# Long-range effects in asymptotic fields and angular momentum of classical field electrodynamics

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Asymptotic properties of classical field electrodynamics are considered. Special attention is paid to the long-range structure of the electromagnetic field. It is shown that conserved Poincaré quantities may be expressed in terms of the asymptotic fields. Long-range variables are shown to be responsible for an angular momentum contribution which mixes Coulomb and infrared free field characteristics; otherwise angular momentum and energy-momentum separate into electromagnetic and matter fields contributions. © *1995 American Institute of Physics.*

## I. INTRODUCTION

It is well-known that the long-range character of electromagnetic field causes certain peculiarities in quantum electrodynamics. Among them the infraparticle problem and breaking of the Lorentz symmetry are the most spectacular ones, for a review see the book by Haag[1] and an article by Morchio and Strocchi.[2] These properties can be traced back, as shown most clearly by Buchholz,[3] to the fact provable within the standard system of ideas on properties of quantum electrodynamics that the flux of electromagnetic field at spacelike infinity is an essentially classical variable supplying a label for uncountably many superselection sectors.[4] Whether these are ultimate features of the quantum theory of electromagnetic interaction or artifacts due to our unsufficient understanding of its algebraic structure is in our opinion an open question as long as we lack consistent, complete QED beyond Feynman graphs. Doubts about completeness of the present-state knowledge of the long-range structure can also be raised on grounds that it tells nothing about the quantization of charge or the magnitude of the fine-structure constant; see works by Staruszkiewicz on this point.[5,6]

In the present work we try to better understand the long-range structure of electrodynamics in classical field theory. We believe that in this way one can gain new insights into the quantum case as well. The domain in which the classical structure is most likely to be of some relevance for the quantum case is the asymptotic region. Rigorous results on the asymptotics of electromagnetic field are presented in Sec. II and on the asymptotics of Dirac field in Sec. IV. The results are relevant for the interacting theory, as argued in Secs. II and V. In that case, some additional assumptions are made which seem plausible, but remain unproved. Our main objective, when discussing the asymptotic fields, is the description of the specific way how matter and radiation separate in the asymptotic regions. In this respect, the approach of the present paper differs from that of Flato, Simon, and Taflin, who have recently described rigorous results on Cauchy problem and scattering states in classical Maxwell–Dirac theory;[7] see also a comment in Sec. V. Using results on asymptotic fields we express energy momentum and angular momentum of the system in terms of those fields.

We stress that our aim is not a purely mathematical study in classical field theory. Rather, with quantization in mind, we try to get a reasonably well-founded notion of the asymptotic structure of fields and conserved Poincaré quantities.

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Some of the results described in the present work were reported earlier in a letter.[8] Quantization of the long-range variables within this approach in a kind of "adiabatic approximation" was discussed in Ref. 9.

Throughout the article we use the abstract index notation,[10] in which the index of a geometrical object rather indicates its type, then being a set of numbers. This interpretation of indices is especially convenient when spinors are introduced: The two-valence mixed spinors $\rho^{AA'}$ are objects of the same type as complex vectors in Minkowski space, the respective structures being isomorphic. One identifies, accordingly, the compound index $AA'$ with the spacetime index $a$ ($BB'$ with $b$ and so on). We write therefore $\rho^a=\rho^{AA'}$ what in more traditional notation would be written as $\rho^a = \sigma^a_{AA'}\rho^{AA'}$, where $\sigma^a_{AA'}$ are the Infeld–van der Waerden symbols giving a concrete realization of the isomorphism. In this notation the metric tensor is $g_{ab}= \epsilon_{AB}\epsilon_{A'B'}$, where $\epsilon_{AB}$ is the fixed antisymmetric spinor (and $\epsilon_{A'B'} \equiv \bar{\epsilon}_{A'B'}$). The correspondence between an antisymmetric tensor $F_{ab}$ and an equivalent symmetric spinor $\varphi_{AB}$ has the form $F_{ab}= \varphi_{AB}\epsilon_{A'B'} + \bar{\varphi}_{A'B'}\epsilon_{AB}$. For the null vector of the spinor $o_A$ we use fixed notation $l_a=o_A\bar{o}_{A'}$, and for the spinor $\xi_A$ respectively $u_a=\xi_A\bar{\xi}_{A'}$. If below the spinor index in $\bar{o}_{A'}$, $\bar{\xi}_{A'}$ is not suppressed and there is no danger of confusion, the bar sign will be omitted.

## II. NULL ASYMPTOTICS OF THE ELECTROMAGNETIC FIELD

In this section we describe some null asymptotic properties of the electromagnetic fields. Much of the material is not new, the null infinity methods being the standard tool in the relativity theory. However, we do not use the Penrose's conformal compactification, as employed in similar context in Refs. 11 and 12, and use an explicitly Lorentz-covariant description in terms of homogeneous functions. Moreover, we describe some global properties in Minkowski space, which are needed in the discussion of Lorentz generators. The reason for avoiding the conformal compactification is that it contracts the timelike past and future infinity to points. This does not seem a natural setting for the description of massive asymptotic fields living there, which is our concern in Sec. IV.

