamiltonian. Clearly such an investigation has a local behavior.

The fixed points of the motion (or singularities in the phase space) are those values of $(J'_i, \gamma'_i, i = 1..r)$, denoted as $(J'_0, \gamma'_0)$, for which the phase flow is stationary, that is

$$\begin{aligned} \frac{\partial H}{\partial J_k} &= 0, \\ \frac{\partial H}{\partial \gamma_k} &= 0. \end{aligned} \tag{2.57}$$

These can be any number of points $(J'_0, \gamma'_0)$, satisfying these conditions, depending on the precise functional form of $H$. We can examine the behavior of the system in the neighbourhood of these points, and discuss the nature of their stability after identifying them.

### 2.2.3 Nonlinear coupled oscillator

In this section, we investigate an example which has an intrinsical resonance. It is the result of the symmetry of the Hamiltonian.

We consider a pair of two-dimensional coupled harmonic oscillators described by the Hamiltonian,

$$H = H_0 + \lambda H_1, \tag{2.58}$$

$$H = \frac{1}{2}\left(\vec{p_1}^{\,2} + w_1{}^2\vec{q_1}^{\,2} + \vec{p_2}^{\,2} + w_2{}^2\vec{q_2}^{\,2}\right) + \lambda\left((\vec{q_1}^{\,2} + \vec{q_2}^{\,2})^2 + a\,(\vec{q_1} \cdot \vec{q_2})^2\right) \tag{2.59}$$

where $\lambda$ is a small parameter. The significance of $\lambda$ is to remind us that for small amplitude oscillations in $\vec{q_1}$ and $\vec{q_2}$, the terms in the perturbation $H_\lambda$ are smaller than the terms in $H_0$. Moreover, the Hamiltonian $H_0$ is invariant under an independent rotation of the variables $\vec{q_1}$ and $\vec{q_2}$, but the interaction term is only invariant under the simultaneous rotation of the $\vec{q_1}$ and $\vec{q_2}$.

We assume that $w_1 << w_2$. Under this assumption, the motion of $\vec{q_1}$ is much slower than that of $\vec{q_2}$. We shall refer to $\vec{q_1}$ as the slow mode and to $\vec{q_2}$ as the fast mode. In this case, we can introduce the Adiabatic Perturbation Theory to find the adiabatic invariant and to define an effective Hamiltonian depending only on the slow mode. (see Sec. Adiab. Pert. Ther.)

Let us consider only an effective Hamiltonian system in which the slow mode at the lowest approximation can be treated as a constant.

$$H_{adiab} = \frac{1}{2}\left(\vec{p_2}^{\,2} + w_2{}^2\vec{q_2}^{\,2}\right) + \lambda\left((\vec{q_2}^{\,2})^2 + 2\vec{q_1}^{\,2}\,\vec{q_2}^{\,2} + a(q_{1i}q_{2i} + q_{1j}q_{2j})^2\right). \tag{2.60}$$

We transform eq.(2.60) to the appropriate form to apply our general theory by introducing two dimensional action-angle variables for the harmonic oscillator, which are given by

$$q_k = \sqrt{\frac{2J_k}{w_2}}\, sin\,\psi_k \tag{2.61}$$

$$p_k = \sqrt{2J_k w_2}\, cos\,\psi_k, \tag{2.62}$$

yielding

$$H = H_0(J_1, J_2) + \lambda H_1(\vec{J}, \vec{\psi}), \tag{2.63}$$

where

$$H_0 = w_2\,(J_1 + J_2)\,, \tag{2.64}$$

$$\begin{aligned} H_1 = & \left(\frac{2J_1}{w_2}sin^2\psi_1 + \frac{2J_2}{w_2}sin^2\psi_2\right)^2 + 2\vec{y}^2\left(\frac{2J_1}{w_2}sin^2\psi_1 + \frac{2J_2}{w_2}sin^2\psi_2\right) + \\ & a(\sqrt{\frac{2J_1}{w_2}}y_1 sin\psi_1 + \sqrt{\frac{2J_2}{w_2}}y_2 sin\psi_2)^2). \end{aligned} \tag{2.65}$$

The unperturbed frequencies for the two degrees of freedom are then:

$$w_{x_1} = \frac{\partial H}{\partial J_1} = w_2, \qquad w_{x_2} = \frac{\partial H}{\partial J_2} = w_2, \tag{2.66}$$

so that the resonance condition becomes:

$$\frac{w_{x_1}}{w_{x_2}} = 1. \tag{2.67}$$

To remove the intrinsic degeneracy, we choose the generating function

$$F_2 = (\psi_1 - \psi_2)\, I_1 + \psi_2 I_2, \tag{2.68}$$

which defines a canonical transformation from $(\vec{J}, \vec{\psi})$ to $(\vec{I}, \vec{\theta})$ such that

$$\theta_1 = \frac{\partial F_2}{\partial J_1} = \psi_1 - \psi_2 \tag{2.69}$$

$$\theta_2 = \frac{\partial F_2}{\partial J_2} = \psi_2 \tag{2.70}$$

$$J_1 = \frac{\partial F_2}{\partial \psi_1} = I_1 \tag{2.71}$$

$$J_2 = \frac{\partial F_2}{\partial \psi_2} = I_2 - I_1. \tag{2.72}$$

The new variable $\theta_1$ measures the slow deviation from resonance. In removing a degeneracy by using a generating function as in (2.68), there is an arbitrary choice of the original phase variables which is left unchanged. In the case that $w_1/w_2 \approx r/s$, and $w_1 < w_2$, one can introduce a criteriom for this choice, but in the case $w_1 = w_2$ it is still arbitrary. After the canonical transformation, the unperturbed Hamiltonian $H_0$ is a function of $I_2$ alone,

$$H_0 = w_2 I_2; \tag{2.73}$$



and the total Hamiltonian in terms of the new variables is

$$\begin{aligned} H = & w_2 I_2 + \\ & \lambda\frac{4}{w_2^2}\{\frac{3}{8}I_2^2 + \frac{w_2}{2}\vec{y}^2\, I_2 + \frac{a\,w_2}{4}(I_1(y_1^2 - y_2^2) + y_2^2\, I_2) + \\ & \frac{a\,w_2 y_1\, y_2}{2}\sqrt{I_1(I_2 - I_1)}cos\theta_1 + \frac{I_1(I_2 - I_1)}{2}sin^2\theta_1 + \\ & \frac{I_1^2}{8}cos(4\theta_1 + 4\theta_2) - 4cos(2\theta_1 + 2\theta_2)) + \\ & \frac{(I_2 - I_1)^2}{8}(cos(4\theta_2) - 4cos(2\theta_2)) + \\ & \frac{I_1(I_2 - I_1)}{2}\left(\frac{cos(2\theta_1 + 4\theta_2)}{2} - cos(2\theta_1 + 2\theta_2) - cos2\theta_2\right) - \\ & \frac{w_2\, I_2(2\vec{y}^2 + a y_1)}{4}(cos(2\theta_2 + 2\theta_1)) - \\ & \frac{w_2(I_2 - I_1)(2\vec{y}^2 + a y_2)}{4}(cos(2\theta_2)) - \\ & \frac{w_2 a y_1\, y_2\sqrt{I_1(I_2 - I_1)}}{2}(cos(2\theta_2 - \theta_1))\}. \end{aligned} \tag{2.74}$$

The Hamiltonian equations for $\theta_i$ are:

$$\begin{aligned} \dot{\theta_1} &= \frac{\partial H}{\partial I_1} = O(\lambda), \\ \dot{\theta_2} &= \frac{\partial H}{\partial I_2} = O(1). \end{aligned} \tag{2.75}$$

We see that $\theta_1$ is varying slowly as compared to $\theta_2$ and, hence we can average eq. (2.74) over $\theta_2$. Now proceeding as in section Pert. Theor., we average over the fast phase as in eq. (2.42), to obtain the transformed Hamiltonian to first order

$$\begin{aligned} \bar{H} = & w_2 I_2 + \lambda\frac{4}{w_2^2}\{\frac{3}{8}I_2^2 - \frac{I_1(I_2 - I_1)}{2}sin^2\theta_1 + \frac{w_2}{2}\vec{y}^2 I_2 + \\ & \frac{a w_2 y_1 y_2}{2}\sqrt{I_1(I_2 - I_1)}cos\theta_1 + \frac{a w_2}{4}\left(I_1(y_1^2 - y_2^2) + y_2^2 I_2\right)\} \end{aligned} \tag{2.76}$$

Here, we have introduced a generating function:

$$F_2(I_i', \theta_i) = \sum_i I_i'\theta_i + \lambda S_1(I_i'\theta_i), \tag{2.77}$$

where $S_1$ is defined by:

$$S_1(I_i', \theta_i) = -\frac{1}{w_2}\int d\theta_2 \tilde{H}, \tag{2.78}$$

and given by

$$\begin{aligned} S_1(I_i', \theta_i) = & -\frac{1}{w_2}\lambda\frac{4}{w_2^2}\{\frac{I_1^2}{8}\left(\frac{sin(4\theta_1 + 4\theta_2)}{4} - 2cos(2\theta_1 + 2\theta_2)\right) + \\ & \frac{(I_2 - I_1)^2}{8}\left(\frac{sin(4\theta_2)}{4} - 2cos(2\theta_2)\right) + \end{aligned}$$

$$\frac{I_1(I_2-I_1)}{2}\left(\frac{sin(2\theta_1+4\theta_2)}{8}-\frac{sin(2\theta_1+2\theta_2)-sin2\theta_2}{2}\right)-$$
$$\frac{w_2I_2(2\vec{y}^2+ay_1)}{8}sin(2\theta_2+2\theta_1)-$$
$$\frac{w_2(I_2-I_1)(2\vec{y}^2+ay_2)}{8}sin(2\theta_2)-$$
$$\frac{w_2ay_1y_2\sqrt{I_1(I_2-I_1)}}{4}sin(2\theta_2-\theta_1)\}. \qquad (2.79)$$

The lowest order adiabatic invariant is therefore

$$\dot{I'}_2=0\rightarrow I'_2=I_1+I_2+O(\lambda), \qquad (2.80)$$

with $I'_2$ a constant and the $I'_1,\theta'_1$-motion given by Hamiltonian (2.76) is effectively the motion in a single degree of freedom and therefore is integrable.

The stationary points $(I'_1,\theta'_1)_0$ exist in the $I'_1,\theta'_1$-phase plane at

$$\frac{\partial\bar{H}}{\partial I_1}|_{f.p} = 0,$$
$$\frac{\partial\bar{H}}{\partial\theta_1}|_{f.p} = 0. \qquad (2.81)$$

In this case, we obtain an elliptic point which represents a periodic solution for the perturbed Hamiltonian. We can expand the variable $(I'_1,\theta'_1)$ about the elliptic singularity to obtain an approximate solution through the normal perturbation theory near this point. In the given example, we can solve the equation of motion defined by the Hamiltonian (2.76) for the variables $(I'_1,\theta'_1)$ by transforming them to traditional action-angle variables (see ref. [50]). However, instead of doing this, we examine the general character of the solution by linearizing about the elliptic fixed point. We also include higher order terms in power of $\delta\theta_1$ and $\delta J_1$ up to a determined order in the Hamiltonian, where $\delta\theta_1$,$\delta J_1$ are new variables about the stationary point defined by:

$$\delta\theta_1 = \theta_1-\theta_{10}$$
$$\delta J_1 = J_1-J_{10}. \qquad (2.82)$$

We have

$$\bar{H}(\delta\theta_1,\delta J_1) = \bar{H}(\delta\theta_{10},\delta J_{10})+\frac{\partial\bar{H}}{\partial I_1}|_{f.p}\delta I_1+\frac{\partial\bar{H}}{\partial\theta_1}|_{f.p}\delta\theta_1+ \qquad (2.83)$$
$$\frac{1}{2}\left(\frac{\partial^2H}{\partial I_1^2}|_{f.p}(\delta I_1)^2+\frac{\partial^2H}{\partial\theta_1^2}|_{f.p}(\delta\theta_1)^2+2\frac{\partial^2H}{\partial\theta_1I_1}|_{f.p}\delta I_1\,\delta\theta_1\right)+\sum_k V_k(\delta I_1,\delta\theta_1). \qquad (2.84)$$

The linear terms in $\bar{H}$ drop out in virtue of the eq. (2.81)

Applying (2.81) we find the following condition for $\theta_1$:

$$-sin(\theta_1)\sqrt{I_1(I_2-I_1)}\left(cos(\theta_1)+\frac{aw_2y_1y_2}{2\sqrt{I_1(I_2-I_1)}}\right)=0 \qquad (2.85)$$



and the following values:

$$\theta_1=0,\pi \qquad \text{and} \qquad cos(\theta_1)=-\frac{aw_2y_1y_2}{2\sqrt{I_1(I_2-I_1)}}. \qquad (2.86)$$

With these values for $\theta$'s we obtain for $I_1$ the following equations which define their values:

$$\frac{w_2a(y_1^2-y_2^2)}{4}+\frac{w_2ay_1y_2(I_2-2I_1)}{4\sqrt{I_1(I_2-I_1)}}\xi=0, \qquad (2.87)$$

where $\xi=+/-$ correspond to $\theta=0,\pi$ and

$$\frac{w_2a(y_1^2-y_2^2)}{4}-\frac{(I_2-2I_1)}{2}=0. \qquad (2.88)$$

The fixed points are given by:

$$I_{10}^{(+/-)} = \frac{I_2}{2}\left(\frac{\vec{y}^2(+/-)sign(y_1^2-y_2^2)}{\vec{y}^2}\right), \qquad (2.89)$$
$$I_{10} = \frac{I_2}{2}+\frac{w_2a(y_2^2-y_1^2)}{4}. \qquad (2.90)$$

In order to study the nature of a singular point, we define the following expressions

$$f = \frac{\partial^2H}{\partial I_1^2}|_{f.p} \qquad (2.91)$$
$$g = \frac{\partial^2H}{\partial\theta_1^2}|_{f.p} \qquad (2.92)$$
$$h = \frac{\partial^2H}{\partial\theta_1I_1}|_{f.p}. \qquad (2.93)$$

A criterion to determine the nature of a singular point states: if $f>0$ or $g>0$ and $fg-h^2>0$, then the singular point is elliptic [51] and the Hamiltonian can be used to plot the phase orbit in the reduced phase space. For an n-dimensional system, the elliptic point is defined by the condition that all eigenvalues of the quadratic term of the linearization are complex. Now, the frequency of the quadratic part is then given by:

$$w_{f.p}^2=fg-h^2. \qquad (2.94)$$

One can show that only the fixed point defined through the second conditions (2.86) and (2.90) satisfies the condition for an elliptic point and it is a stable fixed point for our Hamiltonian. Moreover, the other conditions defined hyperbolic points.

Finally the reduced Hamiltonian (up to quadratic order in the expansion about the fixed point) is

$$\bar{H}=w_2I_2+\lambda\frac{4}{w^2}\left(\frac{1}{4}I_2^2+\frac{w(1+a/4)}{2}\vec{y}^2\,I_2-\frac{(a\,w)^2(\vec{y}^2)^2}{32}+w_{f.p}\mathcal{I}_1\right), \qquad (2.95)$$

where the second term is the Hamiltonian (2.83) evaluated at the fixed point, and the last term corresponds to the harmonic oscillator. The $\mathcal{I}_1$ is the new action variable for the oscillator defined at the fixed point and $w_{f.p}$ becomes

$$w_{f.p}^2=\frac{4I_2^2-(wa)^2(\vec{y}^2)^2}{16}. \qquad (2.96)$$

### 2.2.4 Higher Order Perturbation Theory

Now, we discuss the correction of the next order of the invariants derived from the higher order nonlinear terms. This comes from two different sources: 1. the nonlinear term results when we expand $H$ in the neighbourhood of the equilibrium position. The Hamiltonian up to order k can be expressed as

$$\bar{H} = w_{f,p}(\delta I^2 + \delta\theta^2)/2 + H_1 + ..... + H_k, \tag{2.97}$$

where $H_k$ is a form of degree k in the phase variables $\delta I_1, \delta\theta_1$.
2. A nonlinear term appears when we consider the term of order $\lambda^2$, which can be written as

$$H = \bar{H}(I_1, I_2) + \lambda^2 H_2(I_1, I_2, \theta_1, \theta_2). \tag{2.98}$$

If we take into account the effect of the second order in $\lambda$ to modify the adiabatic invariant, we can find in this case a secondary resonance which is present in the Hamiltonian (2.98). These resonances are between the harmonic frequency of the $\delta I_1, \delta\theta_1$-phase oscillator and the frequency $w_2$ obtained from the Hamiltonian (2.98). In general, these resonances are accidental and are present in the case of a non-small $\lambda$-parameter. They can be removed in a manner analogous to the one used in the last section. We postpone this discussion, in order to treat the first source of nonlinearity.

These nonlinear terms in the Hamiltonian (2.97) can be investigated using canonical Perturbation Theory or applying a Birkhoff normal form. Let us introduce the following definition: A Birkhoff normal form of degree L for a Hamiltonian is a polynomial of degree L in the symplectic phase variables $\delta I_1, \delta\theta_1$, which is generally a polynomial of degree L/2 in the variable $\tau = \sqrt{(\delta I_1^2 + \delta\theta_1^2)}/2$. With this definition one can state the following theorem:

**Theorem** [Birkhoff]: There exists a symplectic change of coordinates $(\delta I_1, \delta\theta_1)$ to (P,Q) in the neighbourhood of the equilibrium position such that in the new coordinates the Hamiltonian is reduced to a Birkhoff normal form of degree L up to the terms of order L+1,

$$H(P,Q) = \mathcal{H}_\mathcal{L} + \mathcal{R} \tag{2.99}$$

$$R(P,Q) = O(|P| + |Q|)^{l+1}. \tag{2.100}$$

If we neglect the remainder $R$ in (2.99) we obtain a completely integrable system for which the symplectic polar coordinates $\tau, \psi$ given by

$$Q_k = \sqrt{\frac{2J}{w}} sin\psi_k \qquad P_k = \sqrt{2Jw} cos\psi_k, \tag{2.101}$$

are action-angle variables.

Birkhoff normalization reduces to the "average procedure" (Lindstedt's) for eliminating fast variables (up to order L), upon rescaling the deviation from the equilibrium position by a small quantity $\epsilon$ and introducing symplectic polar coordinates.

We let the transformed Hamiltonian be $K_0(I)$. With $I$ we indicate that $K_0$ is a form of degree $l$ in the (P,Q) variables or order $l/2$ in the $\tau$-variable. The new Hamiltonian has the form,

$$K_0 = H_0 + \lambda\{ H_1(I_2, I_{10}, \theta_{10}) + w_{f,p}\mathcal{I}_1 + \mathcal{H}_1(\tau) + .....\mathcal{H}_l(\tau)\} \tag{2.102}$$



where $w_{f,p}$ and $\mathcal{H}_l$ are functions of $I_2$. The Hamiltonian (2.102) is the formal solution up to first order in $\lambda$ . It is independent from the angles so that $I_1, I_2$ are two constants of the motion.

To complete this general discussion, we take into account the effect of the next order perturbation term, which modifies the invariant to this order. Let us write the total Hamiltonian beyond the first order perturbation theory,

$$H = K_0(I_1, I_2) + \lambda^2 H_2(\vec{I}, \vec{\theta}), \tag{2.103}$$

which has a form similar to eq.(4), so that we can use the method of the average over all phase variables $\theta_1, \theta_2$. The new frequencies are defined by

$$\begin{aligned} w_1' &= \frac{K_0}{\partial I_1} \\ w_2' &= \frac{\partial K_0}{\partial I_2}. \end{aligned} \tag{2.104}$$

If the frequencies are incommensurable, then we can apply the canonical perturbation theory to obtain the new invariant. In the case that the frequencies satisfy a resonance condition

$$\frac{w_{x_1}}{w_{x_2}} = \frac{p}{q}, \tag{2.105}$$

we transform then to new variables as in the sect.2 , where the generating function is given by:

$$F_2 = (p\theta_1' - q\theta_2')\, I_1 + \theta_2' I_2, \tag{2.106}$$

and $\theta_1' = p\theta_1 - q\theta_2$ is the slow variable while $\theta_2'$ is the fast variable. To obtain the new invariant of the motion, one can then repeat the process of the last section.

We must investigate the existence of a resonance for our calculation to the second order. The "unperturbed" frequencies up to $O(\lambda^2)$ are defined as

$$w_2' = \frac{\partial K_0}{\partial I_2} = w_2 + O(\lambda), \tag{2.107}$$

$$w_1' = \frac{\partial K_0}{\partial I_1} = O(\lambda). \tag{2.108}$$

We can see that the resonance is accidental and is only relevant in the case that $\lambda$ is not small. Assuming that the coupling constant is small, this resonance condition need not be considered. We can calculate to second order the adiabatic invariant with this result, as discussed in the sec. (Pert. Ther.) A generating function is introduced and is given by the power series in $\lambda$ up to order $O(\lambda^3)$,

$$F_2(I_i', \theta_i) = \sum_i I_i'\theta_i + \lambda S_1(I_i', \theta_i) + \lambda^2 S_2(I_i', \theta_i). \tag{2.109}$$

In terms of the new variables, we obtain

$$H(\theta_i', J_i') = H_0(J_2') + \lambda\left(w2\frac{\partial S_1}{\partial I_2} + H_1(I_i', \theta_i')\right) + \tag{2.110}$$

$$\lambda^2\left(w_2\frac{\partial}{\partial\theta_2}\left(S_2 - \frac{1}{2}\frac{\partial S_1}{\partial I_i}\frac{\partial S_1}{\partial I_i}\right) + H_2\right), \tag{2.111}$$

with

$$H_2(\theta_1', J_1') = \left[ S_1, w_2 \frac{\partial S_1}{2\partial\theta_2} + H_1(I_1', \theta_1') \right], \tag{2.112}$$

where we chose $S_1$ to cancel the oscillatory part of $H_1$ with respect to $\theta_2$. Given $S_1$, we can compute $H_2$. We average over the fast variable $\theta_2$ and obtain to order $\lambda^2$ the new Hamiltonian

$$H = H_0(I_2) + \lambda H_1(\vec{I}, \theta_1) + \lambda^2 H_2(\vec{I}, \theta_1), \tag{2.113}$$

which is dependent on the phase variable. In order to obtain general information about the solutions, we can repeat the step used in the investigation in the last section in order to find the correction to first order, i.e. we can find an elliptic point and expand around it. Finally, we can introduce a canonical transformation to express our reduced Hamiltonian as a Birkhoff normal form of order k.
The singular points in this order are given by:

$$\frac{\partial \bar{H}}{\partial I_1}|_{f.p} = \frac{\partial H_1}{\partial I_1}|_{f.p} + \lambda \frac{\partial H_2}{\partial I_1}|_{f.p} = 0 \tag{2.114}$$

$$\frac{\partial \bar{H}}{\partial \theta_1}|_{f.p} = \frac{\partial H_1}{\partial \theta_1}|_{f.p} + \lambda \frac{\partial H_2}{\partial \theta_1}|_{f.p} = 0. \tag{2.115}$$

To find the singular points, we consider the following statement: The effect of the second order term modifies the singular points derived from the first order for a quantity of order $\lambda$,

$$\begin{aligned} I_{f.p} &= I_{10} + \lambda I_{11} \\ \theta_{f.p} &= \theta_{10} + \lambda \theta_{11}, \end{aligned} \tag{2.116}$$

which is based in the Theorem of the inverse function. obtaining up to quadratic order in the expansion around the fixed point

$$\bar{H} = w_2 I_2 + \lambda \frac{4}{w^2} \left( \frac{1}{4} I_2^2 + \frac{w(1+a/4)}{2} \vec{y}^2 I_2 - \frac{(a\, w_2)^2 (\vec{y}^2)^2}{32} + w_{f.p}^* \mathcal{I}_1 \right) \tag{2.117}$$

$$\lambda^2 f(|\vec{y}|), \tag{2.118}$$

where $w^*$ is the frequency to the second order and $f(|\vec{y}|)$ is

$$f(|\vec{y}|) = \frac{1}{128} \{ I_2 (\vec{y}^2)^2 (32 + 16a\, w_2 + a^2\, w_2) + I_2 \vec{y}^2 (62 + 16a) + 32 \frac{I_2^2}{w_2^2} \}. \tag{2.119}$$

## 2.3 Lie transformation Method.

The expansion of the Perturbation Theory and the calculation of the invariants to higher than the first orde becomes increasingly tedious when the averaging procedure of Kruskal [46] and Bogoliubos [52] are used. The reason is that the inverse transformations become increasingly complicated when both angle and action variables are considered together and any relationship that exists in the physic may lie hidden. It is more convenient to obtain the expression to higher order term of an expansion of the transformation in a power series in $\epsilon$ with the Lie Algrebraic Method introduced by Hori [53] and Gourido [54]. No function of mixed variables



appears when the Lie Method is used and all the terms in the series are repeating Poisson brackets, making the theory canonically invariant.

Contributions to improve the method were also made by Depril [55] and others. A presentation of the Lie formalism can be found in ref. [33]. In the following section we develop the basic Lie formalism to obtain the series for adiabatic invariants. We conclude by describing briefly the modifications to higher order for constraint in the case that the system admits a symmetry.

We now consider the autonomous systems. The generalization to time-dependent Hamiltonian can be studied in the context of extended Hamiltonian formulation.

Let $\mathbf{x} = (\mathbf{p}, \mathbf{q})$ be the vector of generalized momenta and coordinates representing the system point in the phase space. We take the function $w(\mathbf{x}, \epsilon)$, called a Lie generating function which satisfies the condition

$$\frac{\delta \bar{\mathbf{x}}}{\delta \epsilon} = [\mathbf{x}, w]. \tag{2.120}$$

These equations are just Hamilton's equations in Poisson bracket notation which are applied to the "Hamiltonian" $w$ with the parameter $\epsilon$ as the time, and therefore generate a canonical transformation for any $\epsilon$, i.e. the solution for any initial system point $\mathbf{x}$

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{x}, \epsilon), \tag{2.121}$$

represent a transformation for $\mathbf{x}$ to $\bar{\mathbf{x}}$, which satisfies the Poisson bracket condition

$$[\bar{q}_i, \bar{q}_j] = [\bar{p}_i, \bar{p}_j] = 0, \qquad [\bar{q}_i, \bar{p}_j] = \delta_{ij}. \tag{2.122}$$

Corresponding to this transformation, one introduces the evolution operator $T$, which evaluates any function $g$ at the transformed point $\bar{\mathbf{x}}(\mathbf{x}, \epsilon)$

$$f = Tg \qquad \text{and} \qquad f(x) = g(\bar{x}(x, \epsilon)), \tag{2.123}$$

and if $g$ is the identical function, then it follows the operator $T$, which evaluates any function $g$ at the transformed point $\bar{\mathbf{x}}(\mathbf{x}, \epsilon)$

$$\bar{\mathbf{x}} = T\mathbf{x}. \tag{2.124}$$

To find the transformation $T$ explicitly, we introduce the Lie operator $L$

$$L = [w, ], \tag{2.125}$$

and note from (2.120) and (2.124) that

$$\frac{\delta T}{\delta \epsilon} = -LT, \tag{2.126}$$

with the formal solution

$$T = exp - \int^{\epsilon} d\epsilon' L(\epsilon') = exp - C(\epsilon). \tag{2.127}$$

For any canonical transformation, particularly that which is generated by $w$, the new Hamiltonian $\bar{H}$ is related to the old Hamiltonian by

$$\bar{H}(\bar{x}(x, \epsilon)) = H(x). \tag{2.128}$$

Comparing this to equation (2.123) and applying (2.127) we have

$$T^{-1} = 1 + \frac{[C,\,]}{1!} + \frac{[C,[C,\,]]}{2!} + \ldots . \tag{2.129}$$

Now for slow perturbation, we have two groups of variables: $(J_i, \theta_i, i = 1..n_f)$ are the fast action-angle variables; and $\bar{y} = (p_i, q_i, i = 1...(n - n_f))$ are the generalized coordinates for the remaining slow degrees of freedom. In order to obtain a perturbation series with the condition that derivatives with respect to the y's and t are smaller by an order $\epsilon$, we must expand $w, T, L, H$ and $\bar{H}$ as power series in $\epsilon$ [20] and in this case of slow perturbation we have to modify the Lie Operator. It takes the form:

$$L = L_f + \epsilon L_s, \tag{2.130}$$

where the fast part is:

$$L_f = (\frac{\delta w}{\delta \theta_i}\frac{\delta}{\delta J_i} - \frac{\delta w}{\delta J_i}\frac{\delta}{\delta \theta_i}), \tag{2.131}$$

and the slow part is:

$$L_s = (\frac{\delta w}{\delta p_i}\frac{\delta}{\delta q_i} - \frac{\delta w}{\delta q_i}\frac{\delta}{\delta p_i}). \tag{2.132}$$

The power series are:

$$H = \sum_{n=0} \epsilon^n H_n \tag{2.133}$$

$$\bar{H} = \sum_{n=0} \epsilon^n \bar{H}_n \tag{2.134}$$

$$L = \sum_{n=0} (\epsilon^n L^f_{n+1} + \epsilon^{n+1} L^s_{n+1}). \tag{2.135}$$

An advantageous property of the Lie Algrebraic Method is that, we arrive automatically at the expression of H in terms of the new variables. Since there are no dangers of mixing the new and the old variables, we shall omit the $(q, p)$ dependence on $H$ and keep in mind that $H, L$ depend only on the new variables.

We have seen that the new Hamiltonian is given in eq.(2.128). Inserting the power series with the separation in $L_s, L_f$, expanding and equating like power of $\epsilon$, we obtain up to third order

$$\begin{aligned}
\bar{H}_0 &= H_0 \\
\bar{H}_1 &= H_1 + L^f_1 H_0 \qquad (2.136)\\
\bar{H}_2 &= \{\ H_2 + L^f_1 H_1 + \frac{1}{2!} L^f_1 L^f_1 H_0 + L^s_1 H_0\} + L^f_2 H_0 \\
\bar{H}_3 &= \{\ H_3 + L^f_1 H_2 + \frac{1}{2!}(L^f_1 L^f_1 H_1 + L^f_2 L^f_1 H_0 + L^f_1 L^f_2 H_0) + \\
&\quad \frac{1}{3!} L^f_1 L^f_1 L^f_1 H_0 + L^s_1 H_1 + L^s_2 H_0 + \frac{1}{2!}(L^f_1 L^s_1 H_0 + L^s_1 L^f_1 H_0)\} + L^f_3 H_0. \qquad (2.137)
\end{aligned}$$

It is useful to define a new term $H^*_n$, in order to rewrite the last equations in the following generic compact form,

$$\begin{aligned}
\bar{H}_0 &= H_0 \\
\bar{H}_n &= H^*_n + L^f_n H_0, \qquad (2.138)
\end{aligned}$$



and where the $H^*_n$ are defined for the terms in the brackets in the equations (??). We choose $\bar{H}^*_n$ in the usual way, so that we eliminate from $\bar{H}$ the dependence on the fast angle. If we separate $H^*_n$ in an oscillatory part and one averaged, we can obtain the new Hamiltonian with the condition that define the Lie operator.

$$\begin{aligned}
\bar{H}_n &= < H^*_n >_f \qquad (2.139)\\
L^f_n H_0 &= -\tilde{H}^*_n. \qquad (2.140)
\end{aligned}$$

The new Hamiltonian $\bar{H}$ depends only on the slow variables and on the action variables associated with the fast variables, which are adiabatic invariant in this approximation.

Explicitly the term $\bar{H}$ up to third order are given by

$$\begin{aligned}
\bar{H}_0 &= H_0 \qquad (2.141)\\
\bar{H}_1 &= < H_1 >_f \qquad (2.142)\\
\bar{H}_2 &= < H_2 >_f + \frac{1}{2!} < L^f_1 \tilde{H}_1 >_f \qquad (2.143)\\
\bar{H}_3 &= < H_3 >_f + \frac{1}{2} < L^f_1 \tilde{H}_2 - L^f_2 \tilde{H}_1 >_f + \frac{1}{12} < L^f_1 L^f_1 \tilde{H}_1 >_f + \frac{1}{2} < L^s_1 \tilde{H}_1 >_f . \qquad (2.144)
\end{aligned}$$

To compare the $C_n$ and the $S_i$ see ref. [20]. The set of equations, which give the first three members, are formally correct for any number of degrees of freedom. However, for more than one degree of freedom, resonant denominator between the fast variables appear; as they did in the mixed variables perturbation techniques. In the next part we discuss the formal method of eliminating resonances to each order.

In the last section the elimination of $n_r$-resonance conditions a first order through a canonical transformation as well as the application of the standard Perturbation Theory over the $n - n_r$ new fast variables are discussed. However the reduced Hamiltonian, describes a new system with $n - n_r$ adiabatic constant and $n_r$ degrees of freedom. If the resonance is intrinsic, then the dynamic is of order $\epsilon$, and an expansion about an elliptic point lets us apply the Perturbation Theory, where the integrable part is the linearizable term, and the higher order can be considered as perturbation. Finally, we reduced the System to a Birkhoff normal form, giving information over the integrability of the system. The calculation of adiabatic invariant to higher order in case of resonance can be carried out in the same way as to that of the first order. Introducing the Lie operator and expanding in the small parameter $\epsilon$ we arrive at equation (2.136). In the case that the frequencies of the fast variables satisfy some resonance conditions, one can transform this to the resonance variables.

In terms of the resonance variables $(I_i, \theta_i; i = 1, .., n_r)$ and $(J_i, \psi_i; i = 1, ..., (n - n_r))$, the integrable part $H_0$ can be written as:

$$H_0 = H_0(I_1, ..., I_{n_r}). \tag{2.145}$$

After this resonance transformation, one can choose the $L^f_n$, given by eq. (2.140), such that the oscillatory part in the fast $\theta_i$ variables is eliminated from the new Hamiltonian, i.e. we choose the condition that $\bar{H}_n = \bar{H}(\mathbf{I}, \mathbf{J}, \psi)$, thus we found the conditions

$$\begin{aligned}
\bar{H}_n &= < H^*_n >_\theta \qquad (2.146)\\
\sum w_i \frac{\delta w_n}{\delta \theta_i} &= -\tilde{H}^*_n. \qquad (2.147)
\end{aligned}$$

Up to order $n$ we obtain the reduced Hamiltonian without resonances,

$$H = H_0(I_i) + \sum_{m=0}^{n} \epsilon^m < H_m^* >_\theta (I_i, J_i, \psi_i). \tag{2.148}$$

The result is an effective Hamiltonian with $n_r$ degrees of freedom. Assuming the existence of elliptic fixed point, we can expand the Hamiltonian function into a Taylor series around this point, where the action-angle variables are well defined and the integrability of the system can be discussed.

The existence and calculation of the fixed point for a Hamiltonian with many degrees of freedom are not straightforward and difficult. Lie Method up to order $k$ allows the following condition to find the fixed point:

$$\frac{\delta \bar{H}}{\delta \psi_i}|_{(\psi^*, J^*)} = 0 \qquad , \frac{\delta \bar{H}}{\delta J_i}|_{(\psi^*, J^*)} = 0, \tag{2.149}$$

and specifically we have

$$\frac{\delta \bar{H}}{\delta \psi_i} = \frac{\delta}{\delta \psi_i}(< H_1^* >_{\theta,} + \epsilon^2 < H_2^* >_{\theta,} + ... + \epsilon^k < H_k^* >_{\theta,})|_{(\psi^*, J^*)} = 0, \tag{2.150}$$

$$\frac{\delta \bar{H}}{\delta J_i} = \frac{\delta}{\delta J_i}(< H_1^* >_{\theta,} + \epsilon^2 < H_2^* >_{\theta,} + ... + \epsilon^k < H_k^* >_{\theta,})|_{(\psi^*, J^*)} = 0. \tag{2.151}$$

For our investigation we will assume that the fixed point can be expanded in a $\epsilon$-Taylor series,

$$J_i^* = \sum_{n=0} \epsilon^n J_{i,n}^* \tag{2.152}$$

$$\psi_i^* = \sum_{n=0} \epsilon^n \psi_{in}^*, \tag{2.153}$$

$$\tag{2.154}$$

where the first term of the series is given by

$$\frac{\delta \bar{H}_1}{\delta \psi_i}|_{(\psi_i^*, J_i^*)} = 0 \qquad , \frac{\delta \bar{H}_1}{\delta J_i}|_{(\psi_i^*, J_i^*)} = 0, \tag{2.155}$$

and the higher order corrections $(\psi_k^*, J_k^*)$ are defined through conditions derived from equating the same order in the $\epsilon$ conditions (2.150,2.151).

## 2.4 Perturbation Theory with Symmetry

Let us consider that $\Omega$ is a constant of motion, which satisfies the condition

$$[\Omega, H] = 0. \tag{2.156}$$

If the Hamiltonian is integrable, then $\Omega = \Omega(F_i)$ where the $F$'s are the action variables of the system. Now, we will discuss the case that the Hamiltonian is near integrable, i.e

$$H = H_0 + \epsilon H_1. \tag{2.157}$$



and that $\Omega$ is still a constant of the motion for the total Hamiltonian.

The Hamiltonian, itself is a constant of motion being time independent. However, unless it exhibits some other symmetry or is integrable, then there are no other obvious constants. Likewise, associated with the fact that the system is nearly integrable, it is possible to find another integral of the motion, expressed as a power series in $\epsilon$. Invariant of this type had been discussed previously [59]. In the context of the Canonical or Adiabatic Perturbation Theory, an invariant can be calculated explicitly in terms of the generating function or the Lie operator $L$, i.e. the invariants $\Omega_i$, $i = 1...n_\Omega$ are given by

$$\bar{\Omega}_i = T\Omega_i. \tag{2.158}$$

Now, if the system has a symmetry, then the Hamiltonian admits a constraint $\Omega^\alpha$ on the phase space. This constraint is a constant of motion and one can note that: in the case that the constraint are holonomic, the Hamiltonian can be reduced to an effective $n - n_\Omega$ degrees of freedom system. Moreover, if the constraint is non-holonomic, then one can understand this constraint as a condition over the solutions, i.e only the classical solutions which satisfy the condition $\Omega^\alpha = const.$ are physical solutions. For gauge systems, one finds an invariant which is a non-holonomic constraint and satisfies the condition

$$\dot{\Omega} = [\Omega, H] = 0, \tag{2.159}$$

where $H$ is the total Hamiltonian.
Considering that the Hamiltonian and the constraint are expanded as

$$H = \sum_n \epsilon^n H_n$$

$$\Omega = \sum_n \epsilon^n \Omega_n \tag{2.160}$$

it is then evident, that the expansion of the invariants and the condition (2.159) is satisfied only to a finite order in $\epsilon$.

Inserting the expansion in equation (2.159), we obtain this sequence of equations:

$$\begin{aligned} &[H_0, \Omega_0] = 0, \\ &[H_1, \Omega_0] + [H_0, \Omega_1] = 0, \\ &\sum_{k=0}^{N} [H_{N-k}, \omega_k] = 0. \end{aligned} \tag{2.161}$$

In the case that $\Omega$ is not known, one can use these equations in order to construct an invariant. This had been discussed in a different context in ref. [60].

We will discuss the modification of $\Omega$ in the context of the Lie approach. A remarkable feature of this transformation is that any function of the new or old variables can be transformed into the other variables by the same generating function, which depends only on the final variables and the Poisson bracket becoming invariant. With this properties we have

$$\dot{\Omega} = [H, \Omega] = [\bar{H}, \bar{\Omega}] = 0, \tag{2.162}$$

where

$$H(p(\bar{p},\bar{q}),q(\bar{p},\bar{q})) = \bar{H} = \sum \epsilon^n \bar{H}_n$$
$$\Omega(p(\bar{p},\bar{q}),q(\bar{p},\bar{q})) = \bar{\Omega} = \sum \epsilon^n \bar{\Omega}_n. \tag{2.163}$$

When $\bar{H} = \bar{H}^*(\bar{I})$, one can obtain explicitly the Lie operator. In this case, the constraint after the Lie transformation become;

$$\bar{\Omega}^\alpha = T^{-1}\Omega^\alpha = \Omega^\alpha + [L,\Omega^\alpha] + \frac{1}{2}[L,[L,\Omega^\alpha]] + \dots . \tag{2.164}$$

Introducing the $\epsilon$ expansion, we obtain

$$\bar{\Omega}^\alpha = \Omega_0^\alpha + \{\Omega_1^\alpha + [L_1,\Omega_0^\alpha]\} + \{\Omega_2^\alpha + [L_1,\Omega_1^\alpha] + \frac{1}{2}[L_1,[L_1,\Omega_0^\alpha]]\} + \dots . \tag{2.165}$$

Finally, the new adiabatic invariant in terms of the old variables can be found by the condition

$$\bar{x}_i = Tx_i, \tag{2.166}$$

where $\bar{I}_i = TI_i$ and we obtain to second order

$$\bar{I}^i = I^i - \epsilon[\Omega_1, I^i] + \frac{1}{2}[\Omega_1,[\Omega_1, I^i]] + \dots . \tag{2.167}$$



# Chapter 3

# Renormalization in a classical scalar field theory.

Our purpose is to introduce a Real Space renormalization group (RSRG) scheme which can be applied to a Hamiltonian system

In ref. [33] the momentum-space RG-approach for a Classical Scalar $\phi^4$-Theory on a lattice was investigated. Here, following the idea of Kogut and Wilson [27] on renormalization group transformation it was possible to formulate and describe a renormalization group transformation for this classical field theory on a lattice. The idea is to divide the angle variables of the oscillator into fast ones and slow ones and then average over the fast ones. This results in an effective Hamiltonian for the remaining slow variables, which can be compared with the original Hamiltonian. It was possible to derive a fixed-point condition and obtain a scaling law for those classical solutions for which the renormalization step can be iterated. Now, we will use techniques and ideas of this paper to study the RS-RG approach for a classical system with large number of degrees of freedom. We consider in particular a classical Hamiltonian system of $N$ weakly coupled harmonic oscillators as in a classical field theory on a spatial lattice in a finite volume. We study the origin of the "Block-variables" (slow variables) and the method of eliminating (averaging) the "fast variables" in order to obtain an effective Hamiltonian. This effective Hamiltonian can be compared with the starting Hamiltonian, but this depends only on the "Block" variables being constant in the fast variable (adiabatic invariant). We see that the adiabatic invariant and the canonical perturbation theory describe the different steps of the RSRG-transformation for the classical system.

The starting point of the RSRG is the definition of the theory on a spatial lattice. The model to treat is a classical scalar $\phi^4$ field theory defined on each site $\vec{r}$ of the d-dimensional lattice, together with its canonically conjugate variables $\pi_{\vec{r}}$.

