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BOUNDARY-CONDITIONS IN PATH INTEGRALS

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ABSTRACT

An overlook is given, how to implement point interactions and boundary-conditions in the path integral. It involves perturbation expansion techniques and renormalization methods. The starting point is the path integral representation for the one-dimensional Dirac particle with a point perturbation. The method is illustrated with some examples.

1. Introduction.

Boundary-conditions are of inherent importance in the set-up of a physical problem. Solving the Schrödinger or the Dirac equation without imposing a proper boundary-condition makes no sense and does not even define the relevant Hilbert space. For a well-defined Hilbert space in turn boundary-conditions at infinity, i.e., the vanishing of the wavefunctions at infinity, are usually sufficient. This kind of boundary-conditions are contained in the usual path integral formulation [15] in a homogeneous space for a regular potential in a natural way. Boundary-conditions connected with singular potentials require something different and can be taken into account by using a functional weight formulation [27]. The single valuedness conditions for the quantum motion on spheres is taken into account by periodic boundary-conditions [27]. Boundary-conditions at infinity, however, are a very specific idealization and for many more physical situations not appropriate. Typical experimental situations require boundary-conditions at finite distance form the origin, for instance motion in a half-space [8, 11, 22, 23, 27], in a box [10, 23], or some boundary-condition at a singular point [1, 2]. Here Dirichlet and Neumann boundary-conditions come into play as particular cases and can be incorporated into the path integral by considering the infinite strength limit of point interactions. As discussed in e.g. [17, 18, 21]–[26, 34] point interactions in turn can be incorporated in the path integral by, e.g., a simple δ-function perturbation. The path integral with this perturbation can be evaluated by the summation of a perturbation expansion, giving in the general case the energy-dependent Green function. The corresponding propagator can be obtained only in specific cases, e.g., for the free motion subject to a point interaction. The whole procedure can be repeated to incorporate arbitrarily many point interactions. The limit of infinitely repulsive δ-function perturbations gives Dirichlet boundary-conditions at the location of the point interaction. Repeating the procedure, one can state the Green function for a particle in a box, where an otherwise arbitrary well-behaving potential may be included.

In this contribution I would like to demonstrate on the one hand side how one can incorporate the four parameter family of the one-dimensional relativistic point interaction into the path integral, and one the other summarize its applications in non-relativistic quantum mechanics. Point interactions are often used to simplify more complicated interactions by a simple solvable model, may it quark-quark interactions in elementary particle physics [2] (and references therein), or electron-lattice interactions in solid state physics [2, 35, 39]. The non-relativistic limit includes the usual δ-function [4, 18, 21, 22, 36] perturbation, the δ'-function perturbation [2, 26], or more general the four parameter family of boundary-conditions on the real line [1, 9]. Dirichlet and Neumann boundary-conditions follow from limiting cases, where the strength of the point interaction becomes infinitely repulsive.

The emerging four parameter family for the one-dimensional point interaction can be understood in the following way. One considers a matrix valued point interaction for the one-dimensional Dirac particle

$$\mathbf{V}(x) = (\mathbb{1} g_V + \boldsymbol{\sigma}_z g_S)\delta(x-a) \ , \tag{1}$$

where the quantities g_V and g_S correspond to the vector and scalar coupling strength, and $\boldsymbol{\sigma}_z$ is a Pauli matrix. This specific form of the potential gives rise to a boundary-condition of the spinor wave functions at the location $x = a$ of the point interaction according to [13] ($\boldsymbol{\psi}$ is a two-dimensional spinor)

$$\boldsymbol{\psi}(a^+) = \boldsymbol{\Lambda}\boldsymbol{\psi}(a^-) \ , \qquad \boldsymbol{\Lambda} = \cos\sqrt{g_V^2 - g_S^2}\begin{pmatrix} 1 & -\mathrm{i}\alpha_- \\ -\mathrm{i}\alpha_+ & 1 \end{pmatrix} \ . \tag{2}$$

where one has introduced $\alpha_\pm = (g_V \pm g_S)\tan\sqrt{g_V^2 - g_S^2}/\sqrt{g_V^2 - g_S^2}$. From this representation it can be seen that there exists a four parameter family self-adjoint extension of the corresponding Dirac Hamiltonian with a point interaction: i) In the case that $g_V = g_S$ we have $\alpha_- = 0$ and $\alpha_+ = 2g_V$. This yields in the non-relativistic limit a δ-function perturbation. ii) In the case that $g_V = -g_S$ we have $\alpha_+ = 0$ and $\alpha_- = 2g_V$. This yields in the non-relativistic limit a δ'-function perturbation. iii) If $|g_V| \neq |g_S|$ and g_V, g_S real we get a matrix according to $\Lambda = \begin{pmatrix} \cos\lambda & \mathrm{i}\sin\lambda \\ \mathrm{i}\sin\lambda & \cos\lambda \end{pmatrix}$ with $\lambda = \sqrt{g_V^2 - g_S^2}$. iv) Finally we get for $|g_V| \neq |g_S|$ and g_V, g_S imaginary a matrix according to $\Lambda = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}$ with $\lambda = -\mathrm{i}\sqrt{|g_V^2 - g_S^2|}$ [5, 13, 38].

By a proper combination it is therefore possible to cover all the relevant parameters in this family with the cases i) and ii) as the building blocks. The corresponding point perturbations are additive and therefore it is possible to first evaluate a perturbative expansion for one point interaction, and in the second step for the other. Consequently, I can construct by a subsequent consideration of each of the four members of the family the Green's function of the entire problem. Taking the non-relativistic limit then gives the corresponding case of the four parameter family of the boundary-conditions at a point on the real line [1, 9]. However, because I want to give an overview how to incorporate boundary-conditions and point interactions in the path integral not every detail is worked out and will instead given elsewhere.

In the following the technique to achieve this is outlined. I concentrate on the two most important cases of $g_V = g_S$ and $g_V = -g_S$ which correspond to a δ- and δ'-function perturbation in the non-relativistic limit. After a short presentation of the relevant perturbation expansion and the proper path integral representation

of the propagator for the one-dimensional Dirac particle I use, these two specific cases are discussed in some detail. First each case independently, and then second a combination of them. Some of the results have already been announced in [26].