Let us fix the origin in the affine Minkowski space and denote by $x$ a general point-vector. Let $A(x)$ be a continuous field and suppose it has well defined asymptotics $\lim_{R\to\infty}RA(x+Rl)\equiv b(x,l)$ for every point $x$ and null vector $l$ (vector and spinor indices will be often suppressed if no ambiguity arises). $b(x,l)$ is a homogeneous function of degree $-1$ in $l$. Suppose now that $y$ is a vector lying in the hyperplane $y\cdot l=0$. If $y\propto l$ then obviously $b(x+y,l)=b(x,l)$. If $y\not\propto l$, then it is spacelike, and there always exists a null vector $n$ such that $n\cdot y=0$, $n\cdot l=1$. Then, $l+y/R-(y^2/2R^2)n$ is a future null vector and

$$b(x+y,l)=\lim_{R\to\infty} RA\left(x+\frac{y^2}{2R}\,n+R\left(l+\frac{y}{R}-\frac{y^2}{2R^2}\,n\right)\right).$$

Therefore, if $A(x)$ is sufficiently regular, one should expect that again $b(x+y,l)=b(x,l)$, for all $y\cdot l=0$. This means that

$$\lim_{R\to\infty} RA(x+Rl)=\chi(x\cdot l,l), \tag{2.1}$$

where $\chi(s,l)$ is a homogeneous function of degree $-1$: $\chi(\kappa s,\kappa l)=\kappa^{-1}\chi(s,l)$. We shall show that (2.1) is indeed satisfied for a large class, concerning us here, of solutions of the wave equation (both homogeneous and inhomogeneous). Instead of null vectors $l_a=o_Ao_{A'}$ we shall use as independent variables the spinors $o$ and $\bar{o}$, adding further conditions of invariance under the change of the overall spinor phase factor. Thus $\chi(s,o,\bar{o})$ will satisfy

$$\chi(\alpha\bar{\alpha}s,\alpha o,\bar{\alpha}\bar{o})=(\alpha\bar{\alpha})^{-1}\chi(s,o,\bar{o}) \tag{2.2}$$

for any complex number $\alpha\neq 0$. The former notation $\chi(s,l)$ will be used for functions invariant under the overall spinor phase factor change as a shorthand.

Let $A(x)\in C^2$ be a global solution of the wave equation

$$\Box A(x)=0. \tag{2.3}$$

By the classical Kirchhoff integral formula (see, e.g., Ref. 10) the field $A(x)$ inside the future lightcone may be recovered from its values on the cone itself. If these values are represented with the use of a homogeneous function $\eta(p,l)$ according to

$$A(Rl)=R^{-1}\eta(R^{-1},l), \quad \eta(\kappa p,\kappa l)=\kappa^{-1}\eta(p,l), \tag{2.4}$$

then the formula takes on an especially simple form

$$A(x)=-\frac{1}{\pi x^2}\int \dot{\eta}\left(2\,\frac{x\cdot u}{x^2},u\right)d^2u, \tag{2.5}$$

where the dot over $\eta$ denotes the derivative with respect to the first argument and $d^2u$ is the standard invariant measure on the set of null directions discussed in Appendix A. From the homogeneity property of $\eta$ it follows that the integrand is a homogeneous of degree $-2$ function of $u$, which is the condition for the applicability of $d^2u$. Suppose now, that $RA(Rl)$ has a limit for $R\to\infty$ for all $l$ and that this limit is achieved without sharp oscillations, which can be expressed as $\partial_R(RA(Rl))\sim R^{-1-\epsilon}$ for some $\epsilon>0$. Then $A(x)$ has the anticipated asymptotic behavior in the whole future lightcone, and moreover a fall-off property of the asymptotic is implied. More precisely, we have the following proposition (a $t$-gauge is a scaling of the spinor $o$ for which $t\cdot l=1$, $t^a$ being a timelike unit vector; see Appendix A).

*Proposition 2.1: If $A(x)\in C^2$ is a solution of Eq. (2.3) inside the future lightcone with the data on the cone given by (2.4), with $\eta$ in the t-gauge satisfying the bound*

$$|\dot{\eta}(p,l)|<\frac{\text{const.}}{p^{1-\epsilon}}$$

*when $0<p<p_t$ (for some $\epsilon>0$, $p_t>0$).*
*Then for all x inside the future lightcone the asymptotics (2.1) holds with*

$$\chi(s,l)=-\frac{1}{2\pi s}\int \dot{\eta}\left(\frac{l\cdot u}{s},u\right)d^2u. \tag{2.6}$$

*$\chi(s,l)\equiv\chi(s,o,\bar{o})$ has the scaling property (2.2) and falls off according to*

$$|\chi(s,l)|<\frac{\text{const.}}{s^{\epsilon}} \tag{2.7}$$

*for $s>s_t\equiv 2/p_t$ in the t-gauge.*

We note that the form of the bounds on homogeneous functions as those appearing in this proposition (and in what follows) is independent of the choice of the vector $t$ (gauge-independent), only the bounding constants and $p_t$ (and $s_t$) do change. This is easily seen with the use of the inequalities $t\cdot l\leq e^{\psi}\tilde{t}\cdot l$ and $\tilde{t}\cdot l\leq e^{\psi}t\cdot l$ for any null vector $l$ and any two unit, timelike, future-pointing vectors $t$ and $\tilde{t}$, where $t\cdot\tilde{t}=\cosh\psi$.

*Proof:* Fix $x^a=\lambda z^a$, $z^2=1$, $z^0>0$, and choose $l$ and $u$ in $z$-gauge. Parametrize $u$ by $(\rho,\varphi)$ as in Appendix A (with $z$ playing the role of the time-vector) and change the $\rho$ variable to $\rho_0>0$ by $\rho_0^2=(2R\rho^2+\lambda)/(2R+\lambda)$. Then, by (2.5),

$$RA(x+Rl)=\frac{-1}{2\pi\lambda}\int_0^1 2\,d\rho_0^2\int d\varphi\dot{\eta}\left(\frac{2}{\lambda}\rho_0^2,\ u(\rho(\rho_0),\varphi)\right)\theta\left(\rho_0^2-\frac{\lambda}{\lambda+2R}\right).$$

The integral is bounded in module by $\theta(p_z-2/\lambda\rho_0^2)\text{const.}/(\rho_0^2)^{1-\epsilon}+\text{const.}\ \theta(2/\lambda\rho_0^2-p_z)$, hence by the Lebesgue theorem

$$\lim_{R\to\infty} RA(x+Rl)=\frac{-1}{2\pi\lambda}\int_0^1 2d\rho^2\int d\varphi\dot{\eta}\left(\frac{2}{\lambda}\rho^2,u(\rho,\varphi)\right),$$

which is (2.1) with $\chi$ (2.6) in the $z$ gauge and $(\rho,\varphi)$ parametrization. The bound for $\chi$ is easily obtained.