By defining the theory on a spatial lattice with spacing $a$ and with finite volume (length of the d-dimensional lattice is $L = aM, M = 2N + 1$ ), the Hamiltonian takes the form

$$H = H_0 + H_{int}, \tag{3.1}$$

where

$$H_0 = \frac{1}{2}\sum_{\vec{l}}\{\pi_{\vec{l}}^2 + m^2\phi_{\vec{l}}^2 + \sum_{\vec{r}}^{n_d}\phi_{\vec{r}}d(\vec{r}-\vec{l})\phi_{\vec{l}}\} \tag{3.2}$$

and

$$H_{int} = \frac{\lambda}{4}\sum_{\vec{l}}\phi_{\vec{l}}^4, \tag{3.3}$$

here the summation extends over all the lattice points $\vec{l}$, and $d(\vec{r}-\vec{s})$ is a chosen lattice Laplacian. Using a nearest-neighbor definition of the gradient, it is known that $H_0$, the free part of the Hamiltonian, can be diagonalized through a Fourier transformation of the variables $\pi_{\vec{k}}$ and $\phi_{\vec{k}}$,

$$\phi_{\vec{l}} = \frac{1}{L^D}\sum_{\vec{k}} e^{i\vec{k}.\vec{l}}\phi_{\vec{k}}\,, \tag{3.4}$$

where

$$\vec{k} = \frac{2\pi}{L}\vec{n},\ \vec{n} = (n_1,...,n_d), \qquad |n_i| < N \tag{3.5}$$

and $H_0$ takes the form

$$H_0 = \frac{1}{2L^D}\sum_{\vec{k}}\{\pi_{\vec{k}}\pi_{-\vec{k}} + \omega^2(\vec{k})\phi_{\vec{k}}\phi_{-\vec{k}}\}, \tag{3.6}$$

where $\omega$ is defined as follows,

$$\omega^2(\vec{k}) = m^2 + 4\sum_{i=1}^{d}\sin^2\frac{k_i}{2}. \tag{3.7}$$

The complex-valued variables $\phi_{\vec{k}}$ and $\pi_{\vec{k}}$ satisfy $\phi_{-\vec{k}} = \phi_{\vec{k}}^*$ and $\pi_{-\vec{k}} = \pi_{\vec{k}}^*$ .

Following the RSRG-scheme [?] the lattice $\Lambda_a$ is divided into blocks of $n_s^d$ sites, such that these cells form the new lattice $\Lambda_{a'}$ having the same geometry as $\Lambda$. The Hamiltonian $H_0$ can then be divided into two parts, $H_{block}$ and $V^{int}$, where $H_{block}$ is the block Hamiltonian and $V^{int}$ contains the coupling part between different blocks.

The block-Hamiltonian can be diagonalized by a coordinate transformation which depends on the variables defining the corresponding block-Hamiltonian. Taking into account that each block-Hamiltonian describes $n_s^d$ harmonic oscillators, it is possible to distinguish between the slow ($\phi$) and fast ($\psi$) degrees of freedom. The former ones belong to the lower frequencies of the oscillator, the latter ones to the higher frequencies. The slow variables, in the quantum field theory language, is the Block-Spin-variable.

We can treat our perturbed system by assuming that this approach is valid in the classical formulation and that the frequencies obey the following relation $\omega_\psi >> \omega_\phi$ for all $k$ in the region of pertinent momentum. One observes that the motion of $\phi(k)$ is much slower than the of $\psi(k)$. After this distinction between the slow and fast degrees of freedom, we will use Canonical Perturbation Theory [20] which lead to the finding of an effective block-Hamiltonian by averaging only over the fast modes. As a result, the effective block-Hamiltonian depends only on the slow degrees of freedom where the fast modes are constants (adiabatic invariants). In principle, one can introduce a time scale $\epsilon = \frac{\omega_\phi}{\omega_\psi}$ which satisfies the condition $\epsilon < 1$, and one can then use the Adiabatic Perturbation Theory [20]. In this time scale, all slow degrees of freedom are "frozen". However, in the global sense, the adiabatic supposition is violated for the system treated here, because $\omega_\psi >> \omega_\phi$ is not valid for all region of momenta due to the



existence of the mass term in the definition of the frequencies. In the case of a massless system, we find the Infra-Rot problem. Therefore, we use the Canonical Perturbation Theory to study the perturbed system.

This chapter is organized as follows: section 1 describes the RSRG transformation for a 1-dimensional system illustrating our approximations. Sec. II contains the discussion of systems in higher dimensions.

## 3.1 1-dimensional model and description

Let us begin considering the one-dimensional $\phi^4$-Theory on the lattice in order to describe the method in a classical theory. The 1-dimensional model can be described by the Hamiltonian:

$$H = H_0 + H_{int}, \tag{3.8}$$

where

$$H_0 = \frac{1}{2}\sum_i\left(\pi_i^2 + m^2\phi_i^2 + (\phi_{i+i} - \phi_i)^2\right). \tag{3.9}$$

For the specific case of the Hamiltonian defined in equation (3.8) we will discuss blocks containing two sites each, as show in Fig.1.

These blocks are labeled by the integer $p$ and the sites within each block by $2p + r$ where $r$ taking the values 0 or 1 (so that the $p$-th block contains the degrees of freedom $\phi_{2p}$ and $\phi_{2p+1}$ of the original lattice.).
In this way the Hamiltonian can be rewritten as

$$H_0 = \frac{1}{L^D}\sum_{p\epsilon\Lambda_{2a}}\left(H_{block}(p) + V_{int}(p)\right), \tag{3.10}$$

where $H_{block}(p)$ is given by

$$H_{block}(p) = \frac{1}{2}\{\pi_{2p}^2 + \pi_{2p+1}^2 + (m^2+2)(\phi_{2p}^2 + \phi_{2p+1}^2) - \phi_{2p}\phi_{2p+1}\}, \tag{3.11}$$

and $V_{int}$ contains the coupling part between different blocks and is given by

$$V_{int}(p) = -\frac{1}{2}\phi_{2p+1}\phi_{2(p+1)}. \tag{3.12}$$

Corresponding to this blocking of sites, we group the variables of the block into a two-component vector. The two-component vector is:

$$\hat{\Phi}_p = \begin{pmatrix}\phi_{2p}\\ \phi_{2p+1}\end{pmatrix} \tag{3.13}$$

and similar for $\hat{\Pi}_{\vec{p}}$. Now, we rewrite the Hamiltonian $H_{block}$ in terms of the new variable $\hat{\Phi}_p$ and $\hat{\Pi}_p$ so that it is reducible by a linear symplectic real transformation to the normal form

$$H_{block}(p) = \frac{1}{2}\left(\hat{\Pi}_p^T\hat{\Pi}_p + \hat{\Phi}_p^T D^2\hat{\Phi}_p\right), \tag{3.14}$$

where $D^2 = Diag(\omega_1^2, \omega_2^2)$, the $\omega_i$ are the eigenvalues of $V_{int}$ and the Jacobian of the transformation is constant. The relation (3.14) is equivalent to

$$H_{block} = \frac{1}{2}\sum_{l=1}^{2}(\eta_l^2 + \omega_l^2\Psi_l^2) \tag{3.15}$$

so that $H_{block}$ generates two uncoupled harmonic oscillators. The canonical transformation is achieved by the matrix R, $\hat{\Psi} = R\hat{\Phi}$ and

$$R = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}. \tag{3.16}$$

That is, the new variables are:

$$\Psi_p^{(1)} = \Phi_p = \frac{1}{\sqrt{2}}\{\phi_{2p} + \phi_{2p+1}\} \tag{3.17}$$

$$\Psi_p^{(2)} = \Psi_p = \frac{1}{\sqrt{2}}\{\phi_{2p} - \phi_{2p+1}\} \tag{3.18}$$

$$\Pi_p^{(1)} = \frac{1}{\sqrt{2}}\{\pi_{2p} + \pi_{2p+1}\} \tag{3.19}$$

$$\Pi_p^{(2)} = \frac{1}{\sqrt{2}}\{\pi_{2p} - \pi_{2p+1}\}. \tag{3.20}$$

In terms of the new variables $H_{block}$ becomes

$$H_{block}(p) = \frac{1}{2}\left(\Pi_l^{(1)} + (m^2+1)\Phi_p^2 + \Pi_l^{(2)} + (m^2+3)\Psi_p^2\right), \tag{3.21}$$

and the term $V_{int}(p)$ is given by

$$V_{int}(p) = -\{\Phi_p\Phi_{p+1} - \Psi_p\Phi_{p+1} + \Phi_p\Psi_{p+1} - \Psi_p\Psi_{p+1}\}. \tag{3.22}$$

The variable defined in eq.(3.17) is the Block-Spin-variable in quantum field theory [?]:

$$\Phi_p = \frac{1}{\sqrt{n_s}}\sum_{i \in Block(p)} \phi_i. \tag{3.23}$$

Now, the "fast" and "slow" variables are: $\Psi$ and $\Phi$ respectively. The reason for this definition is clear, since from the point of view of the $H_{block}$ the $\Psi$-oscillator has a higher-frequency than the $\Phi$-oscillator (i.e. $\omega_\Psi = (m^2+3)^{\frac{1}{2}}$ and $\omega_\Phi = (m^2+1)^{\frac{1}{2}}$). In this formulation, one can start with this variables and then formulate a RGT, which in the region of low energy reproduces the same properties of the theory, as we will discuss later on.

Let us now discuss the validity of this approximation, since this separation of slow and fast variables is carried out in the real-space.

Since we are working in the momentum space on the block lattice with spacing $\hat{a} = 2a$, we can rewrite the Hamiltonian in this basis (the momentum are defined on the region $\Lambda_{2a}^* = (-\frac{\pi}{\hat{a}}, \frac{\pi}{\hat{a}})$) as

$$H_0 = \frac{1}{2L}\sum_k \left(\hat{\Theta}_{-k}^T\hat{\Theta}_k + \hat{\Psi}_{-k}^T D(k)\hat{\Psi}_k\right). \tag{3.24}$$



We use the following Fourier transformation for the new variables in the block lattice,

$$\Phi(l) = \frac{1}{V}\sum_{k \in \Lambda_{2a}^*} e^{ikl}\Phi_k, \tag{3.25}$$

where $V = L^d$ is the volume of the system. From now we consider $a = 1$.

For the two variable scheme, the Laplacian-matrix appearing in eq.(3.24) has the components:

$$d_{\Phi,\Phi}(k) = m^2 + 2\sin^2(k) = \omega_\Phi^2 \tag{3.26}$$

$$d_{\Psi,\Psi}(k) = m^2 + 2(1+\cos^2(k)) = \omega_\Psi^2 \tag{3.27}$$

$$d_{\Phi,\Psi}(k) = -i\sin(k)\cos(k), \tag{3.28}$$

and in the momentum basis the new variables assume the form

$$e^{-i\frac{k}{2}}\phi(k) = \sqrt{2}\left(\cos(\frac{k}{2})\Phi(2k) + i\sin(\frac{k}{2})\Psi(2k)\right) \tag{3.29}$$

$$e^{-i\frac{k}{2}}\pi(k) = \sqrt{2}\left(\cos(\frac{k}{2})\Theta^1(2k) + i\sin(\frac{k}{2})\Theta^2(2k)\right). \tag{3.30}$$

To get the blocking relation in momentum basis, we start from eqs.(3.17-3.20)

$$\phi_{2p} = \frac{1}{\sqrt{2}}\{\Phi_p + \Psi_p\} \tag{3.31}$$

$$\phi_{2p+1} = \frac{1}{\sqrt{2}}\{\Phi_p - \Psi_p\}, \tag{3.32}$$

and multiply both side of each equation by $e^{i2kn}$ and $e^{i(2n+1)k}$ and sum over $n$.

The Hamiltonian given by eq.(3.24) can be transformed into a normal form through variable transformations.

$$H_k^0 = \frac{1}{2}\left(\pi_{1,k}\pi_{1,-k} + \omega_1(k)^2\Phi_{1,k}\Phi_{1,-k} + \pi_{2,k}\pi_{2,-k} + \omega_2(k)^2\Phi_{2,k}\Phi_{2,-k}\right) \tag{3.33}$$

where

$$\omega_1(k)^2 = m^2 + 4sin^2\frac{k}{2} \tag{3.34}$$

$$\omega_2(k)^2 = m^2 + 4\cos^2\frac{k}{2}, \tag{3.35}$$

and

$$\Phi_1(k) = e^{\theta(k)}\{\cos(\frac{k}{2})\Phi(2k) + i\sin(\frac{k}{2})\Psi(2k)\} \tag{3.36}$$

$$\Phi_2(k) = e^{\theta(k)}\{\sin(\frac{k}{2})\Phi(2k) + i\cos(\frac{k}{2})\Psi(2k)\}. \tag{3.37}$$

We compare the eqs.(3.34,3.35) which define the frequency for the variables which diagonalize $H_0$. In this case, the following relations can be found:

$$\omega(k)^2 = \omega_1(k)^2 \tag{3.38}$$

$$\omega(k+\pi)^2 = \omega_2(k)^2, \tag{3.39}$$

and the momentum is defined on the block lattice. Likewise, a relation between the normal coordinate and our original variables defined by the eq.(3.29) can also be found. In this case, considering the eqs.(3.36,3.37) and eq.(3.29) it follows that

$$\Phi_1(k) = \phi(k) \tag{3.40}$$
$$\Phi_2(k) = \phi(k+\pi). \tag{3.41}$$

Therefore $\Phi_1(k)$ are the slow modes and $\Phi_2$ the fast mode, and these relations lead us to conclude that the normal variables correspond to the slow and fast variables introduced in ref. [33] in the momentum-space renormalization.

Considering the case where the momenta are small, it can be observed that the slow variables $\Phi_1$ are well described by the block-spin variables, i.e.

$$\begin{aligned}\Phi_1(k) &\cong \Phi(k) + O(k/2)\Psi(k),\\ \Phi_2(k) &\cong \Psi(k) + O(k/2)\Phi(k),\end{aligned} \tag{3.42}$$

and the difference in this region is of to order O(k). By introducing the scaling of these variables, it can be seen that this contribution is not important.

At this point, we have two possibilities to continue: first, one can state a RGT for the slow variables $\Phi_1$ defined by the diagonalization of the total $H_0$ term. Therefore, in the classical formulation of the RGT, the average of the fast variables must be done over this variable $\Phi_1$. However, considering the relation given by equation (3.40), it can be concluded that the slow modes are the same as considered in the paper of Bartels and Chang. Moreover, it is interesting to discuss the alternative way which is to assume the validity of the approximation that the variables introduced in the real space defined our starting slow and fast variables. We will then formulate a RGT in term of these block spin variables. In this case, the $H_0$ is separated into two parts, a block part which is integrable in terms of the blocks variables and a second term $V_{int}$ which plays the role of a new perturbed term. Here, we are confronted with the problem of two perturbation terms. In the Perturbation Theory up to first order, it can be observed that the generating function which define the canonical transformation can be separated into two contributions: the first is derived from the first term and the other comes from the second perturbation term. Considering this fact, we will apply the Perturbation Theory in two steps: the first one, the Canonical Perturbation Theory is applied to the $V_{int}$ in order to define an effective Hamiltonian. In this case, it is clear that the only contribution derived from the Perturbation Theory must be a modification of the frequency for the block-variables. If this approach is correct, one must obtain in the limit $k \to 0$, the expected behavior of the frequency for the slow modes since $H_0$ is integrable, i.e.

$$w^2 \to m^2 + \sum_i k_i^2. \tag{3.43}$$

Furthermore, since the new frequencies satisfy the condition $w_\Phi < w_\Psi$ after the Perturbation Theory, we can go to the last step and introduce the perturbation term and investigate its effect on the integrable part. Here, we assume that the new "fast-modes" (higher-frequency oscillator) can be averaged, allowing an affective Hamiltonian for the slow modus with renormalized parameter. The validity of this step is supported by an adiabatic treatment.



In the following, we treat the first perturbative term derived by blocking using the Canonical Perturbation Theory [32], with the expansion parameter

$$\lambda_k = i \sin 2k. \tag{3.44}$$

It represents the nondiagonal coupling between the new variables defined in each block. We rewrite $H_0$ as

$$H_0 = \frac{1}{2(L/2)} \sum_k \left( H^0_{block}(k) + V_{int}(k) \right), \tag{3.45}$$

where

$$H^0_{block}(k) = (\pi_{\Phi,k}\pi_{\Phi,-k} + \omega^2_\Phi(k)\,\Phi_k\Phi_{,-k} + \pi_{\Psi,k}\pi_{\Psi,-k} + \omega^2_\Psi(k)\Psi_k\Psi_{-k}) \tag{3.46}$$

and

$$V_{int}(k) = \sin 2k\, \Phi_k\Psi_{-k}. \tag{3.47}$$

In principle we could write our Hamiltonian in terms of real and imaginary part of $\phi$ and $\pi$, and all calculation would give real-valued results. However, it turns out to be more convenient to introduce complex-valued variables a(k) and $a^*(k)$ for the $\Phi$ variables and b(k) and $b^*(k)$ for the $\Psi$ variables, which are the classical analogues of the annihilation and creation operators in quantum field theory [58].

For each $k$, one defines

$$a(k) = \frac{(L/2)^{\frac{1}{2}}}{2}\{\sqrt{\omega_\Phi(k)}\,\Phi_k + \frac{i}{\sqrt{\omega_\Phi(k)}}\pi_{\Phi,k}\} \tag{3.48}$$

$$b(k) = \frac{(L/2)^{\frac{1}{2}}}{2}\{\sqrt{\omega_\Psi(k)}\,\Psi_k + \frac{i}{\sqrt{\omega_\Psi(k)}}\pi_{\Psi,k}\}, \tag{3.49}$$

with analogous expressions for the complex conjugates $a^*(k)$ and $b^*(k)$. The canonical transformation which takes $(\pi, \Phi)$ to (ia*(k),a(k)) has the generating function:

$$\begin{aligned}F_2(a^*, b^*, \Phi, \Psi) = \sum_{k\Psi\Lambda_{2a}} \{&\frac{-i}{2}(L/2)^{-1}\omega_\Phi(k)\,\Phi_k\Phi_{-k} + i\sqrt{2\omega_\Phi(k)}(L/2)^{-D/2}\,a^*(k)\Phi(k)\\ &-\frac{1}{2}a^*(k)a(-k) + \frac{-i}{2}(L/2)^{-1}\omega_\Psi(k)\,\Psi_k\Psi_{-k} + i\sqrt{2\omega_\Psi(k)}(L/2)^{-D/2}\,b^*(k)\Psi(k)\\ &-\frac{1}{2}b^*(k)b(-k)\}.\end{aligned} \tag{3.50}$$

The new variables have canonical Poisson brackets, e.g.

$$\{a(k), a^*(k')\} = -i\delta_{k,k'} \tag{3.51}$$
$$\{b(k), b^*(k')\} = -i\delta_{k,k'} \tag{3.52}$$
$$\{a(k), b^*(k')\} = 0, \tag{3.53}$$

and the effective free part $H^0_{block}$ takes the form

$$H^0_{block} = \sum_k (\, w_\Phi\, a(k)a^*(k) + w_\Psi\, b(k)b^*(k)\,). \tag{3.54}$$

For these new variables, we now introduce the action and angle variables $(I(k), \theta(k))$ and $(J(k), \varphi(k)\ )$, both real-valued, such that

$$a(k) = \sqrt{I(k)}\, e^{-i\theta(k)} \tag{3.55}$$

$$b(k) = \sqrt{J(k)}\, e^{-i\varphi(k)} \tag{3.56}$$

giving

$$H^0 = \sum_k (w_\Phi\, I(k) + w_\Psi\, J(k)\ ). \tag{3.57}$$

The connection with the old variables is:

$$\Phi(k) = \frac{(L/2)^{1/2}}{\sqrt{2\omega_\Phi(k)}} \left( \sqrt{I(k)}\, e^{-i\theta(k)} + \sqrt{I(-k)}\, e^{i\theta(-k)} \right) \tag{3.58}$$

$$\pi(k) = (-i)\,(L/2)^{1/2} \sqrt{\frac{\omega_\Phi(k)}{2}} \left( \sqrt{I(k)}\, e^{-i\theta(k)} - \sqrt{I(-k)}\, e^{i\theta(-k)} \right). \tag{3.59}$$

Now, we apply the idea of averaging over the "fast" variables and performing a canonical transformation to remove the angle variables $\varphi(k)$ associated to the fast variables $\Psi(k)$ in $H_0$.

The first step is to decompose $V_{int}$ into an averaged part (over fast variables) and an oscillatory part.

$$V_{int,k}(I, J, \theta, \varphi) = < V_{int,k}(I, J, \theta, \varphi) >_\varphi + \tilde{V}_{int,k}(I, J, \theta, \varphi). \tag{3.60}$$

We introduce a generating function

$$F_{2,k}(I', J', \theta, \varphi) = I'(k)\theta(k) + J'(k)\varphi(k) + \lambda_k S_1(I', J', \theta, \varphi) + \lambda_k^2 S_2(I', J', \theta, \varphi), \tag{3.61}$$

which leads to the canonical transformation

$$\begin{aligned} I(k) &= I'(k) + \lambda_k \frac{\partial S_1}{\partial \theta(k)} + \lambda_k^2 \frac{\partial S_2}{\partial \theta(k)} \\ \theta'(k) &= \theta(k) + \lambda_k \frac{\partial S_1}{\partial I'(k)} + \lambda_k^2 \frac{\partial S_2}{\partial I'(k)}, \end{aligned} \tag{3.62}$$

and analogous equations for J(k) and $\varphi$'(k). Inserting the eqs.(3.62) into (3.45) and expressing everything in terms of the primed-variables, we find

$$\begin{aligned} H_{0,k} = H_{0,k}(I', J') + \lambda_k \left( \{S_1, H_k^0\} + V_{int,k} \right) \\ + \lambda_k^2 \left( \{S_2', H_0\} + \{S_1, V_{int,k}\} + \frac{1}{2}\{S_1, \{S_1, H^0\}\} \right), \end{aligned} \tag{3.63}$$

where the Poisson brackets are with respect to the primed action-angle variables. We perform a decomposition similar to eq.(3.60), and the Hamiltonian becomes

$$H_0 = \sum_k H_{0,k}(I', J') + < V_{1,k} >_{\varphi(k)} + < V_{2,k} >_{\varphi(k)}, \tag{3.64}$$

where the following conditions for $V_2$ and S hold:

$$\{S_{1,k}, H_k^0\} = -\tilde{V}_{1,k}, \tag{3.65}$$

$$V_2(k) = \{S_1, V_1(k)\} + \frac{1}{2}\{S_1, \{S_1, H^0\}\}, \tag{3.66}$$

$$S_2' = S_2 - \frac{1}{2} \sum_k \left( \frac{\partial S_1}{\partial I'(k)} \frac{\partial S_1}{\partial \theta'(k)} + \frac{\partial S_1}{\partial J'(k)} \frac{\partial S_1}{\partial \varphi'(k)} \right). \tag{3.67}$$



The main feature of the resulting Hamiltonian is that the dependence upon the $\varphi(k)$ has been removed; i.e the $J'(k)$ are constant (adiabatic invariants) up to order $\lambda_k^3$.

The averaging procedure which we use to eliminate the coupling between the slow and the fast variable leads to the following results:

$$\begin{aligned} S_{1,k} &= \frac{\sin 2k}{\sqrt{2\omega_\Phi(k) 2\omega_\Psi(k)}} \{a(k)\{\frac{b^*(k)}{\omega_\Psi(k) - \omega_\Phi(k)} - \frac{b(-k)}{\omega_\Psi(k) + \omega_\Phi(k)}\} + \\ &\quad a^*(-k)\{\frac{b^*(k)}{\omega_\Psi(k) + \omega_\Phi(k)} - \frac{b(-k)}{\omega_\Psi(k) - \omega_\Phi(k)}\}\}, \end{aligned} \tag{3.68}$$

$$< V_{1,k} >_{\varphi(k)} = 0, \tag{3.69}$$

$$< V_{2,k} >_{\varphi(k)} = \frac{1}{2}\frac{1}{2\omega(k)} (\, a(K) + a^*(-k)\,)(\, a(-k) + a^*(k)\,)\Sigma(k) + O(J'(k)), \tag{3.70}$$

where $\Sigma$ is given by

$$\Sigma(k) = (-\sin^2 k) = \frac{i^2}{2^2} \frac{\lambda(k)\lambda(-k)}{\omega_{\Psi_1}^2(k) - \omega_{\Phi_1}^2(k)}. \tag{3.71}$$

Now we bring the Hamiltonian $H_0$ into a form which allows it to be compared with the starting Hamiltonian. For this purpose we drop those terms of the averaged Hamiltonian which depend only upon the $J(k)$, but not on $I$ or $\theta$. This defines an affective Hamiltonian $H_{eff}^0$ which describes the motion of the slow-momentum (Block-variables) degrees of freedom in the presence of a set of adiabatic invariants for the large-momentum modes:

$$H_0 = \sum_k \omega(k) a'(k) a'^*(k) + \frac{1}{2} \sum_k \frac{1}{2\omega(k)} (\, a'(K) + a'^*(-k)\,)(\, a'(-k) + a'^*(k)\,)\Sigma(k). \tag{3.72}$$

*To simplify our calculation we use the a'(k),a"'(k) variables. Following the same steps as in* ref. [33], we perform a Bogoliubov transformation ,

$$\begin{aligned} a"(k) = a'(k)\alpha(k) + a'^*(-k)\beta(k) \\ \alpha^2(k) - \beta^2(k) = 1, \end{aligned} \tag{3.73}$$

and eq.(3.72) becomes

$$H_0 = \sum_k \omega"(k) a"(k) a"^*(k) \tag{3.74}$$

with

$$\omega"^2(k) = \omega^2(k) + \Sigma(k, J), \tag{3.75}$$

*and* $\Sigma(k, J)$ *is given by*

$$\Sigma(k, J) = -\sin^2 k. \tag{3.76}$$

The transformation defined in eq.(3.73) has the property that

$$\frac{1}{\sqrt{2\omega(k)}} (\, a(k) + a^*(-k)\,) = \frac{1}{\sqrt{2\omega"(k)}} (\, a"(k) + a"^*(-k)\,). \tag{3.77}$$

This implies that the other perturbation terms will not change under this transformation.

The result of this transformation is a change in the frequency for the slow variables:

$$\omega''^2(k) = m^2 + \sin^2 k \tag{3.78}$$

and the main property of this result is that in the region of $k$ very small, eq.(3.78) will be approximately:

$$\omega''^2(k) = m^2 + k^2. \tag{3.79}$$

This approximation becomes valid after repeating the RS-RGT $n$ times, since this will transfer the momentum into the $\Lambda''_{2^n a}$-region. For small momenta $k$, $\omega''^2(k)$ is of the same form as the "bare" frequency $\omega^2(k)$ of the starting Hamiltonian $H_0$. As we have emphasized, these transformations are motivated by the demand for the new Hamiltonian to be in a form where it can be compared with the starting Hamiltonian.

## 3.2 Perturbation Theory

Let us now introduce the interaction term $H_{int,\lambda}$ in our system and assume the coupling $\lambda$ to be small. We apply the Classical Canonical Perturbation Theory in order to find an effective Hamiltonian for the block variables with renormalized parameter. Our idea is to bring this Hamiltonian into a form that it can be compared with the starting Hamiltonian ( 3.6). This can be approximately done through different canonical transformations. The first transformation is done in order to eliminate the angle dependence of the fast variables by averaging over the fast degrees of freedom. In order to bring the effective Hamiltonian to the some form as the original Hamiltonian, we need to perform one more canonical transformation similar to the the canonical transformations discussed in ref. [33]. Then, we made a canonical transformation in order to redefine the frequency, this we will call "mass renormalization", and another transformation which we call "wave function renormalization". The latter, will be discussed further below. The averaging procedure and the canonical transformations define what we call a "real-space renormalization group" transformation.

We begin this section with a study of the interaction term which is given by

$$\begin{aligned} H_{int} &= \frac{\lambda}{4}\sum_l \phi_l^4 && (3.80)\\ &= \frac{\lambda}{4L^3}\sum_{k_1,k_2,k_3,k_4} \delta(\sum_i k_i)\prod_{k_i} \varphi_{k_i}. && (3.81) \end{aligned}$$

Now, we introduce the relation between the new and the old variables which is given in the eqs.(3.36,3.37), and the interaction term takes the form:

$$H_{int} = \frac{\lambda}{4L^3}\sum_{k_1,k_2,k_3,k_4}\sum_{i=0}^{4} H_{4-i,i}(\Phi,\Psi|k_1,k_2,k_3,k_4), \tag{3.82}$$

where

$$H_{4-i,i}(\Phi,\Psi|k_1,k_2,k_3,k_4) = f_{4-i,i}(k_1,k_2,k_3,k_4)\prod_{l=1}^{i}\Phi_{k_l}\prod_{j=i+1}^{4}\Psi_{k_j} \qquad i=1,.,3 \tag{3.83}$$



$$H_{4,0}(\Phi,\Psi|k_1,k_2,k_3,k_4) = f_{4,0}(k_1,k_2,k_3,k_4)\prod_{l=1}^{4}\Phi_{k_l} \tag{3.84}$$

$$H_{0,4}(\Phi,\Psi|k_1,k_2,k_3,k_4) = f_{0,4}(k_1,k_2,k_3,k_4)\prod_{l=1}^{4}\Psi_{k_l}, \tag{3.85}$$

and the functions $f_{i,j}$ are given in Appendix A. The eqs.(A.4-A.6) show that the vertices between the different fast and slow variables have a $k$-dependence. After the first canonical transformations discussed in the previous section, it follows that the new variables take the form:

$$\Phi_k(a,\omega) = \Phi_k(a'',\tilde{\omega}) + O(\lambda_k). \tag{3.86}$$

Inserting this relation in the expressions for the vertices, we can show that the contributions derived form this correction are not relevant to the considered order.

We begin the second step of the RSRGT by averaging over the fast variables. After the first canonical transformation, the original Hamiltonian can be written as

$$H = H_0^{eff}(I'',J'') + \lambda H_{int}(I'',J'',\theta'',\varphi''), \tag{3.87}$$

and $H_{int}$ decomposes into an averaged part over the $\varphi''$ variables and an oscillatory part.

As in the first section, here we seek for a canonical transformation from $(I'',J'',\theta'',\varphi'')$ to $(I,J,\theta,\varphi)$ such that the effective Hamiltonian is independent from the $\varphi$ up to order $\lambda^3$. In this approximation the actions $J$ are constant, i.e. there are adiabatic invariants. Therefore, the Hamiltonian to second order in Perturbation Theory takes the form

$$H_{effec} = H_{0,k}(I,J) + \lambda < H_{int}(I,J,\theta,\varphi) >_{\varphi(k)} + \lambda^2 < H_2(I,J,\theta,\varphi) >_{\varphi(k)}, \tag{3.88}$$

with the condition

$$\begin{aligned} \{S_1,H^0\} &= -\tilde{H}_{int}, && (3.89)\\ H_2 &= \{S_1,H_{int}\} + \frac{1}{2}\{S_1,\{S_1,H^0\}\}, && (3.90)\\ S_2' &= S_2 - \frac{1}{2}\sum_k\left(\frac{\partial S_1}{\partial I(k)}\frac{\partial S_1}{\partial\theta(k)} + \frac{\partial S_1}{\partial J(k)}\frac{\partial S_1}{\partial\varphi(h)}\right), && (3.91) \end{aligned}$$

where $S_1$ and $S_2$ defined the generating function.

### 3.2.1 First order Perturbation Theory and One-loop discussion

To first order, the correction to the integrable term are derived from the term

$$H^1_{effec} = < H_{int}(I,J,\theta,\varphi) >_{\varphi(k)}. \tag{3.92}$$

It may be useful to give some explanation about the interaction term and the average over the fast variables. From the structure of $H_{int}$, one observes that it has a vertex with zero, one,two, three and four $\Psi$ variables. The average over the angle variables associated to these fast variables eliminates the contribution from the vertex with one and three fast variables.

Only the contribution of the vertex with an even number of fast variables give a contribution to $< H_{int} >$. In this case, we see that in momentum space only those terms in the fast variables which satisfy the condition $k = -k'$ have a non zero contribution. For detailed derivation see ref. [33] and appendix B. With this consideration, we begin the calculation of the average over the fast variables, i.e $< H_{int} >$. The contributions from the vertex with an even number variables are illustrated in Fig.2. We introduce the action and angle variables which are given by

$$a(k) = \sqrt{I(k)}\, e^{-i\theta(k)} \tag{3.93}$$

$$b(k) = \sqrt{J(k)}\, e^{-i\varphi(k)}. \tag{3.94}$$

These variables leads us to compute the averaging over the term with an equal number of fast modes b and $b^*$. In this case, we have that the following relations are important for our calculation:

$$\begin{aligned} < b(k_1)b(k_2)b^*(k_3)b^*(k_4) >_\varphi &= < b(k_1)b^*(k_3) >< b(k_2)b^*(k_4) > + \\ &\quad < b(k_1)b^*(k_4) >< b(k_2)b^*(k_3) > \qquad (3.95) \\ < b(k)b^*(q) >_\varphi &= \delta(k-q)J(k) \qquad (3.96) \\ < b(k)a^*(q) >_\varphi &= 0, \qquad (3.97) \end{aligned}$$

with them we can compute $< H_{int} >$ straightforwards

$$< H_{int} >= H^4(\Phi) + H^2(\Phi) + H_{vacuum} \tag{3.98}$$

where:

$$H^4(\Phi) = \frac{\lambda}{4L^3} \sum_{k_1,k_2,k_3,k_4} f_{4,0}(k_1,k_2,k_3,k_4) \prod_{l=1}^{4} \Phi_{k_l}, \tag{3.99}$$

$$H^2(\Phi) = \frac{\lambda}{4L^3} \sum_{k_1,k_2,k_3,k_4} f_{2,2}(k_1,k_2;k_3,k_4) \prod_{l=1}^{2} \Phi_{k_l} < \prod_{j=3}^{4} \Psi_{k_j} > \delta(k_3-k_4), \tag{3.100}$$

$$H_{vacuum} = \frac{\lambda}{4L^3} \sum_{k_1,k_2,k_3,k_4} f_{0,4}(k_1,k_2,k_3,k_4) < \prod_{l=1}^{2} \Psi_{k_l} >^2 \delta(k_1-k_2)\delta(k_3-k_4). \tag{3.101}$$

We ignore a term due to $k_1 = k_2 = k_3 = k_4$, because this term is negligible in the infinite volume limit.

The new vertex for four $\Phi$ variables takes a nontrivial k-dependence in comparison to the original vertex. Since eventually, we will be interested only in the region of small external momenta, we approximate this momentum dependence by a Taylor expansion around the renormalization point $k_1^2$=$k_2^2$=$k_3^2$=$k_4^2$= 0, and it takes the form

$$f_{4,\Phi}(k_1,k_2,k_3,k_4) = \delta(\sum_i (k_i)) \left(1 + O(k_i k_j, k_i^2)\right). \tag{3.102}$$

The leading term of this expansion becomes the original term, and the higher term are proportional to $k_i^2$ or $k_i k_j$. Following Wilson's argumentation on irrelevant operators, we shall argue below that such terms are not important near the fixed point.



In the following steps, we carry out canonical transformations (mass and wave function renormalization) which allow the reduction of the Hamiltonian to its original form and allows us to define of the renormalized mass and the frequency in the region for small momenta. To show the origin of this definition, let us now consider the free term from H and all those terms in $< H_1 >$ which generate an effective mass term (two external block variables). The only term which contributes in this case, comes from $H_2$ (see eq.102), and is represented in Fig.2b.

$$H_0 = \sum_k \omega(k) a(k) a^*(k) + \frac{1}{2} \sum_k \frac{1}{2\omega(k)} (\, a(K) + a^*(-k)\,)(\, a(-k) + a^*(k)\,)\Sigma(k,J), \tag{3.103}$$

where

$$\Sigma(k,J) = \frac{\lambda}{L} \sum_k \frac{J(k)}{2\omega_\Psi(k)}. \tag{3.104}$$

It becomes natural to perform a new Bogoliubov transformation such that the eq.(3.103) becomes

$$H_0 = \sum_k \omega"(k) a"(k) a"^*(k) \tag{3.105}$$

with

$$\omega"^2(k) = \omega^2(k) + \Sigma(k,J), \tag{3.106}$$

and an analogous relation (see eq.(3.77)) for the variable $a$ and $a"$. In this case, the Bogoliubov transformation implies, that the other contribution of $< H_{int} >$ and $< H^2 >$ are not changed under this transformation up to order $\lambda^3$.

Expanding in the region of small momenta one obtains

$$\omega"^2(k) = Z^{-1} \left[m^2 + \Sigma(0,J))Z + k^2\right], \tag{3.107}$$

with

$$Z^{-1} = 1 + \Sigma'(0,J)). \tag{3.108}$$

So we define the renormalized mass and frequency:

$$m_r^2 = (m^2 + \Sigma(0,J))Z, \tag{3.109}$$

$$\omega_r^2(k) = Z\omega"^2 = m_r^2 + k^2. \tag{3.110}$$

For small momenta $k, \omega_r^2(k)$ assumes the same form as the original frequency. Inserting this result into the original Hamiltonian (3.54), we obtain

$$H_{0\,slow} = Z^{-1/2} \sum_k \omega_r(k) I"(k). \tag{3.111}$$

In order to make the last equation look like the free part of the starting Hamiltonian, one has to perform another canonical transformation whose generating function is

$$S(I'", \theta") = Z^{1/2} \sum_k I'"(k)\theta"(k). \tag{3.112}$$

Finally we obtain

$$H_{0\,slow} = \sum_k \omega_r(k) I'"(k). \tag{3.113}$$

The transformation is called "wave function renormalization", and it affects those terms which have slow external lines. In this case, it can be noted that such lines and each of them has a factor $\sqrt{2I''(k)/\omega''(k)}$. However, considering the definition of the renormalization of the frequency and the generating function it can be obtained that each line becomes a factor of $Z^{1/2}$. Let us define with this result the renormalized four point interaction term as:

$$V_{4ren} = Z^2 V_4. \tag{3.114}$$

These canonical transformation looks like the classical analogous of renormalization in quantum field theory.

Finally, the last contribution to first order is denoted as $H_{vacuum}$ (see Fig. 2c) and corresponds to a graphs with zero external slow lines ($\Phi$ variables). It comes from the average of the term $H_{0,4}$. This graph is irrelevant for our investigation, because they have a contribution to the background energy, and depends only on the adiabatic invariant.

### 3.2.2 Second order Perturbation Theory.

For the calculation of the correction on the 4-vertex, we need to go to the second order of the Perturbation Theory and then we need to calculate the term $< H_2 >$ given by

$$< H_2 >= \frac{1}{2} < \{S_1, \tilde{H}_1\} >, \tag{3.115}$$

which is a function of $S_1$. Therefore, let us now calculate this term. The oscillatory part $\tilde{H}_1$ can be written as

$$\tilde{H}_1 = H_1 - < H_1 > . \tag{3.116}$$

From $\tilde{H}_1$, we can compute $S_1$ and $H_2$ through the eqs.(3.89-3.91). Introducing the notation

$$c_\Psi(k,1) = b(k), \tau = 1 \tag{3.117}$$

$$c_\Psi(k,-1) = b^*(k), \tau = -1, \tag{3.118}$$

one finds that the function $S_1$ takes the following form:

$$S_1 = \frac{\lambda}{4L^3} \sum_{k_1 \ldots k_4}{}' \sum_{i=1}^{4} \sum_{\{\tau_i\}} f_{4-i,i}(k_1,k_2,k_3,k_4) \prod_{l=1}^{i} \frac{c_\Phi(k_l,\tau)}{\sqrt{2\omega_\Phi}} \prod_{j=i+1}^{4} \frac{c_\Psi(k_j)}{\sqrt{2\omega_\Psi}} \frac{-i}{\sum \omega_{\Phi,i}\tau_i + \sum \omega_{\Psi,j}\tau_j}$$

$$S_1 = \sum_{k_1,k_2,k_3,k_4}{}' \sum_{i=0}^{4} S_{4-i,i}(c_\Phi, c_\Psi | k_1,k_2,k_3,k_4), \tag{3.119}$$

where the symbol $\sum'$ indicates that we have subtracted from $H_{int}$ terms associated with $< H_{int} >$. We now investigate the term $< H_2 >$, where $H_2$ is defined in eq.(3.91). With the relation (eq.3.89) $H_2$ is rewritten as:

$$H_2 = \frac{1}{2}\{S_1, \tilde{H}_1\} + \{S_1, < H_1 >\}. \tag{3.120}$$

The second term in the eq.(3.120) vanishes after averaging over the fast variables. So we need only to consider the first term $\frac{1}{2}\{S_1, \tilde{H}_1\}$ :

$$< H_2 >= \frac{1}{2} < \{S_1, \tilde{H}_1\} >_\varphi . \tag{3.121}$$



To calculate $< H_2 >$ we must follow two steps. The first one is the calculation of the Poisson bracket over the fast or the slow variables, and the other one is an average over the fast angles. From the structure of the $S_1$ and the $\tilde{H}_1$, we obtain as a result a new vertex with six slow variables. It comes from the Poisson bracket between the terms $H_{4,0}$ and $S_{4,0}$ over the slow variables and from the term $H_{3,1}$ and $S_{3,1}$ over the fast variables. In general, when we calculate $< H_2 >_\varphi$, we find three classes of diagrams: graphs with zero external slow variables (vacuum diagrams) which are not considered: graphs with two slow variables which provides a two-loop contribution to the function $\Sigma(k,J)$. However, in order to be consistent with our treatment of the four point function, this two-loop contribution will not be kept, and finally, graphs with four external slow variables coming from the terms $H_{2,2}$ and $S_{2,2}$ when we consider the Poisson bracket and average over the fast variables, and $H_{3,1}$ and $S_{3,1}$ when we consider the Poisson bracket over the slow variables and average over the fast variables. The representation is given in the Fig.3. In order to find a final expression for the vertex with four variables, we must take the term defined in eq.(3.121) and all contributions from the diagram 3a. and 3b. with different time ordering for the external lines. The final result for the effective Hamiltonian takes the form:

$$H_0 = \sum_k{}' \omega(k) I(k) + V_{4ren} + V_{6ren} \tag{3.122}$$

where $V_{4ren}$ and $V_{6ren}$ must be included to first order.

## 3.3 Fixed Points and Scaling Law.

In order to investigate the conditions needed to find the fixed points of the parameters of our Hamiltonian and the scaling law for the variable $\phi$, let us compare our effective Hamiltonian, which has been found after a first application of the RS-transformation, with the original Hamiltonian. The first condition is derived from the analysis on the two-point vertex. In this case, we compare the frequencies $\omega(k)$ and $\bar{\omega}(k)$ defined in the eqs.(3.7 and 3.110). Since at the fixed point the condition $m^* = m$ is satisfied, one obtain the following relation:

$$m^2 = 4Z(m^2 + \Sigma(0,J')), \tag{3.123}$$

where $\Sigma$ depends upon $m^2$ and is of order $\lambda$.