In a third Section I give some explicit formulæ for the non-relativistic case, i.e., point interactions and boundary-conditions for the one-dimensional Schrödinger particle. In the fourth Section I present some of the corresponding propagators explicitly including the case of the motion in a box. The fifth Section contains a summary.

2. Perturbation Expansions for a Dirac Particle.

The general method for the time-ordered perturbation expansion is quite simple. Let us assume that we are given a potential $W(x) = V(x) + \tilde V(x)$ in the path integral and suppose that W is so complicated that a direct path integration is not possible. However, the path integral $K^{(V)}$ corresponding to $V(x)$ is assumed to be known. We expand the integrand of the path integral containing $\tilde V(x)$ in a perturbation expansion about $V(x)$. The result has a simple interpretation on the lattice: the initial kernel corresponding to V propagates during the short-time interval ϵ unperturbed, then it interacts with $\tilde V$ in order to propagate again in another short-time interval ϵ unperturbed, and so on, up to the final state. One then obtains the following series expansion (Feynman and Hibbs [15], Devreese et al. [18]–[20] ($\mathbf{x} \in \mathbb{R}^D$)

$$\begin{aligned} K(\mathbf{x}'',\mathbf{x}';T) &= K^{(V)}(\mathbf{x}'',\mathbf{x}';T) + \sum_{n=1}^{\infty}\Big(-\frac{\mathrm{i}}{\hbar}\Big)^n\left(\prod_{j=1}^{n}\int_{t'}^{t_{j+1}}dt_j\int_{-\infty}^{\infty}d\mathbf{x}_j\right) \\ &\times K^{(V)}(\mathbf{x}_1,\mathbf{x}';t_1-t')\tilde V(\mathbf{x}_1)K^{(V)}(\mathbf{x}_2,\mathbf{x}_1;t_2-t_1)\times\ldots \\ &\times \tilde V(\mathbf{x}_{n-1})K^{(V)}(\mathbf{x}_n,\mathbf{x}_{n-1};t_n-t_{n-1})\tilde V(\mathbf{x}_n)K^{(V)}(\mathbf{x}'',\mathbf{x}_n;t''-t_n) \ . \end{aligned} \tag{3}$$

Here I have ordered time as $t' = t_0 < t_1 < t_2 < \ldots < t_{n+1} = t''$ and paid attention to the fact that $K(t_j - t_{j-1})$ denotes the retarded propagator and thus is different from zero only if $t_j \geq t_{j-1}$. Several problems in path integration which are definitely non-Gaussian, non-Besselian or non-Legendrian can be addressed by a perturbation expansion approach. Let us mention the incorporation of point interactions (Bauch [4], Goovaerts et al. [18, 19] and Refs. [21]–[26]) and boundary-conditions at finite distances [22, 23]. Also $1/r$- [20, 34] and $1/r^2$-potentials [34] can be treated by means of an exact summation of a perturbation expansion. Particularly in the case of the Coulomb potential this perturbation expansion is an expansion in powers of the coupling of the Coulomb interaction strength [20].

We consider the path integral representation for the matrix-valued kernel

$\mathbf{K}^{(V)}(T)$ for the one-dimensional Dirac equation [15, 16, 28]-[30] ($p_x = -\mathrm{i}\hbar\partial_x$)

$$\begin{aligned}\mathbf{K}^{(V)}(x'',x';T) &= \left\langle x''\middle|\exp\left[-\frac{\mathrm{i}}{\hbar}T\left(c\boldsymbol{\sigma}_x p_x + mc^2\boldsymbol{\sigma}_z + \mathbf{V}(x)\right)\right]\middle|x'\right\rangle \\ &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}\boldsymbol{\nu}(t)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\mathbf{V}(x)dt\right) \ .\end{aligned} \tag{4}$$

$\mathbf{V}$ may be a matrix-valued potential. The support property of the measure $\mathcal{D}\boldsymbol{\nu}$ is defined in such a way that the motion it is describing selects paths of N steps each of length $c\epsilon$ ($\epsilon = T/N$ in the lattice representation) that start at x' in the direction α, and end at x'' in the direction β, where α and β take the values "right" and "left". The path integration then is a summation over all reversings of directions [15]. $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_z$ are the Pauli matrices. We introduce the Green function $\mathbf{G}^{(V)}(E)$ with its matrix representation

$$\mathbf{G}^{(V)}(x'',x';E) = \begin{pmatrix} G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'',x';E) \\ G_{21}^{(V)}(x'',x';E) & G_{22}^{(V)}(x'',x';E)\end{pmatrix} \ . \tag{5}$$

We first consider a δ-function perturbation in the electron (= "+"-) component, i.e., $\hat{\mathbf{V}} = -\alpha\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\delta(x-a)$. We obtain by inserting it into the path integral and summing the perturbation expansion

$$\begin{aligned}\mathbf{G}^{(\delta+)}(x'',x';E) &= \mathbf{G}^{(V)}(x'',x';E) - \frac{1}{G_{11}^{(V)}(a,a;E) - 1/\alpha} \\ &\quad\times\begin{pmatrix} G_{11}^{(V)}(a,x';E)G_{11}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,x';E)G_{12}^{(V)}(x'',a;E) \\ G_{21}^{(V)}(a,x';E)G_{11}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,x';E)G_{12}^{(V)}(x'',a;E)\end{pmatrix}\end{aligned} \tag{6}$$

$$\begin{aligned}&= \frac{1}{G_{11}^{(V)}(a,a;E) - \frac{1}{\alpha}} \\ &\times\begin{pmatrix} \begin{vmatrix} G_{11}^{(V)}(x'',x';E) & G_{11}^{(V)}(a,x';E) \\ G_{11}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,a;E) - \frac{1}{\alpha}\end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'',x';E) & G_{11}^{(V)}(a,x';E) \\ G_{12}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,a;E) - \frac{1}{\alpha}\end{vmatrix} \\ \\ \begin{vmatrix} G_{21}^{(V)}(x'',x';E) & G_{21}^{(V)}(a,x';E) \\ G_{11}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,a;E) - \frac{1}{\alpha}\end{vmatrix} & \begin{vmatrix} G_{22}^{(V)}(x'',x';E) & G_{21}^{(V)}(a,x';E) \\ G_{12}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,a;E) - \frac{1}{\alpha}\end{vmatrix}\end{pmatrix} \ .\end{aligned} \tag{7}$$