The next proposition gives the field itself from its asymptotic, by a slightly strengthened fall-off condition (no sharp oscillations).

*Proposition 2.2: Let $\chi(s,l)$ and its derivatives with respect to s of up to the third order be continuous functions of s and l for $s\in R$. Suppose $\chi(s,l)$ and $\dot{\chi}(s,l)$ satisfy respectively (2.7) and*

$$|\dot{\chi}(s,l)|<\frac{\text{const.}}{s^{1+\epsilon}} \tag{2.8}$$

*for $s>s_t>0$ in the t-gauge.*
*Then $\chi(s,l)$ is the asymptotic (2.1) of the field*

$$A(x)=-\frac{1}{2\pi}\int \dot{\chi}(x\cdot l,l)d^2l, \tag{2.9}$$

*which satisfies the wave equation. For a given x there is $|RA(x+Rl)|<$const. for all $R\geq 0$ and l in the t-gauge.*
*If in addition we demand that also $\ddot{\chi}(s,l)$ satisfies (2.8), then uniqueness of $A(x)$ with the given asymptotic is guaranteed.*

*Proof:* The wave equation is obviously satisfied. Further $RA(x+Rl)=(-1/2\pi)\times\int\dot{\chi}(x\cdot u+Rl\cdot u,u)d^2u$. Choose $l$ and $u$ in the $t$-gauge, use $(\rho,\varphi)$-parametrization for $u$ as in Appendix A and replace the $\rho$ variable by $\beta=2R\rho^2$. Then $RA(x+Rl)=(-1/2\pi)\int_0^{2\pi}d\varphi\int_0^{2R}d\beta\dot{\chi}(x\cdot u+\beta,u)$, where $u=u(\rho,\varphi)=u(\sqrt{\beta/2R},\varphi)$. For $\beta>s_t+|x^0|+|\mathbf{x}|$ the condition (2.8) implies $|\dot{\chi}(x\cdot u+\beta,u)|<\text{const.}(\beta-|x^0|-|\mathbf{x}|)^{-1-\epsilon}$. For $0\leq\beta\leq s_t+|x^0|+|\mathbf{x}|$ there is $-(|x^0|+|\mathbf{x}|)\leq x\cdot u+\beta\leq 2(|x^0|+|\mathbf{x}|)+s_t$, so, for fixed $x$, by continuity $|\dot{\chi}(x\cdot u+\beta,u)|<$const. in this case. The asymptotic (2.1) and the bound follow now easily. If (2.8) is assumed for $\ddot{\chi}(s,l)$, then also the field $\nabla_a A(x)=-(1/2\pi)\int l_a\ddot{\chi}(x\cdot l,l)d^2l$ has similar asymptotic properties. By the Kirchhoff formula $A(x)$ can be uniquely recovered from the values of $A$ on any past-directed lightcone, such that $x$ lies inside the cone; the formula involves the field itself on the cone and its derivative along the generating lines of the cone. If one tends with the vertex of the cone to the future timelike infinity, then the integrands tend to respective null asymptotics. The dominated convergence given by the proposition gives then (2.9) as the limit of the Kirchhoff formula, which implies uniqueness.

Up to now we have considered the null asymptotic in the future direction only. In exactly the same way the past null asymptotic can be considered. Proposition 2.2 holds again, with (2.7), (2.8), (2.1), and (2.9) replaced respectively by

$$|\chi'(s,l)|<\frac{\text{const.}}{|s|^{\epsilon}} \tag{2.10}$$

and

$$|\dot{\chi}'(s,l)|<\frac{\text{const.}}{|s|^{1+\epsilon}}, \tag{2.11}$$

for $s < s_t' < 0$ in the $t$-gauge,

$$\lim_{R\to\infty} RA(x-Rl)=\chi'(x\cdot l,l), \tag{2.12}$$

$$A(x)=\frac{1}{2\pi}\int \dot{\chi}'(x\cdot l,l)d^2l. \tag{2.13}$$

The null asymptotics $\chi$ and $\chi'$ are not independent, as for any $x$ the representations (2.9) and (2.13) have to agree:

$$\int \Sigma(x\cdot l,l)d^2l=0, \tag{2.14}$$

where

$$\Sigma(s,l)=\dot{\chi}(s,l)+\dot{\chi}'(s,l). \tag{2.15}$$

*Lemma 2.3: If $\Sigma(s,l)$ is continuous, satisfies (2.14), and $|\Sigma(s,l)|$ is bounded by some polynomial in s (in some t-gauge), then $\Sigma(s,l)=\Sigma_{k=0}^{N}s^k\Sigma_k(l)$ with $N<\infty$ and $\int l_{a_1}\cdots l_{a_k}\Sigma_k(l)d^2l = 0$.*

Before giving a proof we fix our conventions for the Fourier transformations. For $f(x)$ a function of the spacetime point and $g(s,o,\bar{o})$ a function of the spinor $o$ and a real variable $s$ we denote

$$\hat{f}(p)=\frac{1}{2\pi}\int f(x)e^{ip\cdot x}\,d^4x, \tag{2.16}$$

$$\tilde{g}(\omega,o,\bar{o})=\frac{1}{2\pi}\int g(s,o,\bar{o})e^{i\omega s}\,ds. \tag{2.17}$$

If $g(\alpha\bar{\alpha}s,\alpha o,\bar{\alpha}\bar{o})=\alpha^p\bar{\alpha}^q g(s,o,\bar{o})$ then $\tilde{g}(\omega/\alpha\bar{\alpha},\alpha o,\bar{\alpha}\bar{o})=\alpha^{p+1}\bar{\alpha}^{q+1}\tilde{g}(\omega,o,\bar{o})$.