The next condition for the fixed point comes from the analysis of the four-point vertex defined in eq.(3.122). In this case, the effective four-point vertex is represented up to order $\lambda$ in the Fig.3. It is the sum of the $k$-dependent vertex for four $\Phi$ defined in Appendix B. and the contribution of all orders of the external lines in Fig.4. The effective vertex depends upon the external momenta $k_1, ..., k_4$.
In order to arrive at the final condition on the coupling, we have investigated our fixed point around the renormalization point $k_1^2 = k_2^2 = k_3^2 = k_4^2 = 0$. Making an expansion about this point, we obtain the following results:
a) The k-dependence of the vertex in $H_{4,0}$ becomes

$$f_{4,0}(k_1,..,k_4) = \lambda\delta(\sum k_i)\{1 + O(k^2)\}. \tag{3.124}$$

b) The contribution of the second diagram in the Fig.3 vanishes, because the effective vertex in $H_{3,1}$ is reduced to the form

$$f_{3,1}(k_1,..,k_4) = \lambda\delta(\sum k_i)\{\sum k_i + O(k^2)\}. \tag{3.125}$$

The non-zero contribution for these diagrams is of order $k_i k_j$, and since we consider Wilson's idea on the "irrelevant operator", we shall argue that such terms are not important near a fixed point.
c) At the renormalization point, the effective vertex in $H_{2,2}$ takes the form:

$$f_{2,2}(k_1,..,k_4) = \lambda\delta(\sum k_i)\{1 + O(k^2)\}. \tag{3.126}$$

Let us assume that our lattice is taken so large, that the sums over momenta are approximated by integration, and that at the fixed point the value for the mass becomes small. With this approximation and with the "mass" and "wave function" renormalization on the external variables the effective coupling becomes :

$$\lambda_{eff} = 2^2 Z^2\{\lambda - \lambda^2 I(J')\}, \tag{3.127}$$

where

$$I(J') = \int_{q\in\Lambda^*_{2a}} \frac{dq}{2\pi}\frac{1}{2\omega_{\tilde{c}}(q)}\frac{1}{2\omega_{\tilde{c}}(q-k)}\frac{J(q)}{\omega_{\tilde{c}}(q-k)+\omega_{\tilde{c}}(q-k)}. \tag{3.128}$$

Then we have as a condition for the fixed point the equations:

$$m^2 = 4Z(m^2 + \Sigma(0,J')) \tag{3.129}$$
$$\lambda_{effec} = 2^{3-D}Z^2\{\lambda - \lambda^2 I(J')\}, \tag{3.130}$$

where $\Sigma, I(J')$ have a dependence on the adiabatic invariant of the fast variables. It is clear from eq.(3.130) that a trivial point $m = \lambda = 0$ exists as a solution, but in this case, we can find other non trivial point. It is given by:

$$m^2 = -\frac{3}{4}\Sigma(0,J') \tag{3.131}$$
$$\lambda_{effec} = \frac{1 - 1/2^{3-D}}{I(J')}. \tag{3.132}$$

In what follows we suppose that $\lambda$ and $m^2$ have been chosen to satisfy the fixed point condition for a certain set of $J$'s. Now, we will discuss the condition on the variables $J'$, necessary to remain on the fixed point $m^*, \lambda^*$ after many iterations of the RSRG transformation. Let us now apply the first step of the RS-approach and investigate what happens to our variable $\phi$, when we iterate again the transformation described in the previous section. The initial variable $\phi$ is given by:

$$\phi(k) = \sqrt{2}\left(\cos(\frac{k}{2})\,\Phi(2k) + i\sin(\frac{k}{2})\,\Psi(2k)\right), \tag{3.133}$$

where the block-variable takes after the first group of canonical transformation the form

$$\Phi(k) = \frac{(L/2)^{1/2}}{\sqrt{2\omega_\phi(k)}}\left(\sqrt{I(k)}\,e^{-i\theta(k)} + \sqrt{I(-k)}\,e^{i\theta(-k)}\right) + O(\lambda_k) \tag{3.134}$$
$$\Phi(k) = (L/2)^{D/2} f_\Phi(k,\omega_\phi,\theta,J). \tag{3.135}$$



The remaining canonical transformations (mass and wave-function) and the rescaling of momenta change the form $\Phi(k)$ into

$$\Phi(k) = Z^{1/2}\Phi(\bar{k},\bar{\omega}_\phi,\bar{\theta},\bar{J}) + O(\lambda), \tag{3.136}$$

where $k = \frac{\bar{k}}{2}$ and the original variable takes the form:

$$\phi(k) = \left(Z^{1/2}2^{1/2}\cos(\frac{\bar{k}}{4})\,\Phi(\bar{k}) + 2^{1/2}i\sin(\frac{\bar{k}}{4})\,\Psi(\bar{k})\right). \tag{3.137}$$

The equation for $\Phi$ is the starting point for the second renormalization group approximation on the block lattice. Here, we introduce a new blocking by defining a new lattice $\Lambda_{2^2a}$. We average over the new fast variables with a change of our expression for $\phi$. Then the variable is transformed to

$$\phi(k) = \left((Z^{1/2}2^{1/2})^2\cos(\frac{\bar{k}}{8})\,\cos(\frac{\bar{k}}{4})\,\Phi(\bar{k}) + iZ^{1/2}2\cos(\frac{\bar{k}}{8})\sin(\frac{\bar{k}}{4})\,\Psi^{(1)}(\bar{k}) + i\sin(\frac{\bar{k}}{8})\,\Psi(\bar{k})\right). \tag{3.138}$$

where $k = \frac{\bar{k}}{4}$ .

After many iterations of the RSRG-transformation, the region of the momenta become small. In this region, we can consider that the $J$'s are not modified under a new transformation. The *generalization of the eq.( 3.138) to n iterations gives us the following expression:*

$$\begin{aligned}\frac{\phi(k)}{\sqrt{V}} = \frac{1}{\sqrt{\tilde{V}}}\{&(Z^{1/2}2^{D/2})^n\cos(\frac{\bar{k}}{4})\ldots\cos(\frac{\bar{k}}{2^{n+1}})\,\Phi(\bar{k}) + \\ &i(Z^{1/2}2^{1/2})^{n-1}\sin(\frac{\bar{k}}{4})\cos(\frac{\bar{k}}{8})\ldots\cos(\frac{\bar{k}}{2^{n+1}})\,\epsilon^{(n-1)}(\bar{k}) + \ldots + \\ &i(Z^{1/2}2^{1/2})\cos(\frac{\bar{k}}{2^{n+1}})\sin(\frac{\bar{k}}{2^n})\,\Psi^{(1)}(\bar{k}/2^n) + 2^{(1/2)}i\sin(\frac{\bar{k}}{2^{n+1}})\,\Psi(\bar{k}/2^n)\},\end{aligned} \tag{3.139}$$

where $\bar{k} = 2^n k$, and we have taken the volume factor to the other side. $(L/2^n)^D$ is the lattice volume after $n$-step.

Expanding the last expression in the region for small momentum one finds a scaling behavior for $\phi(k)$. It is given by

$$\frac{\phi(\bar{k})}{\sqrt{V}} = ((Z^{1/2}2^{1/2})^n f_\Phi(\bar{k}), \tag{3.140}$$

with $f$ being independent from $n$ only after many iterations, and the scaling behavior in the small $k$ approximation for the $\phi(k)$ variable is:

$$\phi(k) \approx k^{-\frac{1}{2}-\frac{log Z}{2log 2}}\,\Phi(\bar{k}). \tag{3.141}$$

To close this section we can investigate the scaling behavior of the action variables J's, which produces a restriction on them. In this case one finds that the scaling law is given by:

$$J \approx k^{-\frac{log Z}{log 2}}. \tag{3.142}$$

We conclude that the introduction of an effective Hamiltonian with a dependence on the variable $\Phi$ through the RSRG-transformation lead us to obtain a similar information about the fixed point and the scaling law for the $\phi$ variables and for the action variables as found in the paper of Bartels and Chang.

## 3.4 Extension to more dimensions

In this section we discuss the generalization of our RSRG-approach to the case of (2+1) and (3+1) dimensions for the Classical Field Theory on the lattice. We investigate the theory with scalar variables, so that the phase space have real valued functions $\phi_{\vec{x}}$ and $\pi_{\vec{x}}$, which are defined on a lattice $\Lambda_a$. It is identified with the subset $\{\vec{x}\epsilon aZ^D/\ -L_\mu \leq x_\mu < L_\mu\,;\mu = 1,..,D\}$. Now, we introduce a new lattice $\Lambda_{2a}$, which is defined as $\Lambda_{2a} = \Lambda_a \cap 2aZ^D$. We divide $\Lambda_a$ into blocks B(y) parametrized by the points of $\Lambda_{2a}$, and $B(y) = \{\vec{x}\epsilon\Lambda_a/\ -y_\mu \leq x_\mu < y_\mu + a\,;\mu = 1,..,D\ \vec{y}\epsilon\Lambda_{2a}\}$, containing $2^D$ variables $\phi_{\vec{x}}$ with $\vec{x}\epsilon B(y)$.

The first step of the renormalization group approach is to rewrite in term of the variables in each block, and the Hamiltonian is separated into a term depending only on the variables of each block ($H_{block}$) and a term which has the coupling between the nearest block ($V^{int}$).

The block Hamiltonian can be diagonalized through a symplectic change of variables which is given by:

$$\Phi_{\vec{y}} = \frac{1}{\sqrt{2^D}} \sum_{\vec{x}\epsilon B(y)} \phi_{\vec{x}} \tag{3.143}$$

$$\Psi^{(i)}(\vec{y}) = A_{i,\vec{x}}\phi_{\vec{x}} \quad \left| \begin{array}{l} i = 1,\ldots,(2^D-1) \\ \vec{x}\epsilon B(\vec{y}) \end{array} \right. \tag{3.144}$$

and a particular choice for $A_{i,\vec{x}}$ is :
for D=2

$$A = \frac{1}{\sqrt{4}} \begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{vmatrix}, \tag{3.145}$$

and for D=3

$$A = \frac{1}{\sqrt{8}} \begin{vmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{vmatrix}. \tag{3.146}$$

The eigenvalues associated to each new variables (in the real-space) are:
for D=2

$$\omega^2_\Phi = m^2 + 2, \tag{3.147}$$

$$\omega^2_{\Psi^1} = m^2 + 6, \tag{3.148}$$

$$\omega^2_{\Psi^i} = m^2 + 4 \quad i = 2,3, \tag{3.149}$$

and for D=3

$$\omega^2_\Phi = m^2 + 3, \tag{3.150}$$

$$\omega^2_{\Psi^i} = m^2 + 5 \quad i = 1,2,3, \tag{3.151}$$

$$\omega^2_{\Psi^j} = m^2 + 7 \quad j = 4,5,6, \tag{3.152}$$

$$\omega^2_{\Psi^7} = m^2 + 9. \tag{3.153}$$



The eq.(3.143) define what we call "block-variable", which has the lower eigenvalue as compared with the other variables. In terms of the new variables, one obtains

$$H_0^{D=2} = \sum_{\vec{y}\epsilon\Lambda_{2a}} \left( H(\Phi(\vec{y}),\omega_\Phi) + \sum_i H^i(\Psi^i(\vec{y}),\omega_{\Phi^{(i)}}) + V_{int}(\Phi;\Psi^{(2)},\Psi^{(3)}) \right.$$
$$\left. V_{int}(\Psi^{(1)};\Psi^{(2)},\Psi^{(3)}) \right), \tag{3.154}$$

$$H_0^{D=3} = \sum_{\vec{y}\epsilon\Lambda_{2a}} \left( H(\Phi(\vec{y},\omega_\Phi) + \sum_i H^i(\Psi(\vec{y}),\omega_{\Phi^{(i)}}) + V_{int}(\Phi;\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}) + \right.$$
$$\left. V_{int}(\Psi^{(7)};\Psi^{(4)},\Psi^{(5)},\Psi^{(6)}) \right), \tag{3.155}$$

where $V_{int}(\Phi;\Psi^{(a)},\ldots,\Psi^{(p)})$ contains the couplings between $\Phi$ and $\Psi^{(l)}$ but not between $\Psi^{(i)}$ and $\Psi^{(k)}$.

The next step for our RSRGT is to average over the fast variables in order to eliminate the coupling between $\phi$ and $\Psi_j$ (variables with higher eigenvalues). There is a correspondence between high eigenvalues and high frequencies, when we introduce a Fourier Transformation. In higher dimensions this step may not be clear because some new variables have the same eigenvalues (see eqs.(3.149, 3.152)). We will show that this is not really a problem. Then, since after the first averaged, for example over $\Psi^2$ for D=2 or over $\Psi^7$ for D=3, one finds new couplings between the variables $\Psi^i$ (i=1,2) or couplings between the variables $\Psi^i$ (i=4,5,6) are generated. This fact allows us to carry out a variable transformation in this "subset" of variables obtaining a splitting for the eigenvalues ( frequency in momentum-space). Now one can eliminate the new fast variables through an average. This step is repeated until we can find an effective Hamiltonian depending only on the block-variable. The main result of this approach is, that the effective frequency associated with the slow variables in the region for small momentum takes the form:

$$\tilde{\omega}^2(k) = m^2 + \sum_i^D k_i^2. \tag{3.156}$$

Now, we will apply this method to a D=2 dimensional scalar classical theory defined on the lattice. For D=3 this extension is straightforward.

### 3.4.1 Two Dimensional Model.

In order to show the different steps of our RS-approach let us consider our theory defined by the term:

$$H_0 = \frac{1}{2}\sum_{\vec{l}} \left( \Pi^2_{\vec{l}} + m^2\Phi_{\vec{l}} + \sum_{\vec{i}}^D (\Phi_{\vec{l}+\vec{i}} - \Phi_{\vec{l}})^2 \right). \tag{3.157}$$

We choose a block with four variables, which are given by

$$\Phi_{2\vec{l}} = \Phi_1(\vec{y}) \qquad \Phi_{2\vec{l}+\vec{i}} = \Phi_2(\vec{y})$$
$$\Phi_{2\vec{l}+\vec{j}} = \Phi_3(\vec{y}) \quad \Phi_{2\vec{l}+\vec{i}+\vec{j}} = \Phi_4(\vec{y}), \tag{3.158}$$

where we use a notation given in the Fig.5. In terms of these variables we can rewrite:

$$H_{block} = \frac{1}{2}\sum_{\vec{y}\epsilon\Lambda_{1a}}\Big(\sum_i\Big(\Pi_i^2(\vec{y}) + (m^2+2D)\Phi_i^2(\vec{y})\Big) - 2\left(\Phi_1(\vec{y})\Phi_2(\vec{y}) + \Phi_1(\vec{y})\Phi_3(\vec{y}) + \Phi_4(\vec{y})\Phi_2(\vec{y}) + \Phi_4(\vec{y})\Phi_3(\vec{y})\right), \quad (3.159)$$

and

$$V^{int} = -\sum_{\vec{y}}\Big((\Phi_1(\vec{y}+\vec{i})\Phi_2(\vec{y}) + \Phi_3(\vec{y}+\vec{i})\Phi_4(\vec{y}) + \Phi_1(\vec{y}+\vec{j})\Phi_3(\vec{y}) + \Phi_2(\vec{y}+\vec{j})\Phi_4(\vec{y})\Big). \quad (3.160)$$

$H_{block}$ is diagonalized with the new variables defined in eq.( 3.145). Now, we introduce a Fourier transformation to the new variables defined on the block lattice. In momentum-space the Hamiltonian takes the form

$$H_{block} = \frac{1}{2L^D}\sum_{\vec{y}}\Big(\sum_{i=1,..3}\Big(\Pi_i(\vec{k})\Pi_i(-\vec{k}) + \omega_{\Psi_i}^2(\vec{k})\Psi_i(\vec{k})\Psi_i(-\vec{k})\Big) + \Pi_\Phi(\vec{k})\Pi_\Phi(-\vec{k}) + \omega_\Phi^2(\vec{k})\Phi(\vec{k})\Phi(-\vec{k}) + \Phi(-\vec{k})\sum_{l=2,3}\Psi_l(\vec{k})\lambda_{\Phi,l} + \Psi_1(-\vec{k})\sum_{l=2,3}\Psi_l(\vec{k})\lambda_{\psi,l}\Big), \quad (3.161)$$

where

$$\omega_\Phi^2(\vec{k}) = m^2 + 2\sum_\mu \sin^2 k_\mu \quad (3.162)$$

$$\omega_{\Psi_1}^2(\vec{k}) = m^2 + 4 + 2\sum_\mu \cos^2 k_\mu \quad (3.163)$$

$$\omega_{\Psi_2}^2(\vec{k}) = m^2 + 4 + 2(\sin^2 k_2 - \sin^2 k_1) \quad (3.164)$$

$$\omega_{\Psi_3}^2(\vec{k}) = m^2 + 4 + 2(\sin^2 k_1 - \sin^2 k_2). \quad (3.165)$$

The couplings are:

$$\lambda_{\Phi,2} = 2i\sin 2k_1 \quad \lambda_{\Phi,3} = 2i\sin 2k_2$$
$$\lambda_{\Psi,2} = 2i\sin 2k_2 \quad \lambda_{\Psi,3} = 2i\sin 2k_1, \quad (3.166)$$

and the following relation is satisfied

$$\omega_\Phi^2(\vec{k}) \le \omega_{\Psi_{2or3}}^2(\vec{k}) \le \omega_{\Psi_1}^2(\vec{k}).$$

Now, we perform an average over the "fast variable" $\Psi_1$ by the method described in the first section. In order to obtain an effective Hamiltonian which depends on the adiabatic invariant associated with the fast variable ($J_1$), one needs to introduce the following canonical transformation:

$$F_2(J_i', \theta_i) = \sum_{\vec{k}}\Bigg(\sum_{i=1,..3} J_i'(k)\theta_i(k) - \frac{i}{2L^D}\frac{\lambda_{1,2}(\vec{k})}{\sqrt{2\omega_{\Psi_1}(k)2\omega_{\Psi_2}(k)}}\Bigg(a_1(k)\{\frac{a_2^*(k)}{\omega_{\Psi_1}(k)-\omega_{\Psi_2}(k)} - \frac{a_2(-k)}{\omega_{\Psi_1}(k)+\omega_{\Psi_2}(k)}\} + a_1^*(-k)\{\frac{a_2^*(k)}{\omega_{\Psi_1}(k)+\omega_{\Psi_2}(k)} - \frac{a_2(-k)}{\omega_{\Psi_1}(k)-\omega_{\Psi_2}(k)}\}\Bigg) - \frac{i}{2L^D}\frac{\lambda_{1,3}(\vec{k})}{\sqrt{2\omega_{\Psi_1}(k)2\omega_{\Psi_3}(k)}}\Bigg(a_1(k)\{\frac{a_3^*(k)}{\omega_{\Psi_1}(k)-\omega_{\Psi_3}(k)} - \frac{a_2(-k)}{\omega_{\Psi_1}(k)+\omega_{\Psi_2}(k)}\} + a_1^*(-k)\{\frac{a_2^*(k)}{\omega_{\Psi_1}(k)+\omega_{\Psi_3}(k)} - \frac{a_3(-k)}{\omega_{\Psi_1}(k)-\omega_{\Psi_3}(k)}\}\Bigg)\Bigg), \quad (3.167)$$



and the effective Hamiltonian becomes:

$$H_0 = \frac{1}{2L^D}\sum_{\vec{y}}\Bigg(\sum_{i=2,3}\omega_{\Psi_i}^2(\vec{k})a_i(\vec{k})a_i^*(\vec{k}) + \omega_\Phi^2(\vec{k})a_\Phi(\vec{k})a_\Phi^*(\vec{k}) + \Phi(-\vec{k})\sum_{l=2,3}\Psi_l(\vec{k})\lambda_{\Phi,l} + H_0(J') + \sum_{l=2,3}\frac{1}{2\omega_{\Psi_l}(k)}(a_l(\vec{k}) + a_l^*(-\vec{k}))(a_l(-\vec{k}) + a_l^*(\vec{k}))\Sigma_l(\vec{k}) + \frac{1}{\sqrt{2\omega_{\Psi_2}(k)2\omega_{\Psi_3}(k)}}(a_2(\vec{k}) + a_2^*(-\vec{k}))(a_3(-\vec{k}) + a_3^*(\vec{k}))\Delta_{2,3}(\vec{k})\Bigg) \quad (3.168)$$

where

$$\Sigma_l(\vec{k}) = \frac{i^2}{2^2}\frac{\lambda_{1,l}(\vec{k})\lambda_{1,l}(-\vec{k})}{\omega_{\Psi_1}^2(\vec{k}) - \omega_{\Psi_l}^2(\vec{k})} \quad (3.169)$$

$$\Delta_{2,3}(\vec{k}) = \frac{i^2}{2^2}\left[\frac{\lambda_{2,1}(\vec{k})\lambda_{3,1}(-\vec{k})}{\omega_{\Psi_1}^2(\vec{k}) - \omega_{\Psi_2}^2(\vec{k})} + \frac{\lambda_{3,1}(\vec{k})\lambda_{2,1}(-\vec{k})}{\omega_{\Psi_1}^2(\vec{k}) - \omega_{\Psi_3}^2(\vec{k})}\right]. \quad (3.170)$$

After a Bogoliubov transformation on the variables $\Psi_2$ and $\Psi_3$, a renormalization of the corresponding frequencies for the variables is found:

$$\tilde{\omega}_l^2(\vec{k}) = \omega_l^2(\vec{k}) + \Sigma_l(\vec{k}), \quad (3.171)$$

and the coupling between $\Psi_2$ and $\Psi_3$ are not modified under this transformation.

The next step is to carry out a change of variables of $\Psi_2$,$\Psi_3$, which diagonalize the Hamiltonian in this set of variables. The result is a splitting of the frequencies given in the eqs.(3.164 and 3.165) and a new coupling between the $\Phi$ variable and the new variables $\bar{\Psi}$, leading us to the definition of the fast variables. One can show that the following relation is satisfied

$$\omega_\Phi^2(\vec{k}) \le \bar{\omega}_{\Psi_2}^2(\vec{k}) \le \bar{\omega}_{\Psi_3}^2(\vec{k}).$$

Finally, we average over the new fast variables with the result that the frequency associated to the slow variable is modified. It becomes

$$\tilde{\omega}_\Phi^2(\vec{k}) = \omega_\Phi^2(\vec{k}) + \sum_{l=2,3}\tilde{\Sigma}_l(\vec{k}), \quad (3.172)$$

and the last term takes the form:

$$\tilde{\Sigma}_l(\vec{k}) = \frac{i^2}{2^2}\frac{\bar{\lambda}_{\Phi,l}(\vec{k})\bar{\lambda}_{\Phi,l}(-\vec{k})}{\bar{\omega}_{\Psi_l}^2(\vec{k}) - \bar{\omega}_\Phi^2(\vec{k})}, \quad (3.173)$$

where the $\bar{\omega}$ are the eigenfrequencies of the new normal variables $\bar{\Psi}_j$ and the $\bar{\lambda}$ are the new effective couplings.

Since eventually we will be interested only in the region of small momenta, we expand around the point $\vec{k} = 0$. In this approximation, we find that the frequency for the slow variable have the following behavior:

$$\tilde{\omega}^2(k) = m^2 + \sum_i^D k_i^2 \quad (3.174)$$

We have shown with this result, that our original Hamiltonian can be described approximately (in the region of small momenta), through an effective Hamiltonian, which has a dependence on the Block-variable and the adiabatic invariant of the fast modes.

The next part of the RS-approach is to investigate the effect of an interaction term, which has a dependence on a small coupling constant (for example $\lambda\phi^4$). In this case, the application of our approach is similar to the discussion for the 1-dimensional system. Since this part gives us no new information, we do not include this in this chapter.

The main purpose of this first part is a reformulation and a detailed description of the RG-transformation for this classical system in the real space. This model let us carry out low order calculation easily and allows us to learn the method in order to apply this idea in formulating a RG-transformation in real space for a classical lattice gauge system. It should be stated clearly that our analysis does not attempt to provide any rigorous proof for the existence of these special classical solutions, but rather we intended to formulate the idea and to work out in some details how renormalization might work in a classical lattice field theory.

To conclude this section, one can interpret our results as follow: Among the KAM tori which are preserved in the presence of the interaction term $H_\lambda$, there seems to exist a special set of regular periodic classical solutions, which behave in the same way as the quantum system does near the critical point. In this sense, the classical system (i.e some of its solutions) seems to "know" about the critical behavior of the quantum system. However, there are many other solutions to which our method can not be applied. We can not say at the moment how important these solutions are for the computation of the partition function.



# Chapter 4

# Background Material.

In this chapter, the SU(2)-Hamiltonian lattice gauge theory is briefly presented and discussed in terms of a new set of variables. The transformation properties of the electric term of the Hamilton Operator in term of this change of variables is found and the choice of the new variables is performed taking in account some ideas of block spin transformation. In the last section we discuss our program for the formulation of a RGT for a classical gauge system.

## 4.1 The Lattice Hamiltonian.

The SU(2) lattice Hamiltonian [12] may be derived by starting from the Wilson action in a four dimensional Euclidean space-time lattice and by computing the Transfer Matrix in the temporal gauge. This corresponds to fixing the gauge of the time-like links to unity. The result is,

$$\hat{H} = \frac{g^2}{2a}\hat{T} + \frac{2}{ag^2}\hat{V}, \tag{4.1}$$

where the first term $\hat{T}$ is an elliptic differential operator in the group parameters of the link variables and represents the electric field squared. It plays the role of the kinetic energy. The magnetic term $\hat{V}$ provides the potential which is the traditional Wilson term for a spatial lattice. In this case, one deals with a conventional quantum mechanical system where $\hat{H}$ is the Hamiltonian Operator depending on the link variables $U_l$ and their conjugate momenta which are defined in the spatial lattice, g is the coupling constant and $a$ the lattice constant.

The system is thus described by the time-independent Schrödinger equation:

$$\left(\frac{g^4}{4}\hat{T} + 2\hat{V}\right)\Psi[\{U_l\}] = g^2 E\Psi[\{U_l\}], \tag{4.2}$$

with the subsidiary condition

$$\Gamma\Psi[\{U_l\}] = 0 \tag{4.3}$$

accounting for Gauss' law.

The wave function $\Psi$ depends on the set of all link variables $U_l$, and $\Gamma$ is the generator of the time-independent gauge transformation. The physical solution $\Psi_{phy}$ is a gauge invariant *eigenstate of $\hat{H}$, which obey the Gauss' law for a Yang Mills theory on the lattice.*

Here, the Hamiltonian formulation is presented in terms of links variables. First the gauge group is parametrized. In a $d$-dimensional lattice with lattice spacing a, the link connecting site $a\vec{n}$ ( $\vec{n} \epsilon Z^d$ ) with the site $a\vec{n} + \vec{e_j}a$ is uniquely identified by the lattice site $\vec{n}$ and the direction $\hat{j}$. Denoting this combination by a single link index $l = (\vec{n}, \hat{j})$, a dynamical link variable U, defined on each link of the lattice, is labeled $U_{\vec{n},\hat{j}}$, or more simply $U_l$. It is an element of the SU(2)-Group. Any element $U_l$ ($U_l \epsilon SU(2)$) of the fundamental representation can be written as

$$U_l = e^{\frac{i}{2}\vec{\sigma}\vec{r}} = x^0 1 + ix^a\sigma^a, \tag{4.4}$$

with $|\vec{r}| < 2\pi$.
The Cartesian coordinates $x^a$ are connected with the $r^a$ through the equations

$$x^0 = \cos\frac{|\vec{r}|}{2} \tag{4.5}$$

$$x^a = \frac{r^a}{|\vec{r}|}\sin\frac{|\vec{r}|}{2}, \tag{4.6}$$

and thus the following relation is satisfied

$$x^0 = \sqrt{1 - x^a x^a}. \tag{4.7}$$

The $\sigma^a$ are the usual Pauli matrices and a summation convention is adopted.

The terms of the lattice Hamiltonian for the SU(2)-group are:

$$\hat{T} = \sum_{l,links} E_l^a E_l^a, \tag{4.8}$$

$$\hat{V} = \sum_{p,plaq} Tr(1 - U_p), \tag{4.9}$$

where $U_p$ denotes the product of four U's corresponding to a chain of four links around an elementary plaquette or square and $E_l^2$ is the Casimir Operator of SU(2). The Casimir operator is described in terms of the left ($\vec{E}_L(l)$) and the right ($\vec{E}_R(l)$) generators of SU(2) transformation, having the effect

$$\vec{E}_L(l)e^{i\vec{J}\vec{r}} = \vec{J}e^{i\vec{J}\vec{r}} \tag{4.10}$$

$$\vec{E}_R(l)e^{i\vec{J}\vec{r}} = e^{i\vec{J}\vec{r}}\vec{J}, \tag{4.11}$$

where $\vec{J}$ is any representation of the Lie algebra.

The $\vec{E}$'s obey the commutation relation

$$\begin{array}{ll} [E_R^i(l), E_R^j(l')] = \delta_{l,l'}\epsilon^{ijk}E_R^k(l) & [E_L^i(l), E_R^j(l')] = 0 \\ [E_L^i(l), E_L^j(l')] = -\delta_{l,l'}\epsilon^{ijk}E_L^k(l) & [E_L^i(l), E^L(l')] = 0 \end{array} \tag{4.12}$$

and they are related by

$$\sigma^a E_L^a = \left(U\sigma^b U^{-1}\right) E_R^b. \tag{4.13}$$



Introducing a rotation matrix $R^{ab}(U)$ by

$$\sigma^a R^{ab}(U) = U\sigma^b U^{-1}, \tag{4.14}$$

eq.(4.13) yields for

$$E_L^a(U) = R^{ab}(U)E_R^b(U). \tag{4.15}$$

Here, the Casimir operator $\vec{E}^2$, which appears in the Hamiltonian, is in virtue of the eq.(4.15), the same for either $E_L^a$ or $E_R^a$. It is simply the Laplace-Bertrami operator $-\Delta$ on the group manifold SU(2) [11]. In this case it is isomorphic to the three dimensional sphere $S^3$.

*As mentioned above, the Hamiltonian is derived in the temporal gauge. This derivation still* leaves the freedom to perform time-independent gauge transformation $g$, which is parametrized by an arbitrary function $w^a(n)$

$$\Omega(w^a(\vec{n})) = exp(i\sum_a w^a(\vec{n})\Gamma^a(\vec{n})), \tag{4.16}$$

where $\Gamma^a$ are the generators of such a transformation at an arbitrary site $\vec{n}$ on the lattice and commuting with the Hamiltonian. $\Gamma^a$ reads

$$\Gamma^a(\vec{n}) = \sum_{j=1}^{d}[E_L^a(\vec{n}, j) - E_R^a\left(n - \vec{e}_j, j\right)], \tag{4.17}$$

where $E_R^a(\vec{n}, j) = E_R^a(U_{\vec{n},j})$.
As the wave function should be gauge invariant we have

$$\Omega(w^a(\vec{n}))\Psi[\{U_l\}] = \Psi[\{U_l\}] \rightarrow \Gamma\Psi[\{U_l\}] = 0 \qquad \text{for all a },\vec{n} \tag{4.18}$$

The effect of this transformation (see eq. 4.17) is to multiply each $U_{\vec{n},\hat{e_j}}$ link variables on the positive direction link of $\vec{n}$ at the left by $g(\vec{n}) = e^{\frac{i}{2}\sigma^a w(\vec{n})}$ and each U on the negative direction links of $\vec{n}$ at the right by $g^{-1}(\vec{n})$. The net result is to shift

$$U(\vec{n}, j) \rightarrow g(\vec{n})\,U(\vec{n}, j)\,g^{-1}\left(\vec{n} + \vec{j}\right), \tag{4.19}$$

*which is the correct gauge transformation.*

## 4.2 Transformation of the electric Operator

In order to express the Hamilton operator in terms of a new set of variables, one need to study the transformation properties of the operators $E_L^a, E_R^a$ under a a general coordinate transformation.

To discuss the change of variables, initially, the Hamiltonian is written in terms of the group element $U_l$ on every link in the spatial lattice. These degrees of freedom are now replaced by a new set of variables, which can be separated in two groups; the first group are constituted from a subset of the original variables denoted as $U_l'$, which are on a path (block-gauge-path,

which are discussed later on), and in the second group are the variables $\{y_l\}$, which are parallel transport along determined paths. They are defined as

$$y_l = \Pi_{l\in P_s} U_l^{\sigma_s}, \tag{4.20}$$

where $P_s$ is an oriented path on the lattice, and $\sigma_s = +1$ or $-1$ depending on whether $P_s$ passes through link $l$ in the positive or negative directions. The next step is to rewrite the Hamiltonian in terms of the new variables set. The terms arising from the space-space plaquette (the magnetic term in H) are in general easily re-expressed, and they depend, in this case, only on the $\{y_l\}$ variables. The electric term, being a differential operator in the group parameter of the $U_l$, is more difficult to rewrite, but in order to obtain a close expression of the left- right operators in terms of the new variables let us investigate the transformation properties of $E_{R/L}$ under this change of variables.
The link variables set $\{U_l\}$ is replaced by one of the new sets, which is written generically as $\{u'_l, y_l\}$, where $u'_l$ indicates those that are only on the "gauge-path".

Let us consider $u_{l,\alpha}$ as the components of the parameter for the link:

$$u_l = e^{\frac{i\sigma^\alpha u_{\alpha,l}}{2}}. \tag{4.21}$$

By equation (4.10) the $E_l^\alpha$ has the form

$$E_L^\alpha(u_l) = \sum_{\alpha,\beta} A_{\alpha,\beta}(u_l)\frac{\partial}{\partial u_{l\beta}}. \tag{4.22}$$

In terms of the new variables and using the chain rule one finds:

$$E_L^\alpha(u_l) = \epsilon_l E_L^\alpha(u_l) + \sum_{\alpha,\beta,\gamma,s} A_{\alpha,\beta}(u_l)\frac{\partial y_{s,\gamma}}{\partial u_{l\beta}}\frac{\partial}{\partial y_{s\gamma}}. \tag{4.23}$$

where $\epsilon = 1$ if $l$ is on the gauge-path and is equal to zero otherwise. Thus

$$E_L^\alpha(u_l) = \epsilon_l E_L^\alpha(u_l) + \sum_{\alpha,\beta,\gamma,s} A_{\alpha,\beta}(u_l)\frac{\partial y_{s,\gamma}}{\partial u_{l\beta}} A^{-1}_{\gamma,\delta}(y_s)\, E_L^\delta(y_s). \tag{4.24}$$

The transformation Kern $A(u_l)\frac{\partial y}{\partial u}A^{-1}(y_l)$ is evaluated by noting the structure of the $\{y_l\}$. The $y_s$ is a variable which makes the positive or negative transit of the link $u_l$. The dependence of $y_s$ on $u_l$ is then expressed in this way:

$$e^{i\vec{J}\vec{y}_s} = e^{i\vec{J}\vec{L}_\epsilon} e^{i\vec{J}\vec{u}_l} e^{i\vec{J}\vec{R}_\epsilon} e^{-i\vec{J}\vec{u}_l} e^{i\vec{J}\vec{R}} \tag{4.25}$$

If $y_s$ makes only a positive transit of the link $l$, the last two factors of the above equation are absent and if only a negative transit is made, the first two factors are absent. Noting that

$$\vec{E}_L e^{-i\vec{J}\vec{y}_s} = -e^{-i\vec{J}\vec{y}_s}\vec{J}, \tag{4.26}$$

and

$$e^{i\vec{J}\vec{h}} J_\sigma e^{-i\vec{J}\vec{h}} = \sum_\mu D^1_{\mu\sigma}(h) J_\mu, \tag{4.27}$$



where $D^1_{\mu\sigma}$ is the spin-1 representation of SU(2) in the Cartesian basis, one can show that the following relation yields

$$E_L^\alpha(u_i) = \epsilon_l E_L^\alpha(u_l) + \sum_{\beta,s}^{P} D^1_{\beta,\alpha}(L)\, E_L^\beta(y_s) - \sum_{\beta,s}^{N} D^1_{\alpha,\beta}(R)\, E_R^\beta(y_s), \tag{4.28}$$

where we taken into account that the generators $J_\mu$ are linearly independent and the relation

$$E_R^\alpha(u) = \sum_\mu D^1_{\mu\alpha}(u) E_L^\mu(u). \tag{4.29}$$

Here $\sum^P$ is the sum over all new variables making a positive transit of the links $l$ and $L$ is the path which is defined by the first factor of eq.(4.25). $\sum^N$ is the similar sum for the negative transit and $R$ is defined by the last factor of the eq.(4.25).

Now, let us consider the wave function $\Psi[y_s, u'_l]$ in terms of the new variables set. A gauge transformation is made at the site s, and the element of the SU(2)-group is denoted by $\lambda_s = e^{i\vec{J}\vec{\alpha}_s}$, producing

$$\Omega_s\Psi(u_s, u_e) = \Psi\left(\lambda_s u_s, u_e\lambda_s^{-1}\right) \tag{4.30}$$

where $u_s$ is the generic symbol of a link started at site s and, $u_e$ stands for a link ending at site s. All other variables are unaffected by the gauge transformation. The physical state due to the gauge invariance satisfy the condition

$$\Omega_s\Psi = \Psi. \tag{4.31}$$

Considering a gauge transformation at all sites except at the sites at which the variables $y_s$ start and end, can be arrived to the relation

$$\Psi(y_s, u'_l) = \Psi\left(y_s, \lambda_s u'_l \lambda_E^{-1}\right), \tag{4.32}$$

where $\lambda_s$ is the gauge transformation at the site in which link $l$ starts and $\lambda_E^{-1}$ is the gauge transformation at the site wherein the link $l$ ends.
In this case, $\lambda_E$ along the gauge-path can be chosen always, such that $\lambda_s u'_l \lambda_E = 1$ ( $\lambda_s$ will have been chosen from a previous step). Thus

$$\Psi(y_s, u'_e) = \Phi(y_s). \tag{4.33}$$

The result is that the general state is independent on the link variable on the gauge-path, and is invariant under local gauge transformation for all new variables.

After this change of variables, the Hamiltonian is separated into three terms: the first term is dependent only on the $y_s$ variables, the second term has a dependence on the $u'_s$ variables, and the last term has a dependence on the two set of variables. However, from the condition (4.33) on the wave function, it is concluded that the relevant physical information are included in the first term, which is independent on the links variables defined on the tree path. In this way can be defined an effective Hamilton Operator which depends only on the new link variables set. The analytic form of the Hamiltonian is differently by different choice of the coordinates and gauge-path (gauge fixing), but it must contain the same physical information. A normal choice of gauge-path is the complete gauge fixing on a maximal tree on the lattice [13] and the $y_s$ variables become loops. However we will introduce another change of variables and choice of the gauge fixing, which is motivated from renormalization ideas [61] and block-spin [57] formulation. This set of variables is discussed in the section 4.4.

## 4.3 RGT-idea

Both in the Wilson formulation and Kogut-Susskind Hamiltonian formulation of lattice gauge theories, the Action or Hamiltonian is written in terms of group elements $u_l$ of every link in the lattice. The selection of appropriate degrees of freedom and the introduction of the new variables set through a change of variables has been discussed earlier in different approaches [62], wherein the properties of the Hamiltonian were analyzed and the general form of the wave function was stated. The formulation of the Hamiltonian in terms of variables which are gauge invariant was made after a complete gauge fixing (i.e. all gauge arbitrariness are eliminated [63, 64]).

In order to apply the same schema of the RGT for gauge field theory now to a classical gauge theory, i.e. all the steps of the renormalization method of quantum field theory at the classical level must be implemented to formulate a RGT for classical gauge system, let us briefly discuss the steps of the "block spin" approach in QFT to fix the schema which must be applied in the classical formulation. The first step in this approach for a lattice gauge theory is the block construction and definition of lattice average. The average field is used to define the low momentum degrees of freedom on a particular length scale. In the case of a classical formulation, the average field corresponded to the new variable of the slow degrees of freedom.

In this context, the gauge theory defined on $a$-lattice is covered by cubes (or blocks) with side length $La$ and containing $L^d$ sites (see Fig 6.). The lattice of the centers of these cubes is an $La$-lattice. Let us denote $B(y)$ the block of the $a$-lattice containing the site $y$ in the new lattice. Gauge variables $\bar{U}$ are assigned on bonds of the $La$-lattice, corresponding to the block averages of the gauge fields $U$ in the $a$-lattice. This average must be defined in such a way that the gauge invariance on the $La$-lattice is preserved and the averaged field must be a group element. There are several proposals for them, each of which may be advantageous in different contexts, but all involve consideration of parallel transport of bonds between the centers $y, y'$ in the block lattice. By introducing a set of contour $\{\Gamma_{y,y'}\}$ from $y$ to $y'$ such average is implemented. The definition for the average $\bar{U}(y,y')$ is given by

$$\bar{U}(y,y') = \sum_{x\epsilon B(y),x'\epsilon B(y')} C(y,x;x',y')\Gamma(y,x:x',y')\Gamma(y,y'), \tag{4.34}$$

where $\Gamma$ is in general composed of three segments: the first is a path $\Gamma_{xy}$ from $y$ to a given $x \epsilon B(y)$, the second segment is a straight line $x, x'$ parallel to $y, y'$, and the final segment is from $x'$ to $y'$ (inverse of the path $\Gamma^{-1}_{x'y'}$ from $y'$ to $x'$). These contour in each block are chosen to move along a direction in a specified order. Each is called tree in $B(y)$. However, this choice of the tree and the definition of $C$ are not unique. For example, definitions of $C$ operator can be found in the papers of Wilson [?], Federbusch [65], Balaban [34], and Mack [66].

The following crucial step is the implementation of the "renormalization transformation". This is an integral transformation on the function $\rho$ by fixing block averages of the gauge fields $U$; i.e.

$$\rho_{effect}(V) = T\rho(V) = \int \delta(\bar{U}V^{-1})\rho(U)dU, \tag{4.35}$$

where $\delta$ denotes the product of Dirac measures on the symmetry group $G$ and $\rho$ is defined by

$$Z = \int \rho(U)dU. \tag{4.36}$$



Requiring the invariance on the $La$-lattice, is introduced a local gauge fixing. A complete gauge fixing in the lattice is not used, or else the gauge invariance of the new Hamiltonian in the block lattice is spoiled, thus, the block gauge invariance in the original lattice is preserved. In order to eliminate the gauge degrees of freedom with the condition to preserve the block lattice invariance, one must introduce a partial gauge fixing in each block. However the choice of this gauge fixing is not unique. One can select a determined gauge fixing to investigate some specific properties. For example, to study the regularity properties of the effective action ($\rho_{effect}$) is used the local Landau gauge [67], but to implement a concrete picture of the transformation of variables, it is advantageous to introduce a local axial gauge, which is defined by setting $U(b) = 1$ for each $b$ in one of the maximal trees $P_{T_{max}}$ on each block. A maximal tree is a tree wherein by adding any bond in this block gives a multiple connected sets of bonds. The result of this fixing gauge is that the gauge freedom within each block is limited as much as possible. In the case of a two dimensional lattice with $L = 3$, there are in each block, hence, maximal 9 gauge matrices to be fixed. Since, one of these gauge matrices must be identified with a block transformation, then only 8 remain to be fixed. A procedure as such, which is very simple, destroys in general the lattice symmetry. Therefore, one can choose an axial gauge path in such a way that the 8 links are distributed in a "most" symmetrical form along the coordinate axes and in rotating direction. There are only two axial gauges satisfying these conditions and they are shown in Figure 7a. and 7b.

To construct the maximal tree which defines the axial gauge: let $b$ a bond (the corresponding link variable is denoted by $U(b)$) with an extreme in the block center y, and the endpoint $x \epsilon B(y)$ in the block. Then a gauge transformation $h$ is chosen in the $a$ lattice, where the function $h$ is one, except at the end point of $b$, which is not attached to $y$, so that the gauge transformation with $h$ leaves to set $U(b) = 1$. In this way, $U(b)$ is fixed as 1 on any tree, and in particular on the maximal tree where the gauge is in $B(y)$ completely specified, except at site $y$ which defines the new lattice site. The remaining gauge freedom in the $a$-lattice, which consists of general gauge transformation on the block center, determines the gauge transformation in the $La$ lattice such that are completely unconstrained.

Finally, one can perform the integration in order to find the effective action. In the Wilson formulation this step is interpreted as a constrained approach, which is shown in the equation (4.35). Different methods were considered to perform this integration [66] However in the Hamiltonian formulation and the corresponding classical formulation the implementation of these ideas of renormalization is carried out in the context of change of variables whose definition is depending on the axial gauge fixing, and the integration over the variables with high frequencies up to a determined order corresponds in the classical formulation to an averaging.

## 4.4 Choice of variables

Taking in account the idea of RGT discussed in the above section, we will now introduce a new set of variables on the lattice which are parallel transport between near block sites and plaquette. These new variables let us define in the classical limit the fast and the slow modes of the system.