Similarly for the positron (= "−"-) component, i.e., $\hat{\mathbf{V}} = (4m^2\beta c^2/\hbar^2)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\delta(x-b)$ (the constants have been chosen for convenience, $\hat{\beta} = 4m^2\beta c^2/\hbar^2$)

$$\begin{aligned}\mathbf{G}^{(\delta-)}(x'',x';E) &= \mathbf{G}^{(V)}(x'',x';E) - \frac{1}{\hbar^2/4m^2c^2\beta + G_{22}^{(V)}(b,b;E)} \\ &\quad\times\begin{pmatrix} G_{12}^{(V)}(b,x';E)G_{21}^{(V)}(x'',b;E) & G_{12}^{(V)}(b,x';E)G_{22}^{(V)}(x'',b;E) \\ G_{22}^{(V)}(b,x';E)G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,x';E)G_{22}^{(V)}(x'',b;E)\end{pmatrix}\end{aligned} \tag{8}$$

$$\begin{aligned}&= \frac{1}{G_{22}^{(V)}(b,b;E) + \frac{1}{\beta}} \\ &\times\begin{pmatrix} \begin{vmatrix} G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(b,x';E) \\ G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E) + \frac{1}{\beta}\end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'',x';E) & G_{12}^{(V)}(b,x';E) \\ G_{22}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E) + \frac{1}{\beta}\end{vmatrix} \\ \\ \begin{vmatrix} G_{21}^{(V)}(x'',x';E) & G_{22}^{(V)}(b,x';E) \\ G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E) + \frac{1}{\beta}\end{vmatrix} & \begin{vmatrix} G_{22}^{(V)}(x'',x';E) & G_{22}^{(V)}(b,x';E) \\ G_{22}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E) + \frac{1}{\beta}\end{vmatrix}\end{pmatrix} \ .\end{aligned} \tag{9}$$

Let us assume for simplicity that the component $G_{11}^{(V)}(E)$ in (5) is known and $\mathbf{V}$ is a scalar, then I can derive

$$G_{12}^{(V)}(x,y;E) = \frac{c}{mc^2 - V + E}p_x G_{11}^{(V)}(x,y;E) \ , \tag{10}$$

$$G_{22}^{(V)}(x,y;E) = \frac{-1}{mc^2 - V + E}\left(\frac{c^2}{mc^2 - V + E}p_x p_y G_{11}^{(V)}(x,y;E) + \delta(x-y)\right) \ . \tag{11}$$

From these representations it is easily seen that if $G_{11}^{(V)}(E)$ is of $O(1)$ for $c \to \infty$, $G_{12}^{(V)}(E)$ and $G_{22}^{(V)}(E)$ vanish according to $\propto 1/c$ and $\propto 1/c^2$ for $c \to \infty$, respectively.

In the next step we want to incorporate more than just one relativistic point interaction. Let us first study the case were we have two δ-function perturbations in the electron component with strength α_1, α_2 located at $x = a_1, x = a_2$, respectively. We obtain for the (11)-component

$$G_{11}^{(\delta_+,\delta_+)}(x'',x';E) = \frac{\begin{vmatrix} G_{11}^{(V)}(x'',x';E) & G_{11}^{(V)}(x'',a_1;E) & G_{11}^{(V)}(x'',a_2;E) \\ G_{11}^{(V)}(a_1,x';E) & G_{11}^{(V)}(a_1,a_1;E) - 1/\alpha_1 & G_{11}^{(V)}(a_1,a_2;E) \\ G_{11}^{(V)}(a_2,x';E) & G_{11}^{(V)}(a_2,a_1;E) & G_{11}^{(V)}(a_2,a_2;E) - 1/\alpha_2\end{vmatrix}}{\begin{vmatrix} G_{11}^{(V)}(a_1,a_1;E) - 1/\alpha_1 & G_{11}^{(V)}(a_1,a_2;E) \\ G_{11}^{(V)}(a_2,a_1;E) & G_{11}^{(V)}(a_2,a_2;E) - 1/\alpha_2\end{vmatrix}} \ . \tag{12}$$

Let us abbreviate

$$D_{\alpha_1\alpha_2}(a_1,a_2;E) = \begin{vmatrix} G_{11}^{(V)}(a_1,a_1;E_n) - 1/\alpha_1 & G_{11}^{(V)}(a_1,a_2;E_n) \\ G_{11}^{(V)}(a_2,a_1;E_n) & G_{11}^{(V)}(a_2,a_2;E_n) - 1/\alpha_2\end{vmatrix} \ . \tag{13}$$

The energy-levels are determined by the poles of the denominator and implicitly given by

$$D_{\alpha_1\alpha_2}(a_1,a_2;E_n) = 0 \ . \tag{14}$$

For the (12)-component I get

$$G_{12}^{(\delta_+,\delta_+)}(x'',x';E)$$
$$= \frac{1}{[G_{11}^{(V)}(a_2,a_2;E) - \frac{1}{\alpha_2}]D_{\alpha_1\alpha_2}(a_1,a_2;E)}$$
$$\times \begin{vmatrix} \begin{vmatrix} G_{12}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'';a_2;E) \\ G_{11}^{(V)}(a_2,x'E) & G_{11}^{(V)}(a_2,a_2;E) - \frac{1}{\alpha_2} \end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'',a_1;E) & G_{12}^{(V)}(a_2,a_1;E) \\ G_{11}^{(V)}(x'',a_2;E) & G_{11}^{(V)}(a_2,a_2;E) - \frac{1}{\alpha_2} \end{vmatrix} \\ \begin{vmatrix} G_{11}^{(V)}(a_1,x';E) & G_{11}^{(V)}(a_2,x';E) \\ G_{11}^{(V)}(a_1,a_2;E) & G_{11}^{(V)}(a_2,a_2;E) - \frac{1}{\alpha_2} \end{vmatrix} & D_{\alpha_1\alpha_2}(a_1,a_2;E) \end{vmatrix} \tag{15}$$

It is not possible to rewrite the determinant in the determinant into just one 3×3-determinant as for the (11)-component because the number of relevant entries is too large. The other components are similar.