*Proof of the lemma:* We integrate the condition (2.14) with a function of fast decrease $f(x)$. The result can be rewritten as $\int\tilde{\Sigma}(\omega,l)\overline{\hat{f}(\omega l)}d\omega\, d^2l = 0$. Fix a $t$-gauge and assume that $\hat{f}(\omega l) = g(\omega)h(l)$; this can be extended to a Schwartz function $\hat{f}(p)$ if $h(l)$ is infinitely differentiable and $g(\omega)$ is a Schwartz function vanishing in some neighborhood of $\omega=0$. Going over all possible $h(l)$ we obtain $\int\tilde{\Sigma}(\omega,l)\overline{\tilde{g}(\omega)}d\omega = 0$ for all $l$. Thus, for every $l$, $\tilde{\Sigma}(\omega,l)$ is a distribution concentrated in $\omega=0$, hence a finite linear combination of derivatives of $\delta(\omega)$. By polynomial boundedness the supremum over $l$ of the degree of the highest derivative of $\delta(\omega)$ is finite, hence $N<\infty$. Inserting the expansion into (2.14) one obtains the constraints on $\Sigma_k(l)$.

The result of the lemma gives via (2.15) the relation between the future and the past null asymptotics. We add now the physical condition that the energy of the field be finite. It will be seen below, that it leads for the electromagnetic field to the condition which corresponds here to the integrability of $|\dot{\chi}(s,l)|^2$ over all $s\in(-\infty,+\infty)$. By the result of the lemma and the fall-off conditions for $\dot{\chi}(s,l)$ and $\dot{\chi}'(s,l)$ it follows now that $\Sigma(s,l)$ vanishes identically. Thus we have

$$\dot{\chi}(s,l)+\dot{\chi}'(s,l)=0.$$

This implies that both $|\dot{\chi}(s,l)|$ and $|\dot{\chi}'(s,l)|$ satisfy both fall-off conditions (2.8) and (2.11). This implies also that there exist limits $\chi(-\infty,l)\equiv\lim_{s\to-\infty}\chi(s,l)$, $\chi'(+\infty,l)\equiv\lim_{s\to+\infty}\chi'(s,l)$ and

$$\chi(s,l)+\chi'(s,l)=\chi(-\infty,l)=\chi'(+\infty,l). \tag{2.18}$$

If we think of $A(x)$ as an analog of the electromagnetic potential, or in fact a component of the latter, then the fields with nonvanishing $\chi(-\infty,l)$ are exactly those infrared singular in the usual sense (as observed in Ref. 12). This is easily seen, when the connection between (2.9) and the usual Fourier representation is clarified. This is simply achieved if $\dot{\chi}(x\cdot l,l)$ in (2.9) is represented by its transform (2.17). If we write the Fourier representation of the field $A(x)$ as

$$A(x)=\frac{1}{\pi}\int a(k)\,\delta(k^2)\,\epsilon(k^0)e^{-ix\cdot k}\,d^4k, \tag{2.19}$$

then

$$a(\omega l)=-\tilde{\chi}(\omega,l)/\omega. \tag{2.20}$$

But $\tilde{\chi}(0,l)=-(1/2\pi)\chi(-\infty,l)$, as easily seen from (2.17). If this does not vanish, then $a(\omega l)\sim\omega^{-1}$ at the origin. We note that the function $\chi(-\infty,l)$ is not only [as $\chi(s,l)$] Lorentz-frame independent, but also independent of the choice of the origin in Minkowski space. It describes uniquely the spacelike asymptotic of the field $A(x)$:

$$\lim_{R\to\infty} RA(x+Ry)=\frac{1}{2\pi}\int\chi(-\infty,l)\,\delta(y\cdot l)d^2l, \tag{2.21}$$

where $\delta$ is the Dirac distribution. This is true both point-like in $y$ for $y^2\neq0$ (for timelike $y$ yielding simply 0), and distributionally when integrated with a test function $f(y)$, for any fixed $x$. One proves this by a method similar to that used in the proof of the Proposition 2.2.

In the next step we want to take into consideration fields with nonvanishing sources, satisfying equation

$$\Box A(x)=4\pi J(x). \tag{2.22}$$

Some restrictions on the current density have to be assumed. As the scattering aspects are those which concern us here, we want the free radiation field $A^{\mathrm{rad}}=A^{\mathrm{ret}}-A^{\mathrm{adv}}$ to fall into the class of fields considered up to now. The Pauli–Jordan function $D(x)=(1/2\pi)\epsilon(x^0)\,\delta(x^2)$ can be written in the representation (2.9) as

$$D(x)=-\frac{1}{8\pi^2}\int\delta'(x\cdot l)d^2l.$$

Using it, we obtain $A^{\mathrm{rad}}(x)$ in the representation (2.9) with the integrand $\dot{\chi}^{\mathrm{rad}}(s,l)=\dot{c}(s,l)$,

$$c(s,l)=\int\delta(s-l\cdot y)J(y)d^4y. \tag{2.23}$$

We assume therefore that this function is well defined, and that $\dot{c}(s,l)$ satisfies the premises of Proposition 2.2. Note, however, that $c(+\infty,l)$ need not vanish, and the future null asymptotic of $A^{\mathrm{rad}}$ is given by $\chi^{\mathrm{rad}}(s,l)=c(s,l)-c(+\infty,l)$. Suppose further that the support of the current is bounded in spacelike directions, that is for every $x$ the set $\{y|y^2\leq0,J(x+y)\neq0\}$ is bounded. This condition can be relaxed to some decay in spacelike directions, but we dot study this problem in detail. The asymptotics of the retarded and advanced solutions is easily found

$$\lim_{R\to\infty} RA^{\text{ret}}(x+Rl)=\lim_{R\to\infty} RA^{\text{adv}}(x-Rl)=c(x\cdot l,l). \tag{2.24}$$