To define the new variables, we divide first the lattice into blocks with three lattice sites

long. This division in two dimension is shown in Fig.6. The points of the block lattice are shown as heavy dots, and some of the new variables, the "block links", can be defined to connect these block points. After the blocking on the $a$-lattice the link variables are divided into two group of links: the first group; the links are defined within the block; i.e. $u_l$ with $l = (n, \mu)$, where $n$ and $n + \mu$ are in the same block, and the links between the blocks define the other group of links, e.d. $u_l$, $l = (n, \vec{\mu})$ where $n \epsilon B(y)$ and $n + \vec{\mu} \epsilon B(y + \mu)$. Moreover, the links within each block can be separated in links on the maximal tree $u'_l$ from the rests that are not on the maximal axial tree $u_l$. In this case, the maximal axial tree gauge fixed 8 gauge matrix, therefore, one must fix 8 links to unit from the 12 links in the block.

Finally, let us denote in a compact form this group of variables as: $\{u'_l, u_l\}_{\Lambda'}$ where $\{u'_l\}_{\Lambda'}$ denotes the links variables on the maxima block tree, and $\{u_l\}_{\Lambda'}$ represent the other links in the block and between the nearest neighbor blocks.

Now, we defined on the starting lattice the following set of variables:
links on the maximal axial trees $u'_l$ (see Fig 7a.):

$$\begin{aligned} u'_{s,i} &= u_{n,i} & u'_{s+j,i} &= u_{n+j,i} \\ u'_{s,j} &= u_{n,j} & u'_{s-i,j} &= u_{n-i,j} \\ u'_{s-i,i} &= u_{n-i,i} & u'_{s-i-j,i} &= u_{n-i-j,i} \\ u'_{s-j,j} &= u_{n-j,j} & u'_{s+i-j,i} &= u_{n+i-j,j}, \end{aligned} \tag{4.37}$$

parallel transport in direction $i$ (see Fig 8a.):

$$\begin{aligned} y^1_{s,i} &= u_{n,j}\, u_{n+j,i}\, u_{n+j+i,i}\, u^\dagger_{n+2i,j}\, u_{n+2i,i} \\ y^2_{s,i} &= u_{n,i}\, u_{n+i,i}\, u_{n+2i,i} \\ y^3_{s,i} &= u_{n,i}\, u^\dagger_{n+i-j,j}\, u_{n-j+i,i}\, u_{n+2i-j,i}\, u_{n+3i-j,j}, \end{aligned} \tag{4.38}$$

the parallel transport in direction $j$ (see Fig 8b.):

$$\begin{aligned} y^1_{s,j} &= u^\dagger_{n-i,i}\, u_{n-i,j}\, u_{n-i+j,j}\, u_{n+2j-i,i}\, u_{n+3j,j} \\ y^2_{s,j} &= u_{n,j}\, u_{n+j,j}\, u_{n+2j,j} \\ y^3_{s,j} &= u_{n,j}\, u_{n+j,i}\, u_{n+j+i,j}\, u_{n+2j+i,j}\, u^\dagger_{n+3j,i}, \end{aligned} \tag{4.39}$$

and the loops variables (see Fig 8c.),

$$\begin{aligned} y^1_s &= u_{n,i}\, u_{n+i,j}\, u^\dagger_{n+j,i}\, u^\dagger_{n,j} \\ y^2_s &= u^\dagger_{n-i,i}\, u_{n-i,j}\, u_{n+j-i,i}\, u^\dagger_{n,j} \\ y^3_s &= u_{n,i}\, u^\dagger_{n-j+i,j}\, u^\dagger_{n-j,i}\, u_{n-j,j} \\ y^4_s &= u^\dagger_{n-j,j}\, u_{n-j,i}\, u_{n+i-j,j}\, u^\dagger_{n,i}, \end{aligned} \tag{4.40}$$

which are the path-ordered product of the links variables fixed on the paths shown in Fig. 8a-8c. Therefore, the new variables are divided into three groups: the first one containing the links variables between the nearest neighbor sites in the $3a$-lattice and which are represented as $y_L$ and $L = (y, y + \vec{\mu})$. In the second group loops variables $y_s$ within each block are contained



and the remaining links variables $u'_l$ are defined on the maximal tree gauge. Finally, $y_s$ is noted to represent the new links and loops variables. These set of variables $\{u'_l, y_s\}$ define our change of variables.

In order to obtain an analytical form of the Hamilton operator in terms of the new variables set, the form for the operators $E_L$ and $E_R$ after the change of variables must be found. In this case the transformation properties of the operator $E_L$ illustrated in the eq. ( 4.28) and the relation between these operators given in eq. (4.29) are used. Then the following expressions are found:

$$\begin{aligned} E^\beta_L(u_{n,i}) &= E^\beta_L(u'_{n,i}) + \{ E^\beta_L(y^2_{s,i}) + E^\beta_L(y^3_{s,i}) + E^\beta_L(y^{l1}_s) - E^\beta_R(y^{l3}_s) - E^\beta_R(y^3_{s-j,j})\} \\ E^\beta_L(u_{n+i+j,i}) &= D^1_{\beta,\alpha}(L_1) E^\alpha_L(y^1_{s,i}) \\ E^\beta_L(u_{n+i,i}) &= D^1_{\beta,\alpha}(L_2) E^\alpha_L(y^2_{s,i}) \\ E^\beta_L(u_{n-j+i,i}) &= D^1_{\beta,\alpha}(L_3) E^\alpha_L(y^3_{s,i}) \\ E^\beta_L(u_{n+i,j}) &= D^1_{\beta,\alpha}(L_4) E^\alpha_L(y^{l1}_s) \\ E^\beta_L(u_{n+2i-j,j}) &= D^1_{\beta,\alpha}(L_5) E^\alpha_L(y^{l4}_{s+i}) \\ E^\beta_L(u_{n+2i,j}) &= E^\beta_L(u'_{n+2i,j}) + D^1_{\beta,\alpha}(L_6)\{-E^\alpha_L(y^1_{s,i}) + E^\alpha_R(y^{l2}_{s+i}) + E^\alpha_R(y^1_{s+i,j})\} \\ E^\beta_L(u_{n+i-j,j}) &= E^\beta_L(u'_{n+i-j,j}) + D^1_{\beta,\alpha}(L_7)\{-E^\alpha_L(y^3_{s,i}) + E^\alpha_L(y^{l2}_{s-j,j}) + E^\alpha_L(y^3_{s-j,j})\} \\ E^\beta_L(u_{n+2i,i}) &= E^\beta_L(u'_{n+2i,i}) + D^1_{\beta,\alpha}(L_8)\{E^\alpha_R(y^1_{s,i}) + E^\alpha_R(y^2_{s,i}) - E^\alpha_L(y^{l2}_{s+i}) \\ &\quad -E^\alpha_L(y^{l4}_{s+i}) - E^\alpha_L(y^1_{s+i-j,j})\}, \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} E^\beta_L(u_{n,j}) &= E^\beta_L(u'_{n,j}) + \{E_L(y^2_{s,j}) + E^\beta_L(y^2_{s,j}) + E^\beta_L(y^3_{s,j}) + E^\beta_L(y^1_{s,i}) - E^\beta_R(y^{l1}_s) - E^\beta_R(y^{l2}_s) \\ E^\beta_L(u_{n-i+j,j}) &= D^1_{\beta,\alpha}(J_1) E^\alpha_L(y^1_{s,j}) \\ E^\beta_L(u_{n+j,j}) &= D^1_{\beta,\alpha}(J_2) E^\alpha_L(y^2_{s,j}) \\ E^\beta_L(u_{n+j+i,i}) &= D^1_{\beta,\alpha}(J_3) E^\alpha_L(y^3_{s,j}) \\ E^\beta_L(u_{n-i+j,i}) &= -D^1_{\beta,\alpha}(J_4) E^\alpha_L(y^{l2}_s) \\ E^\beta_L(u_{n+2i,i}) &= -D^1_{\beta,\alpha}(J_5) E^\alpha_L(y^{l3}_{s+j}) \\ E^\beta_L(u_{n+2j-i,i}) &= E^\beta_L(u'_{n+2j-i,i}) + D^1_{\beta,\alpha}(J_6)\{E^\alpha_L(y^1_{s,j}) - E^\alpha_L(y^{l2}_{s-i,j}) + E^\alpha_L(y^3_{s+j-i,i})\} \\ E^\beta_L(u_{n+j,i}) &= E^\beta_L(u'_{n+j,i}) + D^1_{\beta,\alpha}(J_7)\{E^\alpha_L(y^3_{s,j}) + + E^\alpha_L(y^{l2}_{s-i-j,i}) + E^\alpha_L(y^1_{s,i})\} \\ E^\beta_L(u_{n+2j,j}) &= E^\beta_L(u'_{n+2j,j}) + D^1_{\beta,\alpha}(J_8)\{E^\alpha_R(y^1_{s,j}) + E^\alpha_R(y^2_{s,j}) + E^\alpha_R(y^3_{s+j-i,i}) - \\ &\quad E^\alpha_L(y^{l4}_{s+j}) - -E^\alpha_L(y^{l3}_{s+j})\}, \end{aligned} \tag{4.42}$$

where the links are defined in the region $A$ and $B$ as illustrated in Fig. 9. and the expression $D^1_{\beta,\alpha}(L_i/J_i)$ is the orthogonal matrix representation of the SU(2)-parallel transport. One can shown that these terms can be re-expressed in term of the links $u'_l$.

It is observed too that as the new variable set is introduced, the magnetic term depends only on the plaquette variables and on the new links in the block lattice while the electric term depends on the all variables. In the discussion of gauge invariance of the wave function, the axial gauge fixing is introduce to eliminate the gauge degrees of freedom within each block. This is equivalent to set the $u'_l$ links variables to unit matrix whereas the other variables are not

modified. Since the wave function are only dependent on the $y_l$ variables, a reduced Hamiltonian is defined as one which depends only on the loops and links variables between the neighboring block sites. In particular, considering the feature that the matrices $D^1(L)$ are orthogonal in the Cartesian basis, the electric term takes the deneral form:

$$\hat{H}_E = \frac{g^2}{2}\sum_{s,s'}\sum_{i,j}\sum_{\beta,\alpha}\{E_L^\beta(s,i)T_1^{\beta,\alpha}{}_{i,j}(s,s')E_L^\alpha(s',j) + E_L^\beta(s,i)T_2^{\beta,\alpha}{}_{i,j}(s,s')E_R^\alpha(s',j) + E_R^\beta(s,i)T_3^{\beta,\alpha}{}_{i,j}(s,s')E_L^\alpha(s',j) + E_R^\beta(s,i)T_4^{\beta,\alpha}{}_{i,j}(s,s')E_R^\alpha(s',j)\}. \quad (4.43)$$

The advantage of the new set of variables is that under gauge transformation in the lattice, except on the block center, they are not modified. Otherwise can be observed that the link variables defined not on the maximal tree in the lattice after the gauge fixing, are the same as the new set of variables.

Finally, the discrete version of the Gauss" law for zero external charge static is

$$\Omega_s^i\Psi[\{U_l\}] = 0, \quad (4.44)$$

for all $s$ site in the $a$-lattice and i denote the color index. The generator of gauge transformation takes the for:

$$\Omega_s^i = \sum_{\hat{\mu}}\{E_L^i(u_{s,\hat{\mu}}) - E_R^i(u_{s-\hat{\mu},\hat{\mu}})\}. \quad (4.45)$$

Introducing the new set of variables discussed in the above part, one can show that the following law on the $3a$-lattice yields,

$$\Omega_n^a = \sum_{\hat{\mu}}\{\sum_{\alpha=1,2,3}(E_L^i(y_{s,\hat{\mu}}^\alpha) - E_r^i(y_{s-\hat{\mu},\hat{\mu}}^\alpha)) + \sum_{\alpha=1,2,3,4}(E_L^i(y_s^\alpha) - E_r^i(y_s^\alpha))\}, \quad (4.46)$$

where this expression correspond to the generator of gauge transformation on the $3a$-lattice at the site $n$. The first term defines the transformation law of the new parallel transport on the $3a$-lattice, and the last term the transformation law for the loops variables, i.e. under a gauge transformation on the block lattice one find

$$\begin{aligned} y^\alpha(\vec{n},j) &\rightarrow g(\vec{n})\, y^\alpha(\vec{n},j)\, g^{-1}\left(\vec{n}+\vec{j}\right), \\ y^\alpha(\vec{n}) &\rightarrow g(\vec{n})\, y^\alpha(\vec{n})\, g^{-1}(\vec{n}). \end{aligned} \quad (4.47)$$

## 4.5 RGT for a Classical Gauge System

In this section we will discuss our program for the formulation of a RGT for a classical gauge system.

Let us consider in this discussion a classical gauge Hamiltonian system derived from the lattice Hamilton operator with periodic boundary conditions and $n^d$ lattice points. This system is invariant under local gauge transformation and it has constraints defined in each point in the lattice. In the context of constrained Systems, one can eliminate the gauge degrees of freedom through a canonical transformation or equivalently introducing a set of "gauge"



constraints. As a result of this method, one can define an effective Hamiltonian in terms of physical degrees of freedom. On the other hand, one can interpret these constraints, which are of first-class, as a condition for the general solutions of the equation of motion. In this case, one must first investigate the question of integrability of this system. In general, this study is not straightforward for a system with many degrees of freedom, for example $3d(n-1)^d$ in the case of a SU(2) gauge Hamiltonian. However, we can obtain a considerable amount of information by using the methods of the classical perturbation theory since the Hamiltonian can be separated into two parts, an integrable term, and a perturbation term characterized by a coupling parameter. As it was discussed before, if the coupling constant is small, the KAM-Theorem states that many of the tori (solutions of the integrable term of the Hamiltonian) whose frequencies are noncommensurable, are preserved or slightly deformed. In contrast those solution whose frequencies are commensurable are strongly modified; hence, they should be investigated in detail. In the case of a classical gauge system, one finds that the symmetry of the integrable term is the origin of a set of resonances between the different degrees of freedom. In particular, if we consider the SU(2)- gauge symmetry, it can be shown that the integrable term is reduced to a set of three dimensional harmonic oscillators.

Now, the main goal of the second part of this thesis is the extension of the idea of a classical RGT developed for scalar classical theory to classical gauge Hamiltonian. In the formulation of a RGT for gauge system we are confronted with the problem of the " gauge" degrees of freedom and the resonances. These problems complicate the formulation in comparison to the scalar case. Thus, this formulation can be divided into two important parts: in the first part, we performed a set of canonical transformations which enabled us to eliminate some gauge degrees of freedom and define the slow and fast variables, and in the second part we averaged over the fast degrees of freedom in order to obtain an effective Hamiltonian, which describes the dynamic of the slow variables in term of adiabatic invariant corresponding to the fast variables. As a general result, one can observe that the different steps in the classical formulation are very analogous to the quantum formulation.

Let us discuss the different steps of this approach. We start this investigation by considering the classical SU(2) gauge Hamiltonian on a two dimensional lattice with $6(n-1)^2$ degrees of freedom and $3(n-1)^2$ constraints. We denoted the variables as $(u_s^i, p_s^i), i = 1,2,3$ where $s$ corresponds to the $2n^2$ lattice points, and $\Omega_s^i = 0$ represent the Gauss' Law on each lattice point.

The choice of a new set of variable is the first key step. We will now state the strategy to choose this new variables. We start dividing the original lattice in blocks with size $L = 3$, and the block center defines our new lattice called block lattice. In this case, each block has 9 lattice points. Since we will derive a new Hamiltonian defined on a block-lattice which is still invariant under gauge transformation in each block point, we can introduce gauge fixing constraints on the remaining 8 lattice point within each block separately, which defined a gauge path in the block. In order to define this partial gauge fixing let us introduce a canonical transformation which is defined considering the variables in each block and between the near neighboring blocks. Our change of variables is defined as: the 8 variables defined on the path gauge are not transformed, and the remaining variables are transformed in new "link variables" which are defined between two neighboring point, and "loop variables" defined in each block. One can show that these new variables have a simple behavior under a block gauge transformation. in particular the "link variables" transform like an ordinary link variable connecting the blocks. As

a result of this transformation, one can identify the gauge degrees of freedom (variables defined on the path gauge) and introduce a gauge fixing in order to eliminate these gauge degrees of freedom. Therefore one find a reduced Hamiltonian which depends on the $3*10*2(n/3-1)^2$ remaining degrees of freedom. One can observe that the choice of which of the original variables be left untransformed depends on the choice of the gauge fixing. In terms of the new variables, the free part of the reduced Hamiltonian is a $10 \times 10$ quadratic form which can be diagonalized introducing normal-modes in the Fourier space. As a result of this first part, one finds that the integrable term is separated in three dimensional harmonic oscillators and there is a ordering in the eigenvalues (frequencies). Moreover, we found a "mass gap", which allows us to separate, in the Fourier space for fixed momenta, the modes into two group of variables: a set of 2 slow modes and a group of 8 fast modes. In particular one finds that the following condition is satisfied:

$$w_{fast} >> w_{slow}. \tag{4.48}$$

Under this condition the motion of the slow modes is much more slower that the fast modes. In other words, this condition define a time scale which allows us to consider that the evolution of the slow variables are constant in comparison to the evolution of the set of fast variables. Therefore, one can investigate the effect of the fast modes on the slow modes in the presence of a perturbation term which couple the different fast and slow modes. This investigation can be performed in the context of adiabatic perturbation theory by taking the averaging over the fast modes. In particular, in chapter 5, we investigated the integrable term of the Hamiltonian, and discussed our change of variables and the choice of the gauge fixing in order to find the normal modes. The result is that the slow and fast modes are separated by a "mass gap".

The presence of resonance between the fast variables implies that the normal canonical perturbation theory must be modified, because it destroys the convergence of the perturbed serie. The resonance can be handled in the context of the degenerate perturbation theory. In this case, the resonant variables can be eliminated from the unperturbed Hamiltonian by a canonical transformation to a frame of reference that rotates with the resonance frequencies. In other words, we introduce a new set of variables defined around the resonance. The new coordinate then measure the slow oscillation of the variables about their values at resonance. In the case of SU(2)-classical system, one finds that the fast normal modes satisfy the resonance condition

$$w^{\alpha\, i}(k) = w^{\alpha}(k) \qquad i = 1,2,3 \qquad \alpha = 1,....8. \tag{4.49}$$

Let us now describe the differents step in order to average over the all fast modes. We will perform this investigation in the context of adiabatic perturbation theory, because there is a clear time scale, where the slow modes can be consider that their evolution are slow. Moreover, in this case we must not express the slow variables in Action-Angle variables. As discussed previously, the free part of the Hamiltonian is separable into three dimensional slow and fast harmonic oscillators. Introducing Action-Angle variables for the fast variables the unperturbed Hamiltonian depends only of the action variables $I^{\alpha\, i}$, and the perturbed term is a periodic function of the angles variables $\psi^{\alpha\, i}$. The resonance condition ( 4.49) can be removed by applying a transformation that eliminate two of the original action variables for each fast modes from the unperturbed Hamiltonian. For the angle variables, the transformation define two new angles variables which measure the slow deviation from the resonance and the third is unchanged. In this definition of new variables, there is an arbitrary choice of which of the original fast phase variables to leave unchanged. We think that this arbitrarity is irrelevant



for this investigation. Therefore, we leave $\psi^{\alpha 3}$ unchanged. In terms of these variables, the Hamiltonian depends only of the $I'^{\alpha 3}$ variables, but the interaction term is still dependent on the all fast action-angles. Now, we can average the Hamiltonian over the $\psi'^{\alpha 3}$. This averaging is valid near the resonance.

Since the averaged Hamiltonian is independent of $\psi^{\alpha 3}$, we have the result that $I'^{\alpha 3}$ are constants of motion. This term can be understand as the first term of the serie for the adiabatic invariant of the Hamiltonian. In this case, one can shown easily that

$$I'^{\alpha 3} = I^{\alpha 1} + I^{\alpha 2} + I^{\alpha 3} \tag{4.50}$$

The effect of this canonical transformation is to exhibit explicitly the modified invariant of the system near the resonance.

As a result of this first averaging over $I'^{\alpha 3}$, one find that the averaged Hamiltonian is still dependent on the remaining fast angle variables. Since this reduced Hamiltonia has many degrees of freedom $(2*8*2(n/3-1)^2)$, one can not directly state whether this reduced system is integrable or not. However, to obtain some general information on the structure of the reduced phase space, one can investigate the stationary points of the averaged Hamiltonian. In the neighborhood of the fixed point one notices that there are two distinct types of behavior which are dependent of the feature of the fixed point. By transforming to coordinates system fixed on an elliptic singular point, we are able to investigate systematically the region of regular solution in the phase space. On the other hand, the behavior of the system in the neighborhood of hyperbolic point is unstable. In this case can be shown that the homoclinic point let us understand the generation of stochasticity near a separatrix.

Thus, in order to eliminate the remaining fast degrees of freedom, we must study our reduced Hamiltonian in the neighborhood of an elliptic point, where the Action-Angle variables are well defined. As last step, we can average over these angles, which is valid only in the vicinity of the elliptic point, where the solution are periodic and regular. In this approximation, we examine the general character of the solution by linearizing about this stationary point. The quadratic term can be diagonalized and the perturbed terms derived from this expansion can be handled in the context of Birkhoff-normal-form, which is equivalent to perturbation theory. Here, it is possible that a new set of resonance can be satiesfied between harmonic of the phase oscillator derived near the elliptic point. They can be removed in a manner analogous to the one used in the last discussion.

The expected result of the last step is an effective Hamiltonian which describes the dynamic of the slow modes. Therefore, one could investigate the behavior of the effective vertices under a sucessive application of this RG-transformation, and conclude about the evolution of the coupling constant with the number of degrees of freedom.

Let us discuss the new Gauss' Law on the new lattice. In general, this constraint is a Taylor expansion in the coupling constant. Therefore, to one loop calculation one can find a zero and a first order term. After these series of canonical transformation, the Gauss' Law is modified to first order by a contribution derived from the canonical transformations. It is given by the Poisson bracket between the zero order term in the Gauss' Law and the first order term of the generating function. The final analytic expression depends on the structure of this generating function and thetransformation matrix for the fast modes, which can be found after one step of the RG-transformation. However, one can expecte that the new Gauss' Law is reduced to

the original constraint, because in the construction of this transformation we have preserved the invariance on the block lattice.

In particular, the effect of the perturbation term on the integrable Hamiltonian and the first averaging over the fast modes ($\psi^{\alpha i}$) in order to find a reduced Hamiltonian is presented in chapter 6. As a result from this investigation, we found the general structure of the effective vertices of the averaged Hamiltonian. At first order (tree level), we find that the effective Hamiltonian reduces to the original Hamiltonian. In particular, one finds that the effective Hamiltonian defined by averaging over $\psi^{\alpha i}$ is independent on the other fast degrees of freedom. To higher order, we calculate the two, three point function and the vacuum energy contribution (zero slow external lines) using the adiabatic perturbation theory up to third order which is equivalent, in the language of Quantum Field Theory, to one loop calculation. Moreover, an expression for the derivation of the fixed point are obtained. In general, we showed that all these results are dependent on the transformation matrix used to define the normal modes which is in this case a $10 \times 10$ matrix. The analytic structure of the total matrix was not possible to calculate, and only in the approximation of small momenta $|\vec{k}| << 1$ an analytic form for this matrix can be stated, in the sector of slow modes. But, a numerical calculation of this matrix was possible to perform.

Due to this complication, the calculation for an analytic expression of the fixed point of the reduced Hamiltonian, the last averaging over the action-angles defined at the elliptic point and the discussion of the new Gauss' Law was not possible to perform. We think that through a numerical investigation, one is able to obtain the information needed in order to study the behavior of the vertices and to state the behavior of the different parameter. Therefore, we included a discussion about a numerical implementation to calculate the behavior of the coupling constant in chapter 7.



# Chapter 5

# The Free Hamiltonian

In the first section of this chapter we will define our classical gauge system which is the starting point for the formulation of the RG-transformation. Following the steps discussed previously, we introduce a canonical transformation in order to distinguish slow and fast degrees of freedom. Moreover, the integrable part of our Hamiltonian is discussed.

## 5.1 Weak coupling expansion

To define a classical Hamiltonian corresponding to the lattice gauge Hamilton operator with the property that it is near integrable, the weak expansion approximation is discussed.
The Hamilton Operator is given by:

$$\hat{H} = \frac{g^2}{2a} \sum_{l,links} E_l^a E_l^a + \frac{2}{ag^2} \sum_{p,plaq} Tr(1 - U_p). \tag{5.1}$$

Aiming for an expansion of small g, it is observed that the $H_{mag}$ reaches its minimum only when all $U_l$ are equal to unit group element $\mathbf{1}$. An expansion around this minimum requires a chart from the group manifold which contains $e$, therefore the fundamental representation defined in eq.(4.4) is selected. Using a vector notation for the 3-generators of the Lie algebra in the left- and right representations yields,

$$\begin{aligned} \vec{E}_L(l) &= \frac{1}{2}\{y^0\vec{p} + \vec{L}\}, \\ \vec{E}_R(l) &= \frac{1}{2}\{y^0\vec{p} - \vec{L}\} \end{aligned} \tag{5.2}$$

respectively, with the short notation $p = \frac{1}{i}grad$, and $\vec{L} = \vec{y} \times \vec{p}$. An asymptotic g-small expansion is now performed by Taylor expansion in the $\vec{u}_l$ by introducing scaled variables

$$u_i \to \frac{g}{2} u_i, \qquad p_i \to \frac{2}{g} p_i. \tag{5.3}$$

Thus the following expansion of the Hamilton Operator is obtained

$$\hat{H} = \hat{H}_0 + \frac{g}{2}\hat{H}_1 + \frac{g^2}{4}(\hat{H}_{2,el} + \hat{H}_{2,mg}) + \ldots\ldots . \tag{5.4}$$

This expansion can be understood as a Hamilton operator $\hat{H}_0$ which is soluble with small perturbation terms of order $g, g^2, \ldots$. The zero order term $H_0$ is given by

$$\hat{H}_0 = \frac{1}{2a}\{\sum_l \vec{p}_l{}^2 + \sum_{n,\mu,\nu}(\vec{u}_{n,\mu} - \vec{u}_{n+\nu,\mu} + \vec{u}_{n+\mu,\nu} - \vec{u}_{n,\nu})^2\}. \tag{5.5}$$

It is only a quadratic elliptic differential operator, which has a soluble Schrödinger equation. This system can be diagonalized after a gauge fixing [62] such that $H_0$ is brought to a canonical form of uncoupled 3-dimensional harmonic oscillators. Based on this, the correction to the spectrum and the wave function to high order in the coupling constant by standard perturbation theory was investigated. To investigate in this direction see ref. [62, 64]

The expressions for the first two high order terms take the form,
first order

$$\begin{aligned} H_1 \;&= -\textstyle\sum_l\{\; \mathbf{x}_2\cdot(\mathbf{x}_3\times\mathbf{x}_4) + \mathbf{x}_3\cdot(\mathbf{x}_4\times\mathbf{x}_1) + \\ &\qquad \mathbf{x}_4\cdot(\mathbf{x}_1\times\mathbf{x}_2) + \mathbf{x}_1\cdot(\mathbf{x}_2\times\mathbf{x}_3)\}, \end{aligned} \tag{5.6}$$

second order

$$\begin{aligned} H_2 \;=\; & \frac{1}{2}\sum_l\{(\mathbf{x}_l\cdot\mathbf{p}_l)(\mathbf{x}_l\cdot\mathbf{p}_l)\} + \sum_{s,i,j}\{\frac{1}{8}(\mathbf{x}_1^2 - \mathbf{x}_2^2 + \mathbf{x}_3^2 - \mathbf{x}_4^2)^2 - \\ & \frac{1}{4}\left[(\mathbf{x}_1+\mathbf{x}_2)^2(\mathbf{x}_3+\mathbf{x}_4)^2 - (\mathbf{x}_1-\mathbf{x}_3)^2(\mathbf{x}_3-\mathbf{x}_1)^2 + (\mathbf{x}_1+\mathbf{x}_4)^2(\mathbf{x}_3+\mathbf{x}_3)^2\right]\}, \end{aligned} \tag{5.7}$$

where:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_{s,i} \qquad \mathbf{x}_3 = -\mathbf{x}_{s+i+j,i} \\ \mathbf{x}_2 &= \mathbf{x}_{s+i,j} \qquad \mathbf{x}_4 = -\mathbf{x}_{s+j,i}. \end{aligned} \tag{5.8}$$

Now we will define a classical system which is derived from the Kogut-Susskind Hamilton Operator.

In the quantum formalismus the evolution is defined either by operator equations in the Heisenberg picture:

$$\begin{aligned} \delta_t\hat{x} &= \left[\hat{H},\hat{x}\right] \\ \delta_t\hat{p} &= \left[\hat{H},\hat{p}\right], \end{aligned} \tag{5.9}$$

which are basically Hamilton's equations of motion in the case of a classical system, or by a Schrödinger equation for the wave function,

$$i\,\hbar\,\delta_t\Psi = \hat{H}\Psi. \tag{5.10}$$

The transition from the quantum formulation to the corresponding classical formulation is described by the correspondence principle, which states that in the limit $\hbar \to 0$ or large quantum number in the quantum formulation are obtained as limit a classical formulation [69]. In the Heisenberg picture one can assume that the classical limit is defined considering the following correspondence: the operator equations are equations of complex numbers. In this simple picture the classical limit is reached by this prescription, which is equivalent to the opposite



postulate in the canonical quantization. It is clear that the problem in this approach is to estimate the accuracy of this classical limit. But this is relevant in the case for calculation of correction to classical results. Since our interest is the investigation of chaotic behavior of the classical system, and because that is only a classical property, this problem is in this case not relevant. Chaos can be understood as an exponential sensitivity to initial conditions, and on account of the Uncertainty principle this effects is not present in the quantum mechanic formulation. However the description of quantum system whose corresponding classical system is chaotic, is still not a well understood problem and is an actual theme of investigation. In the Schrödinger picture, one must consider the wave function to study the semiclassical limit.

As was discussed, the rigorous limit to a classical formulation is given by the semiclassical limit, i.d. when $\hbar \to 0$. In this context the Hamilton Operator is considered as a conventional quantum mechanic system, where the square of the coupling plays the role of the Planck's constant. Moreover in the continuum limit, the bare lattice coupling constant becomes small, therefore the semiclassical approximation is applied. Imposing periodic boundary condition on a cubic lattice Bartels and Wu [70] found the following classical Hamiltonian for the special case of constant fields:

$$H_{class} = \frac{1}{2}\{\sum_s g^{ab}p_s^a p_s^b + \sum_s\sum_{i,j} V_{s(i,j)}\}, \tag{5.11}$$

where

$$\begin{aligned} V_{s(i,j)} \;=\; & \frac{1}{4}\{1 - x_1^0x_2^0x_3^0x_4^0 \\ & + (\,x_1^0x_2^0\,\mathbf{x}_3\cdot\mathbf{x}_4 + x_1^0x_2^0\,\mathbf{x}_3\cdot\mathbf{x}_4 + x_1^0x_3^0\,\mathbf{x}_2\cdot\mathbf{x}_4 + x_1^0x_4^0\,\mathbf{x}_2\cdot\mathbf{x}_3 \\ & + x_2^0x_3^0\,\mathbf{x}_1\cdot\mathbf{x}_4 + x_2^0x_4^0\,\mathbf{x}_1\cdot\mathbf{x}_3 + x_3^0x_4^0\,\mathbf{x}_1\cdot\mathbf{x}_2) - (x_1^0\,\mathbf{x}_2\cdot(\mathbf{x}_3\times\mathbf{x}_4) \\ & + x_2^0\,\mathbf{x}_3\cdot(\mathbf{x}_4\times\mathbf{x}_1) + x_3^0\,\mathbf{x}_4\cdot(\mathbf{x}_1\times\mathbf{x}_2) + x_4^0\,\mathbf{x}_1\cdot(\mathbf{x}_2\times\mathbf{x}_3)) \\ & - (\mathbf{x}_1\cdot\mathbf{x}_2)(\mathbf{x}_3\cdot\mathbf{x}_4) + (\mathbf{x}_1\times\mathbf{x}_2)\cdot(\mathbf{x}_3\times\mathbf{x}_4)\}, \end{aligned} \tag{5.12}$$

and

$$g^{ab} = \frac{1}{4}(\delta^{ab} - x^a x^b), \tag{5.13}$$

with the abbreviations:

$$\begin{aligned} x_1 &= x_{s,i} \qquad x_3 = -x_{s+i+j,i} \\ x_2 &= x_{s+i,j} \qquad x_4 = -x_{s+j,i}\;, \end{aligned} \tag{5.14}$$

for the ansatz $\Psi(x) = Ae^{\frac{-gW(x)}{g^2}}$ for the ground state. The $x_{s,i}$ are the group variables in the fundamental representation. In this limit, it is found out that the Hamiltonian is independent of the coupling constant and the quadratic term derived from the potential is zero.

In order to define the classical Hamiltonian we assume the classical limit in the picture of Heisenberg, i.e., we define the classical system in the region of the small coupling constant through the following correspondence,

$$\hat{q} \longleftrightarrow q \qquad \hat{p} \longleftrightarrow p; \tag{5.15}$$

and the classical Hamiltonian is defined by the equivalence:

$$\hat{H}(\hat{p},\hat{q}) \longleftrightarrow H(p,q). \tag{5.16}$$

The classical system is defined so that for the canonical quantization reproduces the Hamilton operator in the expansion of small coupling constant.

One explicit expression of the classical Hamiltonian is derived from the corresponding quantum form. However, one can consider that the scaling condition defined by (5.3) is equivalent to performing this expansion in the region of small fields $|x_l| << 1$ [70]. In this case, one can, as a first approximation, neglect the nontrivial piece of the metric tensor.This means that one can neglect the term $(x^a p^a)(x^b p^b)$ of the second order perturbation term. Then our Hamiltonian assumes the following structure:

$$H = H_0 + \frac{g}{2}H_1 + \frac{g^2}{4}H_2 + ..., \tag{5.17}$$

where the zero order is given as

$$H_0 = \frac{1}{2a}\{\sum_l \vec{p}_l{}^2 + \sum_{n,\mu,\nu}(\vec{u}_{n,\mu} - \vec{u}_{n+\nu,\mu} + \vec{u}_{n+\mu,\nu} - \vec{u}_{n,\nu})^2\}. \tag{5.18}$$

Expressions to high order are given by the corresponding classical expressions of the perturbative term defined by equations (5.6, 5.7). Because we shall assume periodic boundary condition for the $a$-lattice, the variables satisfy

$$\vec{u}_{n,\mu} = \vec{u}_{n+L_\mu,\mu}; \tag{5.19}$$

where $L$ is the large of the Lattice. This periodicity makes it convenient to introduce Fourier component defined by the equations:

$$u_{s,\mu} = \frac{1}{L^2}\sum_{\vec{k}} e^{ikx}\bar{u}_{k,\mu}, \tag{5.20}$$

where the vector $k_i$ may only take the values in the first Brillouin zone defined by

$$\Lambda_k^a = \{\vec{k}\,| -\frac{\pi}{a} < k_i < \frac{\pi}{a}\}. \tag{5.21}$$

Introducing the Fourier transformation in the Hamiltonian, one finds:

$$H_0 = \frac{1}{2aL^2}\{\sum_k \vec{p}_{-k,\mu}\vec{p}_{k,\mu} + \sum_{k,k',\mu,\nu}\vec{u}_{k',\mu}V(k,k')^{\mu,\nu}\vec{u}_{k,\nu}\}, \tag{5.22}$$

where the matrix term in the last expression takes the form

$$V(k) = (\tilde{k}^2\delta_{\mu,\nu} - k_\mu k_\nu)e^{i(k_\mu - k_\nu)}\delta(k+k'). \tag{5.23}$$

The higher order expressions are given by:
first order:

$$H_1 = \frac{1}{L^4}\sum_{\{k\}}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\Gamma^{abc}_{3\,ijk}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)u^a_{k_1\,i}u^b_{k_2\,j}u^c_{k_3\,k}, \tag{5.24}$$

second order:

$$H_2 = \frac{1}{L^6}\sum_{\{\mathbf{k}\}}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\Gamma^{abcd}_{4\,ijkl}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)u^a_{k_1\,i}u^b_{k_2\,j}u^c_{k_3\,k}u^d_{k_4\,l}. \tag{5.25}$$



The non-linear terms can then be interpreted as vertices of three or four variables, and are defined as:

$$\begin{aligned}\Gamma^{abc}_{3\,ijk}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \epsilon^{abc}\{(\widetilde{k_3-k_2})_\mu\cos(\frac{k_\nu a}{2})\delta_{\nu,\lambda}\\ &+ (\widetilde{k_1-k_3})_\nu\cos(\frac{k_\mu a}{2})\delta_{\mu,\lambda}\\ &+ (\widetilde{k_2-k_1})_\lambda\cos(\frac{k_\mu a}{2})\delta_{\mu,\nu}\}e^{-i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)/2},\end{aligned} \tag{5.26}$$

where $(\widetilde{k_3-k_2})_\mu$ stand for

$$(\widetilde{k-q})_\mu = \frac{2}{a}\sin(\frac{(k-q)_\mu a}{2}), \tag{5.27}$$

and the expression to second order $\Gamma^{abcd}_{4\,ijkl}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$, which is very long, is equivalent (see Appendix C, equation C.10) to the expression in the spatial lattice given in the paper by Kawai et al. [71].

*These expression are derived by using the cyclic symmetry of the structure constant* $\epsilon^{abc}$, and the symmetry under the permutation of the pair indices $(a,\mu)$. Notice that because of the appearance of the phase factor in the expression (5.23), the quantity is a periodic function in each component of the momenta with periodicity $2\pi/a$. One can easily verify that $\Gamma_3$ and $\Gamma_4$ become a phase, which lets that the vertices are periodic functions in all components of the momenta with periodicity $2\pi/a$. The phase factor can now be absorbed in the Fourier transform of the gauge degrees of freedom, which amounts to redefining the variables at the midpoint of the link connecting two neighbouring lattice sites, however we will nevertheless prefer to carry them along in order to exhibit the periodic structure of the above mentioned *expressions. But when computing the contribution of the perturbation theory one finds that* the phases associated with the interaction vertices cancel at each order. In this case the phase factors complicate only the structure of each interaction vertices.

The Hamiltonian $H$ is invariant under $SU(2)$ rotation in the color indices. An arbitrary gauge transformation $g$ is defined in terms of group elements $\alpha_s$ attached to the lattice site $s$ such that the link variable transform according to

$$U(\vec{n},j) \to g(\vec{n})\,U(\vec{n},j)g^{-1}\left(\vec{n}+\vec{j}\right), \tag{5.28}$$

and the generator of the transformation is given by

$$\Omega(g) = \sum_{s,j}\alpha^i_s\{x^0_{s,j}p^i_{s,j} - x^0_{s-j}p^i_{s-j,j} + \epsilon^{ibc}(x^b_{s,j}p^c_{s,j} + x^b_{s-j,j}p^c_{s-j,j})\}. \tag{5.29}$$

Because we are working with the expansion of small coupling, we can restrict our attention to infinitesimal gauge transformation given by

$$g_s = 1 + i\epsilon\alpha^a_s\sigma_a, \tag{5.30}$$

*up to quadratic terms in the small parameter* $\epsilon$. *The differences* $\Delta x^b_{s,j}$ and $\Delta p^b_{s,j}$ up to $O(\epsilon)$ are between the gauge transformed and the original coordinates and momenta [38], that is,

$$\begin{aligned}\Delta x^a_{s,j} &= \epsilon\{x^0_{s,j}(\alpha^a_s - \alpha^a_{s+j}) + \epsilon^{abc}x^b_{s,j}(\alpha^c_s + \alpha^c_{s+j})\}\\ &= \{x^a_{s,j}, \epsilon\Gamma\},\end{aligned} \tag{5.31}$$

and

$$\begin{aligned}\Delta p^a_{s,j} &= \epsilon\{\epsilon^{abc}p^b_{s,j}(\alpha^c_s+\alpha^c_{s+j})\}\\ &= \{x^a_{s,j},\epsilon\Gamma\}.\end{aligned} \tag{5.32}$$

This Hamiltonian $H$ serves as the starting point of our investigation. It contains two terms: the first term is the integrable part, which is a superposition of the 3 dimensional harmonic oscillators, and the second term is a series of small perturbative terms, which modified the integrability of the first term. Considering the KAM-theorem can be stated, the switch of a small perturbation modified only the stable solutions which are incommensurable. In order that the perturbation theory describes the feature of the systems, one must state that the total Hamiltonian admit regular solutions and that there is a finite measure in the phase space of the stable solutions. In this investigation we assume the existence of this kind of solutions which will be well described by the perturbation theory. However, the study of the validity of perturbative methods and in particular the investigation of the transition to chaos lets us conclude the existence of a critical coupling parameter $\lambda_c$ [22], which determines the point of breaking of the the perturbation theory. Moreover, for near-integrable autonomous systems, energy is a constant of motion, but it can be stated that this constant regulates the integrability behavior. The study of integrability through, for example, Poincaré mapping for some models at different energy but constant $\lambda$ let to conclude the existence of an SST-energy (Strong-Stochasticity-Threshold), i.e, over this threshold of energy, it can be found that the system has only chaotic solutions. It is to expect that this SST-energy is model-dependent, and in general it is accepted that the system is chaotic to high energy. In this context we can understand the result of ref. [72], where the Lyapunov exponents for the SU(2)-lattice gauge theory was calculated at high energy, and they are all positive; that means that the theory is chaotic. But it is important to note that the solutions for the numerical calculation of the Lyapunov exponents was chosen from a thermal bath, which is an ergodic generator of configuration in the lattice simulation. Since we will consider that the perturbation theory is still relevant to describe the regular solution, we must accept the following consideration: the coupling constant is smaller than the critical values, and that we consider the region of low energy.

## 5.2 Change of variables

We will start the formulation of the renormalization for the classical gauge system discussing here the choice of new variables.