Let us finally combine a point interaction in the electron component with a point interaction in the positron component. The (11)-component has the form

$$G_{11}^{(\delta_+,\delta_-)}(x'',x';E)$$
$$= \frac{1}{[G_{22}^{(V)}(b,b;E) + \frac{1}{\beta}]D_{\alpha\beta}(a,b;E)}$$
$$\times \begin{vmatrix} \begin{vmatrix} G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'';b;E) \\ G_{21}^{(V)}(b,x'E) & G_{22}^{(V)}(b,b;E) + \frac{1}{\beta} \end{vmatrix} & \begin{vmatrix} G_{11}^{(V)}(x'',a;E) & G_{12}^{(V)}(b,a;E) \\ G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E) - \frac{1}{\beta} \end{vmatrix} \\ \begin{vmatrix} G_{11}^{(V)}(a,x';E) & G_{12}^{(V)}(b,x';E) \\ G_{21}^{(V)}(a,b;E) & G_{22}^{(V)}(b,b;E) + \frac{1}{\beta} \end{vmatrix} & D_{\alpha\beta}(a,b;E) \end{vmatrix} \tag{16}$$

Here I have abbreviated

$$D_{\alpha_1\alpha_2}(a,b;E) = \begin{vmatrix} G_{11}^{(V)}(a,a;E_n) - 1/\alpha & G_{11}^{(V)}(a,b;E_n) \\ G_{11}^{(V)}(b,a;E_n) & G_{11}^{(V)}(b,b;E_n) + 1/\beta \end{vmatrix} . \tag{17}$$

The energy-levels are determined by the poles of the denominator and implicitly given by $D_{\alpha\beta}(a,b;E_n) = 0$. The other components are similar. The special case that the point interaction is proportional to $\mathbb{1}$ or $\boldsymbol{\sigma}_z$ has been discussed in [38] and is in accordance with our results. In this case the point perturbations of each component contribute additively to the Green's function.

3. The Non-Relativistic Limit.

3.1. δ-Functions.

We consider the incorporation of δ-function perturbations, i.e., a δ-function as an additional potential located at $x = a$ with strength γ. Only a closed formula for the corresponding Green's function can be stated; an explicit result for the propagator can only be obtained in the simplest or in some exceptional cases, e.g., for $V \equiv 0$. One obtains [21]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x) + \gamma\delta(x-a)\right]dt\right\}$$
$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E) - 1/\gamma} . \tag{18}$$

Here $G^{(V)}(E)$ denotes the Green's function for the unperturbed problem ($\gamma = 0$). Possible bound states are determined by the poles of $G(E)$, i.e., by the equation $G^{(V)}(a,a,E_n) = 1/\gamma$.

3.2. δ′-Functions.

The next case incorporates a δ'-function perturbation. Taking the non-relativistic limit of $G_{11}^{(\delta_-)}(E)$ one obtains for a δ'-function perturbation in the path integral the representation

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x) + \beta\delta'(x-a)\right]dt\right\}$$
$$= G^{(V)}(x'',x';E) - \frac{G_{,x'}^{(V)}(x'',a;E)G_{,x''}^{(V)}(a,x';E)}{\widehat{G}_{,x'x''}^{(V)}(a,a;E) + 1/\beta} . \tag{19}$$

$$\widehat{G}_{,xy}^{(V)}(a,a;E) = \left(\frac{\partial^2}{\partial x \partial y}G^{(V)}(x,y;E) - \frac{2m}{\hbar^2}\delta(x-y)\right)\Big|_{x=y=a} . \tag{20}$$

Note that in the path integral (19) the formal expression "$G_{,xy}\,(a,a;E)$" is automatically regularized by the removal of an ultraviolet divergence. This regularization prescription is not put in "by hand" but is *a result*.

3.3. Combination of δ and δ′-Functions.

From the above considerations it is obvious how to obtain the Green's function representation of a combined δ- and δ′-function perturbation. We get

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)+\alpha\delta(x-a)+\beta\delta'(x-b)\right]dt\right\}$$
$$=\frac{\begin{vmatrix}G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}(x'',a;E)\\ G^{(V)}_{,x''}(b,x';E) & \widehat{G}^{(V)}_{,x'x''}(b,b;E)+1/\beta & G^{(V)}_{,x''}(b,a;E)\\ G^{(V)}(a,x';E) & G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E)-1/\alpha\end{vmatrix}}{\begin{vmatrix}\widehat{G}^{(V)}_{,x'x''}(b,b;E)+1/\beta & G^{(V)}_{,x''}(b,a;E)\\ G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E)-1/\alpha\end{vmatrix}} \quad . \tag{21}$$

Setting $a=b$ yields a special case (of boundary-conditions).

3.4. Dirichlet Boundary-Conditions.

The case of (Dirichlet) boundary-conditions, respectively the motion in a half-space, have been addressed by several authors in order to develop a method to incorporate them into the path integral, e.g., Barut and Duru [3], Clark et al. [11], Carreau [8], and [22, 23]. In our formalism Dirichlet boundary-conditions are obtained when we consider in (18) the limit $\gamma\to-\infty$. This has the consequence that an impenetrable wall appears at $x=a$. The result then is for the motion in the half-space $x>a$, say, [22, 23]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(D)}_{(x>a)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=G^{(V)}(x'',x';E)-\frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)} \quad . \tag{22}$$

Possible bound states are determined by the poles of $G(E)$, i.e., by the equation $G^{(V)}(a,a,E_n)=0$. Furthermore, for the motion inside a box with boundaries at $x=a$ and $x=b$ and Dirichlet boundary-conditions at both sides one obtains ($a<x<b$) [10, 22, 23]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(DD)}_{(a<x<b)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$

$$=\frac{\begin{vmatrix}G^{(V)}(x'',x';E) & G^{(V)}(x'',b;E) & G^{(V)}(x'',a;E)\\ G^{(V)}(b,x';E) & G^{(V)}(b,b;E) & G^{(V)}(b,a;E)\\ G^{(V)}(a,x';E) & G^{(V)}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}}{\begin{vmatrix}G^{(V)}(b,b;E) & G^{(V)}(b,a;E)\\ G^{(V)}(a,b;E) & G^{(V)}(a,a;E)\end{vmatrix}} \quad . \tag{23}$$