Combined with the asymptotics of the radiation field this also gives

$$\lim_{R\to\infty} RA^{\text{ret}}(x-Rl)=c(-\infty,l), \quad \lim_{R\to\infty} RA^{\text{adv}}(x+Rl)=c(+\infty,l). \tag{2.25}$$

The incoming and outgoing fields are defined as usual by $A=A^{\text{in}}+A^{\text{ret}}=A^{\text{out}}+A^{\text{adv}}$ and they are assumed to belong to the class of free fields considered here. Denoting $\chi$ and $\chi'$ the future and past null asymptotics of $A$ we have the relations

$$\chi'(s,l)=\chi'^{\text{in}}(s,l)+c(-\infty), \quad \chi(s,l)=\chi^{\text{out}}(s,l)+c(+\infty).$$

The full relation (2.18) is lost now, but it remains true, that

$$\chi(-\infty,l)=\chi'(+\infty,l). \tag{2.26}$$

The extension of the preceding discussion to the case of the electromagnetic fields involves some physically important modifications. The Maxwell equations in the spinor form read

$$\nabla^{A}_{B'}\varphi_{AB}(x)=2\pi J_b(x) \tag{2.27}$$

with a real conserved current $J_b$. If complex $J_b$ is admitted, its imaginary part is the magnetic current of the generalized Maxwell equations in tensor form. To see the physical consequences of the absence of magnetic currents in the context of asymptotic fields we take this condition only later into account. We shall see later that in order that the radiated energy-momentum and angular momentum be well defined not only $\varphi_{AB}(x)$ but also $\varphi_{AB}(x)x^{B}_{A'}$ should have the asymptotic behavior discussed above. As the latter field appears repeatedly in the present context it is convenient to denote

$$\varrho_{AA'}(x)=\varphi_{AB}(x)x^{B}_{A'}. \tag{2.28}$$

This field satisfies

$$\nabla^{A}_{B'}\varrho_{AA'}(x)=2\pi J_{BB'}(x)x^{B}_{A'}. \tag{2.29}$$

From (2.27) and (2.29) the inhomogeneous wave equations follow

$$\Box\varphi_{AB}(x)=4\pi\nabla_{AC'}J^{C'}_{B}(x), \tag{2.30}$$

$$\Box\varrho_{AA'}(x)=4\pi\nabla_{AC'}(J^{C'}_{B}(x)x^{B}_{A'}). \tag{2.31}$$

In the free field case we demand therefore that

$$\varphi_{AB}(x)=-\frac{1}{2\pi}\int \dot{f}_{AB}(x\cdot l,o,\bar{o})d^2l,$$

$$\varrho_{AA'}(x)=-\frac{1}{2\pi}\int \dot{h}_{AA'}(x\cdot l,o,\bar{o})d^2l,$$

where $|\dot{f}_{AB}(s,o,\bar{o})|$ and $|\dot{h}_{AA'}(s,o,\bar{o})|$ are bounded by const.$|s|^{-1-\epsilon}$ for large $|s|$ and both $|f_{AB}(s,o,\bar{o})|$ and $|h_{AA'}(s,o,\bar{o})|$ vanish for $s\to+\infty$. (Here and in what follows such bounds on spinor and tensor functions are to be understood in some $t$-gauge, component-wise in some Minkowski frame in which time axis is parallel to $t$, and in the associated spinor frame; for fixed vector $t$ the bounds do not depend on the choice of the spacelike frame.) Setting these formulas into (2.27) and (2.29) respectively (with $J_b=0$) and taking null asymptotics one obtains $o^A f_{AB}(s,o,\bar{o})=0$, $o^A h_{AA'}(s,o,\bar{o})=0$, i.e., $f_{AB}(s,o,\bar{o})=o_A o_B f(s,o,\bar{o})$, $h_{AA'}(s,o,\bar{o})=o_A h_{A'}(s,o,\bar{o})$. (These formulas follow also from the generalized Kirchhoff formula.[10]) Contracting the above representation of $\varphi_{AB}(x)$ with $x^B_{A'}$ and using (A8) we have

$$\begin{aligned}\varphi_{AB}(x)x^B_{A'}&=\frac{1}{2\pi}\int o_A x_{A'B} o^B \dot{f}(x\cdot l,o,\bar{o})d^2l\\&=\frac{1}{2\pi}\int o_A(\partial_{A'}-\partial'_{A'})f(x\cdot l,o,\bar{o})d^2l\\&=-\frac{1}{2\pi}\int o_A\partial'_{A'}f(x\cdot l,o,\bar{o})d^2l,\end{aligned}$$

where $\partial_{A'}\equiv\partial/\partial o^{A'}$ and $\partial'_{A'}f(x\cdot l,o,\bar{o})\equiv\partial_{A'}f(s,o,\bar{o})|_{s=x\cdot l}$.