In view of the classical formulation, canonical transformations must be found in order to reproduce the steps from the quantum formulation of the block spin variable. The degrees of freedom in the Hamiltonian are three dimensional vectors in each direction $x'_{s,\mu}$ which are the vectorial components of the fundamental representation. This is interpreted that each link is described by a vector. Based on the new variables and the transformation law for the left and right operators in the Hamiltonian formulation discussed in the last chapter, we can introduce the following canonical transformation $(\{p_{s,\mu},u_{s,\mu}\}\rightarrow\{p'_{s,\mu},u'_{s,\mu},\pi^c_{s,\mu},y^c_{s,\mu}\})$, which is given through the following relations between the canonically conjugated variables:
the link variables on the block-path:

$$\vec{u'}^{\,1}_s = \vec{u}_{n,i} \qquad \vec{u'}^{\,5}_s = \vec{u}_{n+j,i}$$



$$\begin{aligned}\vec{u'}^{\,2}_s &= \vec{u}_{n,j} & \vec{u'}^{\,6}_s &= \vec{u}_{n-i,j}\\ \vec{u'}^{\,3}_s &= \vec{u}_{n-i,i} & \vec{u'}^{\,7}_s &= \vec{u}_{n-i-j,i}\\ \vec{u'}^{\,4}_s &= \vec{u}_{n-j,j} & \vec{u'}^{\,8}_s &= \vec{u}_{n+i-j,j},\end{aligned} \tag{5.33}$$

and the another new variables are defined by:

$$\begin{aligned}\vec{y}^{\,1}_{s,i} &= \vec{u}_{n,j}+\vec{u}_{n+j,i}+\vec{u}_{n+j+i,i}-\vec{u}_{n+2i,j}+\vec{u}_{n+2i,i}\\ \vec{y}^{\,2}_{s,i} &= \vec{u}_{n,i}+\vec{u}_{n+i,i}+\vec{u}_{n+2i,i}\\ \vec{y}^{\,3}_{s,i} &= \vec{u}_{n,i}-\vec{u}_{n+i-j,j}+\vec{u}_{n-j+i,i}+\vec{u}_{n+2i-j,i}+\vec{u}_{n+3i-j,j}\\ \vec{y}^{\,3}_{s,j} &= \vec{u}_{n,j}+\vec{u}_{n+j,i}+\vec{u}_{n+j+i,j}+\vec{u}_{n+2j+i,j}-\vec{u}_{n+3j,i}\\ \vec{y}^{\,2}_{s,j} &= \vec{u}_{n,j}+\vec{u}_{n+j,j}+\vec{u}_{n+2j,j}\\ \vec{y}^{\,1}_{s,j} &= -\vec{u}_{n-i,i}+\vec{u}_{n-i,j}+\vec{u}_{n-i+j,j}+\vec{u}_{n+2j-i,i}+\vec{u}_{n+3j,j}\\ \vec{y}^{\,1}_{s} &= \vec{u}_{n,i}+\vec{u}_{n+i,j}-\vec{u}_{n+j,i}-\vec{u}_{n,j}\\ \vec{y}^{\,2}_{s} &= -\vec{u}_{n-i,i}+\vec{u}_{n-i,j}+\vec{u}_{n+j-i,i}-\vec{u}_{n,j}\\ \vec{y}^{\,3}_{s} &= -\vec{u}_{n-j,j}-\vec{u}_{n-j-i,i}+\vec{u}_{n-j,i}+\vec{u}_{n-j-i,j}-\vec{u}_{n+i,j}\\ \vec{y}^{\,4}_{s} &= \vec{u}_{n-j,j}-\vec{u}_{n-j,i}-\vec{u}_{n+i-j,j}+\vec{u}_{n,i},\end{aligned} \tag{5.34}$$

Here $s$ denotes the site in block lattice or the equivalent block center on the old lattice, and the variables $\vec{y}_{s,j}{}^{\alpha}$ are the vector variables of the new link variables and $\vec{y}_s{}^{\alpha}$ are the vector variables of the new loop variables. The transformation for the momenta is:

$$\begin{aligned}p^\beta(u_{n,i}) &= p^\beta(u'_{s,i})+p^\beta(y^2_{s,i})+p^\beta(y^3_{s,i})+p^\beta(y^{l1}_{s,i})-p^\beta(y^{l2}_{s-j,j})-p^\beta(y^3_{s-j,j})\\ p^\beta(u_{n+i+j,j}) &= p^\beta(y^1_{s,i})\\ p^\beta(u_{n+i,i}) &= p^\beta(y^2_{s,i})\\ p^\beta(u_{n-j+i,i}) &= p^\beta(y^3_{s,i})\\ p^\beta(u_{n+i,j}) &= p^\beta(y^{l1}_{s,i})\\ p^\beta(u_{n+2i-j,j}) &= p^\beta(y^{l2}_{s,i})\\ p^\beta(u_{n+2i,j}) &= p^\beta(u'_{n+2i,j})-p^\beta(y^1_{s,i})+p^\beta(y^{l1}_{s+i,j})+p^\beta(y^1_{s+i,j})\\ p^\beta(u_{n+i-j,j}) &= p^\beta(u'_{n+i-j,j})-p^\beta(y^3_{s,i})+p^\beta(y^{l2}_{s-j,j})+p^\beta(y^3_{s-j,j})\\ p^\beta(u_{n+2i,i}) &= p^\beta(u'_{n+2i,i})+p^\beta(y^2_{s,i})+p^\beta(y^2_{s,i})-p^\beta(y^{l4}_{s+i})-p^\beta(y^{l2}_{s+i})-p^\beta(y^1_{s+i,j}),\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}p^\beta(u_{n,j}) &= p^\beta(u'_{n,j})+p^\beta(y^2_{s,j})+p^\beta(y^3_{s,j})-p^\beta(y^{l1}_{s})-p^\beta(y^{l2}_{s})+p^\beta(y^1_{s,i})\\ p^\beta(u_{n-i+j,j}) &= p^\beta(y^1_{s,j})\\ p^\beta(u_{n+j,j}) &= p^\beta(y^2_{s,j})\\ p^\beta(u_{n+j+i,i}) &= p^\beta(y^3_{s,j})\\ p^\beta(u_{n-i+j,i}) &= -p^\beta(y^{l1}_{s,j})\\ p^\beta(u_{n+2j,i}) &= -p^\beta(y^{l2}_{s,j})\\ p^\beta_L(u_{n+2j-i,i}) &= p^\beta(u'_{n+2j-i,i})+p^\beta(y^1_{s,j})-p^\beta(y^{l2}_{s-i})+p^\beta(y^3_{s+j-i,i})\\ p^\beta(u_{n+j,i}) &= p^\beta(u'_{n+j,i})+p^\beta(y^3_{s,j})+p^\beta(y^{l1}_{s})+p^\beta(y^1_{s,i})\\ p^\beta(u_{n+2j,j}) &= p^\beta(u'_{n+2j,j})+p^\beta(y^1_{s,j})+p^\beta(y^2_{s,j})+p^\beta(y^3_{s+j-i,i})-p^\beta(y^{l2}_{s+j})-p^\beta(y^{l2}_{s+j})\end{aligned} \tag{5.36}$$

One can show that the new variables satisfy:

$$\{y_s^\beta, p^\alpha(y_{s'})\} = \delta^{\beta\alpha}\delta^{ss'} \tag{5.37}$$

In this change of variables we have conserved the periodicity of the variables and, in particular, the new variables satisfy the following condition,

$$\vec{y}_{n,\mu} = \vec{y}_{n+L'_\mu,\mu}, \tag{5.38}$$

where $L'$ is the total number of lattice sites which is reduced by a a factor $1/3$. The periodicity let us introduce the Fourier component defined by the equations:

$$u_{s,\mu} = \frac{1}{L'^2}\sum_{\vec{k}'} e^{ik'x}\tilde{u}_{k',\mu}, \tag{5.39}$$

where the vector $k'_i$ may only take the values in the first Brillouin zone defined by

$$\Lambda'_k = \{\vec{k}\,|\, -\frac{\pi}{a'} < k_i < \frac{\pi}{a'}\}, \tag{5.40}$$

and $a'$ is the lattice spacing in the $3a$-lattice. The Fourier transform is defined in similar form for the new variables as in the original lattice, with $n$ replaced by $n'$.

Finally introducing the new variables the Hamiltonian takes the form

$$H = H_0(\pi_y, y, \pi_{u'}, u') + V(y). \tag{5.41}$$

This Hamiltonian is still invariant under SU(2) local gauge transformation and leads conserved quantities $\Omega^a_n, a = 1,2,3$ in every lattice point. Due to this Gauss' law is this system a constrained dynamical system, i.e., the dynamic of the system is restricted to a subspace of the phase space defined by these constraints. Consequently, only the solution of the equation of motions derived from the last Hamiltonian are of physical interest, which also satisfy these constrains. In order to define the physical phase space or reduced phase space one must eliminate the non-dymanical gauge degrees of freedom. However we will not eliminate all the gauge degrees of freedom of the theory by using the constraint formalismus of Dirac, but only the gauge variables in each block. The advantage of this procedure is the conservation of some constraint on the new variables which means that the gauge invariance on the new lattice is conserved.

In order to perform the renormalization approach we must eliminate the gauge variables before we begin with the first steps. But this elimination of these gauge degrees of freedom using gauge fixing is not a trivial process, which we will discuss in the next section.

## 5.3 The reduce phase space.

In order to perform the renormalization approach we must eliminate some gauge variables before we begin with the first steps of the RG transformation. In this section, we discuss the elimination of these gauge degrees of freedom using gauge fixing. Here, we briefly present a theorem which clarifies the dynamical situation in a Hamiltonian system with constraints, and allows for the identification of the reduced phase space.

**Theorem** [[73],[74]]: Let $\{\psi_\alpha : \alpha = 1...r\}$ be a set of independent constraints where:



$$rank||\{\psi_\alpha, \psi_\beta\}|| = s < r.$$

Then there are canonical variables $(w, (Q,P), \phi)$ such that:
- $w$ is a set of pairs of canonically conjugated coordinates and momenta,
- $P$ are momenta canonically conjugated to the coordinate Q, and $\phi$ is a set of pairs of canonical conjugate variables.
- $\Omega \approx (P, \phi)$ is equivalent to the constraints $\psi$. The number of momenta in $P$ equals $t = r - s$ and the pairs $\psi$ constitute $s$ coordinates and momenta.

The momenta $P$ represents the $t$-first class constraints of the theory and the $s$-second class constraints are transformed to canonical pairs.

One finds that the equation of motions are given by:

$$\dot{w} = \{w, \tilde{H}_E\}, \tag{5.42}$$
$$\dot{Q} = \{Q, \tilde{H}_E\}, \tag{5.43}$$
$$\Omega = 0, \tag{5.44}$$

where

$$\tilde{H}_E = H_r + \sum_t \lambda_t P_t + \sum_\alpha \lambda_\alpha \phi_\alpha, \tag{5.45}$$

and $H_r(w,Q) = \tilde{H}(w,Q,\Omega)|_{\Omega=0}$. By construction $H_r$ depends on $w$ and $Q$, but the conservation of the constraints $P$ in the time implies

$$\dot{P} = \{P, \tilde{H}_E\} \approx 0, \tag{5.46}$$

and thus $H_r = H_r(w)$. Hence, the dynamics for the variable $w$ are given by equation ( 5.42), and the reduced Hamiltonian is independent from the other coordinates and momenta. In terms of the new variables, the constraint hypersurface $\Gamma_c$ is determined by the equation $\Omega^i_s = 0$ and the pair $\omega$ can be viewed as canonical coordinates and momenta on $\Gamma_c$. Using the invariance of the equation of motion under the canonical transformation they read

$$\dot{w} = \{w, H_r\} \tag{5.47}$$
$$\Omega = 0. \tag{5.48}$$

Furthermore, recalling (5.42) as well as (5.45), it follows that

$$\dot{Q}_i(t) = \lambda_i(t). \tag{5.49}$$

As a consequence, the equation for the variable $Q_i$ solves for any function of $\lambda_i$. Thus, the function $\lambda_i$, the Lagrangian multipliers corresponding to the t-first class constraints of the theory, are not determined by the complete system of equation of motion and its solution is an arbitrary function. Due to this arbitrariness, the motion on the phase-space $\Gamma_c$ (constraint hypersurface) is not completely determined in theory with first class constraints. This degeneracy is then nonphysical, and their extent is given by the number of the first class constraints. To formulate the theory consistently, it is necessary to lift its nonphysical degeneracy which lead to different trajectories in $\Gamma_c$. One therefore identifies these trajectories which in turn leads to

a division of $\Gamma_c$ into equivalence classes. The reduced phase space $\Gamma_r$ is obtained by choosing exactly one representative out of each equivalence class. This may be achieved by choosing an element from each gauge class by implicitly introducing additional constraints $\chi$ on the phase space variables, a procedure usually called "gauge fixing". Formally, it can be established that a classical Hamiltonian system subject to a set of $t$-first class and $s$-second class constraints $(\psi^1, \psi^2)$ is consistently gauge fixed when there are $t$-supplementary conditions on the phase space variables $\chi_j = 0$ such that for $\phi = (\psi, \chi)$

$$det||\{\psi^1_\alpha, \chi_\beta\}|||_{\phi=0} \neq 0. \tag{5.50}$$

This gauge fixing $\chi = 0$ is to be chosen in such a way that the whole set of constraints $\phi = (\psi, \chi)$ are second class. Next, the undetermined parameter still inherent in the equation of motion for $Q_i$ are fixed through the consistency requirements for the supplementary conditions and the degeneracy of the theory is lifted. All physical information is contained in the canonical equation(5.47), since the dynamical equation for the variables $\omega$ as well as the constraint equation $\Omega = 0$ remain unaffected. Therefore, the reduced phase space $\Gamma_r$ is parametrized by the canonical par $\omega$, and we identify $H_r(\omega)$ as the reduced Hamiltonian.

In general, such a formulation does not provide a representation for the reduced phase space, but proves that a canonical transformation exists, i.e. one can again try to isolate the nonphysical variable and construct the reduced Hamiltonian $H_r$. The problem of finding this canonical transformation to the distinguished variable is highly non trivial.

We want to use this formulation to point out that it is possible to apply in each Block this idea and that one can define a reduced Hamiltonian, which is independent from the gauge link variables after we had introduced a gauge fixing.

Now, we present our classical Hamiltonian. In this case we consider first the integrable term with the corresponding Gauss' law in each lattice site s:

$$\begin{aligned} H &= H_0 && (5.51)\\ \Omega_s &= 0. && (5.52) \end{aligned}$$

The integrable terms, in view of the new set of variables, are only a quadratic form given by

$$H_0 = T(\pi_y, y) + T(\pi_{u'}, u') + T(\pi_{u'}, \pi_y,) + V(y), \tag{5.53}$$

and the Gauss' law is the constant of motion, which can be considered as first class constraint, i.e.

$$\dot{\Omega}^i_{n'} = \{\Omega^i_{n'}, H_0\} \approx 0 \qquad \{\Omega^i_{n'}, \Omega^i_{n'}\} \approx 0 \qquad i = 1, 2, 3; \tag{5.54}$$

where the $\Omega^i$ are defined on the Block-points $n'$ (in d=2 with a blocking size 3, there are 8 block points $n'$ and $n' \neq n_B$).

Our aim in the following presentation is to replace the $H_0$ by a reduced Hamiltonian depending only on the new dynamical degree of freedom $y_s$ through gauge fixing. However, we will not eliminate all the gauge degrees of freedom from our system, because we will conserve the gauge symmetry on the block-sites.



Now we want to prove that the variables $u'$ are gauge degrees of freedom. The equations of motion are:

$$\begin{aligned} \dot{\omega}_y &= \{\omega_y, H\}, && (5.55)\\ \dot{\omega}' &= \{\omega', H\}, && (5.56) \end{aligned}$$

whereby the last equation implies that $\pi_{u'} = 0$. The equation for the $y$ variables is reduced to:

$$\begin{aligned} \dot{\pi}_y &= \{\pi, H_r\} && (5.57)\\ \dot{y} &= \{y, H_r\} && (5.58)\\ \dot{u}' &= \{u', H\}, && (5.59) \end{aligned}$$

where $H_r = H|_{\pi_{u'}=0}$ .

Introducing the extended Hamiltonian

$$H_E = H_0 + \sum_{n'} \lambda^i_{n'} \Omega^1_{n'} + \sum_{n_B} \lambda^i_{n_B} \Omega^i_{n_{Block}}, \tag{5.60}$$

one can see that

$$\begin{aligned} \Omega^i_{n'} &= f^i(\pi', \pi_y)\\ \Omega^i_{n_{Block}} &= f^i(\pi_y). && (5.61) \end{aligned}$$

Now we see that the equation of motion for the $u'$ variables has a solution for any function $\lambda^i_{n'}$. These functions are the Lagrangian multipliers. Due to the complexity of this system, there is only a little hope of obtaining a canonical transformation to the distinguished variables described in the Theorem which allows for an immediate identification of the reduced phase space. Therefore, we have to lift the degeneracy of the theory by a consistent gauge fixing. In addition to the 27 initial first-class constraints, we have to find 24 independent supplementary conditions on the phase space in such a way that we end up with a set of 48 second-class constraints.

We can introduce a gauge fixing, which is necessary to lift these nonphysical variables. We consider that our Hamiltonian is gauge invariant under any local gauge transformations on each lattice site, thus we can carry out a gauge transformation on each $s'$-site on the block (but $s'$ is not the block-center) which eliminates the gauge degrees of freedom. The gauge fixing is defined on the gauge-path as:

$$\vec{u'_s} \rightarrow \vec{u'^g_s} = 0. \tag{5.62}$$

This can be achieved when we carry out a gauge transformation on each $s'$-site on the block, but not on the block-center. Finally, the new constraints are denoted as

$$\chi^i_s = u^{i\prime}{}_s = 0. \tag{5.63}$$

The Poisson brackets between the gauge fixing conditions ( 5.63) and the constraints form the $24 \times 24$ matrix

$$M = det||\{\Omega^j_s, \chi^i_s\}||. \tag{5.64}$$

We find that $\{\Omega^j_s, \chi^i_s\} = \delta^{ij}$, then

$$det||M|||_{\Omega=\chi=0} \neq 0. \tag{5.65}$$

Therefore, the reduced phase space $\Gamma_r$ is parametrized by the remaining canonical variables. By simply inserting the second-class constraints into the canonical Hamiltonian we finally get the reduced Hamiltonian $H_r$ on $\Gamma_r$.

Hence, the dynamics for the new variables are given

$$\dot{\omega} = \{w, H_r\} \rightarrow w = (\pi, y), \tag{5.66}$$

where $H_r = H|_{\pi^g_{y'}, u'^g = 0}$
One can see that this gauge is consistent with the Gauss' Law.

Finally, we can see that the integrable reduced Hamiltonian defines a dynamical system on a phase space reduced $\Gamma^g_r$ which is $\Gamma^g_r = \Gamma|_{\Omega^a=0, \Omega_{gauge\,fixing}=0}$.

Next, we consider the effect of the perturbation terms on the integrable term. One can consider that this reduced phase space is not modified when we introduce the perturbation terms which are characterized by a coupling constant, because they are independent from the gauge degrees of freedom. The action of this perturbation term on the integrable Hamiltonian is carried out on this reduced phase space. Our aim is to investigate the modification of the torus when we introduce this perturbation term. This theme is investigated in the next chapter.

## 5.4 Integrable term $H_0$.

Let us first discuss the solution of the integrable term $H_0$ before we explore the effect of the perturbation term by perturbation method.

The integrable term $H_0$ takes the final form before the new variables are introduced,

$$H_0 = \frac{1}{2} \sum_{s,s',\alpha,\beta,a} \{ p^{a,\alpha}_s T^{\alpha,\beta}_{s,s'} p^{a,\beta}_{s'} + u^{a,\alpha}_s V^{\alpha,\beta}_{s,s'} u^{a,\beta}_{s'} \}. \tag{5.67}$$

where we use $a = 1$. To simplify the calculation, we introduce the notation $p^{a,\alpha}_s, u^{'a,\beta}_s$ as components of the one vector $\hat{P}^a_s, \hat{Q}^a_s$ defined in the site $s$ and where $\alpha$ runs from 1 to 10: i.e.

$$\begin{aligned}
(\hat{X}^a_s)^\alpha &= x^{a,k}_{s,i} \qquad \alpha = 1,...,3 \qquad k = 1,...,3 \\
(\hat{X}^a_s)^\alpha &= x^{a,k}_{s,j} \qquad \alpha = 6,...,8 \qquad k = 1,...,3 \\
(\hat{X}^a_s)^\alpha &= x^{a,k}_s \qquad \alpha = 4,5,9,10 \qquad k = 1,3,2,4,
\end{aligned} \tag{5.68}$$

where $X$ denotes $u$ or $p$ respectively. Introducing the Fourier transform of the new coordinates and momenta, we obtain:
kinetics term:

$$H_{kin} = \frac{1}{2L'^2} \sum_{k,\alpha,\beta,a} p^{a,\alpha}_{-k} T^{\alpha,\beta}_k p^{a,\beta}_k = \frac{1}{2L'^2} \sum_{k,a} \hat{P}^a_s T_k \hat{P}'^a_s, \tag{5.69}$$

where $T_k$ is a $10 \times 10$ matrix, given by

$$T = \begin{vmatrix} T_{xx} + T_x(k) & T_{xy} + e^{ik_1} T_1 + e^{-ik_2} T_2 + e^{i(k_1-k_2)} T_2 \\ \text{c.c.} & T_{yy} + T_y(k) \end{vmatrix}, \tag{5.70}$$



where the matrices are defined as:

$$T_{xx} = \begin{vmatrix} 5 & 1 & 0 & -2 & 0 \\ 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ -2 & 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}, \quad T_{yy} = \begin{vmatrix} 5 & 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & 5 & -1 & 0 \\ 2 & -1 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}, \tag{5.71}$$

$$T_x(k) = \begin{vmatrix} 0 & 0 & 0 & 0 & e^{ik_1} \\ 0 & 0 & 0 & 0 & e^{ik_1} \\ 0 & 0 & 0 & 0 & -2e^{ik_1} \\ 0 & 0 & 0 & 0 & 0 \\ e^{-ik_1} & e^{-ik_1} & -2e^{-ik_1} & 0 & 0 \end{vmatrix}, \quad T_y(k) = \begin{vmatrix} 0 & 0 & 0 & 0 & -e^{ik_2} \\ 0 & 0 & 0 & 0 & -e^{ik_2} \\ 0 & 0 & 0 & 0 & 2e^{ik_2} \\ 0 & 0 & 0 & 0 & 0 \\ -e^{-ik_2} & -e^{-ik_2} & 2e^{-ik_2} & 0 & 0 \end{vmatrix}, \tag{5.72}$$

$$T_{xy} = \begin{vmatrix} 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & -1 & -2 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{vmatrix}, \quad T_1 = \begin{vmatrix} -2 & 0 & 0 & -2 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \tag{5.73}$$

$$T_2 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 \end{vmatrix}, \quad T_3 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \tag{5.74}$$

The potential takes the form:

$$H_{pot} = \frac{1}{2L'^2} \sum_{k,\alpha,\beta,a} u^{a,\alpha}_{-k} V^{\alpha,\beta}_k u^{a,\beta}_k = \frac{1}{2L'^2} \sum_{k,a} \hat{Q}^a_s V_k \hat{Q}'^a_s, \tag{5.75}$$

where $V_k$ is a $10 \times 10$ matrix, defined as

$$V = \begin{vmatrix} V_{xx} + V_x(k) & V_{xy} + e^{ik_1} V_1 + e^{-ik_2} V_2 + e^{i(k_1-k_2)} T_2 \\ \text{c.c.} & V_{yy} + V_y(k) \end{vmatrix}, \tag{5.76}$$

where the matrices are defined as:

$$V_{xx} = \begin{vmatrix} 2 & -1 & 0 & 1 & 0 \\ -1 & 2 & -1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}, \quad V_{yy} = \begin{vmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}, \tag{5.77}$$

$$V_x(k) = \begin{vmatrix} 0 & 0 & -e^{ik_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e^{ik_1} \\ -e^{-ik_2} & 0 & 0 & 0 & e^{ik_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-ik_1} & e^{-ik_1} & 0 & 0 \end{vmatrix}, \quad V_y(k) = \begin{vmatrix} 0 & 0 & -e^{-ik_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{ik_2} \\ -e^{ik_1} & 0 & 0 & 0 & -e^{ik_2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-ik_2} & -e^{-ik_2} & 0 & 0 \end{vmatrix}, \tag{5.78}$$

$$V_{xy} = \begin{vmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \qquad V_1(k) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \tag{5.79}$$

$$V_2(k) = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \qquad V_3(k) = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \tag{5.80}$$

In order to integrate the Hamilton's equations, one must perform a change of variables, ordinarily called normal modes. The kinetic and the potential terms are positive definite quadratic forms, therefore they can be diagonalized by a linear change of variables

$$\begin{aligned} \hat{Q}'^{i}_{k} &= R(k)\hat{Q}^{i}_{k} \\ \hat{P}'^{i}_{k} &= D(k)\hat{P}^{i}_{k}. \end{aligned} \tag{5.81}$$

In terms of these new variables the Hamiltonian can be rewritten

$$H_0 = \frac{1}{2L^{\prime 2}}\{\sum_{k,\alpha,a} p'_{-k}{}^{a,\alpha} p'^{a,\alpha}_{k} + u'_{-k}{}^{a,\alpha}\lambda(k)^{\alpha} u'^{a,\alpha}_{k}\}, \tag{5.82}$$

where the $\lambda(k)^{\alpha}$ are the eigenvalues of the form $T_k^{(-1)}$ with respect to $V_k$, which satisfies the characteristic equation

$$det(T_k V_k - \lambda_k) = 0 \qquad det(V_k - T_k^{(-1)}\lambda_k) = 0. \tag{5.83}$$

The symmetry of the matrices $T$ and $V$ impose that all the roots are real, and these roots define the characteristic frequencies of the corresponding harmonic oscillators defined by each term of the Hamiltonian. In this case it is a 3-dimensional harmonic oscillator with a frequency $w^{\alpha}(k) = \sqrt{\lambda^{\alpha}(k)}$.

$$H_0 = \frac{1}{2L^{\prime 2}}\sum_{\alpha}(\sum_{k,a} p'_{-k}{}^{a,\alpha} p'^{a,\alpha}_{k} + w^{\alpha}(k)^2 u'_{-k}{}^{a,\alpha} u'^{a,\alpha}_{k}). \tag{5.84}$$

Since $T, V$ are $10 \times 10$ complex matrices, we can not find an analytical expression for the eigenvalues and eigenvectors. The characteristic polynomial of this problem is real and of order 10. It can be reduced to one of order 9 because one eigenvalue becomes zero.

Numerical solutions in order to find the other eigenvalues have been performed obtaining the eigenvalues that satisfy the following features:
a) The frequencies are real and positives.
b) The frequencies are separated in three groups: with the first group containing the two smaller eigenvalues for fixed $k_i$, which are $w^1(k) = 0$ and $w^2(k) = f(\vec{k})$. It was also possible to demonstrate that the second frequency is a symmetric function and has a dependence only on $|\vec{k}|$. This is represented in Figure 10.
c) The other frequencies take the form (see Fig. 11-13)

$$\begin{aligned} w^{\alpha}(k)^2 &= 3 + f^{\alpha}(k_i) \qquad \alpha = 3, \ldots, 6 \\ w^{\alpha}(k)^2 &= 6 + f^{\alpha}(k_i) \qquad \alpha = 7, \ldots, 10. \end{aligned} \tag{5.85}$$



An analytical form of the function $f^{\alpha}$ was not possible to find.

In Figure 13, the existence of a gap between the first and the other groups of frequencies is clearly observed. This property enables us to introduce the definition of 'slow' and 'fast' degrees of freedom. The oscillators with frequencies in the first group are called slow modes and the oscillators with frequencies $w^{\alpha}, \alpha > 2$ are classified as fast modes, and they satisfy the condition

$$\frac{w^{min}_{fast}}{w^{max}_{slow}} > 1. \tag{5.86}$$

In order to obtain an idea of the structure of the eigenvalues and eigenvectors, we have performed an investigation assuming that the wavelength is large, i.e.

$$|\vec{k}| << 1. \tag{5.87}$$

Retaining only terms up to the fourth order in $k_i$, when we expand the exponentials in the kinetic and potential term, it is only possible to find an analytical expression for the slow modes. The eigenvalues become:

$$\begin{aligned} w^1(k)^2 &= 0, \\ w^2(k)^2 &= k_1^2 + k_2^2, \end{aligned} \tag{5.88}$$

and the eigenvalues are:

$$\begin{aligned} |w_1> &= \frac{k_1}{\sqrt{k_1^2+k_2^2}}\begin{bmatrix} \hat{v}_1 \\ \hat{0} \end{bmatrix} + \frac{k_2}{\sqrt{k_1^2+k_2^2}}\begin{bmatrix} \hat{0} \\ \hat{v}_1 \end{bmatrix} + O(k^2), \\ |w_2> &= -\frac{k_2}{\sqrt{k_1^2+k_2^2}}\begin{bmatrix} \hat{v}_1 \\ \hat{0} \end{bmatrix} + \frac{k_1}{\sqrt{k_1^2+k_2^2}}\begin{bmatrix} \hat{0} \\ \hat{v}_1 \end{bmatrix} + O(k^2), \end{aligned} \tag{5.89}$$

where $\hat{0}$ is the null 5-column vector and $\hat{v}_1$ is

$$\hat{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.90}$$

This solution of the slow modes is not an impossible thing to obtain, especially, when the idea of the RGT is taken into consideration. The aim of the RGT is to define an effective system which describes the physic at large wavelengths or small momenta. In the context of Real-Space-RGT it is equivalent to a formulation of the effective system on a $La$-lattice. Now, when we consider that the effective theory becomes the same analytical form of the original system, but the difference is only on the perturbative terms, one must expect that the integrable term on the new lattice has the same behavior for the eigenvalues as on the original lattice but with a scaling factor.

The integrability of the system can be observed by introducing action-angle variables, which are straightforward in this case, and the Hamiltonian is rewritten as;

$$H_0 = H_{slow} + H_{fast}(I), \tag{5.91}$$

where $I$ are the action variables for the fast modes, and they are constants of motion. If we introduce a shift in the total energy, where in this trivial case an effective Hamiltonian can be defined as:

$$H_0' = H^0_{slow}. \tag{5.92}$$

Now, defining the new variables "block-spin" as

$$\vec{u}_{s,\mu}{}^{BS} = \frac{1}{3}\sum_{\alpha=1}^{3} \vec{y}_{s,\mu}{}^{\alpha}, \tag{5.93}$$

and rewriting the $H_0'$ in terms of these variables, it can be shown that the Hamiltonian takes the following form,

$$H_0 = \frac{1}{2L^2}\{\sum_k \vec{p}_{-k,\mu}{}^{BS}\vec{p}_{k,\mu}{}^{BS} + \sum_{k,\mu,\nu} \vec{u}_{-k',\mu}{}^{BS}V(k)^{\mu,\nu}\vec{u}_{k,\nu}{}^{BS}\}, \tag{5.94}$$

where the matrix term in the last expression takes the form

$$V(k) = (\tilde{k}^2\delta_{\mu,\nu} - \tilde{k}_\mu\tilde{k}_\nu) \tag{5.95}$$

in the approximation of small $k$. This result reflects the property that in the limit, $g \to 0$, the classical block-spin variable defined by the equation (5.93) becomes the same structure of the Block-spin in the quantum formulation within the limit of small field approximation [75]. This can be shown by introducing the representation of the link variables as $u_{s,\mu} = exp(igA_{s,u})$ with $A$ being small, then expanding the corresponding definition of Block-spin, the result is this:

$$\vec{u}_{s,\mu}{}^{BS} = \frac{1}{L^d}\sum_{\alpha} \vec{A}(\Gamma^{\alpha}_{s,s'}). \tag{5.96}$$

A study of the fast variables in quantum formalism is not carried out, because they are integrated out. In our classical formulation the integration is equivalent to an average over the fast variables.

To conclude this section, we will introduce the action-angle variables for the fast variables in order to prepare the Hamiltonian for the application of the perturbation theory when we switch the perturbation. Making a transformation to the complexe variables $a$, $a^*$ defined for each $k$, which are defined as:

$$a^{\alpha,i}(k) = \frac{(L'/2)^{\frac{d}{2}}}{2}\{\sqrt{\omega^{\alpha}(k)}\,u_k^{\alpha,i} + \frac{i}{\sqrt{\omega^{\alpha}(k)}}\pi_k^{\alpha,i}\}, \tag{5.97}$$

with analogous expressions for $a^{*\,\alpha,i}(k)$.
The canonical transformation which takes $(\pi, u)$ to $(ia^*(k), a(k))$ has the generating function:

$$\begin{aligned} F_2(a^*,u) &= \sum_{k\epsilon\Lambda_{2a}}\{\frac{-i}{2}(L/2)^{-D}\omega^{\alpha}(k)\,u_{-k}^{\alpha,i}\,u_k^{\alpha,i} + i\sqrt{2\omega^{\alpha}(k)}(L/2)^{-D/2}\,a^{*\alpha}(k)\,u_k^{\alpha,i} \\ &\quad -\frac{1}{2}a^{*\alpha}(k)a^{\alpha}(-k)\}. \end{aligned} \tag{5.98}$$

The new variables have canonical Poisson brackets

$$\{a^{\beta,i}(k), a^{*\,\alpha,j}(k')\} = -i\delta_{k,k'}\delta^{ij}\delta^{\beta,\alpha}, \tag{5.99}$$



and the effective free part of $H^0$ takes the form

$$H^0 = \sum_{k,\alpha}\{\,w^{\alpha}(k)\,a^{\alpha,i}(k)a^{*\,\alpha,i}(k)\}. \tag{5.100}$$

For these new variables, we now introduce the action and angle variables ( $I^{\alpha,i}(k)$,$\theta^{\alpha,i}(k)$ ), both are real-valued, such that

$$\begin{aligned} a^{\alpha,i}(k) &= \sqrt{I^{\alpha,i}(k)}\,e^{-i\theta^{\alpha,i}(k)} \\ a^{*\,\alpha,i}(k) &= \sqrt{I^{\alpha,i}(k)}\,e^{i\theta^{\alpha,i}(k)}, \end{aligned} \tag{5.101}$$

giving

$$H^0 = \sum_{k,\alpha,i} w^{\alpha}(k)\,I^{\alpha,i}(k) \tag{5.102}$$

being the connection with the old variables

$$u^{\alpha,i}(k) = \frac{(L'/2)^{d/2}}{\sqrt{2\omega^{\alpha}(k)}}\left(\sqrt{I^{\alpha,i}(k)}\,e^{-i\theta^{\alpha,i}(k)} + \sqrt{I^{\alpha,i}(-k)}\,e^{i\theta^{\alpha,i}(-k)}\right), \tag{5.103}$$

$$\pi^{\alpha,i}(k) = (-i)\,(L'/2)^{d/2}\sqrt{\frac{\omega^{\alpha}(k)}{2}}\left(\sqrt{I^{\alpha,i}(k)}\,e^{-i\theta^{\alpha,i}(k)} - \sqrt{I^{\alpha,i}(-k)}\,e^{i\theta^{\alpha,i}(-k)}\right). \tag{5.104}$$

The invariance of $H_0$ under SU(2) rotation in the color index leads to a conserved constant $\Omega_k^i$, which in the approximation $g = 0$ reduces to

$$\Omega_k^i = \sum_{\hat{\mu},\alpha}(1 - e^{-ik\cdot\hat{\mu}})\vec{p}_{k,\hat{\mu}}^{\,i,\alpha}. \tag{5.105}$$

This symmetry is the basis for the generation of the resonance condition between the same modes. This condition can be observed when the frequencies for each modes are calculated, i.e.

$$\frac{\delta H_0}{\delta I^{\alpha,i}(k)} = w^{\alpha,i}(k) = w^{\alpha}(k), \tag{5.106}$$

therefore the frequencies are independent of the color indices, which states the following resonance conditions:

$$\begin{aligned} w^{\alpha,3}(k) - w^{\alpha,2}(k) &= 0 \\ w^{\alpha,3}(k) - w^{\alpha,1}(k) &= 0. \end{aligned} \tag{5.107}$$

Considering the discussion over degenerate systems in Chap.1, we can introduce a canonical transformation in order to eliminate the resonance conditions, given by the transformation from $\{I^{\alpha,i}(k), \theta^{\alpha,i}(k)\}$ to $\{J^{\alpha,i}(k),\psi^{\alpha,i}(k)\}$ with the following generating function

$$F_2 = J^{\alpha,1}(k)\{\theta^{\alpha,1}(k) - \theta^{\alpha,3}(k))\} + J^{\alpha,2}(k)\{\theta^{\alpha,2}(k) - \theta^{\alpha,3}(k))\} + J^{\alpha,3}(k)\theta^{\alpha,3}, \tag{5.108}$$

and the new and old variables are related by

$$\begin{aligned} I^{\alpha,1}(k) &= J^{\alpha,1}(k) \qquad \psi^{\alpha,1}(k) = \theta^{\alpha,1}(k) - \theta^{\alpha,3}(k) \\ I^{\alpha,2}(k) &= J^{\alpha,2}(k) \qquad \psi^{\alpha,2}(k) = \theta^{\alpha,2}(k) - \theta^{\alpha,3}(k) \\ I^{\alpha,3}(k) &= J^{\alpha,3}(k) - J^{\alpha,1}(k) - J^{\alpha,2}(k) \qquad \psi^{\alpha,2}(k) = \theta^{\alpha,3}(k). \end{aligned} \tag{5.109}$$

Inserting in the Hamiltonian the new canonical variables which measure the slow deviation from the resonance, and since the transformation is time independent, $H_0$ can be written as a function of $J^{\alpha,3}$ alone, i.e.

$$H_0 = \sum_{k,\alpha} w^{\alpha}(k) J^{\alpha,3}(k). \tag{5.110}$$

Finally to conclude this chapter, let us rewrite the $a^{\alpha,i}$ variables in terms of the final action-angle coordinates, which are useful when we calculate the effective Hamiltonian. They assume the following form:

$$\begin{aligned} a^{\alpha,1}(k) &= \sqrt{J^{\alpha,1}(k)}\, e^{-i\psi^{\alpha,3}(k)} e^{-i\psi^{\alpha,1}(k)} \\ a^{\alpha,2}(k) &= \sqrt{J^{\alpha,2}(k)}\, e^{-i\psi^{\alpha,3}(k)} e^{-i\psi^{\alpha,2}(k)} \\ a^{\alpha,3}(k) &= \sqrt{J^{\alpha,3}(k) - J^{\alpha,2}(k) - J^{\alpha,1}(k)},\, e^{-i\psi^{\alpha,3}(k)}. \end{aligned} \tag{5.111}$$

with the condition that

$$J^{\alpha,3}(k) \geq J^{\alpha,2}(k) + J^{\alpha,1}(k). \tag{5.112}$$



# Chapter 6

# Perturbation Theory.

The separation of the modes into two classes, slow and fast, was made possible by introducing a gauge fixing and a new set of variables that allows the existence of a gap between the frequencies of the slow and fast variables. This gap is an important feature for the application of the Adiabatic approach, because we can introduce a time scale in the motion where the modes are separated. In a global sense, this is non violated for the system treated here. In the following sections, we will apply the method of averaging over the fast variables in order to define an effective Hamiltonian which in turns describes the dynamics of the slow modes in the presence of adiabatic invariants corresponding to the fast modes. Our aim is to bring this Hamiltonian into a form which allows a comparison with our starting Hamiltonian. This can be done through a few canonical transformations defined by the Adiabatic Perturbation Theory. However, for the case of gauge variables these steps are more complicated because of the presence of some resonance conditions, and the application of the degenerate Perturbation Theory [20] is needed, as discussed in the Chapter 1.

The starting point is the reduced Hamiltonian defined as

$$\begin{aligned} H &= \frac{1}{2L'^2} \sum_{\alpha} \left[ \sum_{k,a} ({p'}_{-k}^{a,\alpha} {p'}_{k}^{a,\alpha} + w(k)^{2\alpha} {u'}_{-k}^{a,\alpha} {u'}_{k}^{a,\alpha}) \right] \\ &\quad \frac{1}{2L'^4} g \sum_{\{\mathbf{k}\}} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \Gamma'^{abc}_{3\,ijk}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) x^a_{k_1\,i} x^b_{k_2\,j} x^c_{k_3\,k} \\ &\quad \frac{1}{4L'^6} g^2 \sum_{\{\mathbf{k}\}} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \Gamma'^{abcd}_{4\,ijkl}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) x^a_{k_1\,i} x^b_{k_2\,j} x^c_{k_3\,k} x^d_{k_4\,l}, \end{aligned} \tag{6.1}$$

which is dependent only on the normal coordinates and where $\Gamma'^{abc}_{3\,ijk}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ and $\Gamma'^{abcd}_{4\,ijkl}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ are the vertices for the normal modes, given in Appendix D and E. In particular, they are the old vertices multiplied by the transformation matrix $R$.

In analogy to the investigation of the scalar theory, it is convenient to rewrite $H$ in terms of variables $a^{\alpha,i}_k, a^{*\alpha,i}_k$. The Hamiltonian becomes:

$$H = H_{slow} + \sum_{\alpha} \left[ \sum_{k,a} (w(k)^{2\alpha} a^{a,\alpha}_k a^{*a,\alpha}_k) \right]$$

$$\frac{1}{2L'^4}g\sum_{\{\mathbf{k}\}}\delta(\sum_i \mathbf{k}_i)\Gamma^{\prime\alpha_1,\alpha_2,\alpha_3}_{3i_1\ldots i_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\prod_{j=1\ldots3}\frac{L'^{d/2}}{\sqrt{2w^{\alpha_j}(k_j)}}\left[a^{\alpha_j,i_j}(k_j)+a^{*\alpha_j,i_j}(-k_j)\right]+$$

$$\frac{1}{4L'^6}g^2\sum_{\{\mathbf{k}\}}\delta(\sum_i \mathbf{k}_i)\Gamma^{\prime\alpha_1,\alpha_2,\alpha_3,\alpha_4}_{4i_1\ldots i_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)\prod_{j=1\ldots4}\frac{L'^{d/2}}{\sqrt{2w^{\alpha_j}(k_j)}}\left[a^{\alpha_j,i_j}(k_j)+a^{*\alpha_j,i_j}(-k_j)\right]. \tag{6.2}$$

having a dependence on the action angle variables through the variables $a$ and $a^*$'s. In the formulation of Perturbation Theory, the action-angle variables must be used but there are some calculations that are more simple to carry out in terms of the $a$ and $a^*$'s. In the following section, we apply the idea of RGT to the Hamiltonian (6.2). The process of averaging over the fast angle variables is for this Hamiltonian with a symmetry more complicated. Thus, it must be carried out in three steps:
1.) Introduction of new variables $\psi_k^{\alpha,i}$ defined around the resonance, that allow us to average over the fast modes, for example, over $\psi_k^{\alpha,3}$.
2.) The averaged Hamiltonian is still dependent on the rest of $\psi$ angle variables. Their elimination can be performed only by averaging around an elliptic point, where one can define new action-angle variables.
3.) Finally, the first two steps lead to the statement of an effective Hamiltonian for the slow modes with new vertices and modified parameters. Let us define the new parameters as renormalized parameters. In order to know the condition into which the new Hamiltonian can take the same form as the starting Hamiltonian, we have to find some conditions on the new vertices and over the renormalized parameters. We will call all these canonical transformations as a step of the renormalization group transformation.

In the following sections, we apply the Adiabatic Perturbation Theory to high order in order to carry out the first averaging over the $\psi_k^{\alpha,3}$ variables and to define the new vertices that describe the perturbative term of the effective Hamiltonian. In section 1, we study the effective Hamiltonian to the first order. In the second section, the contribution to the second order are calculated, and in the last section we discuss the calculation to the third order.

## 6.1 Calculation to first order in perturbation theory.

In order to carry out this calculation, we will use the Lie-transformation approach which was discussed in the chapter 1. The Lie transformation to first order is equivalent to the normal Adiabatic Perturbation Theory, and the transformed Hamiltonian to this order takes the form

$$\bar{H}_1 = H_1 + \{C_1, H_0\}', \tag{6.3}$$

where $\{\ ,\ \}'$ denotes the Poisson brackets with respect to the fast variables, particularly regarding the coordinates $\psi_k^{\alpha,3}$ and momenta $J_k^{\alpha,3}$.