3.5. Neumann Boundary-Conditions.

In an obvious way we can also obtain a path integral representation in the half-space $x>a$, say, with Neumann boundary-conditions at $x=a$ by letting $\beta\to-\infty$ in (19) [25, 26]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(N)}_{(x>a)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=G^{(V)}(x'',x';E)-\frac{G^{(V)}_{,x'}(x'',a;E)G^{(V)}_{,x''}(a,x';E)}{\widehat{G}^{(V)}_{,x'x''}(a,a;E)} \quad . \tag{24}$$

The same procedure as for the motion in a box $a<x<b$ with Dirichlet boundary-conditions at both boundaries, can be applied for Neumann boundary-conditions at both boundaries

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(NN)}_{(a<x<b)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$
$$=\frac{\begin{vmatrix}G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}_{,x'}(x'',a;E)\\ G^{(V)}_{,x''}(b,x';E) & \widehat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x'x''}(b,a;E)\\ G^{(V)}_{,x''}(a,x';E) & G^{(V)}_{,x'x''}(a,b;E) & \widehat{G}^{(V)}_{,x'x''}(a,a;E)\end{vmatrix}}{\begin{vmatrix}\widehat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x'x''}(b,a;E)\\ G^{(V)}_{,x'x''}(a,b;E) & \widehat{G}^{(V)}_{,x'x''}(a,a;E)\end{vmatrix}} \quad . \tag{25}$$

Similarly we obtain for Dirichlet boundary-conditions at $x=a$, and Neumann boundary-conditions for $x=b$ in the box $a<x<b$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT e^{\mathrm{i}ET/\hbar}\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(DN)}_{(a<x<b)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]dt\right\}$$

$$= \frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}(x'',a;E) \\ G^{(V)}_{,x''}(b,x';E) & \hat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x''}(b,a;E) \\ G^{(V)}(a,x';E) & G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E) \end{vmatrix}}{\begin{vmatrix} \hat{G}^{(V)}_{,x'x''}(b,b;E) & G^{(V)}_{,x''}(b,a;E) \\ G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E) \end{vmatrix}} . \tag{26}$$

Radial boxes and rings can be taken into account as well, and potentials with absolute value dependence by combining the results for Dirichlet and Neumann boundary-conditions, i.e. [26]:

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(|x|)\right]dt\right\}$$

$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',0;E)G^{(V)}(0,x';E)}{2G^{(V)}(0,0;E)} - \frac{G^{(V)}_{,x'}(x'',0;E)G^{(V)}_{,x''}(0,x';E)}{2\hat{G}^{(V)}_{,x'x''}(0,0;E)} . \tag{27}$$

3.6. Two- and Three-Dimensional Point Interactions.

It is also possible to incorporate two- and three-dimensional δ-function perturbations in the path integral [2, 24, 31]. In order to do this a ultra-violet regularization must be performed. We incorporate this case for completeness. The idea of the regularization prescription is to modify the original domain of the Hamiltonian in a suitable way, i.e., to take into account the singular modes in such a way that the corresponding extension is self-adjoint. For this purpose one considers the operator by including the point interaction ($r > 0$)

$$\dot{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2}{2m}\frac{\lambda(\lambda-1)}{r^2} + \frac{\eta}{r} + \beta r^{-a} + \hat{V}(r) , \tag{28}$$

and $\dot{H}$ is said to have deficiency index (1,1). The results are summarized in the following theorem [2, 6]

Theorem 1 *Let*

$$F^{(0)}_\lambda(r) = r^\lambda , \qquad G^{(0)}_\lambda(r) = \begin{cases} -\frac{m}{2\pi\hbar^2}\sqrt{r}\ln r , & \lambda = \frac{1}{2} , \\ \frac{m}{2\pi\hbar^2} , & \lambda = 1 , \\ \frac{m}{2\pi\hbar^2}\frac{r^{1-\lambda}}{2\lambda-1} . \end{cases} \tag{29}$$

All self-adjoint extensions of the operator $\dot{H}$ are given by

$$H_\alpha = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2}{2m}\frac{\lambda(\lambda-1)}{r^2} + \frac{\eta}{r} + \beta r^{-a} + \hat{V}(r) , \tag{30}$$

$$\mathcal{D}(H_\alpha) = \left\{ g \in L^2(\mathbb{R}^+) \Big| g, g' \in AC_{loc}(\mathbb{R}^+); \alpha g_{0,\lambda} = g_{1,\lambda}; H_\alpha g \in L^2(\mathbb{R}^+) \right\} , \tag{31}$$

$$-\infty < [\alpha] \le \infty, \quad \frac{1}{2} \le \lambda < \frac{3}{2}, \quad [\beta,\eta] \in \mathbb{R}, \qquad 0 < a < 2 .$$

$\hat{V} \in L^\infty(\mathbb{R}^+)$ is real valued, and $AC_{loc}(\mathbb{M})$ denotes the set of absolutely continuous functions on $\mathbb{M}$ (here $= \mathbb{R}^+$). The regularizing functions $g_{0,\lambda}$ and $g_{1,\lambda}$ are defined by

$$g_{0,\lambda} = \lim_{r\to 0^+}\frac{g(r)}{G^{(0)}_\lambda(r)} , \qquad g_{1,\lambda} = \lim_{r\to 0^+}\frac{g(r) - g_{0,\lambda}G^B_\lambda(r)}{F^{(0)}_\lambda(r)} , \tag{32}$$

where $G^B_\lambda(r)$ denotes the asymptotic expansion of the irregular solution $G_\lambda(r)$ of the unperturbed problem up to order r^t, $t \le 2\lambda - 1$. Generally one has $G^B_\lambda(r) = G^{(0)}_\lambda(r) +$ additional terms, where $G^{(0}_\lambda$ denotes the free particle case. $g(r)$ is a solution of the corresponding unperturbed problem, and α is the (regularized) coupling.