From now on we make a general assumption that the spinor derivatives of the asymptotics up to the order which will appear in the future considerations do not spoil the fall-off properties, so that, e.g., together with $|f(s,o,\bar{o})|$ also $|\partial_{A'}f(s,o,\bar{o})|$ falls off as $s^{-\epsilon}$ for $s\rightarrow+\infty$ and with $|f(s,o,\bar{o})-f(-\infty,o,\bar{o})|$ also $|\partial_{A'}[f(s,o,\bar{o})-f(-\infty,o,\bar{o})]|\sim|s|^{-\epsilon}$ for $s\rightarrow-\infty$. From the homogeneity properties of asymptotics then follows that the differentiation with respect to $s$ increases the rate of fall-off by one inverse power of $|s|$, as, e.g., $o^{A'}\partial_{A'}f+s\dot{f}=-f$.

Comparing now the two above representations of $\varphi_{AB}(x)x^B_{A'}$ and using Lemma 2.3, we have $\partial_{A'}f(s,o,\bar{o})=\dot{h}_{A'}(s,o,\bar{o})$. Contracting this with $o^{A'}$ and using homogeneity we get $\partial_s(sf)=-o^{A'}\dot{h}_{A'}$, or $sf=o^{A'}h_{A'}-g$, where $g=g(o,\bar{o})$ has the homogeneity property $g(\alpha o,\bar{\alpha}\bar{o})=\alpha^{-2}g(o,\bar{o})$. Differentiation on $\partial_{B'}$ yields $s\dot{h}_{B'}=\partial_{B'}(o^{A'}h_{A'})-\partial_{B'}g$. The left-hand side (lhs) vanishes for $|s|\rightarrow\infty$, whereas the right-hand side (rhs) tends to $-\partial_{B'}g$ for $s\rightarrow+\infty$ and to $\partial_{B'}(o^{A'}h_{A'}(-\infty)-g)$ for $s\rightarrow-\infty$. Hence $\partial_{B'}g=\partial_{B'}(o^{A'}h_{A'}(-\infty))=0$, which implies by (A5) $g=o^{A'}h_{A'}(-\infty)=0$. Therefore $f=s^{-1}o^{A'}h_{A'}\sim|s|^{-1-\epsilon}$ for $|s|\rightarrow\infty$, so there is a unique representation $f=\dot{\zeta}$ with $\zeta$ vanishing for $s\rightarrow+\infty$.

Summarizing the free electromagnetic field case we have

$$\varphi_{AB}(x)=-\frac{1}{2\pi}\int o_A o_B\ddot{\zeta}(x\cdot l,o,\bar{o})d^2l,\tag{2.32}$$

$$\varrho_{AA'}(x)=-\frac{1}{2\pi}\int o_A\partial'_{A'}\dot{\zeta}(x\cdot l,o,\bar{o})d^2l,\tag{2.33}$$

$$\lim_{R\rightarrow\infty}R\varphi_{AB}(x+Rl)=o_A o_B\dot{\zeta}(x\cdot l,o,\bar{o}),\tag{2.34}$$

$$\lim_{R\rightarrow\infty}R\varrho_{AA'}(x+Rl)=o_A\partial'_{A'}\zeta(x\cdot l,o,\bar{o}).\tag{2.35}$$

This class of fields admits a class of Lorentz-gauge potentials with properties characterized by Proposition 2.2

$$A_a(x)=-\frac{1}{2\pi}\int\dot{V}_a(x\cdot l,l)d^2l.\tag{2.36}$$

$V_a(s,l)$ is a real vector function with properties of $\chi(s,l)$ of Proposition 2.2 and such that

$$o_{C'}V_A^{C'}(s,l)=o_A\zeta(s,o,\bar{o}). \tag{2.37}$$

Turning now to the asymptotics of the retarded and advanced fields

$$\varphi^{\text{ret,adv}}{}_{AB}(x)=4\pi\int G^{\text{red,adv}}(x-y)\nabla_{AC'}J_B^{C'}(y)d^4y$$

we observe first that the fields $\varphi^{\text{ret}}{}_{AB}(x)x^B_{A'}$ and $\varphi^{\text{adv}}{}_{AB}(x)x^B_{A'}$ may be obtained as respectively retarded and advanced solutions of (2.31). This is seen as follows. Suppressing the labels "ret" or "adv" we have

$$4\pi\int G(x-y)\nabla_{AC'}(J_B^{C'}(y)y^B_{A'})d^4y-\varphi_{AB}(x)x^B_{A'}=4\pi\int G(z)\nabla^{(z)}_{AC'}(J_B^{C'}(x-z)z^B_{A'})d^4z.$$

Using the conservation law of $J_b$ and the rules for transforming spinor into tensor expressions one has (all differentiations on $z$)

$$\begin{aligned}\nabla_{AC'}(J_B^{C'}(x-z)z^B_{A'})&=J_a(x-z)+z^B_{A'}\nabla_{BC'}J_A^{C'}(x-z)\\&=(\tfrac{1}{2}z\cdot\nabla+1)J_a(x-z)+(z_{[a}\nabla_{c]}-ie_{acbd}z^b\nabla^d)J^c(x-z).\end{aligned}$$