To remove the dependence upon the angle $\psi_k^{\alpha,3}$, we average over these variables, and we obtain a condition which define $C_1$

$$\tilde{H}_1 + \{C_1, H_0\}' = 0 \tag{6.4}$$



and the $\tilde{H}_1$ becomes

$$\tilde{H}_1 =< H_1 > . \tag{6.5}$$

The $H_1$ perturbative term can be interpreted as a three-point function with three external lines of slow and fast variables. Therefore, one can introduce the following notation

$$H_1 = \sum_{i=1,\ldots3} H^1_{3-i,i}; \tag{6.6}$$

where $H_{3-k,k}$ denote a term with $(3-k)$-slow modes and $k$-fast modes as external lines. To calculate the average over the fast angle variables in the interaction term $H_1$, it is useful to give a statement about how the averaging is performed and to show some useful relations in order to calculate the contribution derived from the average. Taking into account the above definition of $H_1$, one sees that by averaging all $\psi_k^{\alpha,3}$, many terms are left out. Only those terms with an even number of fast modes as external lines have a non vanishing contribution, i.e. if we consider $H_1$, in principle, it may obtain a non zero contribution only from the terms:

$$< H_1 >=< H^1_{3,0} > + < H^1_{1,2} > . \tag{6.7}$$

The first term is obviously a non zero, because it has only slow external lines. In order to calculate the second term, let us state the general properties of averaging over the $a$ and $a^*$ variables. Considering the expression for the fast variables $a_k^{\alpha,i}$ in terms of the action-angle variables, one can obtain the following relations:

$$\begin{aligned} < a_k^{\alpha,i} > &= 0 \qquad < a_p^{\alpha,i}a_q^{\beta,j}a_r^{\gamma,k} >= 0 \\ < a_p^{\alpha,i}a_r^{\beta,j} > &= 0 \\ < a_p^{\alpha,i}a_q^{*\beta,j} > &= \delta^{\alpha,\beta}\delta_{p,q}\delta^{i,j}(p)\sqrt{\tilde{J}_p^{\alpha,i}\tilde{J}_p^{\alpha,j}}, \end{aligned} \tag{6.8}$$

thus, it can be observed that only those terms where for each $e^{-i(\psi_k^{\alpha,3})}$ we also find that an $e^{i(\psi_{-k'}^{\alpha,3})}$ with $k=-k'$ will contribute to $< H_1 >$. The factor $\delta^{i,j}(k)$ is defined as

$$\begin{aligned} \delta^{i,j}(k) &= e^{-i(\psi_k^{\alpha,i}-\psi_k^{\alpha,j})} \qquad i,j=1,2 \\ \delta^{i,3}(k) &= e^{-i(\psi_k^{\alpha,i})} \qquad i=1,2 \\ \delta^{i,i}(k) &= 1 \qquad i=1,2,3. \end{aligned} \tag{6.9}$$

In other words, in the three external lines of the three points vertex, either all three external modes are slow, or one is slow and the other two must be fast with momenta equal and opposite.

The corresponding expressions are:

$$< H^1_{3,0} >= \frac{g}{2L'^4}\sum_{\{\mathbf{k}\},\alpha'_i=1,2}^{\prime}\Gamma_3^{\prime\alpha'_1,\alpha'_2,\alpha'_3}\epsilon^{i_1i_2i_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\prod_{j=1\ldots3}u^{\alpha'_j,i_j}(k_j), \tag{6.10}$$

and

$$\begin{aligned} < H^1_{1,3} > &= \frac{g}{2L'^2}\sum_{\{\mathbf{k}\},\alpha'_1=1,2}^{\prime}\sum_{\alpha^*>2}^{\prime}\Gamma_3^{\prime\alpha'_1,\alpha^*_2,\alpha^*_3}\epsilon^{i_1i_2i_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)u^{\alpha'_1,i_1}(k_1) \\ &\sum\tau\delta(\mathbf{k}_2+\mathbf{k}_3)\delta^{\alpha^*_2,\alpha^*_3}\delta^{i_2,i_3}(\tau k_2)\frac{\sqrt{\tilde{J}^{\alpha^*_2,i_2}(\tau k_2)\tilde{J}^{\alpha^*_2,i_3}(\tau k_2)}}{2w^*_2(k_2)}. \end{aligned} \tag{6.11}$$

The conservation of the momenta in the last expression leads to the evaluation of the sum at the point $\vec{k} = 0$. But, this term is irrelevant in the limit of large volumen. Finally, this result defines an effective Hamiltonian to first order as

$$H^1_{eff} = H_{0,slow} + H_0(J') + gH^1_{3,0}, \tag{6.12}$$

which describes the motion of the slow modes in the presence of a set of adiabatic invariants ($J_k^{\prime\alpha,3}$) for the fast modes. Furthermore, it can be observed that up to this order, it is independent of the other fast angle variables.

To close this discussion about the form of the effective Hamiltonian, we will investigate the new vertex associated with the perturbative term $H^1_{3,0}$. This term is derived only from the expansion of the potential up to first order in $g$ or third order in the variables. In order to study its properties, we use the form of the vertex calculated in appendix D, where this term was defined by

$$H_1 = \frac{g}{2L^4}\sum_{\alpha=1,2}\left[\sum_{a,b,c}\sum_{\{k\}}\delta(k_1+k_2+k_3)\Gamma'^{abc}_{3\,ijk}(k_1,k_2,k_3)x^a_{k_1\,i}x^b_{k_2\,j}x^c_{k_3\,k}\right], \tag{6.13}$$

where $\Gamma'^{abc}_{3\,ijk}(k_1,k_2,k_3)$ are the vertices for all the modes. This form of the three vertex function describes the interaction among all the variables, but we need only to consider this vertex restricted to the slow modes because this term defines the new vertex to order $g$ of the effective Hamiltonian after the average. Here, we will introduce the vertex $\Gamma$ which is defined by equation (D.11), in order to state our conclusion. Therefore the $H^1_{3,0}$ takes the form

$$\begin{aligned} H^1_{3,0} &= \frac{g}{2L^4}\sum_{\alpha=1,2}\{\sum_{a,b,c=1,2}\sum_{a',b',c'=1,\ldots 10}\sum_{\{k\}}\delta(k_1+k_2+k_3)\,\epsilon^{ijk} \\ &\Gamma^{a'b'c'}(k_1,k_2,k_3)R^{a'a}(k_1)R^{b'b}(k_2)R^{c'c}(k_3)x^a_{k_1\,i}x^b_{k_2\,j}x^c_{k_3\,k}\}. \end{aligned} \tag{6.14}$$

We can not write an analytic expression for the transformation Matrix $R$ because the calculation of this matrix was only possible numerically. However, it can be observed that in this calculation, we need only to know the first two columns of this matrix. Let us denote this matrix $10 \times 2$ as $R^*$. As stated in the last chapter, it is possible to find an analytical form for this matrix in the approximation of small momenta. In this case, this matrix takes the following form:

$$R^*(k) = \begin{vmatrix} f_1(k)\hat{v}_1 & f_2(k)\hat{v}_1 \\ -f_2(k)\hat{v}_1 & f_1(k)\hat{v}_1 \end{vmatrix} \quad \text{where} \quad \hat{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{6.15}$$

and the functions $f_1, f_2$ are

$$f_1(k) = \frac{k_2}{\sqrt{k_1^2+k_2^2}} \quad \text{and} \quad f_2(k) = \frac{k_1}{\sqrt{k_1^2+k_2^2}}. \tag{6.16}$$

This matrix can be separated as

$$R^*(k) = R' * D(k) \tag{6.17}$$



where $R'$ is a constant matrix of $10 \times 2$ and the matrix $D$ is a $2 \times 2$ which are given by:

$$R' = \begin{vmatrix} \hat{v}_1 & \hat{0} \\ \hat{0} & \hat{v}_1 \end{vmatrix} \quad \text{and} \quad D(k) = \begin{vmatrix} f_1(k) & f_2(k) \\ -f_2(k) & f_1(k) \end{vmatrix}. \tag{6.18}$$

Introducing this decomposition in the vertex we obtain

$$\Gamma'^{abc}(k_1,k_2,k_3) = \{\Gamma^{a'b'c'}(k_1,k_2,k_3)R'^{a'a^*}R'^{b'b^*}R'^{c'c^*}\}D^{a^*a}(k_1)D^{b^*b}(k_2)D^{c^*c}(k_3). \tag{6.19}$$

We can bring the expression in the parentheses into a form that is more suitable for the interpretation by denoting it as $\Gamma_0$. This is achieved by evaluating the operation defined in parentheses. The result of this straightforward but rather lengthy algebraic calculation is

$$\Gamma^{abc}_0(k_1,k_2,k_3) = \{\delta_{ab}(k_3-k_2)_c + \delta_{bc}(k_1-k_3)_a + \delta_{ca}(k_2-k_1)_b\} \tag{6.20}$$

where $a,b,c = 1,2$. This result was obtained using the algebraic program "Maple". If one compares this result with the original three vertex on the $a$-lattice after an expansion in the approximation of small momenta, it can be observed that they are identical.

Finally, the effective three point function to first order can be rewritten as

$$\begin{aligned} H_{3,0} &= \sum_{\alpha=1,2}\{\sum_{a,b,c=1,2}\sum_{a',b',c'=1,2} g\sum_{\{k\}}\delta(k_1+k_2+k_3)\epsilon^{ijk} \\ &\Gamma^{a'b'c'}_0(k_1,k_2,k_3)D^{a'a}(k_1)D^{b'b}(k_2)D^{c'c}(k_3)x^a_{k_1\,i}x^b_{k_2\,j}x^c_{k_3\,k}\}. \end{aligned} \tag{6.21}$$

Considering the result in Appendix B, and especially equation (B.1), one can conclude that the structure of the perturbative term takes the same form as the three point function of the original lattice after the introduction of the normal modes which diagonalized the quadratic term. Therefore, the expression $D^{a,b}(k)x^b(k)$ can be interpreted as the definition of the change of variables from the normal modes $x^{\alpha,i}(k)$ to the "original" variables $u^{\alpha,i}(k)$ defined on the $3a$-lattice, which are denoted by

$$\begin{aligned} u^{\alpha=1,a}(k) &= \tilde{u}^a_i(k) \\ u^{\alpha=2,a}(k) &= \tilde{u}^a_j(k) \end{aligned} \tag{6.22}$$

and considering the form of the slow modes $x^{a,i}(k)$, we obtain that

$$\tilde{u}^i_\mu(k) = \sum_{\alpha=1,2,3} u^{\alpha,i}_\mu(k), \tag{6.23}$$

where the $u^{\alpha,i}_\mu(k)$ are the link variables previously introduced.

It can be concluded from this result that the effective Hamiltonian for the slow modes to first order takes a similar form as that of the original one. In particular, if we introduce a new change of variables defined as in the last equations and which is defined now on the $La$-lattice, this effective Hamiltonian takes the same form as that the original, but in this case the new links variables are defined as:

$$\tilde{u}^i_\mu(k) = \sum_{\alpha} u^{\alpha,i}_\mu(k), \tag{6.24}$$

where the $u^{\alpha,i}_\mu(k)$ are links variables defined between the center of two blocks.

To conclude this section, we need to calculate the term $C_1$ because the next step in this program is to investigate the high order corrections to the Hamiltonian (6.12) using the Lie approach. To second order, we must average the redefined Hamiltonian $H_2^*$, which is given in term of the generating function as:

$$H_2^* = H_2 + \frac{1}{2}\{C_1, \tilde{H}_1\} + \{C_1, < H_1 >\} + \{H_0, C_1\}_{slow}. \tag{6.25}$$

In order to find an analytical solution for the generating function $C_1$ we must resolve the equation

$$\sum_{\alpha,k} w^{\alpha}(k)\frac{\delta C_1}{\delta\psi^{\alpha,3}(k)} = -\tilde{H}_1. \tag{6.26}$$

The form of the oscillatory term is then

$$\tilde{H}_1 = \tilde{H}_1 - < H_1 >= \sum_{i=1,2,3} H^1_{3-i,i}. \tag{6.27}$$

Assuming that $C_1$ can be separated as

$$C_1 = \sum_{i=1,2,3} C^1_{3-i,i} \tag{6.28}$$

then each term of the sum satisfies the condition

$$\{C^1_{i,j}, H_0\} = -\tilde{H}^1_{i,j}. \tag{6.29}$$

To solve this equation, it is more convenient to introduce a new set of variables:

$$b^{\alpha,i}(k,\tau) = \begin{cases} a^{\alpha,i}(k) & \tau = 1 \\ a^{*\alpha,i}(-k) & \tau = -1 \end{cases} \tag{6.30}$$

and in terms of this variable $\tilde{H}_1$ takes the compact form

$$\tilde{H}^1_{3-k,k} = \frac{1}{L^{4-k}} \sum_{q_i,\alpha_i,a_i}^{i=1\ldots3-k} \sum_{p_j,\beta_j,b_j}^{j=1\ldots k} \Gamma^{\alpha_i,\beta_j}_{3-k,k\,a_i b_j}(q_i,p_j) \prod_{j=1}^{k} \frac{b^{\beta_j,i_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}} \prod_{j=1}^{3-k} u^{\alpha_j,i_j}(q_j), \tag{6.31}$$

where $k = 1,2,3$. Using the following relations of the Poisson brackets between the products of $b$'s and $H_0$

$$\begin{aligned} i\{b^{\alpha,i}(k,\tau), H_0\} &= \tau w^{\alpha}(k) b^{\alpha,i}(k,\tau) \\ i\{\prod_j b^{\alpha_j,i_j}(k_j,\tau_j), H_0\} &= \sum_i \tau_i w^{\alpha_i}(k_i) \prod_j b^{\alpha_j,i_j}(k_j,\tau_j), \end{aligned} \tag{6.32}$$

it is easy to obtain $C_1$, which is given as

$$\begin{aligned} C_{3-k,k} &= \frac{1}{L^{4-k}} \sum_{q_i,\alpha_i,a_i}^{i=1\ldots3-k} \sum_{p_j,\beta_j,b_j}^{j=1\ldots k} \frac{i}{-\sum_{i'=1\ldots k}\tau_{i'} w^{\beta_{i'}}(p_{i'})} \Gamma^{\alpha_i,\beta_j}_{3-k,k\,a_i b_j}(q_i,p_j) \\ &\quad \prod_{j=1}^{k} \frac{b^{\beta_j,i_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}} \prod_{j=1}^{3-k} u^{\alpha_j,i_j}(q_j). \end{aligned} \tag{6.33}$$

The problem of a small denominator is not present in the first term of $C$, but it is possible that there exists some commensurable relations among the fast frequencies for a determined value of the momentum. However, we will assume that this is not the case. This point will be discussed in the chapter of conclusion.



## 6.2 Second order correction

In the context of the Adiabatic Perturbation Theory, the Hamiltonian $\tilde{H}_2$ up to second order takes the form

$$\tilde{H}_2 = H_2^* + \{C_2, H_0\}_{fast}, \tag{6.34}$$

where

$$H_2^* = H_2 + \frac{1}{2}\{C_1, \tilde{H}_1\} + \{C_1, < H_1 >\} + \{C_1, H_0\}_{slow}. \tag{6.35}$$

To eliminate the fast variables, we proceed as in the last section, and an effective second order Hamiltonian is obtained by averaging over the fast variables

$$\tilde{H}_2 =< H_2^* >=< H_2 > + \frac{1}{2} < \{C_1, \tilde{H}_1\} > + < \{C_1, < H_1 >\} > + < \{C_1, H_0\}_{slow} > . \tag{6.36}$$

Moreover, one gets a condition over the term $C_2$:

$$\tilde{H}_2^* + \{C_2, H_0\}' = 0 \tag{6.37}$$

The last two terms in (6.36) vanish after averaging over the fast modes, because they are oscillatory. Thus, we only need to average over the first terms. These terms generate non trivial contributions to the starting Hamiltonian. In the language of quantum field theory such contributions are understood as one loop correction to the two point function and to the vacuum. Moreover, we obtain to this order a new four vertex function. To continue, we will study the average $< H_2 >$ in the first place, and posterity, we investigate the second contributions to the effective Hamiltonian $\bar{H}$ derived from the term $< \{C_1, \tilde{H}_1\} >$.

Analogous to the discussion of the first order, it can be observed that $H_2$ separates according to the number of external lines as

$$H_2 = \sum_{i=1,\ldots,4} H^1_{4-i,i}, \tag{6.38}$$

where $H_{4-k,k}$ denotes: $4-k$-slow external lines and $k$-fast external lines.

By averaging over the fast modes, many terms vanish, and it can be found that the contribution associated with the average of the perturbation to second order fall into three classes:
1.) All four lines are slow and the contribution comes from the term $H_{4,0}$.
2.) Two of the lines belong to fast modes and the other two belong to slow modes. This case can be represented as a loop, where the internal line is a fast mode. It is derived from the term $H_{2,2}$.
3.) All four lines belong to the fast modes and their contractions give two loops contribution to the vacuum, which is generated from $H_{0,4}$.

The three contributions are illustrated in Fig. (14a-14c), and their corresponding expressions are:

$$< H^2_{4,0} >= \frac{1}{L^6} \sum_{\{k\},\alpha'_i=1,2}^{t} \Gamma'^{\alpha'_1,\alpha'_2,\alpha'_3,\alpha'_4}_{4\;i_1 i_2 i_3 i_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) \prod_{j=1\ldots4} u^{\alpha'_j,i_j}(k_j), \tag{6.39}$$

and

$$< H^2_{2,2} > = \frac{1}{L^4} \sum_{\{\mathbf{k}\},\alpha'_i=1,2}{}' \sum_{\alpha_j=>2}{}' \Gamma^{\prime\alpha'_1,\alpha'_2,\alpha_3,\alpha_4}_{i_1 i_2 i_3 i_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) \prod_{i=1,2} u^{\alpha'_i,i_i}(k_i)$$
$$\delta(\mathbf{k}_4+\mathbf{k}_3)\delta^{\alpha_3,\alpha_4}\delta^{i_4,i_3}(k_3)\frac{\sqrt{\tilde{j}^{\alpha_3,i_3}(k_3)\tilde{j}^{\alpha_3,i_3}(k_3)}}{2w^{\alpha_3}(k_3)}, \quad (6.40)$$

and

$$< H^2_{0,4} > = \frac{1}{L^2} \sum_{\{\mathbf{k}\},\alpha_i>2}{}' \Gamma^{\prime\alpha'_1,\alpha'_2,\alpha'_3,\alpha'_4}_{4\, i_1 i_2 i_3 i_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$$
$$< \prod_{j=1\ldots4} \sum_{\tau} \frac{b^{\alpha'_j,i_j}(k_j,\tau_j)}{\sqrt{2w^{\alpha_j}(k_j)}} > . \quad (6.41)$$

The first term represented in Fig. 14.a, corresponds to the four point function vertex with all of the slow external lines, and it is the standard four point function in the approximation of small momenta.
The next term (Fig. 14.b) provides a contribution to the two point function, which can define a change in the frequencies of the slow modes. Finally, the last term corresponding to Fig. 14.c has zero external slow lines and can be called a vacuum diagram. In principle, it can not be considered in the discussion about the effective Hamiltonian since these diagrams contribute to the background energy density of the system, and do not affect the dynamics of the slow modes. In the scalar case, we used this argument in order to eliminate these contributions. However, for a system with a degeneracy that is not clear, this statement is true, only in the case that these contributions are dependent on the adiabatic invariants $J'^{\alpha_3,i}(k)$, hence, it can be concluded that this term is irrelevant. Moreover, in the case of any dependence on the variables $(J^{\alpha_j,i}(k), \psi^{\alpha_j,i}(k))$, then it is not true. We will discuss this further in the next section.

Now, let us turn to the second group of contributions to the effective Hamiltonian generated from $< \{C_1, \tilde{H}_1\} >$. First, we will give a qualitative description of the calculation and posterity. The details of the calculation are given and the results are stated.

Let us show the structure of the terms generated by the operation of taking the Poisson bracket before averaging over the expression $\{C_1, \tilde{H}_1\}$. Considering the structure of each $C_1, \tilde{H}_1$, the Poisson bracket generates a four-vertex and it is represented in Fig.15, where it can have either fast or slow external lines. It decomposes according to the number of fast and slow modes, which can be observed in the Fig.(16a-16e.). The internal line that connects the three-vertex $\tilde{H}_1$ to the three point function $C_1$, represents the contraction due to Poisson bracket denoted graphically by a cross, and up to this order, it corresponds to a fast line.

Now when we return to the averaging over of the fast angles, we find three classes of diagrams:
1.) Graphs with zero external slow lines obtained from the term $< \{H^1_{0,3}, C^1_{0,3}\} >$ and represented in the Fig. 17a-17b,
2.) Graphs with two external slow lines obtained from the term $< \{H^1_{1,2}, C^1_{1,2}\} >$, and $< \{H^1_{2,1}, C^1_{0,3}\} >$, $< \{H^1_{0,3}, C^1_{1,2}\} >$, and illustrated in Fig. 17c and Fig. 17d respectively.
3.) Graphs with four external slow lines obtained from the term $< \{H^1_{2,1}, C^1_{2,1}\} >$, and represented in Fig. 16b.



Graphs of the first type are not directly dependent on the slow modes, but similar to the other vacuum contributions. One must not immediately state that they are irrelevant. In the case of a dependence on the other fast variables, we must still expand around the elliptic point in order to eliminate these variables and define the other adiabatic invariants. Therefore, there exists a possibility that these terms become a dependent of the slow modes from the expansion at the fixed point. The next graph provides a one loop contribution to the two-point function, which must be considered together with the contribution to the same vertex derived from the averaged over $H_2$, and finally one obtains a contribution to the four vertex function.

### 6.2.1 Vacuum Contribution.

To second order, we found the following contribution to the vacuum energy

$$H_{vacuum} =< H_{0,4} > -\frac{1}{2} < \{H^1_{0,3}, C^1_{0,3}\} >, \quad (6.42)$$

which are represented in the Fig.(14c,17a).

The first contribution corresponds to the expression:

$$H^a_{vacuum} = \frac{1}{L'^2} \sum_{\{\mathbf{k}\},\alpha'_i>2}{}' \Gamma^{\prime\alpha'_1,\alpha'_2,\alpha'_3,\alpha'_4}_{4\, i_1 i_2 i_3 i_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$$
$$< \prod_{j=1\ldots4} \sum_{\tau} \frac{b^{\alpha'_j,i_j}(k_j,\tau_j)}{\sqrt{2w^{\alpha_j}(k_j)}} > . \quad (6.43)$$

In order to evaluate this term we use the following feature:

$$< a^{\alpha,i}_p a^{\beta,j}_q a^{*\gamma,k}_r a^{*\mu,l}_s >=< a^{\alpha,i}_p a^{*\gamma,k}_r >< a^{\beta,j}_q a^{*\mu,l}_s > + < a^{\alpha,i}_p a^{*\mu,l}_s >< a^{\beta,j}_q a^{*\gamma,k}_r > \quad (6.44)$$

which is equivalent to the Wick's expansion. Taking into account this feature and the fact that the averaging generates different contributions from all the possible combinations of the indices, one obtains,

$$H^a_{vacuum} = \frac{12}{L'^2} \sum_{\{\mathbf{k}\},\alpha'_i>2}{}' \Gamma^{\prime\alpha'_1,\alpha'_2,\alpha'_3,\alpha'_3}_{4\, i_1 i_2 i_3 i_4}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)$$
$$\sum_{\tau} \delta(\mathbf{k}_1+\mathbf{k}_2)\delta^{\alpha'_1,\alpha'_2}\delta^{i_1,i_2}(k_1)\frac{\sqrt{\tilde{j}^{\alpha'_1,i_1}(k_2)\tilde{j}^{\alpha'_2,i_2}(k_2)}}{2w^{\alpha'_2(k_2)}}$$
$$\delta(\mathbf{k}_4+\mathbf{k}_3)\delta^{\alpha'_4,\alpha'_3}\delta^{i_4,i_3}(k_3)\frac{\sqrt{\tilde{j}^{\alpha'_3,i_3}(k_3)\tilde{j}^{\alpha'_3,i_4}(k_3)}}{2w^{\alpha'_3(k_3)}}. \quad (6.45)$$

To evaluate this expression analytically, we need to know the analytical form of the transformed four vertex restricted to the sector of fast degrees of freedom. In particular this vertex is only known numerically, thus, one way to state whether these contributions become dependent of the fast angle, is a numerical evaluation of this graph. We will discuss this point in the next chapter.

The second contribution to the vacuum energy are given by the

$$H^b_{vacuum} = \frac{1}{2} < \{\hat{H}_{0,3}, C_{0,3}\} >, \tag{6.46}$$

where

$$\tilde{H}^1_{0,3} = \frac{1}{L'} \sum_{p_j,\beta_j,b_j}^{j=1...3} \Gamma'^{\beta_j}_{3\,b_j}(p_j) \prod_{j=1}^{3} \frac{b^{\beta_j,i_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}}, \tag{6.47}$$

$$C^1_{0,3} = \frac{1}{L'} \sum_{p_j,\beta_j,b_j}^{j=1...3} \frac{i}{-\sum_{i'=1...3} \tau_{i'} w^{\beta_{i'}}(p_{i'})} \Gamma'^{\beta_j}_{3\,b_j}(p_j) \prod_{j=1}^{3} \frac{b^{\beta_j,i_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}}. \tag{6.48}$$

The calculation can be done in two steps: firstly, we look at the Poisson bracket, and then we average over the fast variable.

To calculate the Poisson bracket in the equation (6.46) we consider the identity

$$\{\prod_j b^{\alpha_j,i_j}(k_j,\tau_j), \prod_l b^{\beta_l,i_l}(q_j,\tau_l)\} = \sum_{m,n} \prod_{j\neq m} b^{\alpha_j,i_j}(k_j,\tau_j) \prod_{l\neq n} b^{\beta_l,i_l}(q_l,\tau_l) \{b^{\alpha_m,i_m}(k_m,\tau_m), b^{\beta_n,i_n}(q_n,\tau_n)\} \tag{6.49}$$

and

$$\{b^{\alpha_m,i_m}(k_m,\tau_m), b^{\beta_n,i_n}(q_n,\tau_n)\} = \tau^m \delta^{i_m,i_n} \delta_{k_m,q_n} \delta^{\alpha_m,\beta_n}. \tag{6.50}$$

This expression leads to nine identical contributions which cancel a symmetry factor of the vertex. Considering the non vanishing Poisson bracket, one gets to the expression

$$\{\tilde{H}^1_{0,3}, C^1_{0,3}\} = \frac{1}{L'^2} \sum_{p_j,\beta_j,a_j}^{j=1...3} \sum_{p_j,\beta_j,b_j}^{j=1...3} \epsilon^{ia_1a_2} \epsilon^{ib_1b_2} \Gamma'^{\beta\beta_1\beta_2}_3(p,p_1,p_2) \Gamma'^{\beta\alpha_1\alpha_2}_3(q,q_1,q_2) \delta(\mathbf{p_1}+\mathbf{q_1})$$
$$\frac{\tau^\beta}{2w^\beta(p)} \frac{i}{-\tau w^\beta(q) - \sum_{i'=1,2} \tau_{i'} w^{\beta_{i'}}(p_{i'})} \prod_{j=1}^{2} \frac{b^{\beta_j,a_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}} \prod_{l=1}^{2} \frac{b^{\alpha_l,b_l}(q_l,\tau_l)}{\sqrt{2w^{\alpha_l}(q_l)}}. \tag{6.51}$$

The next step is the averaging over the fast angle. One finds two possible contributions: the first is obtained when we average the external fast lines in the same vertex, and the second is derived by averaging over the fast lines that connect the two different vertices of $H_1$ and $C_1$. The first contribution vanishes because the transfer momenta between the vertices is zero. The contribution of the second graph allows, finally, to state the vacuum value to the $H^b_{vacuum}$, which reads

$$H^b_{vacuum} = \frac{9*4}{L^2} \sum_{p_j,\beta_j,a_j}^{j=1,2} \sum_{p_j,\beta_j,b_j}^{j=1,2} \epsilon^{i_1i_2} \epsilon^{j_1j_2} \frac{i\,\Gamma'^{\beta\beta_1\beta_2}_3(p,p_1,p_2) \Gamma'^{\beta\beta_1\beta_2}_3(q,q_1,q_2)}{-\tau^\beta w^\beta(q) - \tau^{\beta_1} w^{\beta_1}(q_1) - \tau^{\beta_2} w^{\beta_2}(q_2)} \frac{\tau^\beta}{2w^\beta(p)}$$
$$\sum_\tau \delta(\mathbf{p_1}+\mathbf{q_1}) \delta^{i_1,j_1}(\tau^1 p_1) \frac{\sqrt{\tilde{j}^{\beta_1,a_1}(\tau^1 p_1) \tilde{j}^{\beta_1,a_1}(\tau^1 p_1)}}{2w^{\beta_1}(p_1)}$$
$$\delta(\mathbf{p_2}+\mathbf{q_2}) \delta^{i_2,j_2}(\tau^2 p_2) \frac{\sqrt{\tilde{j}^{\beta_2,i_2}(\tau^2 p_2) \tilde{j}^{\beta_2,j_2}(\tau^2 p_2)}}{2w^{\beta_2(\tau^2 p_2)}}. \tag{6.52}$$

Using the property

$$\epsilon^{i_1i_2} \epsilon^{j_1j_2} = \delta^{i_1j_1} \delta^{i_2j_2} - \delta^{i_1j_2} \delta^{i_2j_1}, \tag{6.53}$$



one can note that this term can be separated into a part which is independent on the variables $(I,\psi)$ and a second part which is a function of these variables. Analogous to the study of the first graph, the analytical form of this contribution is limited to knowing the analytical form of the transformed three vertex in the sector of fast degrees of freedom.

It may be concluded from this discussion that the vacuum energy becomes dependent on the adiabatic invariant $I'^{\alpha,3}(k)$ as the other resonant fast degrees of freedom. Therefore, only those terms with a dependence on these variables are considered in the determination of the fixed points. However, we still need to investigate the structure of the other redefined term of the Hamiltonian, i.e. the two point function of the slow variables, the three point function, etc., in order to state that these terms are dependent on the resonant fast degree of freedom. In this order of approximation, the second order in Perturbation Theory can be concluded that the only relevant effective term is the two point function. Investigation of this point is necessary to conclude whether the fixed point is a function of the slow variables or not. In order to solve this problem, we will now study the correction to the two point function.

### 6.2.2 Two vertex function.

In this part, let us look at the free part $H_{slow}$ and all those term in the $\bar{H}_2$ where two of the external legs are slow. It takes the form:

$$H^{eff\,c}_0 = \frac{1}{L^2} \sum_{\beta,\alpha=1,2} \{\sum_{k,a,b} (p'_{-k}{}^{a,\alpha} p'^{a,\alpha}_k + w^\alpha(k)^2 u'_{-k}{}^{a,\alpha} u'^{a,\alpha}_k) \delta^{ab} \delta^{\alpha,\beta} +$$
$$g^2 (u'_{-k}{}^{a,\alpha} \Pi^{ab,\alpha\beta}(k) u'^{b,\beta}_k) \}, \tag{6.54}$$

where the second order contribution to the two point function are contained in the term $\Pi^{ab,\alpha\beta}(k)$ *and the diagrammatical representation are shown in Fig. 18. Analogous to the* vacuum contribution, the first diagram is obtained from averaging over the four vertex. Here, the relevant contribution comes from the vertex with two fast and two slow variables. The second graph is obtained by taking the operations of the Poisson bracket and then the averaging. In analogy to quantum field theory, this contribution can be understood as the classical redefinition of the self-energy for the slow variables.

To second order, the contributions to the free Hamiltonian are given by:

$$H^2 = < H^2_{2,2} > -\frac{1}{2} < \{H^1_{1,2}, C^1_{1,2}\} > . \tag{6.55}$$

The contribution to the slow 'self-energy' from the first graph is given by:

$$(H^a_\Pi)^{ab,\alpha\beta}(k) = \frac{6}{L'^2} \sum_{\{\mathbf{k}\},i_j}^{\prime} \sum_{\alpha^*_j>2} \sum_\tau \Gamma'^{\alpha,\beta,\alpha^*_3,\alpha^*_4}_{abi_3i_4}(\mathbf{k},-\mathbf{k},\mathbf{k_3},\mathbf{k_4})$$
$$\delta(\mathbf{k_4}+\mathbf{k_3}) \delta^{\alpha^*_3,\alpha^*_4} \delta^{i_4,i_3}(\tau k_3) \frac{\sqrt{\tilde{j}^{\alpha^*_3,i_3}(\tau k_3) \tilde{j}^{\alpha^*_3,i_4}(\tau k_3)}}{2w^{\alpha^*_3}(k_3)}, \tag{6.56}$$

and the second graph is derived from the term

$$H^b_\Pi = \frac{1}{2} < \{\tilde{H}_{1,2}, C_{1,2}\} >, \tag{6.57}$$

where

$$\bar{H}^1_{1,2} = \frac{3}{L'^2}\sum_{\{p,\beta,b\}q,\alpha,a}\Gamma^{\alpha\beta_1\beta_2}_{ab_1b_2}(\mathrm{q},\mathrm{p}_1,\mathrm{p}_2)u^{\alpha a}(q)\prod_{j=1}^{2}\frac{b^{\beta_j,i_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}} \quad (6.58)$$

$$C^1_{1,2} = \frac{3}{L'^2}\sum_{\{p,\beta,b\}q,\alpha,a}\frac{i}{-\sum_{i'=1,2}\tau_{i'}w^{\beta_{i'}}(p_{i'})}\Gamma^{\alpha\beta_1\beta_2}_{ab_1b_2}(\mathrm{q},\mathrm{p}_1,\mathrm{p}_2)u^{\alpha a}(q)\prod_{j=1}^{2}\frac{b^{\beta_j,i_j}(p_j,\tau_j)}{\sqrt{2w^{\beta_j}(p_j)}} \quad (6.59)$$

and the corresponding result is:

$$(H^b_{\Omega})^{ab,\alpha\beta}(k) = \frac{1}{L}\sum_{p_j,\beta_j,a_j}^{j=1,2}\sum_{p_j,\beta_j,b_j}^{j=1,2}\epsilon^{ia_{i2}}\epsilon^{ib_{j2}}\frac{i\,\Gamma'^{\beta\beta_1\beta_2}_3(\mathrm{p},\mathrm{p}_1,\mathrm{p}_2)\Gamma'^{\alpha\beta_1\beta_2}_3(\mathrm{p},\mathrm{q}_1,\mathrm{q}_2)}{-\tau^{\beta_1}w^{\beta_1}(q_1)-\tau^{\beta_2}w^{\beta_2}(q_2)}$$
$$\sum_{\tau}\delta(\mathrm{p}_1+\mathrm{q}_1)\frac{\tau^{\beta}}{2w^{\beta}(p)}\delta(\mathrm{p}_2+\mathrm{q}_2)\delta^{i_2,j_2}(\tau^{\beta_2}p_2)\frac{\sqrt{\tilde{j}^{\beta_2,i_2}(\tau^{\beta_2}p_2)\tilde{j}^{\beta_2,j_2}(\tau^{\beta_2}p_2)}}{2w^{\beta_2}(\tau^{\beta_2}p_2)}, \quad (6.60)$$

where we used the equations (6.49,6.50) in order to perform the Poisson bracket, which leads to four identical contributions, and finally we averaged over the remaining two fast modes. Analogous to the discussion about the vacuum, the identity between the $\epsilon$-product allows for the separation of the term either in a contribution independent on the fast angle and in another with a dependence on them.

### 6.2.3 Fixed Point

In order to investigate the fixed point, let us rewrite the effective Hamiltonian up to second order as

$$\bar{H}^{eff\,c}_{(2)} = H'_{(2)}(I'^{\alpha,3}_k)+\sum_{\beta,\alpha=1,2}\{\sum_{k,a,b}(u'^{\,a,\alpha}_{-k}\,\Pi^{ab,\alpha\beta}(k,(\mathbf{I}',\psi)_k)\,u'^{\,b,\beta}_k)+H_{vacuum}(\mathbf{I}',\psi)_k)\}, \quad (6.61)$$

and the fixed point condition reads:

$$\frac{\delta\bar{H}^{eff\,c}_{(2)}}{\delta\psi^{\alpha,i}_k}=0 \qquad \frac{\delta\bar{H}^{eff\,c}_{(2)}}{\delta I^{\alpha,i}_k}=0 \qquad \alpha=3...10,\ i=1,2. \quad (6.62)$$

For the fixed point condition, only the last term in the equation (6.61) is relevant, which can be rewritten as

$$H^*(\mathbf{I}',\psi)_k) = \sum_{\beta,\alpha=1,2}\{\sum_{k,a,b}u'^{\,a,\alpha}_{-k}u'^{\,b,\beta}_k\{\frac{6}{L'^2}\sum_{\{\mathrm{k}\},i_j}^{'}\sum_{\alpha^*_j>2}\sum_{\tau}\mathrm{I}^{\prime\alpha,\beta,\alpha^*_3,\alpha^*_4}_{abi_3i_4}(\mathrm{k},-\mathrm{k},\mathrm{k}_3,\mathrm{k}_4)*$$
$$\delta(\mathrm{k}_4+\mathrm{k}_3)\delta^{\alpha^*_3,\alpha^*_4}\delta^{i_4,i_3}(\tau k_3)\frac{\sqrt{\tilde{j}^{\alpha^*_3,i_3}(\tau k_3)\tilde{j}^{\alpha^*_3,i_4}(\tau k_3)}}{2w^{\alpha^*_3}(k_3)}-$$
$$\frac{1}{L}\sum_{p_j,\beta_j,a_j}^{j=1,2}\sum_{p_j,\beta_j,b_j}^{j=1,2}\frac{i\,\Gamma'^{\beta\beta_1\beta_2}_3(\mathrm{p},\mathrm{p}_1,\mathrm{p}_2)\Gamma'^{\alpha\beta_1\beta_2}_3(\mathrm{p},\mathrm{q}_1,\mathrm{q}_2)}{-\tau^{\beta_1}w^{\beta_1}(q_1)-\tau^{\beta_2}w^{\beta_2}(q_2)}*$$



$$\sum_{\tau}\delta(\mathrm{p}_1+\mathrm{q}_1)\frac{\tau^{\beta}}{2w^{\beta}(p)}\delta(\mathrm{p}_2+\mathrm{q}_2)\delta^{a,b}(\tau^{\beta_2}p_2)\frac{\sqrt{\tilde{j}^{\beta_2,a}(\tau^{\beta_2}p_2)\tilde{j}^{\beta_2,b}(\tau^{\beta_2}p_2)}}{2w^{\beta_2}(\tau^{\beta_2}p_2)}\}+$$
$$\frac{12}{L'}\sum_{\{\mathrm{k}\},\alpha'_i>2}^{'}\Gamma'^{\alpha'_1,\alpha'_2,\alpha'_3,\alpha'_3}_{4\,i_1i_2i_3i_4}(\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3,\mathrm{k}_4)*$$
$$\sum_{\tau}\delta(\mathrm{k}_1+\mathrm{k}_2)\delta^{\alpha'_1,\alpha'_2}\delta^{i_1,i_2}(\tau k_1)\frac{\sqrt{\tilde{j}^{\alpha'_1,i_1}(\tau k_2)\tilde{j}^{\alpha'_2,i_2}(\tau k_2)}}{2w^{\alpha'_2(k_2)}}*$$
$$\delta(\mathrm{k}_4+\mathrm{k}_3)\delta^{\alpha'_4,\alpha'_3}\delta^{i_4,i_3}(\tau k_3)\frac{\sqrt{\tilde{j}^{\alpha'_3,i_3}(\tau k_3)\tilde{j}^{\alpha'_3,i_4}(\tau k_3)}}{2w^{\alpha'_3(k_3)}}-$$
$$\sum_{p_j,\beta_j,a_j}^{j=1,2}\sum_{p_j,\beta_j,b_j}^{j=1,2}\frac{i\,\Gamma'^{\beta\beta_1\beta_2}_3(p,\mathrm{p}_1,\mathrm{p}_2)\Gamma'^{\beta\beta_1\beta_2}_3(q,\mathrm{q}_1,\mathrm{q}_2)}{-\tau^{\beta}w^{\beta}(q)-\tau^{\beta_1}w^{\beta_1}(q_1)-\tau^{\beta_2}w^{\beta_2}(q_2)}\frac{\tau^{\beta}}{2w^{\beta}(p)}*$$
$$\sum_{\tau}\delta(\mathrm{p}_1+\mathrm{q}_1)\delta^{j_2,j_1}(\tau^1p_1)\frac{\sqrt{\tilde{j}^{\beta_1,j_2}(\tau^1p_1)\tilde{j}^{\beta_1,j_1}(\tau^1p_1)}}{2w^{\beta_1(p_1)}}*$$
$$\delta(\mathrm{p}_2+\mathrm{q}_2)\delta^{j_1,j_2}(\tau^2p_2)\frac{\sqrt{\tilde{j}^{\beta_2,j_1}(\tau^2p_2)\tilde{j}^{\beta_2,j_2}(\tau^2p_2)}}{2w^{\beta_2}(\tau^2p_2)}. \quad (6.63)$$

Finally, it can be noted that the fixed point condition is dependent on the reduced vertices, therefore, we need to know the matrix transformation $R$. Moreover, the fixed point is a function of the slow variables. That is expected because in the adiabatic approximation these modes are slow variables in comparison to the fast variables, i.e. they are constant in the time scale, where the perturbation theory is valid.

## 6.3 Three vertex function

In order to study the correction for the three vertex function to one loop, we need to get to third order in Adiabatic Perturbation Theory. In the Lie algebraic method, the Hamiltonian up to third order takes the form:

$$\tilde{H}_3 = < H_3 >_f + \frac{1}{2} < L^f_1\tilde{H}_2 - L^f_2\tilde{H}_1 >_f + \frac{1}{12} < L^f_1L^f_1\tilde{H}_1 >_f + \frac{1}{2} < L^s_1\tilde{H}_1 >_f, \quad (6.64)$$

and the relevant contributions to one loop to the coupling constant are derived from the term

$$H_3 = \frac{1}{2} < L^f_1\tilde{H}_2 - L^f_2\tilde{H}_1 >_f + \frac{1}{12} < L^f_1L^f_1\tilde{H}_1 >_f . \quad (6.65)$$

Therefore, we need to calculate the term $C_2$ which is straightforward, but in this case, it is more complicated because it is defined in terms of $\tilde{H}^s_2$. Moreover, in this order of approximation, we obtained graphs with zero, and two slow external lines, which are corrections to the two loop for the vacuum energy, and to the two point function. However, to be consistent with our treatment of the coupling constant, this two loop contributions will not be retained.

The correction to the coupling constant can be represented by the graphs in the Figure 19. It can be noted that these diagrams are similar to the case of the quantum field theory [75].