To take into account the $\beta \neq 0$ contributions complicates the expressions considerably and will not be stated here [6]. $\alpha = \infty$ corresponds to the Friedrichs extension of the operator $\dot{H}$, and $|[\alpha]| < \infty$ describes a point-interaction. Two special cases of $G^B_\lambda(r)$ can be stated for $\lambda = \frac{1}{2}$ and 1, i.e., for the Schrödinger operator in two and three dimensions, respectively, which will be sufficient for our purposes. Then

$$G^B_{\frac{1}{2}}(r) = G^{(0)}_{\frac{1}{2}}(r) = -\frac{m}{\pi\hbar^2}\sqrt{r}\ln r , \tag{33}$$

$$G^B_1(r) = \frac{m}{2\pi\hbar^2}\left(1 - \frac{m\eta}{\hbar^2}r - \frac{2m\eta}{\hbar^2}r\ln r\right) . \tag{34}$$

By these means, the incorporation of a point interaction in two and three dimensions in the path integral is then defined by ($\frac{1}{2} \le \lambda < \frac{3}{2}, \eta \in \mathbb{R}$).

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty dT\, \mathrm{e}^{\mathrm{i}ET/\hbar} \int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''} \mathcal{D}_{\Gamma^{(V)}_{\alpha,\mathbf{a}}}\mathbf{x}(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\mathbf{x}}^2 - V(\mathbf{x})\right]dt\right\}$$

$$= G^{(V)}(\mathbf{x}'',\mathbf{x}';E) + (\Gamma^{(V)}_{\gamma,\mathbf{a}}(E))^{-1}G^{(V)}(\mathbf{x}'',\mathbf{a};E)G^{(V)}(\mathbf{a},\mathbf{x}';E) . \tag{35}$$

$$V(\mathbf{x}) = \frac{\hbar^2}{2m}\frac{\lambda(\lambda-1)}{|\mathbf{x}|^2} + \frac{\eta}{|\mathbf{x}|} , \qquad \Gamma^{(V)}_{\alpha,\mathbf{a}}(E) = \alpha g_{0,\lambda} - g_{1,\lambda} . \tag{36}$$

Generally, only the Green's function can be stated. Only in some very specific cases, it seems possible to state also the the corresponding propagator, c.f. [1, 37]. The form of the propagator in the two-dimensional case is more complicated and involves a double integral [1, 23]. Some examples of more complicated structure have been investigated in [23].

4. Examples.

It is not possible but in the simplest examples to explicitly state the propagator in closed form. Generally only the free particle case can be treated.

4.1. Relativistic Point Interaction.

We consider the unperturbed free particle; the explicit expression for $\mathbf{G}^{(0)}(E)$ has the form [2]:

$$\mathbf{G}^{(0)}(x'',x';E)=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta & \mathrm{sign}(x''-x')\\ \mathrm{sign}(x''-x') & 1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}\ . \tag{37}$$

where $\zeta=(E+mc^2)/ck\hbar$, $ck\hbar=\sqrt{E^2-m^2c^4}$. This yields for a δ-function perturbation in the electron component:

$$\begin{aligned}\mathbf{G}^{(\delta+)}(x'',x';E)&=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta & \mathrm{sign}(x''-x')\\ \mathrm{sign}(x''-x') & 1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}\\ &-\frac{\alpha\mathrm{e}^{\mathrm{i}k(|x''-a|+|a-x'|)}}{4c\hbar(c\hbar-\mathrm{i}\alpha\zeta/2)}\begin{pmatrix}\zeta^2 & \zeta\,\mathrm{sign}(x''-a)\\ \zeta\,\mathrm{sign}(a-x') & \mathrm{sign}(x''-a)\mathrm{sign}(a-x')\end{pmatrix}\ .\end{aligned}\tag{38}$$

For $[\alpha]>0$ there is one bound state with energy $E=mc^2(1-\lambda^2)/(1+\lambda^2)$ ($\lambda=\alpha/2c\hbar$). Similarly for a δ-function perturbation in the positron component

$$\begin{aligned}\mathbf{G}^{(\delta-)}(x'',x';E)&=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta & \mathrm{sign}(x''-x')\\ \mathrm{sign}(x''-x') & 1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}\\ &+\frac{2m^2\beta\mathrm{e}^{\mathrm{i}k(|x''-a|+|a-x'|)}}{\hbar(2\hbar^3+4\mathrm{i}m^2c\beta/\zeta)}\begin{pmatrix}\mathrm{sign}(x''-a)\mathrm{sign}(a-x') & \mathrm{sign}(a-x')/\zeta\\ \mathrm{sign}(x''-a)/\zeta & 1/\zeta^2\end{pmatrix}\ .\end{aligned}\tag{39}$$

For $[\beta]>0$ there is one bound state with energy $E=-mc^2(1-\lambda^2)/(1+\lambda^2)$ ($\lambda=2m^2c\beta/\hbar^3$).

4.2. δ-Function.

Let us consider a simple δ-function potential in the path integral. We obtain the solution [4, 7, 12, 17, 18, 21, 33, 36]

$$\begin{aligned}&\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2+\gamma\delta(x)\right]dt\right\}\\ &=\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left[\frac{\mathrm{i}m}{2\hbar T}(x''-x')^2\right]\\ &\quad+\frac{m\gamma}{2\hbar^2}\exp\left(-\frac{m\gamma}{\hbar^2}(|x''-a|+|x'-a|)+\frac{\mathrm{i}}{\hbar}\frac{m\gamma^2}{2\hbar^2}T\right)\\ &\qquad\times\mathrm{erfc}\left[\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\left(|x''-a|+|x'-a|-\frac{\mathrm{i}}{\hbar}\gamma T\right)\right]\ .\end{aligned}\tag{40}$$

Some more examples have been investigated in [21], and the case of the harmonic oscillator with a δ-function in [32].

4.3. δ'-Function.