As the retarded and advanced Green functions satisfy

$$(z\cdot\nabla+2)G(z)=0,\quad (z_a\nabla_b-z_b\nabla_a)G(z)=0,$$

the above integral vanishes, which ends the proof of our statement. This property implies that one can attach the ret/adv labels to $\varrho_{AA'}$ without risk of ambiguity. The leading asymptotic terms can be now simply represented. If we denote

$$c_A(s,o,\bar{o})=\int \delta(s-x\cdot l)J_A^{C'}(x)d^4x\ o_{C'}, \tag{2.38}$$

then

$$\lim_{R\to\infty}R\varphi^{\text{ret}}{}_{AB}(x+Rl)=\lim_{R\to\infty}R\varphi^{\text{adv}}{}_{AB}(x-Rl)=o_{(A}\dot{c}_{B)}(x\cdot l,o,\bar{o}),$$

$$\lim_{R\to\infty}R\varrho^{\text{ret}}{}_{AA'}(x+Rl)=\lim_{R\to\infty}R\varrho^{\text{adv}}{}_{AA'}(x-Rl)=\partial'_{A'}c_A(x\cdot l,o,\bar{o}).$$

The last two equalities follow from

$$\begin{aligned}\partial_{A'}c_A(s,o,\bar{o})&=\int\{\delta'(s-x\cdot l)x^B_{A'}o_Bo_{C'}J_A^{C'}(x)-\delta(s-x\cdot l)J_a(x)\}d^4x\\&=\int\delta(s-x\cdot l)(J_a+x^B_{A'}\nabla_{BC'}J_A^{C'})d^4x\\&=\int\delta(s-x\cdot l)\nabla_{AC'}(J_B^{C'}(x)x^B_{A'})d^4x.\end{aligned}$$

As in the case of the scalar field, the current is assumed such that $\dot{c}_A(s,o,\bar{o})$ has the required fall-off properties [implying the existence of limits $c_A(\pm\infty,o,\bar{o})$]. The formulas analogous to (2.25) are

$$\lim_{R\to\infty} R\varphi^{\text{ret}}{}_{AB}(x-Rl)=\lim_{R\to\infty} R\varphi^{\text{adv}}{}_{AB}(x+Rl)=0,$$

$$\lim_{R\to\infty} R\varrho^{\text{ret}}{}_{AA'}(x-Rl)=\partial'_{A'}c_A(-\infty,o,\bar{o}),$$

$$\lim_{R\to\infty} R\varrho^{\text{adv}}{}_{AA'}(x+Rl)=\partial'_{A'}c_A(+\infty,o,\bar{o}).$$

There are further conditions on $c_A$ following from the conservation of $J_a$, and from its reality, when this is the case (pure electrodynamics). From the conservation law we have

$$0=\int \delta(s-x\cdot l)\nabla_a J^a(x)d^4x=\int \delta'(s-x\cdot l)J^a(x)d^4x l_a,$$

that is $\dot{c}_A o^A=0$ or $c^A(s,o,\bar{o})o_A=Q=Q_{\text{el}}-iQ_{\text{mag}}$. $Q_{\text{el}}$ and $Q_{\text{mag}}$ are the electric and the magnetic charge of the field respectively. The last equation implies also $\dot{c}_A(s,o,\bar{o})\propto o_A$ and $\partial_{A'}c_A(s,o,\bar{o})\propto o_A$. To see the consequence of reality of $J_a$ we choose an arbitrary spinor $\iota_A$ complementing $o_A$ to a normalized spinor basis $o_A\iota^A=1$ and decompose in the standard null tetrad[10]

$$\int \delta(s-x\cdot l)J_a(x)d^4x=\alpha(s,l)l_a+\beta(s,l)m_a+\gamma(s,l)\bar{m}_a+Qn_a.$$

If $J_a$ is real, then $\alpha(s,l)$ and $Q$ are real and $\gamma(s,l) = \overline{\beta(s,l)}$. The only condition implied in this case for $c_A(s,o,\bar{o})=\beta(s,l)o_A+Q\iota_A$ is the reality of $Q$. Moreover, in that case the retarded (advanced) Lorentz-gauge potentials $A^{\text{ret}}{}_a(x)$ ($A^{\text{adv}}{}_a(x)$) have the required null asymptotic behavior with asymptotics characterized by

$$c_a(s,l)=\int \delta(s-x\cdot l)J_a(x)d^4x. \tag{2.39}$$

We summarize the general field case now easily obtained as a superposition of a free and the ret/adv fields. The necessary terms of the electromagnetic field asymptotics are represented with the use of a spinor function $\zeta_A(s,o,\bar{o})$ with the fall-off

$$|\dot{\zeta}_A(s,o,\bar{o})|<\frac{\text{const.}}{|s|^{1+\epsilon}}, \tag{2.40}$$

for $|s|>s_t>0$, differential properties as assumed for $\zeta$ above, homogeneity

$$\zeta_A(\alpha\bar{\alpha}s,\alpha o,\bar{\alpha}\bar{o})=\alpha^{-1}\zeta_A(s,o,\bar{o}) \tag{2.41}$$

and satisfying in addition

$$\zeta^A(s,o,\bar{o})o_A=Q=Q_{\text{el}}-iQ_{\text{mag}}. \tag{2.42}$$

Then,

$$\lim_{R\to\infty} R\varphi_{AB}(x+Rl)=o_A\dot{\zeta}_B(x\cdot l,o,\bar{o}), \tag{2.43}$$

$$\lim_{R\to\infty} R\varrho_{AA'}(x+Rl)=\partial'_{A'}\zeta_A(x\cdot l,o,\bar{o})\equiv o_A\nu_{A'}(x\cdot l,o,\bar{o}). \tag{2.44}$$