The analytical expressions for the first graph derived of the term $< L_1^f \tilde{H}_2 - L_2^f \tilde{H}_1 >_f$ is:

$$H_{ijk}^{\alpha_1,\alpha_2,\alpha_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)_a = \frac{4}{2L'^6}\{\sum_{p_j,\beta_j,a_j}^{j=1,2}\sum_{p_j,\beta_j,b_j}^{j=1,2}\frac{i\,\Gamma'^{\alpha_1,\beta_1\beta_2}(\mathbf{k},\mathbf{p}_1,\mathbf{p}_2)(\mathbf{k}_2,\mathbf{k}_3,\mathbf{q}_1,\mathbf{q}_2)\epsilon^{iac}}{-\tau^{\beta_1}w^{\beta_1}(p_1)-\tau^{\beta_2}w^{\beta_2}(p_2)}\frac{\tau^{\beta_1}}{2w^{\beta_1}(p_1)} *$$
$$\sum_{\tau^{\beta_2}}\delta(\mathbf{p}_1+\mathbf{q}_1)\delta(\mathbf{p}_2+\mathbf{q}_2)\delta^{bc}(\tau^{\beta_2}p_2)\frac{\sqrt{\tilde{j}^{\beta_2,b}(\tau^{\beta_2}p_2)\tilde{j}^{\beta_2,c}(\tau^{\beta_2}p_2)}}{2w^{\beta_2}(\tau^{\beta_2}p_2)} *$$
$$\left[\Gamma'^{\alpha_2,\alpha_3,\beta_1\beta_2}_{jkab}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{q}_1,\mathbf{q}_2+\tilde{H}^{*\alpha_2,\alpha_3,\beta_1\beta_2}_{jkab}(\mathbf{k}_2,\mathbf{k}_3,\mathbf{q}_1,\mathbf{q}_2)\right]\}, \qquad (6.66)$$

where $\tilde{H}^{*\alpha_2,\alpha_3,\beta_1\beta_2}_{jkab}$ are the effective vertices of the effective Hamiltonian $\tilde{H}^*$, and for the second graphs (derived from the term $< L_1^f L_1^f \tilde{H}_1 >_f$) one obtains:

$$H_{ijk}^{\alpha_1,\alpha_2,\alpha_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)_a = \frac{4*2}{12L'^6}\sum_{p_j,\beta_j,a_j}\sum_{q_j,\alpha_j,b_j}\sum_{k_j,\gamma_j,c_j}\epsilon^{iac}\Gamma'^{\alpha_1,\beta_1\beta_2}(\mathbf{k}_1,\mathbf{p}_1,\mathbf{p}_2)$$
$$\{\frac{i\,\Gamma'^{\alpha_2,\beta_1\beta_3}(\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)}{(-\tau^{\beta_1}w^{\beta_1}(p_1)-\tau^{\beta_3}w^{\beta_3}(q_2))}\frac{i\,\Gamma'^{\alpha_3,\beta_2\beta_3}(\mathbf{k}_3,\mathbf{r}_1,\mathbf{r}_2)}{(-\tau^{\beta_3}w^{\beta_3}(q_2)-\tau^{\beta_2}w^{\beta_2}(p_2))}$$
$$\delta(\mathbf{p}_1+\mathbf{q}_1)\delta(\mathbf{p}_2+\mathbf{r}_1)\delta(\mathbf{r}_2+\mathbf{q}_2)\frac{\tau^{\beta_1}\tau^{\beta_2}}{2w^{\beta_1}(p_1)2w^{\beta_2}(p_2)2w^{\beta_3}(q_2)}$$
$$\epsilon^{jab}\epsilon^{kcd}\delta^{bd}(\tau^{\beta_3}q_2)\sqrt{\tilde{j}^{\beta_3,b}(\tau^{\beta_3}q_2)\tilde{j}^{\beta_3,c}(\tau^{\beta_3}q_2)}+$$
$$\frac{i\,\Gamma'^{\alpha_1,\beta_1\beta_3}(\mathbf{k}_2,\mathbf{q}_1,\mathbf{q}_2)}{(-\tau^{\beta_3}w^{\beta_3}(q_2)-\tau^{\beta_1}w^{\beta_1}(p_1))}\frac{i\,\Gamma'^{\alpha_1,\beta_3\beta_2}(\mathbf{k}_3,\mathbf{r}_1,\mathbf{r}_2)}{(-\tau^{\beta_3}w^{\beta_3}(p_1)-\tau^{\beta_2}w^{\beta_2}(p_2))}$$
$$\delta(\mathbf{p}_1+\mathbf{q}_1)\delta(\mathbf{q}_2+\mathbf{r}_1)\delta(\mathbf{r}_2+\mathbf{p}_2)\frac{\tau^{\beta_1}\tau^{\beta_3}}{2w^{\beta_1}(p_1)2w^{\beta_2}(p_2)2w^{\beta_3}(q_2)}$$
$$\epsilon^{jab}\epsilon^{kbd}\delta^{dc}(\tau^{\beta_2}p_2)\sqrt{\tilde{j}^{\beta_2,d}(\tau^{\beta_2}p_2)\tilde{j}^{\beta_2,c}(\tau^{\beta_2}p_2)}\}. \qquad (6.67)$$



# Chapter 7

# Approach for the calculation of the behavior of the coupling constant

In this chapter, we discuss an approach to estimate the behavior of the coupling constant under a change of the number of degrees of freedom after a RGT. Since the definition of the Kolmogorov entropy is fundamental for this approach, we discuss this parameter in the first section. In the second section we present our approach and finally we discuss a numerical implementation.

## 7.1 Lyapunov exponents and Kolmogorov entropy.

The theory of Lyapunov exponents [78] was applied to characterize stochastic trajectories by Osledec [79]. The connection between Lyapunov exponents and exponential divergence was established by Benettin et. al [80] and by Piesen [81]. To define the Lyapunov exponents let us consider the flow $\mathbf{x}(t)$ generated by an autonomous first-order system

$$\frac{\delta x_i}{\delta t} = V_i(\mathbf{x}) \qquad i = 1,....n \qquad (7.1)$$

where m is the dimension of the $M$-dimensional phase space, which posses a norm $||..||$. Considering a trajectory in this phase space and a nearby trajectory with initial condition $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{\Delta x}_0$, these evolve with the time, yielding the tangent vector $\mathbf{\Delta x}(\mathbf{0},\mathbf{t})$. One defines the magnitude of the displacement as

$$d(\mathbf{x}_0,t) = ||\mathbf{\Delta x}(\mathbf{0},\mathbf{t})|| \qquad (7.2)$$

Defining $\mathbf{w} = \mathbf{\Delta x}(\mathbf{0},\mathbf{t})$, the time evolution for w, is found by linearizing (7.1) to obtain

$$\frac{\delta \mathbf{w}}{\delta t} = \mathbf{M}(\mathbf{x}(\mathbf{t}))\cdot\mathbf{w} \qquad (7.3)$$

where

$$\mathbf{M} = \frac{\delta \mathbf{M}(\mathbf{x}(\mathbf{t}))}{\delta \mathbf{x}} \qquad (7.4)$$

The rate of the exponential growth or decrease of the two initially close trajectories

$$\lambda(\mathbf{x},\mathbf{w}) = lim_{t\to\infty, d(0)\to 0}\frac{1}{t} ln\frac{d(\mathbf{x}_0,t)}{d(\mathbf{x}_0,0)} \tag{7.5}$$

It can be shown that $\lambda$ is finite. Furthermore, there is an $M$-dimensional basis $\hat{\mathbf{e}}_i$ of $\mathbf{w}$ such that the Lyapunov exponent is associated with a determined direction i.e.,

$$\lambda_i(\mathbf{x}) = \lambda(\mathbf{x},\hat{\mathbf{e}}_i) \tag{7.6}$$

For any flow generated by ( 7.1), at least one of the Lyapunov exponents must vanish because in the direction along the flow, $\mathbf{w}$ grows only linearly with the time. For a Hamiltonian flow the exponents satisfy the relation [82]

$$\lambda_i = -\lambda_{2N-i+1} \tag{7.7}$$

where $2N = m$ with $N$ the number of degrees of freedom and in this case, at least two Lyapunov exponents must vanish because on the constant energy surface $M = 2N - 1$.

Finally, it was established by Piesen [81] that there is a relation between the Lyapunov characteristic exponents and the Kolmogorov entropy (KS entropy) [77], which is defined studying the evolution in the time of a partition of the phase space. This connection reads:

$$h = \int_M \sum_{\lambda_i(\mathbf{x})>0} \lambda_i(\mathbf{x}) d\mu, \tag{7.8}$$

where the sum is over all positive Lyapunov exponents and the integral is over a specific region of the phase space $M$, and $d\mu$ denotes an invariant volume element of the energy surface. The KS entropy can be understood either as a measure to a single region of connected stochasticity or to the entire (compact) region of the phase space [80]. It can be noted that $\lambda_i(x)$ is not usually the same constant for all regions, each isolated region of stochasticity generally has a different value of $\lambda_i(x)$.

## 7.2 Behavior of the Coupling Constant.

Here, we will discuss a numerical estimation of the evolution of the coupling constant with the number of degrees of freedom, when we perform a step of the RGT. This is based on the Monte Carlo Renormalization Group [85, ?].

Let us first discuss briefly the main feature of our approach. The idea of the classical RS-RGT is: define a transformation by averaging over the fast degrees of freedom of a Hamiltonian defined on an $a$-lattice, to another one defined on other La-lattice. The basis assumption is that the "long distance properties" of the system are not affected because we are averaging over modes with small long wave. Moreover, the general feature of this classical system are not modified. For a given Hamiltonian whose perturbation terms are described by a few number of small coupling constant, one can state that the application of a step of the RGT will in general lead to a quite complicated effective Hamiltonian which can be understood as a new Hamiltonian with new perturbed terms characterized by a new set of coupling constant. Under



this transformation, it can be expected that the new coupling constants are a function of the older ones.

This transformation can be iterated repeatedly to generate a sequence of effective Hamiltonians defined in larger lattice spacing. These Hamiltonian, in general, do not have the original perturbation terms. However, it is expected that when such a transformation are iterated many times, one finds a fixed point in this transformation, namely an effective Hamiltonian $H^*$ which reproduces itself under a new transformation. Moreover, this transformation let us know the new coupling constant on the new blocked lattice. This feature allows us to study the behavior of a specific coupling constant with the RGT.

Therefore, we think that the following procedure allows the determination of the behavior of the coupling constant for a classical system. We start with a Hamiltonian ($H$) defined on a lattice of spacing $(La)^n$ at a specific value of the energy ($E$) and the coupling constants $(\{g_1,...,g_k\})$. Now, one can iterate the RGT $n$-time, and finally finds an effective $H^n$. One calculates the Kolmogorov entropy, which is a function of the the energy ($E$) and the couplings constant $(\{g_1,...,g_k\})$. Since a blocking step in this classical formulation is only a set of canonical transformations at a constant energy and considering the definition of the entropy, one can proof that this parameter is invariant under a RGT. This point is fundamental for our approach. This procedure is then repeated for a new effective Hamiltonian ($H'$) where the coupling constants $(\{g'_1,...,g'_p\})$ and the energy ($E'$) assume new values on a lattice with a spacing $(La)^{n-1}$. Finally, one calculates the Kolmogorov entropy Let us discuss here the new effective Hamiltonian ($H'$). In general, it must be chosen from a form close to the renormalized Hamiltonian $H^1$, so that after $(n-1)$ blocking steps $H'^{n-1}$ takes the same analytic form that the $H^n$. At this point, one demands that the different entropies take the same values. However, this is only possible when the parameters of the new effective Hamiltonian reach a determined values: i.e. $\{g^*_1,...,g^*_p\}$. In other words, one adjusts the coupling $\{g'_1,...,g'_p\}$ such that one obtains agreement with the entropy. In this case, one can state that the values of the parameters $\{g^*_1,...,g^*_p\}$ corresponds to the effective Hamiltonian $H^1$ on the blocked lattice. Since the topological entropy is a function of the energy, we must calculate the energy $E'$, such that it corresponds to the reduced energy of the effective Hamiltonian $H^1$ derived after one blocking step. We think that this point is important in order to require so that the entropies take the same values at these determined values of the coupling constant.

In a classical gauge Hamiltonian, one considers only a single coupling constant $g$. Therefore, we would like to obtain the behavior of this simple parameter with the RGT or equivalently with the number of degrees of freedom. As a measure of this behavior, we define the quantity

$$\Delta\beta = (g(La) - g(a)), \tag{7.9}$$

where $g(a)$ is the starting coupling and $g(La)$ is calculated using the above mentioned procedure. Since, we have only interest in the general evolution, i.e. in the sign of this $\Delta\beta$, we can consider that the $H'$ takes the same analytic form as that of the starting Hamiltonian.

In analogy to quantum field theory, one can relate $\Delta\beta$ with the beta function of a gauge theory. In particular, one finds in this case that $\Delta\beta$ is positive, which corresponds to asymptotic freedom.

We can not say at the moment, that if $\Delta\beta$ calculated in this classical formulation, has the same behavior as in the quantum formulation. We will perform this numerical investigation in

a future work.

## 7.3 Selection of chaos parameter

An analytical proof of the existence of a region in the phase space where regular solution can be found, is very complicated for a Hamiltonian system with many degrees of freedom. In general, in the investigation of the stability of a region of the phase space one uses a numerical study of the solution of the equation of motion. In order to state, whether a system is integrable or not, one must determine the criteria that will allow us to conclude about this. Presently, there are some criteria for the determination of the stability of a system, which enable us to give conclusion about the integrability properties of the systems. Criteria such as those based on the energy distribution between the harmonics, which are applied to systems with finite number of oscillatoring points, an example of this are those systems with coupled anharmonic oscillators. Here, the equations of motion are numerically integrated and later, the energy distribution in the different harmonics is considered. If there is a periodic energy transfer between several harmonics, then the motion lies on the invariant tori [76], but if the energy is distributed uniformly among the harmonics, then the motion is in an ergodic layer [20]. This criteria is known as Fermi-Pasta-Ulam- approach.

The sign of the maximal Lyapunov exponent is frequently used as another criterium to distinguish between regular systems and chaos, because they ($\lambda_i(x)$) characterize the rate of exponential divergence of close trajectories, i.e. if $\lambda_i(x)$ is zero, then the motion is regular, for $\lambda_i(x) > 0$ then the motion is irregular (stochastic or chaotic.).

Here, we will choose the last indicator to state the stability of classical solution of a lattice gauge theory. In order to find a chaotic motion, we will consider the evolution of infinitesimally separated gauge field configuration, and calculate the corresponding maximal Lyapunov exponent. If we find divergence trajectories, we can then identify the maximal Lyapunov exponent. Moreover, this result allows the evaluation of the Kolmogorov entropy. Therefore, it is important to study first, how the configuration can be selected in order to calculate the Lyapunov exponents, and how the Lyapunov exponents can be defined such that they are gauge invariant. We think that this gauge invariant definition of the Lyapunov exponents is very important in order to state if a set of solutions are independent or they are related by a gauge transformation.

In ref. [72] the maximal Lyapunov exponent has been calculated for a SU(2)-lattice gauge theory. The result was that all the solutions have positive Lyapunov exponent,which is not surprising, because instead of studying the gauge field dynamics in the vicinity of arbitrarily selected configuration, the authors have investigated the dynamical behaviour of random field configuration, corresponding to gauge field selected from a thermal bad. The disadvantage of this is that their configurations are generated by using an algorithmus which is ergodic (on the energy surface.). We think, that in order to state the existence of a stable solution, one has to generate or select the gauge configuration from an ensemble which is non-ergodic. i.e one must use algorithmus which considers the Hamiltonian dynamical evolution to generate gauge configuration.



### 7.3.1 Lyapunov exponents for Gauge theory

The goal of this section is to state a strategy in order to verify our result numerically. From the start, we have assumed the existence of a regular region in the phase space. Now, we will show that this supposition is true. In order to obtain this result, we will calculate the Lyapunov exponents for the Hamiltonian of a gauge theory. This is done by determining the evolution of two nearly trajectories. Note that in this case, one must use the total Hamiltonian in order to study this evolution.

Now, we will introduce a gauge invariant definition of the Lyapunov exponents as

$$\lambda(\{U_{n,\mu}\}) = lim_{t\to\infty, d(0)\to 0} \frac{1}{t} ln \frac{d(\{U_{n,\mu}\}_0, t)}{d(\{U_{n,\mu}\}_0, 0)} \tag{7.10}$$

where the distance is defined by considering the topology of the manifold of the group parameters [?] as

$$d(\{U_{n,\mu}\}_0, t) = |Tr(U_{p,\mu}) - Tr(U_{p,\mu})'| \tag{7.11}$$

where $|a| = \frac{1}{N}\sqrt{\sum_1^N a_i^2}$ and $U_{p,\mu}$ denotes a plaquette.

The quantity $\lambda$ can be calculated by numerical integration of the equation of motion. This can be achieved by different numerical methods, but in order to verify that in each step the numerical evaluation is correct, one must study the evolution of the constant of motion. In this case, we have three parameter which must be constant in each step: the total energy, the property $UU^\dagger = 1$ and finally the Gauss' law. Therefore, the integration of the equation of motion considering these conditions allows us to calculate the maximal Lyapunov exponent and the existence of $\lambda \to 0$ is a clear indication of a stable region in the phase space. The perturbation theory plays in these regions an important role to describe the regular solutions. The calculation of the total spectrum of Lyapunov exponents for a gauge theory is still a mather for investigation [72].

The above discussion helps us to state the existence of a regular solution and to find them numerically. In the context of the Perturbation Theory, these solutions correspond to those obtained by averaging, and the application of the inverse of Lie algebraic transformation made possible the obtainment of the unperturbed trajectory. Finally, the evaluation of the action-angle variables from the numerical solution enable us to state the general form of the effective vertices under the RT-transformation.

# Chapter 8

# Conclusions and Outlook

In this thesis, we have given a formulation of the renormalization group transformation for two different classical systems. Firstly, the application of our approach to the $\lambda\phi^4$-scalar theory enabled us to find a detailed description of the RG-transformation. In this case, this formulation consists mainly of averaging over the set of fast degrees of freedom. The result is an effective Hamiltonian for the remaining slow variables. To be able to compare this new Hamiltonian with the starting one, some other canonical transformations are necessary. These are called mass and wave function renormalization, and the coupling renormalization. It turned out that the details of all these steps are very similar to the analogous RG-transformation in $\lambda\phi^4$ field theory. Finally, the study of the conditions on the parameter, in order to bring the new Hamiltonian in the same form as that of the original one, allowed us to find similar fixed points condition and the scaling law of the variables $\phi_k$. This result is consistent with the result obtained by Bartels and Chang.

In the second part of this thesis, we have extended the idea of the formulation of a classical RGT developed for scalar classical theory to the case of a classical gauge theory. This formulation is more complicated than the scalar case due to the presence of the gauge symmetry and the resonance. The application of the "block spin" RG-transformation idea and the averaging over fast degrees of freedom allows us to define a set of canonical transformation which constitute our renormalization group transformation for classical systems. The analytic implementation of this program can not be concluded. We think that through a numerical investigation, we will be able to conclude this program, but this point is left for further investigation.

The formulation of the RGT for Classical Gauge Theory enabled us to define a series of canonical transformation allowing us to distinguish between gauge and physical degrees of freedom. Finally we eliminate these gauge degrees of freedom by a "fixing gauge" in analogy to the quantum formulation. The existence of a gap in the frequencies is stated introducing the normal modes, which lead to the separation into slow and fast degrees of freedom. Considering this feature we perform the Secular Adiabatic Perturbation Theory to one loop, or equivalent to third order in the expansion parameter. To first order (tree level), we demonstrated that the effective Hamiltonian defined on the blocked lattice, particularly the two and three vertex function, was reduced to the original one. This calculation was only done in the approximation of low-momenta, where one can find analytically the normal mode transformation matrix. The expressions of the corrections on the two point and the three point functions were calculated to



one loop (second order in adiabatic perturbation theory). The result was, that these vertices are dependent either on the action variables for the fast modes and on the transformation matrix for the normal modes restricted in the indices of the fast modes. This matrix can be calculated only numerically. Due to this complication, the last point of this program, can not be performed analytically, namely, the calculation of the elliptic point and the averaging of the fast variables defined around this fixed point.

Therefore, we think that in order to conclude something about the renormalized parameter, the structure of the different $n$-point functions and the nature of the fixed point, a numerical study of the solution of the equation of motions is fundamental. In this case, it is expected that after the evaluation of the vertices and the expansion around the elliptic fixed point, for example, the two point function $\Pi^{\alpha,\beta}_{i,j}$ must be separated as $\Pi^{\alpha,\beta}\delta^{i,j}$. Considering the gauge invariance of the system, the result should be that the two point function admits an eigenvalue zero, and in particular that

$$\Pi^{\alpha,\beta} \to w^2(k)^2\delta^{\alpha,\beta}\delta^{\alpha,2}Z(k), \tag{8.1}$$

where $w^2(k)$ is the frequency of the slow degree of freedom and $Z(k)$ should be the wave function constant.

In the context of a numerical solution of the equation of motion which must be consistent with the different constraints in this system, we describe an approach in order to study the behavior of the coupling constant with the number of the degrees of freedom. Here, considering a gauge invariant definition of the entropy we suggest a procedure in order to determinate the evolution of the coupling constant with the blocking step. In this case, this quantity can consider as gauge invariant expectation values which describes the global feature of the phase space for statement of the integrability of a classical system.

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# Appendix A

# Vertex of the four point function for scalar system.

Let us consider the starting vertex defined by:

$$H_{int} = \frac{\lambda}{4}\sum_{\vec{l}}\phi_{\vec{l}}^4, \tag{A.1}$$

$$= \frac{\lambda}{4L^3}\sum_{k_i}\delta(\sum_i k_i)\prod_{l=1}^{4}\phi_{k_l}. \tag{A.2}$$

Introducing in the Fourier space the decomposition $p = q + \pi r$ where $-\frac{\pi}{a} < p < \frac{\pi}{a}\ mod(2\pi)$ and $-\frac{\pi}{2a} < q < \frac{\pi}{2a}\ mod(\pi)$ and with $r = 0, 1$ one can shown that the vertex is separated in terms of the number of the variables $\phi_{k_i}$ and $\phi_{k_i+\pi}$.

Considering the relations defined by equations (3.36,3.37) the interaction term takes the form

$$H_{int} = \frac{\lambda}{4L^3}\sum_{k_1,k_2,k_3,k_4}\sum_{i=0}^{4} H_{4-i,i}(\Phi,\Psi|k_1,k_2,k_3,k_4), \tag{A.3}$$

where:

$$H_{4-i,i}(\Phi,\Psi|k_1,k_2,k_3,k_4) = f_{4-i,i}(k_1,k_2,k_3,k_4)\prod_{l=1}^{i}\Phi_{k_l}\prod_{j=i+1}^{4}\Psi_{k_j} \qquad i = 1,.,3, \tag{A.4}$$

$$H_{4,0}(\Phi,\Psi|k_1,k_2,k_3,k_4) = f_{4,0}(k_1,k_2,k_3,k_4)\prod_{l=1}^{4}\Phi_{k_l}, \tag{A.5}$$

$$H_{0,4}(\Phi,\Psi|k_1,k_2,k_3,k_4) = f_{0,4}(k_1,k_2,k_3,k_4)\prod_{l=1}^{4}\Psi_{k_l}. \tag{A.6}$$

In order to obtain the expression of the terms $f_{i,j}$, let us introduce the following notation:

$$a(k) = cos\frac{k}{2} \qquad c(k) = a(k+\pi) = -sin\frac{k}{2} \tag{A.7}$$

$$b(k) = -i\,sin\frac{k}{2} \qquad d(k) = b(k+\pi) = -i\,cos\frac{k}{2} \tag{A.8}$$

Then, they are given as:
four slow external lines;

$$\begin{aligned} f_{4,0}(k_1,k_2,k_3,k_4) &= \{\delta(\sum_i k_i)\prod_{l=1}^{4} a(k_l) + \delta(\sum_i k_i + 4\pi)\prod_{l=1}^{4} c(k_l) + \\ &\frac{1}{6}\delta(\sum_i k_i + 2\pi)\,[a(k_1)a(k_2)c(k_3)c(k_4) + permutations] + \\ &\frac{1}{4}\delta(\sum_i k_i + 3\pi)\left[a(k_1)\prod_{l=2}^{4} c(k_l) + permutations\right] + \\ &\frac{1}{4}\delta(\sum_i k_i + \pi)\left[c(k_1)\prod_{l=2}^{4} a(k_l) + permutations\right]\}, \end{aligned} \tag{A.9}$$

two slow and two fast external lines;

$$\begin{aligned} f_{2,2}(k_1,k_2,k_3,k_4) &= \frac{1}{6}\delta(\sum_i k_i)\,[a(k_1)a(k_2)b(k_3)b(k_4) + permutations] + \\ &\frac{1}{6}\delta(\sum_i k_i + 4\pi)\,[c(k_1)c(k_2)d(k_3)d(k_4) + permutations] + \\ &\frac{1}{6}\delta(\sum_i k_i + 2\pi)\,[a(k_1)a(k_2)d(k_3)d(k_4) + c(k_1)c(k_2)b(k_3)b(k_4) + permutations] + \\ &\frac{1}{6}\delta(\sum_i k_i + 3\pi)\,[a(k_1)c(k_2)d(k_3)d(k_4) + c(k_1)c(k_2)b(k_3)d(k_4) + permutations] + \\ &\frac{1}{6}\delta(\sum_i k_i + \pi)\,[a(k_1)a(k_2)b(k_3)d(k_4) + a(k_1)c(k_2)b(k_3)b(k_4) + permutations]\}. \end{aligned} \tag{A.10}$$

three slow and one fast external lines;

$$\begin{aligned} f_{3,1}(k_1,k_2,k_3,k_4) &= \frac{1}{4}\delta(\sum_i k_i)\,[a(k_1)a(k_2)a(k_3)b(k_4) + permutations] + \\ &\frac{1}{4}\delta(\sum_i k_i + 4\pi)\,[c(k_1)c(k_2)c(k_3)d(k_4) + permutations] + \\ &\frac{1}{12}\delta(\sum_i k_i + 2\pi)\,[a(k_1)c(k_2)c(k_3)b(k_4) + c(k_1)a(k_2)a(k_3)d(k_4) + permutations] + \\ &\frac{1}{12}\delta(\sum_i k_i + 3\pi)\,[a(k_1)c(k_2)c(k_3)d(k_4) + c(k_1)c(k_2)c(k_3)d(k_4) + permutations] + \\ &\frac{1}{12}\delta(\sum_i k_i + \pi)\,[a(k_1)a(k_2)a(k_3)d(k_4) + a(k_1)a(k_2)c(k_3)b(k_4) + permutations]\}. \end{aligned} \tag{A.11}$$

three fast and one slow external lines;

$$\begin{aligned} f_{1,3}(k_1,k_2,k_3,k_4) &= \{\frac{1}{4}\delta(\sum_i k_i)\,[a(k_1)b(k_2)b(k_3)b(k_4) + permutations] + \end{aligned}$$



$$\begin{aligned} &\frac{1}{4}\delta(\sum_i k_i + 4\pi)\,[c(k_1)d(k_2)d(k_3)d(k_4) + permutations] + \\ &\frac{1}{12}\delta(\sum_i k_i + 2\pi)\,[a(k_1)b(k_2)d(k_3)d(k_4) + c(k_1)d(k_2)b(k_3)b(k_4) + permutations] + \\ &\frac{1}{12}\delta(\sum_i k_i + 3\pi)\,[c(k_1)d(k_2)d(k_3)b(k_4) + a(k_1)d(k_2)d(k_3)d(k_4) + permutations] + \\ &\frac{1}{12}\delta(\sum_i k_i + \pi)\,[c(k_1)b(k_2)b(k_3)b(k_4) + a(k_1)b(k_2)b(k_3)d(k_4) + permutations]\}. \end{aligned} \tag{A.12}$$

# Appendix B

# Coupling constant to one loop

The four point vertex have contributions derived from the averaging of the $H_2$. They are represented in Fig.3a and Fig.3b and depend upon the external momenta $k_1, ...k_4$. In particular, the contribution of each graph is obtained by:

$$H_a = (\frac{\lambda}{4L^D})^2 \sum_{p,q}' \sum_{k_1,..k_4}' \sum_{i=1}^{4} \sum_{\{\tau_i\}} f_{2,2}(k_1,k_2,p,q) \prod_{l=1}^{2} \frac{c_\Phi(k_l,\tau)}{\sqrt{2\omega_\Psi}} \frac{\tau_p}{\sqrt{2\omega_\Psi(p)}} \frac{I(\tau_q q)}{\sqrt{2\omega_\Psi(q)}} \frac{-i}{\sum_3^4 \omega_{\Phi,i}\tau_i + \sum_{p,q} \omega_{\Psi,j}\tau_j} f_{2,2}(k_3,k_4,-p,-q) \prod_{l=3}^{4} \frac{c_\Phi(k_l,\tau)}{\sqrt{2\omega_\Phi}}, \quad \text{(B.1)}$$

and

$$H_b = (\frac{\lambda}{4L^D})^2 \sum_{p,q}' \sum_{k_1,..k_4}' \sum_{i=1}^{4} \sum_{\{\tau_i\}} f_{3,1}(k_1,k_2,p,q) \prod_{l=1}^{2} \frac{c_\Phi(k_l,\tau)}{\sqrt{2\omega_\Phi}} \frac{\tau_p}{\sqrt{2\omega_\Phi(p)}} \frac{I(\tau_q q)}{\sqrt{2\omega_\Psi(q)}} \frac{-i}{\sum_3^4 \omega_{\Phi,i}\tau_i + \sum_{p,q} \omega_{\Psi,j}\tau_j} f_{3,1}(k_3,k_4,-p,-q) \prod_{l=3}^{4} \frac{c_\Phi(k_l,\tau)}{\sqrt{2\omega_\Phi}}. \quad \text{(B.2)}$$

In order to find the structure of the four point function represented in Fig.4, one must consider all the different time ordering of the external lines in Fig. 2b. We introduce the notation $k_i$ ($q_i$) which is associated with the external (internal) momenta. In particular, one notes that the sum over all the time ordering cannot be rewritten in the same form as $H_1$ with $\lambda$ being replaced by some more complicated one loop expression because the time ordering of the external lines is different. In Fig.4, we denote all the different contribution to the vertex, which can be grouped into three sets with the same amplitudes. The amplitudes from these diagrams are:

$$H_4^a = (\frac{\lambda}{4L^D})^2 \sum_{p,q}' \sum_{\{k_i\}}' \sum_{i=1}^{4} \sum_{\{\tau_i\}} \prod_{l=1}^{4} \frac{1}{\sqrt{2\omega_\Phi(k_i)}} \{ \frac{f_{3,1}(k_1,k_2,p,q)}{\sqrt{2\omega_\Phi(p)}} \frac{f_{3,1}(k_3,k_4,-p,-q)}{\sqrt{2\omega_\Psi(q)}} \frac{\tau_p I(\tau_q q)}{\omega_\Phi(k_1)+\omega_\Phi(k_2)-\tau_q\omega_\Phi(q)-\tau_p\omega_\Psi(p)}$$



$$+ \frac{f_{2,2}(k_1,k_2,p,q)}{\sqrt{2\omega_\Psi(p)}} \frac{f_{2,2}(k_3,k_4,-p,-q)}{\sqrt{2\omega_\Psi(q)}} \frac{\tau_p I(\tau_q q)}{\omega_\Phi(k_1)+\omega_\Phi(k_2)-\tau_q\omega_\Psi(q)-\tau_p\omega_\Psi(p)} \}$$
$$a(k_1)a(k_2) \times [a(k_3)a(k_4) + 2a(k_3)a^*(-k_4) + a^*(-k_3)a^*(-k_4)], \quad \text{(B.3)}$$

$$H_4^b = (\frac{\lambda}{4L^D})^2 \sum_{p,q}' \sum_{\{k_i\}}' \sum_{i=1}^{4} \sum_{\{\tau_i\}} \prod_{l=1}^{4} \frac{1}{\sqrt{2\omega_\Phi(k_i)}} \{ \frac{f_{3,1}(k_1,k_2,p,q)}{\sqrt{2\omega_\Phi(p)}} \frac{f_{3,1}(k_3,k_4,-p,-q)}{\sqrt{2\omega_\Psi(q)}} \frac{\tau_p I(\tau_q q)}{\omega_\Phi(k_1)-\omega_\Phi(k_2)-\tau_q\omega_\Phi(q)-\tau_p\omega_\Psi(p)}$$
$$+ \frac{f_{2,2}(k_1,k_2,p,q)}{\sqrt{2\omega_\Psi(p)}} \frac{f_{2,2}(k_3,k_4,-p,-q)}{\sqrt{2\omega_\Psi(q)}} \frac{\tau_p I(\tau_q q)}{\omega_\Phi(k_1)-\omega_\Phi(k_2)-\tau_q\omega_\Psi(q)-\tau_p\omega_\Psi(p)} \}$$
$$a(k_1)a^*(-k_2) \times [a(k_3)a(k_4) + 2a(k_3)a^*(-k_4) + a^*(-k_3)a^*(-k_4)], \quad \text{(B.4)}$$

$$H_4^c = (\frac{\lambda}{4L^D})^2 \sum_{p,q}' \sum_{\{k_i\}}' \sum_{i=1}^{4} \sum_{\{\tau_i\}} \prod_{l=1}^{4} \frac{1}{\sqrt{2\omega_\Phi(k_i)}} \{ \frac{f_{3,1}(k_1,k_2,p,q)}{\sqrt{2\omega_\Phi(p)}} \frac{f_{3,1}(k_3,k_4,-p,-q)}{\sqrt{2\omega_\Psi(q)}} \frac{\tau_p I(\tau_q q)}{-\omega_\Phi(k_1)-\omega_\Phi(k_2)-\tau_q\omega_\Phi(q)-\tau_p\omega_\Psi(p)}$$
$$+ \frac{f_{2,2}(k_1,k_2,p,q)}{\sqrt{2\omega_\Psi(p)}} \frac{f_{2,2}(k_3,k_4,-p,-q)}{\sqrt{2\omega_\Psi(q)}} \frac{\tau_p I(\tau_q q)}{-\omega_\Phi(k_1)-\omega_\Phi(k_2)-\tau_q\omega_\Psi(q)-\tau_p\omega_\Psi(p)} \}$$
$$a^*(-k_1)a^*(-k_2) \times [a(k_3)a(k_4) + 2a(k_3)a^*(-k_4) + a^*(-k_3)a^*(-k_4)]. \quad \text{(B.5)}$$

These three amplitudes differ only in the energy denominators and the $a$ and $a^*$ factors.

Next, we turn to the comparison of the effective Hamiltonian with the starting Hamiltonian. Therefore, let us study the contribution to the four vertex function, which are represented in Fig2a. and 2b. We have shown that these contributions depend upon the external momenta $k_1, ..., k_4$ and this dependence differs from one time ordering to another. Since we will be interested only in the region of small external momenta, we approximate this momentum dependence by a Taylor expansion around the renormalization point $k_1^2 = .... = k_4^2 = 0$. One can show that the leading term of the first graph vanish because of momentum conservation, and the expansion starts, therefore, with terms proportional to $k_i^2, k_i k_j$. Considering Wilson's $\epsilon$ expansion and arguments on irrelevant operators, we shall argue that such terms are not important near a fixed point.

*Finally, we can state the result from the graph 2b. In the investigation of this graph,* we will consider in the first place the case where the internal lines have a 'positive' ($\tau = 1$) and a 'negative' ($\tau = -1$) energy, and in the last part, the case where the internal lines have either both positive or both negative energies. Ref. [33] contains a discussion about the reason for these contributions to be small. Here, we will resummarized some ideas. In the definition of $\tilde{H}_1$ and $S_1$, we must leave out all those momentum where $q_1 = -q_2$ (see definition of $< H_1 >$), which is satisfied if $k_1 + k_2 = q = 0$. This condition guarantees that there are no divergent terms in the sum over momentum in the generating function. Moreover, the contribution of those graphs takes a form:

$$\sum_{q_1} \frac{f_{2,2}(k_1,k_2,q-q_1,q_1)}{2\omega_\Psi(q-q_1)} \frac{f_{2,2}(k_3,q-k_3,-(q-q_1),-q_1)}{2\omega_\Psi(q)} \frac{\{\omega_\Psi(q-q_1)-\omega_\Psi(q)\}\{I(q-q_1)-I(q)\}}{\Delta^2-(\omega_\Psi(q-q_1)-\omega_\Psi(q))^2}. \quad \text{(B.6)}$$

At $q = 0 = -k_1 - k_2$, we must substract from the equation (B.6) the same expression so the value is zero. Note, that when $q \to 0$, both the numerator and denominator vanish at the same rate and the sum is finite at this point. However, in this limit, the numerator is proportional to $\delta I/\delta q_1^2$, and for a general trajectory this derivative is of the same order as $I$ itself. However, considering the discussion about the fixed point, one finds that only those trajectory with small derivative are important near a fixed point. Therefore, this contribution is smaller than the other contribution from Fig 2b.

Finally, let us consider the second group of graphs where the internal lines are either positive or negative. At the point $k_1^2 = .... = k_4^2 = 0$, the contribution of the external lines to the energy denominator is either $2m^2$ or zero. Since the energy denominator of the fast modes is big as compared to $2m^2$ , and considering the fixed point values of $2m^2$, which is smaller, it is self consistent (under these conditions) to drop $2m^2$ in all graph contributions to Fig. 2b. Now, we obtain:

$$H_2 = -\frac{\lambda^2}{(4L)^2}\prod_{i=1}^{4}\Phi(k_i)\sum_{q_1}\frac{f_{2,2}(k_1,k_2,q-q_1,q_1)}{2\omega_\Psi(q-q_1)}\frac{f_{2,2}(k_3,q-k_3,-(q-q_1),-q_1)}{2\omega_\Psi(q)}\frac{(I(q-q_1)+I(q))}{(\omega_\Psi(q-q_1)+\omega_\Psi(q))} \tag{B.7}$$

Denoting the expression of the last equation by $I$, we obtain,

$$\lambda' = 2Z^2\{\lambda - \lambda^2 I\} = \lambda. \tag{B.8}$$

. A positive coupling constant $\lambda$ of the starting Hamiltonian was chosen because the system must be stable under small perturbation. Now, we can interpret this result as follows: the behavior of the coupling constant is decreasing, i.e. the $\lambda$ parameter is smaller after a RGT-step. Therefore, one can argue that this parameter as a function of the number of degrees of freedom is not decreasing.



# Appendix C

# Classical Theory

In this appendix, we will discuss the Classical Gauge Theory on the lattice but before this discussion, let us state the supposition which we are to be considered. The classical gauge theory on the lattice follows the Gauss' law, which can be considerer as a condition on the solution or a systems with constraint. In the last case, one can introduce a gauge fixing in order to eliminate the gauge degree of freedom and the phase space is reduced to physical phase which is described by physical degrees of freedom. In this appendix we will consider the first case.

In the approximation of small coupling expansion the Hamiltonian take the form:

$$H = H_0 + \frac{g}{2}H_1 + \frac{g^2}{4}H_2 + ... \tag{C.1}$$

and the Gauss' law is

$$\Omega^i(s) = \sum_\mu\{p^i(s) - p^i(s-\mu) + g\left[L^i(s) + L^i(s-\mu)\right]\}, \tag{C.2}$$

where this condition is defined on each lattice site.

The free Hamiltonian $H_0$ is given by

$$H_0 = \frac{1}{2}\left[\sum_l \vec{p_l}^{\,2} + \sum_{n,\mu,\nu}(\vec{u}_{n,\mu} - \vec{u}_{n+\nu,\mu} + \vec{u}_{n+\mu,\nu} - \vec{u}_{n,\nu})^2\right]. \tag{C.3}$$

Since we assume periodic boundary condition for the $a$-lattice, one can introduce a Fourie Transformation. Then the Hamiltonian becomes,

$$H_0 = \frac{1}{2L^d}\left[\sum_k \vec{p}_{-k,\mu}\vec{p}_{k,\mu} + \sum_{k,k',\mu,\nu}\vec{u}_{k',\mu}V(k,k')^{\mu\nu}\vec{u}_{k,\nu}\right], \tag{C.4}$$

where matrix term in the last expression take the form

$$V(k,k')^{\mu\nu} = (\tilde{k}^2\delta_{\mu,\nu} - \tilde{k}_\mu\tilde{k}_\nu)e^{i(k_\mu - k_\nu)}\delta(k+k'). \tag{C.5}$$

The higher order expressions are given by:
first order:

$$H_1 = \frac{-1}{L^{2d}}\sum_{(\mathbf{k})}\delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\Gamma^{abc}_{3\,ijk}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})u^a_{k_1,i}u^b_{k_1,j}u^c_{k_3,k}, \tag{C.6}$$

second order:

$$H_2 = \frac{-1}{L^{3d}} \sum_{\{\mathbf{k}\}} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \Gamma^{abcd}_{4\,ijkl}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) u^a_{k_1\,i} u^b_{k_2\,j} u^c_{k_3\,k} u^d_{k_4\,l}. \tag{C.7}$$

The non linear terms can be then interpreted as vertices of three or four variables, and are defined as

$$\begin{aligned} \Gamma^{abc}_{3\,\mu,\nu,\lambda}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \epsilon^{abc}\{(k_3 \widetilde{-} k_2)_\mu \cos(\frac{k_\nu a}{2})\delta_{\nu,\lambda} \\ &\quad + (k_1 \widetilde{-} k_3)_\nu \cos(\frac{k_\mu a}{2})\delta_{\mu,\lambda} + (k_2 \widetilde{-} k_1)_\lambda \cos(\frac{k_\mu a}{2})\delta_{\mu,\nu}\}, \end{aligned} \tag{C.8}$$

where $(k_3 \widetilde{-} k_2)_\mu$ stand for

$$(\widetilde{k-q})_\mu = \frac{2}{a}\sin(\frac{(k-q)_\mu a}{2}), \tag{C.9}$$

and the second order expression is equivalent to the expression in the spatial lattice given in the appendix of the paper by Kawai et al. (1981) which reads:

$$\begin{aligned} \Gamma^{abcd}_{4,\mu,\nu,\lambda,\rho}(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) &= \epsilon^{abe}\epsilon^{cde}\{\delta_{\mu,\lambda}\delta_{\nu,\rho}\left[cos\frac{a(q-s)_\mu}{2}\,cos\frac{a(p-r)_\nu}{2} - \frac{a^4}{12}\tilde{p}_\nu\tilde{q}_\mu\tilde{r}_\nu\tilde{s}_\mu\right] - \\ &\quad \delta_{\mu,\rho}\delta_{\nu,\lambda}\left[cos\frac{a(q-r)_\mu}{2}\,cos\frac{a(p-s)_\nu}{2} - \frac{a^4}{12}\tilde{p}_\nu\tilde{q}_\mu\tilde{r}_\mu\tilde{s}_\nu\right] + \\ &\quad \frac{a^2}{12}\delta_{\mu,\nu}\delta_{\nu,\lambda}\delta_{\nu,\rho}\sum_\sigma(\tilde{q}_\sigma e^{-i\frac{a}{2}p_\sigma} - \tilde{p}_\sigma e^{-i\frac{a}{2}q_\sigma})(\tilde{s}_\sigma e^{-i\frac{a}{2}r_\sigma} - \tilde{r}_\sigma e^{-i\frac{a}{2}r_\sigma}) - \\ &\quad \frac{a^2}{6}\delta_{\mu,\nu}\delta_{\mu,\lambda}(\tilde{q}_\rho e^{-i\frac{a}{2}p_\rho} - \ tildep_\rho e^{-i\frac{a}{2}q_\rho})\tilde{s}_\mu cos\frac{ar_\mu}{2} + \\ &\quad \frac{a^2}{6}\delta_{\mu,\nu}\delta_{\mu,\rho}(\tilde{q}_\lambda e^{-i\frac{a}{2}p_\lambda} - \tilde{p}_\lambda e^{-i\frac{a}{2}q_\lambda})\tilde{r}_\mu cos\frac{as_\lambda}{2} - \\ &\quad \frac{a^2}{6}\delta_{\mu,\lambda}\delta_{\mu,\rho}(\tilde{s}_\nu e^{-i\frac{a}{2}r_\nu} - \tilde{r}_\nu e^{-i\frac{a}{2}s_\nu})\tilde{q}_\mu cos\frac{ps_\nu}{2} + \\ &\quad \frac{a^2}{6}\delta_{\nu,\lambda}\delta_{\nu,\rho}(\tilde{s}_\mu e^{-i\frac{a}{2}r_\mu} - \tilde{r}_\mu e^{-i\frac{a}{2}s_\mu})\tilde{q}_\nu cos\frac{ps_\mu}{2}\} + \\ &\quad (a \leftrightarrow c, \nu \leftrightarrow \lambda, q \leftrightarrow r) + (b \leftrightarrow d, \nu \leftrightarrow \rho, q \leftrightarrow s) + \\ &\quad \frac{a^4}{12}\{\frac{2}{3}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\}. \end{aligned} \tag{C.10}$$

The zero order term is a quadratic form, which can be diagonalized straightforward. In this case one can introduce the normal modes which are defined by the following transformation:

$$\begin{pmatrix} u^i_\mu(k) \\ u^i_\nu(k) \end{pmatrix} = \hat{R}(k) \begin{pmatrix} x^{i\,1}(k) \\ x^{i\,2}(k) \end{pmatrix}, \tag{C.11}$$

and where the matrix $\hat{R}$ is

$$\hat{R}(k) = \frac{1}{\sqrt{\tilde{k}_1^2 + \tilde{k}_2^2}} \begin{pmatrix} \tilde{k}_1 & \tilde{k}_2 e^{i(k_1 - k_2)} \\ \tilde{k}_2 e^{-i(k_1 - k_2)} & \tilde{k}_1 \end{pmatrix}. \tag{C.12}$$



In terms of these normal modes the $H_0$ take the form

$$H_0 = \frac{1}{2L^d} \sum_{k,i} \left[\vec{p}^{\,i}_{-k,\mu}\vec{p}^{\,i}_{k,\mu} + w^i(k)\vec{x}^{\,i}_{-k}\vec{x}^{\,i}_{k}\right], \tag{C.13}$$

where the frequencies are defined by:

$$\begin{aligned} w^1(k) &= 0 \\ w^2(k) &= \tilde{k}_1^2 + \tilde{k}_2^2. \end{aligned} \tag{C.14}$$

The presence of the eigenvalues zero leads to the conclusion that the system admits a symmetry.