Let us consider a δ'-function potential in the path integral (and the notion δ'-function should not to be taken too literally [2]. We obtain the solution [1, 26]

$$\begin{aligned}&\int\limits_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2+\beta\delta'(x-a)\right]dt\right\}\\ &=\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left(\frac{\mathrm{i}m}{2\hbar T}|x''-x'|^2\right)+\mathrm{sign}(x''-a)\mathrm{sign}(x'-a)\\ &\quad\times\left(\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left[\frac{\mathrm{i}m}{2\hbar T}(|x''-a|+|x'-a|)^2\right]\mathrm{sign}(x''-a)\mathrm{sign}(x'-a)\right.\\ &\quad+\frac{\hbar^2}{2m\beta}\exp\left[-\frac{\hbar^2}{m\beta}\Big(|x''-a|+|x'-a|\Big)+\frac{\mathrm{i}}{\hbar}\frac{\hbar^6}{2m^3\beta^2}T\right]\\ &\qquad\left.\times\mathrm{erfc}\left\{\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\left[\Big(|x''-a|+|x'-a|\Big)-\frac{\mathrm{i}\hbar^3T}{m^2\beta}\right]\right\}\right)\end{aligned}\tag{41}$$

$$\begin{aligned}&=\frac{\hbar^2}{m\beta}\exp\left[-\frac{\hbar^2}{m\beta}\Big(|x''-a|+|x'-a|\Big)+\frac{\mathrm{i}}{\hbar}\frac{\hbar^6}{2m^3\beta^2}T\right]\mathrm{sign}(x''-a)\mathrm{sign}(x'-a)\\ &\quad+\frac{1}{2\pi}\int_{\mathbb{R}}dp\exp\left(-\mathrm{i}\frac{p^2\hbar}{2m}T\right)\left(\sin px'\sin px''+\cos px'\cos px''\right.\\ &\qquad\left.+\frac{\mathrm{i}mp\beta/\hbar^2}{1+\mathrm{i}pm\beta/\hbar^2}\mathrm{e}^{\mathrm{i}p(|x'-a|+|x''-a|)}\mathrm{sign}(x''-a)\mathrm{sign}(x'-a)\right)\ .\end{aligned}\tag{42}$$

4.4. Motion in a Box: Dirichlet-Dirichlet Boundary-Conditions.

Let us consider free motion in a box with Dirichlet boundary-conditions at $x=-b$ and $x=b$. The general method gives for the Green's function ($\kappa=\sqrt{-2mE}/\hbar$)

$$G^{(DD)}(x'',x';E)=\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\cosh\left[\kappa(|x''-x'|-2b))\right]-\cosh\left[\kappa(x''+x')\right]}{\sinh(2\kappa b)}\ . \tag{43}$$

The energy spectrum follows from the poles of the Green's function yielding

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{4b^2} , \qquad n \in \mathbb{N} . \tag{44}$$

By means of the Laplace transformation pair [14, p.224]

$$\Theta_3\Big(\frac{1}{2}+\frac{x}{2l}\Big|\frac{i\pi\tau}{l^2}\Big) \quad \Leftrightarrow \quad \frac{l}{\sqrt{s}}\frac{\cosh(z\sqrt{s})}{\sinh(l\sqrt{s})} , \qquad |x| < l , \tag{45}$$

I obtain for the propagator the following representations

$$K^{(DD)}(x'',x';T) = \sqrt{\frac{m}{2\pi i\hbar T}}\sum_{n\in\mathbb{Z}}\bigg\{\exp\bigg[\frac{im}{2\hbar T}(x''-x'+4nb)^2\bigg] - \exp\bigg[\frac{im}{2\hbar T}(x''+x'+2(2n+1)b)^2\bigg]\bigg\} \tag{46}$$

$$= \frac{1}{4b}\bigg[\Theta_3\Big(\frac{|x''-x'|}{4b}\Big|-\frac{\pi\hbar T}{8mb^2}\Big) - \Theta_3\Big(\frac{x''+x'}{4b}+\frac{1}{2}\Big|-\frac{\pi\hbar T}{8mb^2}\Big)\bigg] \tag{47}$$

$$= \frac{1}{b}\sum_{n=1}^{\infty}\exp\Big(-i\hbar T\frac{\pi^2 n^2}{8mb^2}\Big)\sin\Big[\frac{\pi n}{2b}(x'+b)\Big]\sin\Big[\frac{\pi n}{2b}(x''+b)\Big] , \tag{48}$$

and I have used some properties of the Jacobi-Theta function $\Theta_3(z|q)$.

4.5. Motion in a Box: Neumann-Neumann Boundary-Conditions.

Let us consider as the next example free motion in a box with Neumann boundary-conditions at $x=-b$ and $x=b$. The general method gives for the Green's function ($\kappa = \sqrt{-2mE}/\hbar$)

$$G^{(NN)}(x'',x';E) = \frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\cosh[\kappa(|x''-x'|-2b))] + \cosh[\kappa(x''+x')]}{\sinh(2\kappa b)} . \tag{49}$$

The energy spectrum follows from the poles of the Green's function yielding

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{4b^2} , \qquad n \in \mathbb{N}_0 . \tag{50}$$

By means of the same Laplace transformation pair as before I obtain for the propagator the following representations ($\epsilon_0 = 1, \epsilon_n = 1, n \in \mathbb{N}_0$)

$$K^{(NN)}(x'',x';T) = \sqrt{\frac{m}{2\pi i\hbar T}}\sum_{n\in\mathbb{Z}}\bigg\{\exp\bigg[\frac{im}{2\hbar T}(x''-x'+4nb)^2\bigg] + \exp\bigg[\frac{im}{2\hbar T}(x''+x'+2(2n+1)b)^2\bigg]\bigg\} \tag{51}$$

$$= \frac{1}{4b}\bigg[\Theta_3\Big(\frac{|x''-x'|}{4b}\Big|-\frac{\pi\hbar T}{8mb^2}\Big) + \Theta_3\Big(\frac{x''+x'}{4b}+\frac{1}{2}\Big|-\frac{\pi\hbar T}{8mb^2}\Big)\bigg] \tag{52}$$

$$= \frac{1}{2b}\sum_{n=0}^{\infty}\epsilon_n\exp\Big(-i\hbar T\frac{\pi^2 n^2}{8mb^2}\Big)\cos\Big[\frac{\pi n}{2b}(x'+b)\Big]\cos\Big[\frac{\pi n}{2b}(x''+b)\Big] . \tag{53}$$

4.6. Motion in a Box: Dirichlet-Neumann Boundary-Conditions.

Let us finally consider free motion in a box with Dirichlet boundary-conditions at $x=-b$ and Neumann boundary-conditions at $x=b$. The general method gives for the Green's function ($\kappa = \sqrt{-2mE}/\hbar$)

$$G^{(DN)}(x'',x';E) = -\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\sinh[\kappa(|x''-x'|-2b))] - \sinh[\kappa(x''+x')]}{\cosh(2\kappa b)} . \tag{54}$$

The energy spectrum follows from the poles of the Green's function yielding

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 (n+\frac{1}{2})^2}{4b^2} , \qquad n \in \mathbb{N}_0 . \tag{55}$$

By means of the Laplace transformation pair [14, p.224]

$$\Theta_2\Big(\frac{1}{2}+\frac{x}{2l}\Big|\frac{i\pi\tau}{l^2}\Big) \quad \Leftrightarrow \quad -\frac{l}{\sqrt{s}}\frac{\sinh(z\sqrt{s})}{\cosh(l\sqrt{s})} , \qquad |x| < l , \tag{56}$$

I obtain for the propagator the following representations

$$K^{(DN)}(x'',x';T) = \sqrt{\frac{m}{2\pi i\hbar T}}\sum_{n\in\mathbb{Z}}(-1)^n\bigg\{\exp\bigg[\frac{im}{2\hbar T}(x''-x'+4nb)^2\bigg] - \exp\bigg[\frac{im}{2\hbar T}(x''+x'+2(2n+1)b)^2\bigg]\bigg\} \tag{57}$$

$$= \frac{1}{4b}\bigg[\Theta_2\Big(\frac{|x''-x'|}{4b}\Big|-\frac{\pi\hbar T}{8mb^2}\Big) - \Theta_2\Big(\frac{x''+x'}{4b}+\frac{1}{2}\Big|-\frac{\pi\hbar T}{8mb^2}\Big)\bigg] \tag{58}$$

$$= \frac{1}{b}\sum_{n=0}^{\infty}\exp\Big(-i\hbar T\frac{\pi^2 (n+\frac{1}{2})^2}{8mb^2}\Big)\sin\Big[\frac{\pi(n+\frac{1}{2})}{2b}(x'+b)\Big]\sin\Big[\frac{\pi(n+\frac{1}{2})}{2b}(x''+b)\Big] . \tag{59}$$

4.7. δ-Function in Three Dimensions.

As a miscellaneous result I cite the case of the free motion in $\mathbb{R}^3$ subject to a point-interaction. We have [37]

$$\int\limits_{\mathbf{x}(t')=\mathbf{x}'}^{\mathbf{x}(t'')=\mathbf{x}''}\mathcal{D}_{\Gamma^{(0)}_{\alpha,\mathbf{a}}}\mathbf{x}(t)\exp\Big(\frac{im}{2\hbar}\int_{t'}^{t''}\dot{\mathbf{x}}^2 dt\Big)$$

$$= K^{(0)}(x'',x';T) + \frac{1}{|\mathbf{a}-\mathbf{x}'||\mathbf{x}''-\mathbf{a}|}\int_0^\infty e^{-2\pi\alpha\hbar u/m}(u+|\mathbf{a}-\mathbf{x}'|+|\mathbf{x}''-\mathbf{a}|) \times K^{(0)}(u+|\mathbf{a}-\mathbf{x}'|+|\mathbf{x}''-\mathbf{a}|,0;T)du , \tag{60}$$

$$= K^{(0)}(\mathbf{x}'',\mathbf{x}';T) + \frac{i\hbar T}{m|\mathbf{a}-\mathbf{x}'||\mathbf{x}''-\mathbf{a}|}K^{(0)}(|\mathbf{a}-\mathbf{x}'|+|\mathbf{x}''-\mathbf{a}|,0;T) . \tag{61}$$

$$= K^{(0)}(\mathbf{x}'',\mathbf{x}';T) + \Psi^{(\alpha)}(\mathbf{x}')\Psi^{(\alpha)}(\mathbf{x}'')e^{iE^{(\alpha)}T/\hbar}$$
$$+\frac{1}{|\mathbf{a}-\mathbf{x}'||\mathbf{x}''-\mathbf{a}|}\int_0^\infty e^{-2\pi|\alpha|\hbar u/m}(u-|\mathbf{a}-\mathbf{x}'|-|\mathbf{x}''-\mathbf{a}|)$$
$$\times K^{(0)}(u-|\mathbf{a}-\mathbf{x}'|-|\mathbf{x}''-\mathbf{a}|,0;T)du\ , \tag{62}$$
$$K^{(0)}(\mathbf{x},\mathbf{y};T) = \left(\frac{m}{2\pi i\hbar T}\right)^{3/2}\exp\left(-\frac{m}{2i\hbar T}|\mathbf{x}-\mathbf{y}|^2\right)\ . \tag{63}$$

for $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$, respectively, the bound state wave-function and the energy spectrum $E^{(\alpha)}$

$$\Psi^{(\alpha)}(\mathbf{x}) = \sqrt{-\frac{\alpha\hbar^2}{m}}\,\frac{e^{-2\pi\alpha\hbar|\mathbf{x}-\mathbf{a}|/m}}{|\mathbf{x}-\mathbf{a}|}\ ,\qquad E^{(\alpha)} = -2\pi^2\frac{\alpha^2\hbar^6}{m^3}\ . \tag{64}$$

5. Summary.

In this contribution I have shown the various features of point interactions in the path integral. I have started from the path integral representation of the one-dimensional Dirac particle with a point interaction incorporated. By considering two kinds of point interactions I have been able to derive the corresponding Green's functions by means of an exact summation of a perturbation expansion which served as the building blocks for a further investigation. In the general case of multiple point interactions it seems to be not possible to derive a simple determinant expression as for the non-relativistic case. One obtains a matrix whose entries are determinants within determinants etc. In the non-relativistic limit they have corresponded to a δ- and a δ'-function perturbation, respectively. I could derive the general feature of the Green's function for the four parameter family point interaction for the one-dimensional Dirac particle thus providing a unified approach. Considering the non-relativistic limit, the corresponding (parameterized) point interactions for a one-dimensional Schrödinger particle can been derived. The limit of infinitely repulsive point interactions has yielded Dirichlet and Neumann boundary-conditions, respectively. I demonstrated the technique by several examples, and for the cases where the propagator could be stated explicitly. Some miscellaneous results have also been stated. Therefore it is possible to incorporate general boundary-conditions in the path integral in an explicit way by means of a singular perturbation.

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