After this transformation of coordinates, the perturbation term to first order takes the form

$$H_1 = \frac{1}{L^{2d}} \sum_{a,b,c} \sum_{i,j,k} \sum_{\{\mathbf{k}\}} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \epsilon^{ijk} \Gamma'^{abc}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) x^{a\,i}_{k_1} x^{b\,j}_{k_2} x^{c\,k}_{k_3}, \tag{C.15}$$

where

$$\Gamma'^{abc}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \Gamma^{a'b'c'}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\hat{R}^{a'a}(k_1)\hat{R}^{b'b}(k_2)\hat{R}^{c'c}(k_3). \tag{C.16}$$

One could investigate the solution by the application of the Perturbation Theory and the consistence with the Gauss' law. However, we do not have interest about this point.

# Appendix D

# Three Vertex Function

In this appendix, we will introduce a useful relation for calculating the trace of a product of SU(2)-matrices and for deriving the structure of the three and four vertex functions.

In the fundamental representation of the SU(2) group, a group element can be representented as:

$$U = u_0 + i\vec{\sigma}\cdot\vec{u} \qquad \text{and} \qquad u_0 = \sqrt{1-\vec{u}^2}. \tag{D.1}$$

If we consider that the variables are small, one can expand the last expression and obtain:

$$u_0 = \sqrt{1-\vec{u}^2} \simeq 1 - \frac{1}{2}u^2 - \frac{1}{8}u^4 + \ldots. \tag{D.2}$$

Now let us state the general properties of $Tr$ over a product of matrices.

$$\begin{aligned} \frac{1}{2}TrU_1 &= x_1^0, \\ \frac{1}{2}TrU_1U_2 &= x_1^0x_2^0 - \mathbf{x}_1\cdot\mathbf{x}_2, \end{aligned} \tag{D.3}$$

$$\begin{aligned} \frac{1}{2}TrU_1U_2U_3 &= x_1^0x_2^0x_3^0 - (x_1^0\mathbf{x}_2\cdot\mathbf{x}_3, + x_1^0\mathbf{x}_1\cdot\mathbf{x}_3 + x_3^0\mathbf{x}_1\cdot\mathbf{x}_2) \\ &\quad - \mathbf{x}_2\cdot\mathbf{x}_3\times\mathbf{x}_4, \end{aligned} \tag{D.4}$$

$$\begin{aligned} \frac{1}{2}TrU_1U_2U_3U_4 &= x_1^0x_2^0x_3^0x_4^0 \\ &\quad + \{\, x_1^0x_2^0\mathbf{x}_3\cdot\mathbf{x}_4 + x_1^0x_2^0\mathbf{x}_3\cdot\mathbf{x}_4 + x_1^0x_3^0\mathbf{x}_2\cdot\mathbf{x}_4 + x_1^0x_4^0\mathbf{x}_2\cdot\mathbf{x}_3 \\ &\quad + x_2^0x_3^0\mathbf{x}_1\cdot\mathbf{x}_4 + x_2^0x_4^0\mathbf{x}_1\cdot\mathbf{x}_3 + x_3^0x_4^0\mathbf{x}_1\cdot\mathbf{x}_2\} \\ &\quad - \{\, x_1^0\mathbf{x}_2\cdot(\mathbf{x}_3\times\mathbf{x}_4) + x_2^0\mathbf{x}_3\cdot(\mathbf{x}_4\times\mathbf{x}_1) \\ &\quad + x_3^0\mathbf{x}_4\cdot(\mathbf{x}_1\times\mathbf{x}_2) + x_4^0\mathbf{x}_1\cdot(\mathbf{x}_2\times\mathbf{x}_3)\} \\ &\quad - (\mathbf{x}_1\cdot\mathbf{x}_2)(\mathbf{x}_3\cdot\mathbf{x}_4) + (\mathbf{x}_1\times\mathbf{x}_2)\cdot(\mathbf{x}_3\times\mathbf{x}_4). \end{aligned} \tag{D.5}$$

The perturbation terms can be obtained if we expand the potential after the introduction of the scaling on the vectorial component. To find the perturbation term to order $g$, which is



equivalent to consider a term with three variables, let us rewrite the potential in terms of the new loop and link variables. The potential is defined by

$$V = \frac{2}{ag^2}\sum_p Tr(1-U_p). \tag{D.6}$$

Considering only the plaquette in the block and between the neighboring blocks, we can be found that the potential takes the form:

$$V = \frac{2}{ag^2}\sum_{s,plaq} Tr(1-U_p(s)), \tag{D.7}$$

where the last term is given by

$$\begin{aligned} Tr(U_p(s)) &= \sum_{i=1,\ldots 4} TrU_s^i + TrX_{s,i}^{(2)}X_{s,i}^{(1)\dagger}U_s^{(1)\dagger} + \\ &\quad TrX_{s,i}^{(2)}U_{s+i}^{(3)\dagger}X_{s,i}^{(3)\dagger} + TrY_{s,j}^{(2)}Y_{s,j}^{(1)\dagger}U_s^{(2)} + TrY_{s,j}^{(3)}U_{s+j}^{(4)\dagger}Y_{s,j}^{(2)\dagger} + \\ &\quad TrX_{s,i}^{(1)}Y_{s+i,j}^{(1)}X_{s+j,i}^{(3)\dagger}Y_{s,j}^{(2)\dagger}. \end{aligned} \tag{D.8}$$

Considering the form of the potential and using the above relations, it can be stated that the contributions to the three vertex function come from the potential which contain three or four link variables in the definition of a loop. Therefore, the potential up to first order take the form

$$\begin{aligned} H_1 &= \frac{g}{2}\sum_s \{\, \mathbf{U}_s^{(1)}\cdot\mathbf{X}_{s,i}^{(2)}\times\mathbf{X}_{s,i}^{(1)} + \mathbf{U}_{s+i}^{(3)}\cdot\mathbf{X}_{s,i}^{(2)}\times\mathbf{X}_{s,i}^{(3)} \\ &\quad \mathbf{U}_s^{(2)}\cdot\mathbf{Y}_{s,j}^{(2)}\times\mathbf{Y}_{s,j}^{(1)} - \mathbf{U}_{s+j}^{(4)}\cdot\mathbf{Y}_{s,j}^{(2)}\times\mathbf{Y}_{s,j}^{(3)} \\ &\quad \mathbf{X}_{s,i}^{(1)}\cdot\mathbf{Y}_{s+i,j}^{(1)}\times\mathbf{X}_{s+j,i}^{(3)} + \mathbf{Y}_{s+i,j}^{(1)}\cdot\mathbf{X}_{s+j,i}^{(3)}\times\mathbf{Y}_{s,j}^{(3)} \\ &\quad \mathbf{X}_{s+j,i}^{(3)}\cdot\mathbf{Y}_{s,j}^{(3)}\times\mathbf{X}_{s,i}^{(1)} - \mathbf{Y}_{s,j}^{(3)}\cdot\mathbf{X}_{s,i}^{(1)}\times\mathbf{Y}_{s+i,j}^{(1)}\,\}. \end{aligned} \tag{D.9}$$

Now introducing a column vector of $10\times 1$, which was defined in chapter 3, leads to a vertex of the form

$$H_1 = \frac{g}{2L^{12d}}\sum_{a,b,c}\sum_{i,j,k}\sum_{\{k\}}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\epsilon^{ijk}\Gamma^{abc}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)u_{k_1,i}^a u_{k_2,j}^b u_{k_3,k}^c, \tag{D.10}$$

where $\Gamma^{abc}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ is given by

$$\begin{aligned} \Gamma^{abc}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) &= \tfrac{1}{3!}\{\; -\delta^{a1}\delta^{b2}\delta^{c3} + permutations \\ &\quad -\delta^{a2}\delta^{b3}\delta^{c2}e^{i\mathbf{k}_3\cdot\mathbf{i}} + permutations \\ &\quad +\delta^{a6}\delta^{b7}\delta^{c9} + permutations \\ &\quad +\delta^{a7}\delta^{b8}\delta^{c10}e^{i\mathbf{k}_3\cdot\mathbf{j}} + permutations \\ &\quad +\delta^{a1}\delta^{b3}\delta^{c6}e^{\mathbf{k}_1\cdot\mathbf{j}+\mathbf{k}_3\cdot\mathbf{i}} + permutations \\ &\quad +\delta^{a1}\delta^{b3}\delta^{c8}e^{i\mathbf{k}_2\cdot\mathbf{j}} + permutations \\ &\quad -\delta^{a3}\delta^{b6}\delta^{c8}e^{\mathbf{k}_1\cdot\mathbf{j}+\mathbf{k}_2\cdot\mathbf{i}} + permutations \\ &\quad -\delta^{a1}\delta^{b6}\delta^{c8}e^{\mathbf{k}_1\cdot\mathbf{i}} + permutations\}. \end{aligned} \tag{D.11}$$

where permutation denotes all possible combinations of the index $a, i$ with $k$. To clear this point the following example is given: Let us consider the term with the indices $\{1, 3, 8\}$, then in this vertex we can find the following terms:

$$\begin{aligned}\Gamma^{\{136\}}(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3) = \{ \; & \delta^{a1}\delta^{b3}\delta^{c6} e^{\mathrm{k}_2\cdot \mathrm{j}+\mathrm{k}_3\cdot \mathrm{i}} + \delta^{a3}\delta^{b6}\delta^{c1} e^{\mathrm{k}_1\cdot \mathrm{j}+\mathrm{k}_2\cdot \mathrm{i}} + \\ & \delta^{a6}\delta^{b1}\delta^{c3} e^{\mathrm{k}_3\cdot \mathrm{j}+\mathrm{k}_1\cdot \mathrm{i}} - \delta^{a1}\delta^{b6}\delta^{c3} e^{\mathrm{k}_3\cdot \mathrm{j}+\mathrm{k}_2\cdot \mathrm{i}} - \\ & \delta^{a3}\delta^{b1}\delta^{c6} e^{\mathrm{k}_2\cdot \mathrm{j}+\mathrm{k}_1\cdot \mathrm{i}} - \delta^{a6}\delta^{b3}\delta^{c1} e^{\mathrm{k}_1\cdot \mathrm{j}+\mathrm{k}_3\cdot \mathrm{i}} \}. \end{aligned} \tag{D.12}$$

To conclude this appendix let us express this interaction part in terms of the new variables derived from the diagonalization of the quadratic form. In this case we must only introduce the matrix transformation between the old and the new variables. This linear change of variables can be denoted as

$$u^{a,i}(k) = R^{a,b}(k) x^{b,i}(k), \tag{D.13}$$

where the $u$ are the loops and links and the $x$ are the normal modes. Therefore, the new vertex $\Gamma'$ which describe the interactions between the normal modes corresponding to the expression

$$H_1 = \frac{g}{23L^{/2d}} \sum_{a,b,c} \sum_{i,j,k} g \sum_{\{\mathrm{k}\}} \delta(\mathrm{k}_1 + \mathrm{k}_2 + \mathrm{k}_3) \epsilon^{ijk} \Gamma'^{abc}(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3) x^a_{k_1\, i}, x^b_{k_2\, j} x^c_{k_3\, k} \tag{D.14}$$

is in a compact form defined as

$$\Gamma'^{abc}(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3) = \Gamma^{a'b'c'}(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3) R^{a'a}(k_1) R^{b'b}(k_2) R^{c'c}(k_3). \tag{D.15}$$



# Appendix E

# Four Vertex Function

In order to obtain an expression of the vertex to second order in the coupling constant, let us consider the expansion to quartic order in the variable x of the product of SU(2)-matrices.

$$\begin{aligned} \frac{1}{2} TrU_1|_4 &= u_0 = -\frac{1}{8}(\mathrm{x}_1^2)^2, \\ \frac{1}{2} TrU_1U_2|_4 &= -\frac{1}{8}(\mathrm{x}_1^2 - \mathrm{x}_2^2)^2, \end{aligned} \tag{E.1}$$

$$\begin{aligned} \frac{1}{2} TrU_1U_2U_3|_4 &= -\frac{1}{8}(\mathrm{x}_1^2 - \mathrm{x}_2^2 + \mathrm{x}_3^2)^2 \\ &\quad + \frac{1}{4}\left[(\mathrm{x}_1 + \mathrm{x}_2)^2 \mathrm{x}_3^2 - (\mathrm{x}_1 - \mathrm{x}_3)^2 \mathrm{x}_3^2 + (\mathrm{x}_3 + \mathrm{x}_3)^2 \mathrm{x}_1^2\right], \end{aligned} \tag{E.2}$$

$$\begin{aligned} \frac{1}{2} TrU_1U_2U_3U_4|_4 &= -\frac{1}{8}(\mathrm{x}_1^2 - \mathrm{x}_2^2 + \mathrm{x}_3^2 - \mathrm{x}_4^2)^2 \\ &\quad + \frac{1}{4}\{(\mathrm{x}_1 + \mathrm{x}_2)^2 (\mathrm{x}_3 + \mathrm{x}_4)^2 - (\mathrm{x}_1 - \mathrm{x}_3)^2 (\mathrm{x}_3 - \mathrm{x}_1)^2 \\ &\quad + (\mathrm{x}_1 + \mathrm{x}_4)^2 (\mathrm{x}_3 + \mathrm{x}_3)^2\}. \end{aligned} \tag{E.3}$$

Finally, one can rewrite the last expression in the following compact form

$$\begin{aligned} \frac{1}{2} TrU_1U_2U_3U_4|_4 &= -\frac{1}{8}\sum_i (\mathrm{x}_i^2)^2 + \frac{1}{4}\sum_{i<j} \mathrm{x}_i^2 \mathrm{x}_j^2 \\ &\quad + \frac{1}{2}\sum_{i<j, j\neq k} (\mathrm{x}_i \cdot \mathrm{x}_j)\mathrm{x}_k^2 + \sum_{i,j,k,l} \mathrm{x}_i \cdot \mathrm{x}_j \, \mathrm{x}_k \cdot \mathrm{x}_l. \end{aligned} \tag{E.4}$$

Analogous to the calculation of the three **vertex function**, we consider the plaquette defined in the block and in the neighboring blocks, **then it can** be found that the potential is given as:

$$\begin{aligned} Tr(U_p(s)) = \sum_{i=1,\ldots,4} \; & TrU_s^i + Tr\boldsymbol{X}^{(2)}_{s,i} \boldsymbol{X}^{(1)}_{s,i}{}^{\dagger} U_s^{(1)\dagger} + TrX^{(2)}_{s,i} U^{(3)}_{s+i}{}^{\dagger} X^{(3)\dagger}_{s,i} + \\ & TrY^{(2)}_{s,j} Y^{(1)\dagger}_{s,j} \boldsymbol{U}^{(2)}_s + \boldsymbol{Tr} Y^{(3)}_{s,j} U^{(4)}_{s+j}{}^{\dagger} Y^{(2)\dagger}_{s,j} + \\ & TrX^{(1)}_{s,i} Y^{(1)}_{s+i,j} \boldsymbol{X}^{(3)}_{s+j,i}{}^{\dagger} \boldsymbol{Y}^{(2)\dagger}_{s,j}. \end{aligned} \tag{E.5}$$

The calculation is in general very long, so we give only the final result. The vertex with four modes is defined as;

$$H_2 = \frac{-g^2}{4L^{3d}} \sum_{a,b,c,d} \sum_{i,j,k} \sum_{\{\mathbf{k}\}} \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\Gamma^{abcd}_{ijkl}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) u^a_{k_1 i} u^b_{k_2 j} u^c_{k_3 k} u^d_{k_4 l}, \tag{E.6}$$

it can be rewritten as,

$$\Gamma^{abcd}_{ijkl} = \Gamma^{abcd}_{1\,ijkl} + \Gamma^{abcd}_{2\,ijkl} + \Gamma^{abcd}_{3\,ijkl} + \Gamma^{abcd}_{4\,ijkl}, \tag{E.7}$$

where the first term is given as:

$$\Gamma^{abcd}_{1\,ijkl} = -\frac{1}{12}\{\delta^{ab}\delta^{ac}\delta^{ad}[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}]\}, \tag{E.8}$$

the second term is given as:

$$\begin{aligned}
\Gamma^{abcd}_{2\,ijkl} = \tfrac{1}{4*6}\{ & \delta^{ab}\delta^{dc}\delta^{a1}\delta^{c2}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a1}\delta^{c4}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a2}\delta^{c9}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a2}\delta^{c3}\delta_{ij}\delta_{kl} + permutation \\
& \delta^{ab}\delta^{dc}\delta^{a2}\delta^{c5}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a3}\delta^{c5}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a6}\delta^{c7}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a6}\delta^{c9}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a7}\delta^{c9}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a8}\delta^{c7}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a8}\delta^{c5}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a7}\delta^{c5}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a1}\delta^{c3}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{j}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a1}\delta^{c6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a1}\delta^{c7}\delta_{ij}\delta_{kl} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a3}\delta^{c6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{j}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a3}\delta^{c7}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{j}} + permutation + \\
& \delta^{ab}\delta^{dc}\delta^{a6}\delta^{c7}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{i}} + permutation\ \}.
\end{aligned} \tag{E.9}$$



and the third contribution is:

$$\begin{aligned}
\Gamma^{abcd}_{4\,ijkl} = \tfrac{1}{2*12}\{ & -\delta^{cd}\delta^{a1}\delta^{b2}\delta^{c4}\delta_{ij}\delta_{kl} + permutation \\
& -\delta^{cd}\delta^{a1}\delta^{b4}\delta^{c2}\delta_{ij}\delta_{kl} + permutation \\
& -\delta^{cd}\delta^{a2}\delta^{b4}\delta^{c1}\delta_{ij}\delta_{kl} + permutation \\
& -\delta^{cd}\delta^{a2}\delta^{b3}\delta^{c5}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{i}} + permutation \\
& -\delta^{cd}\delta^{a2}\delta^{b5}\delta^{c3}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{i}} + permutation \\
& -\delta^{cd}\delta^{a3}\delta^{b5}\delta^{c2}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{i}} + permutation \\
& +\delta^{cd}\delta^{a6}\delta^{b7}\delta^{c9}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{j}} + permutation \\
& -\delta^{cd}\delta^{a6}\delta^{b9}\delta^{c7}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{j}} + permutation \\
& -\delta^{cd}\delta^{a7}\delta^{b9}\delta^{c6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{j}} + permutation \\
& +\delta^{cd}\delta^{a7}\delta^{b8}\delta^{c10}\delta_{ij}\delta_{kl} + permutation \\
& +\delta^{cd}\delta^{a7}\delta^{b10}\delta^{c8}\delta_{ij}\delta_{kl} + permutation \\
& -\delta^{cd}\delta^{a8}\delta^{b10}\delta^{c7}\delta_{ij}\delta_{kl} + permutation \\
& -\delta^{cd}\delta^{a1}\delta^{b3}\delta^{c6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{j}+(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation \\
& +\delta^{cd}\delta^{a1}\delta^{b6}\delta^{c3}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{i}+(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{j}} + permutation \\
& +\delta^{cd}\delta^{a6}\delta^{b3}\delta^{c1}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1)\cdot\mathbf{i}+(\mathbf{k}_2)\cdot\mathbf{j}} + permutation \\
& -\delta^{cd}\delta^{a1}\delta^{b3}\delta^{c8}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{j}} + permutation \\
& +\delta^{cd}\delta^{a1}\delta^{b8}\delta^{c3}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{j}} + permutation \\
& +\delta^{cd}\delta^{a3}\delta^{b8}\delta^{c1}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1)\cdot\mathbf{j}} + permutation \\
& +\delta^{cd}\delta^{a1}\delta^{b6}\delta^{c8}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{i}} + permutation \\
& -\delta^{cd}\delta^{a1}\delta^{b8}\delta^{c6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{j}} + permutation \\
& +\delta^{cd}\delta^{a6}\delta^{b8}\delta^{c1}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1)\cdot\mathbf{j}} + permutation \\
& -\delta^{cd}\delta^{a6}\delta^{b3}\delta^{c8}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1)\cdot\mathbf{i}+(\mathbf{k}_2)\cdot\mathbf{j}} + permutation \\
& -\delta^{cd}\delta^{a6}\delta^{b8}\delta^{c3}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1)\cdot\mathbf{j}+(\mathbf{k}_3)\cdot\mathbf{i}} + permutation \\
& +\delta^{cd}\delta^{a3}\delta^{b8}\delta^{c6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_1)\cdot\mathbf{j}+(\mathbf{k}_3+\mathbf{k}_4)\cdot\mathbf{i}} + permutation\},
\end{aligned} \tag{E.10}$$

and finally, the last term is:

$$\begin{aligned}
\Gamma^{abcd}_{4\,ijkl} = \tfrac{1}{1*24}\{ & \delta^{a1}\delta^{b6}\delta^{c3}\delta^{d8}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{i}+(\mathbf{k}_3)\cdot\mathbf{j}} + permutation - \\
& \delta^{a1}\delta^{b3}\delta^{c6}\delta^{d8}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_2)\cdot\mathbf{j}+(\mathbf{k}_3)\cdot\mathbf{i}} + permutation - \\
& \delta^{a1}\delta^{b8}\delta^{c3}\delta^{d6}\delta_{ij}\delta_{kl}e^{i(\mathbf{k}_4)\cdot\mathbf{i}+(\mathbf{k}_3)\cdot\mathbf{j}} + permutation\ \},
\end{aligned} \tag{E.11}$$

where permutation denotes all possible combinations of the index $a, i$ with $k$. In particular, one can note that in the second, third and in the last, there are 6, 12 and 24 different contributions, respectively

Finally, we define the effective four point function as:

$$\Gamma'^{abcd}_{ijkl}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \Gamma^{a'b'c'd'}_{ijkl}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) R^{a'a}(k_1) R^{b'b}(k_2) R^{c'c}(k_3) R^{d'd}(k_4). \tag{E.12}$$

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## Figure Captions

**Figure 1:** Graphical representation of the lattice in one dimension and definition of variables in each block.

**Figure 2:** Diagrams derived by averaging the four point function $< H_4 >$:
Fig. 2a : all external lines correspond to slow modes.
Fig. 2b : two of the lines are slow modes and the other two are fast. The contraction of the fast lines gives rise to one loop contribution to the two point function.
Fig. 2c : all external lines correspond to fast modes and their contributions give rise to two loops to the vacuum energy.

*Figure 3: Graphs with four slow external lines for scalar modes.*
Fig. 3a: Graph with a slow and fast mode as internal lines.
Fig. 3b: Graph with only fast mode as internal lines.

**Figure 4:** These graphs defined the contributions to the four point function and they have different external-leg propagation directions according to the time ordering.

**Figure 5:** Graphical representation of the lattice in two dimension and definition of variables in each block.

**Figure 6:** Representation of the new block-lattice in two dimension. The boundary of the blocks are denoted by blocken lines, and the block site by heavy dots.

**Figure 7a,7b:** Local Gauge Fixing in each block, and choice of the link variables on the axial path-gauge.

**Figure 8:** Representation of the new set of variables, which are defined on path that run between the block sites and include some links on the local axial path gauge fixing, or are plaquette in each block site. The number correspond to the notation used in the calculation.
Fig. 8a. New link variables in direction $i$.
Fig. 8a. New link variables in direction $j$.
Fig. 8a. New block-variables.

**Figure 9:** Representation of the region where the links variables are considerate.

**Figure 10:** Representation of the eigenvalues for the slow modes. Here $\vec{k} = (x, y)$ and $-0.2 < |k_i| < 0.2$.

**Figure 11:** Representation of the eigenvalues for the fast modes. Here $\vec{k} = (x, y)$ and $-0.2 < |k_i| < 0.2$.

**Figure 12:** Representation of the eigenvalues for the fast modes. Here $\vec{k} = (x, y)$ and $-0.2 < |k_i| < 0.2$.

**Figure 13:** *Representation of the eigenvalues for the fast modes and slow modes. Here* $\vec{k} = (x, y)$ and $-0.2 < |k_i| < 0.2$.



**Figure 14:** Diagrams derived by averaging the four point function $< H_4 >$:
Fig. 14a : all external lines correspond to slow modes.
Fig. 14b : two of the lines are slow modes and the other two are fast. The contraction of the fast lines gives rise to one loop contribution to the two point function.
Fig. 14c : all external lines correspond to fast modes and their contributions give rise to two loops to the vacuum energy.

**Figure 15:** Compact representation of the term $\{\tilde{H}_1, C1\}$ before contraction.

**Figure 16:** Graphs derived from the Poisson bracket between $\hat{H}_1$ and $C_1$ .

**Figure 17:** Graphs obtained by taking the Poisson bracket and then averaging over the fast modes. These diagrams contribute to the vacuum energy (a,b) of the system and to the two point function (c,d).

**Figure 18:** Graphs with two external (slow) lines contribute to the two point function ( self-energy diagrams).

**Figure 19:** Graphs with three external lines are know as three point vertices.

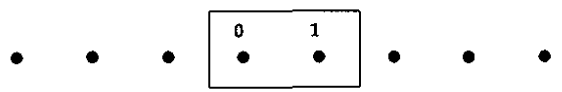



Figure 1.

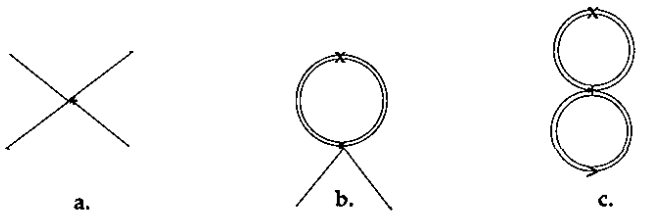
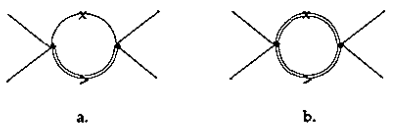



Figure 2.

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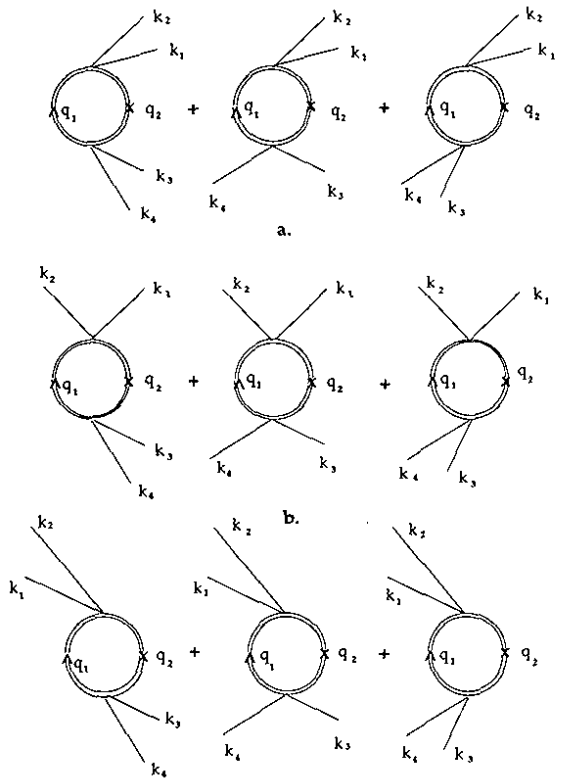


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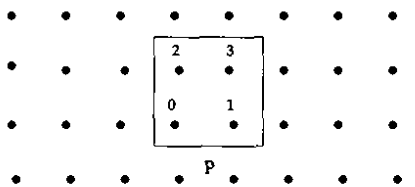


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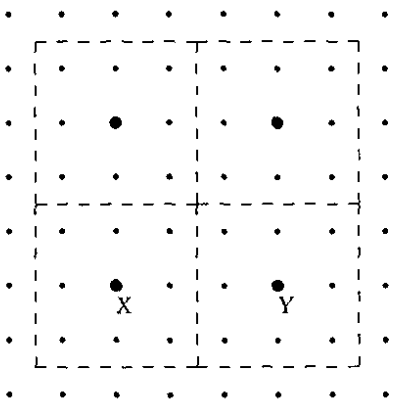


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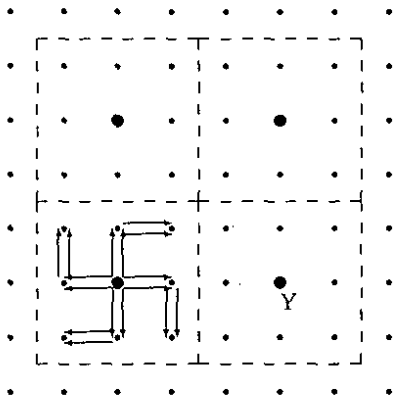



Figure 7a.

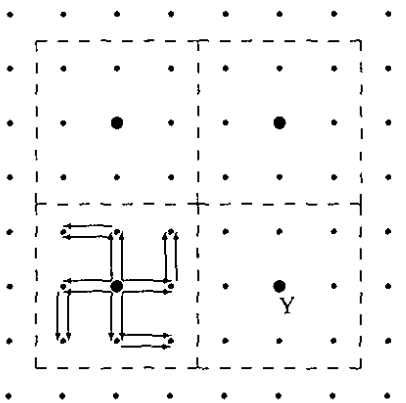



Figure 7b.

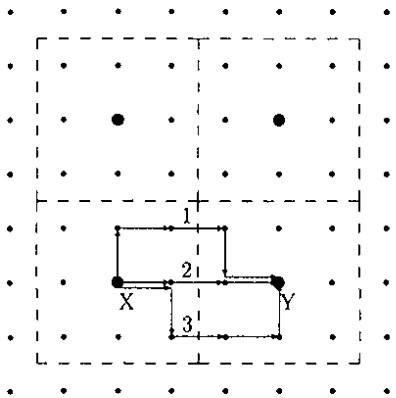



Figure 8a.

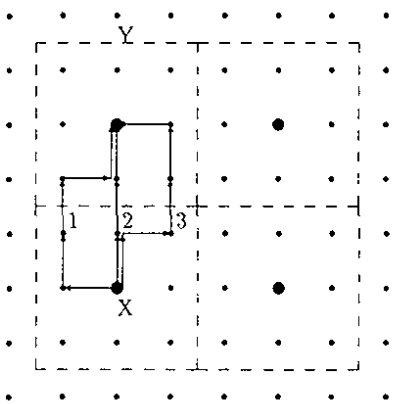



Figure 8b.

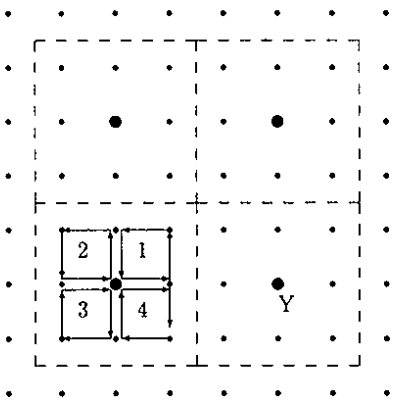
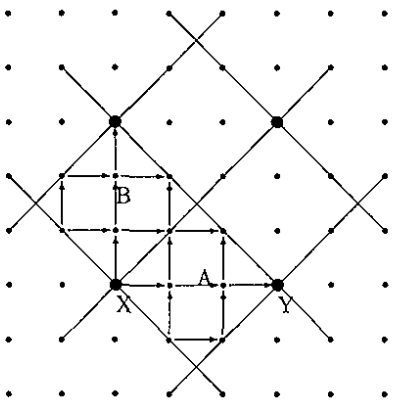



Figure 8c.

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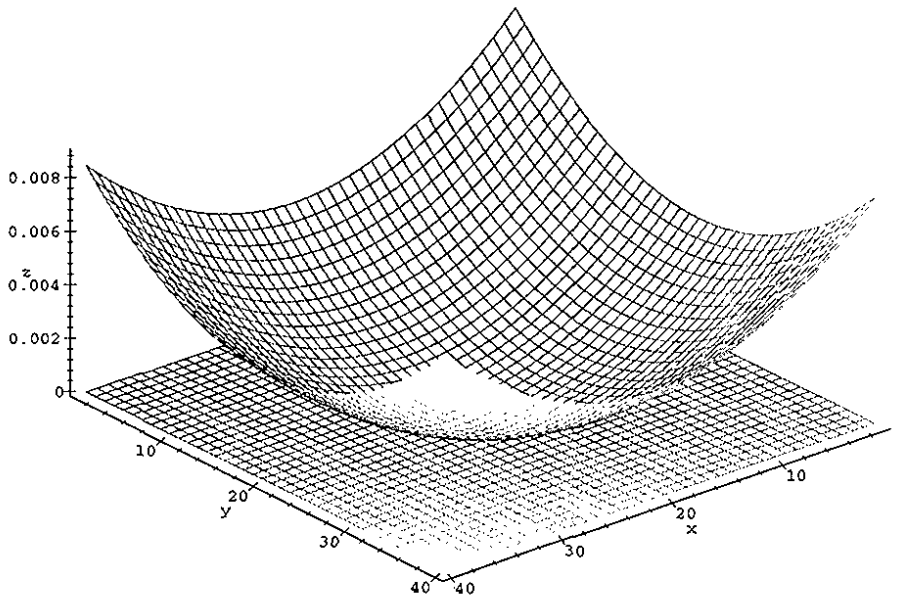


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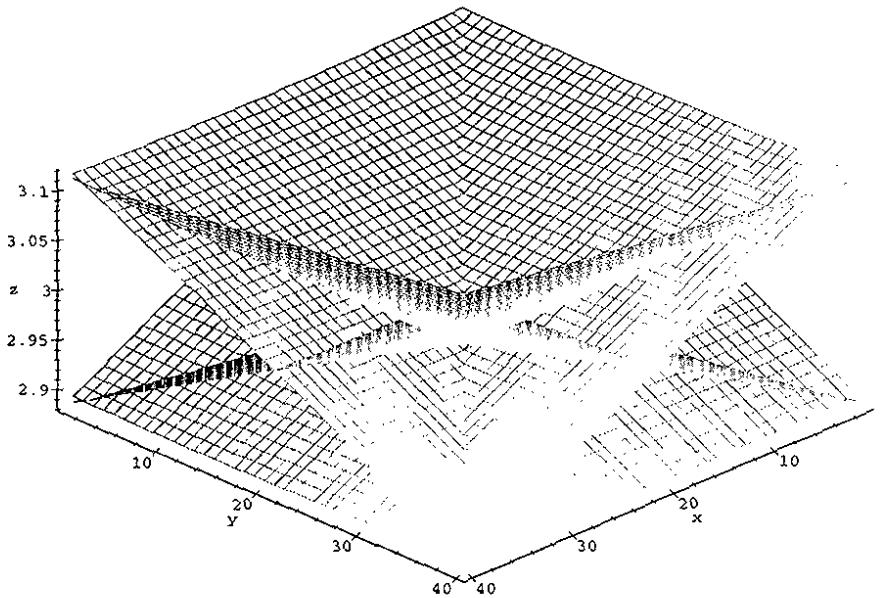


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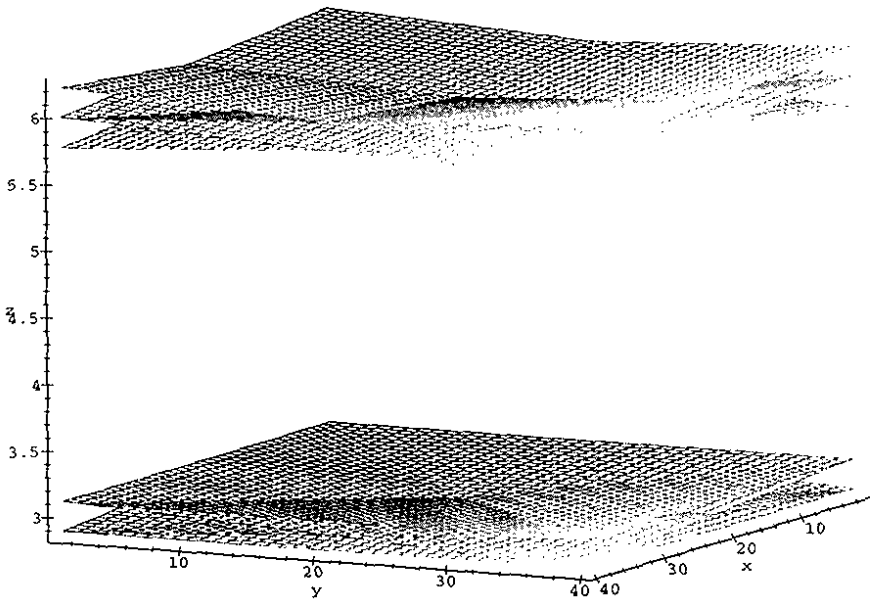


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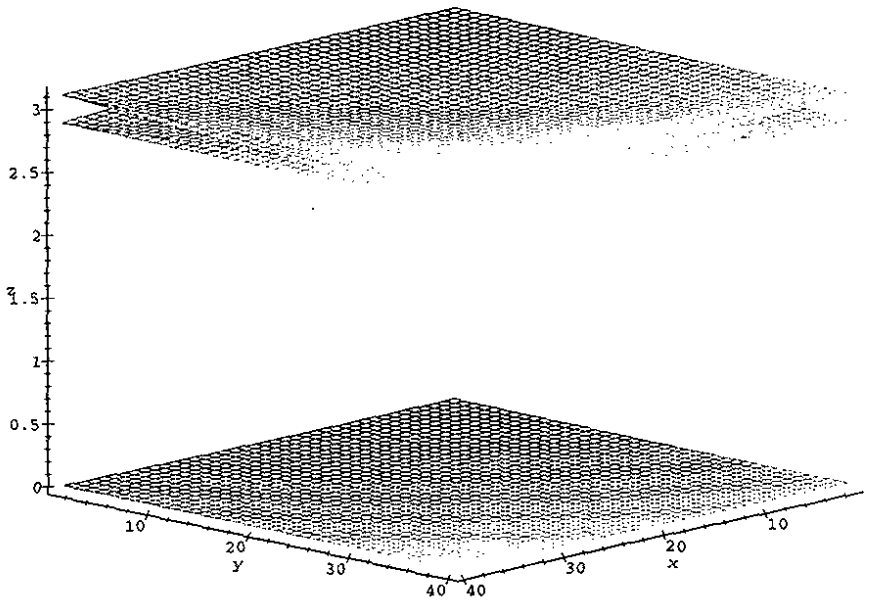


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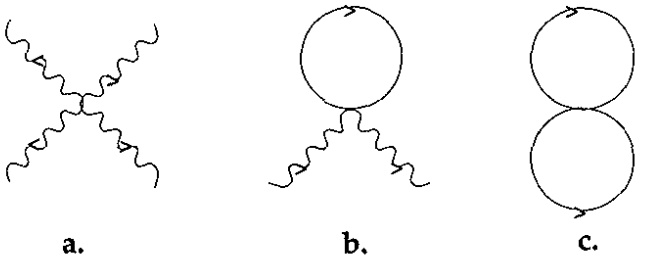


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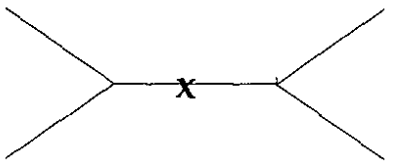


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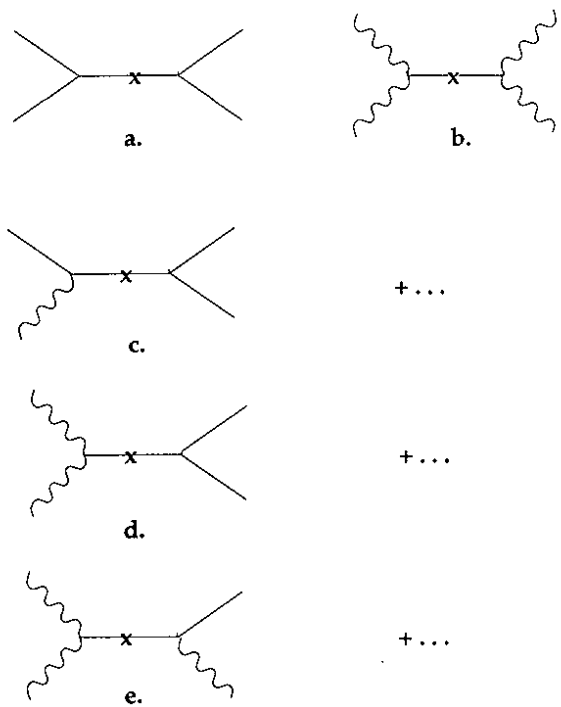


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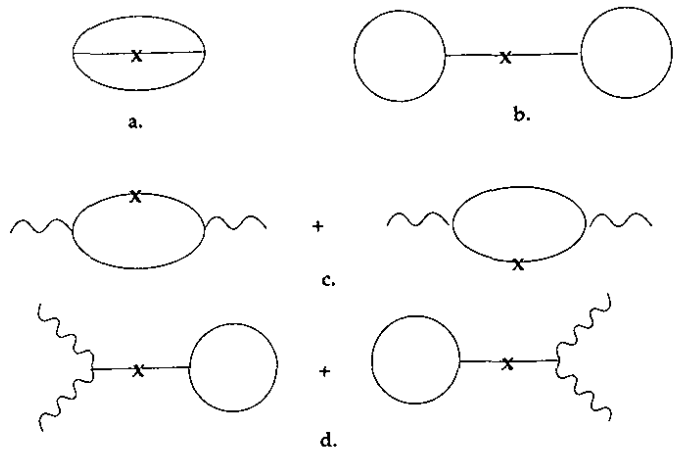


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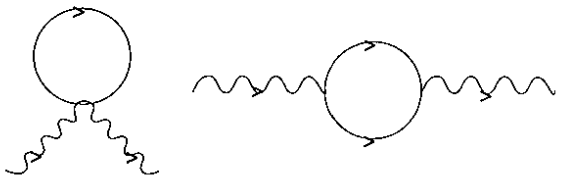


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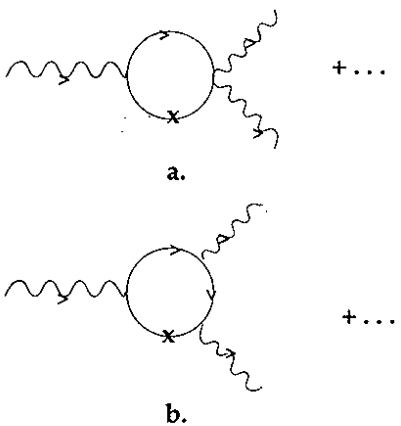


Figure 19.