The last identity is the definition of $\nu_{A'}$. If $J_a(x)$ is real than there exists a class of Lorentz-gauge potentials with null asymptotics

$$\lim_{R\to\infty} RA_a(x+Rl)=V_a(x\cdot l,l), \tag{2.45}$$

where $V_a(s,l)$ has the properties of $\chi(s,l)$ of the scalar case and satisfies

$$o_{C'}V_A^{C'}(s,l)=\zeta_A(s,o,\bar{o}). \tag{2.46}$$

Past null asymptotics are similarly given by another function $\zeta'_A(s,o,\bar{o})$ with the same properties. As in the scalar case there is

$$\zeta_A(-\infty,o,\bar{o})=\zeta'_A(+\infty,o,\bar{o}). \tag{2.47}$$

The future null asymptotic of the free outgoing field is given by

$$\zeta_A(s,o,\bar{o})-\zeta_A(+\infty,o,\bar{o})\equiv o_A\zeta^{\text{out}}(s,o,\bar{o}), \tag{2.48}$$

which is the definition of $\zeta^{\text{out}}(s,o,\bar{o})$ at the same time. Similarly, the past null asymptotic of the incoming field is supplied by

$$\zeta'_A(s,o,\bar{o})-\zeta'_A(-\infty,o,\bar{o})\equiv o_A\zeta'^{\text{in}}(s,o,\bar{o}). \tag{2.49}$$

One observes that the asymptotically relevant (needed for determination of the radiated angular momentum, as will be seen later) information on the asymptotics of the electromagnetic field is not fully contained in the free outgoing or incoming fields. The remaining terms

$$\zeta_A(+\infty,o,\bar{o})=c_A(+\infty,o,\bar{o})=\lim_{s\to+\infty}\int\delta(s-x\cdot l)J_A^{C'}(x)d^4x\ o_{C'}$$

and

$$\zeta'_A(-\infty,o,\bar{o})=c_A(-\infty,o,\bar{o})=\lim_{s\to-\infty}\int\delta(s-x\cdot l)J_A^{C'}(x)d^4x\ o_{C'}$$

are connected with the Coulomb fields of the outgoing and incoming currents respectively.

The physical significance of the limit values $\zeta_A(\pm\infty,o,\bar{o})$ and $\zeta'_A(\pm\infty,o,\bar{o})$ is revealed by considering the spacelike limit of the electromagnetic field. For a free field (2.32) one obtains by the method used already in the scalar case

$$\lim_{R\to\infty} R^2\varphi^{\text{free}}{}_{AB}(a+Ry)=\frac{1}{2\pi}\int\delta'(y\cdot l)o_Ao_B\zeta(-\infty,o,\bar{o})d^2l \tag{2.50}$$

for any point $a$ and spacelike vecor $y$. For a general field we use a trick, which will be useful also in the next section. Decompose the field $\varphi_{AB}$ into the retarded and free outgoing fields $\varphi_{AB}=\varphi^{\text{ret}}{}_{AB}[J]+\varphi^{\text{out}}{}_{AB}$, where the source $J_a$, according to our earlier assumptions, has finite extension in spacelike directions. Choose an arbitrary point $a$ and a time axis through $a$ in the direction of a unit timelike, future-pointing vector $t$. For real positive $c$ denote by $\mathscr{C}_s^{\rm fut}(-c)$ the solid future lightcone with the vertex in $a-ct$, and by $\mathscr{C}_s^{\rm past}(c)$ the solid past lightcone with the vertex in $a+ct$. Choose $c$ such that $J_b=0$ in $R(c)\equiv M\backslash\{\mathscr{C}_s^{\rm fut}(-c)\cup\mathscr{C}_s^{\rm past}(c)\}$. The retarded field is not influenced in $R(c)$ by the values of the current in $\mathscr{C}_s^{\rm fut}(-c)$. We make advantage of this fact to replace $J_b$ by a current $J'_b$ which is identical with $J_b$ in the past of $\mathscr{C}_s^{\rm fut}(-c)$ but to the future of $\mathscr{C}_s^{\rm past}(c)$ represents a point charge $Q$ (possibly both electric and magnetic) sitting on the time axis. This is always possible, since the charge is the only characteristic of a current which cannot be deformed without violation of the continuity equation. Thus in the region $R(c)$ we can write $\varphi_{AB}=\varphi^{\rm ret}{}_{AB}[J']+\varphi^{\rm out}{}_{AB}$. However, if $J_b$ belongs to the class of admitted currents, so does $J'_b$, and the radiated field $\varphi^{\rm rad}{}_{AB}[J']=\varphi^{\rm ret}{}_{AB}[J']-\varphi^{\rm adv}{}_{AB}[J']$ is an admissible free field. On the other hand $\varphi^{\rm adv}{}_{AB}[J']$ is identical in $R(c)$ with the Coulomb field of a point charge $Q$ sitting on the time axis. Summarizing, the field $\varphi_{AB}$ can be represented in $R(c)$ by $\varphi_{AB}=\varphi^{Q}{}_{AB}+\varphi^{\rm free}{}_{AB}$, where $\varphi^{Q}{}_{AB}$ represents this Coulomb field and $\varphi^{\rm free}{}_{AB}$ is a free field. The region $R(c)$ is large enough for this representation to be used for determination of (i) future null asymptotics for $s<s_1$, for some $s_1$, in $t$-gauge; (ii) past null asymptotics for $s>s_2$, for some $s_2$, in $t$-gauge; (iii) spacelike asymptotics from the point $a$.

For the (generalized—with possible magnetic charge) Coulomb field

$$\varphi^{Q}{}_{AB}(x)=\frac{Q}{([(x-a)\cdot t]^2-(x-a)^2)^{3/2}}\, t_{(A}^{C'}(x-a)_{B)C'}, \tag{2.51}$$

one has

$$\lim_{R\to\infty} R^2\varphi^{Q}{}_{AB}(a+Ry)=\frac{Q}{((y\cdot t)^2-y^2)^{3/2}}\, t_{(A}^{C'}y_{B)C'}.$$

The null asymptotics are most easily found with the use of (2.38)

$$\zeta^{Q}{}_A(s,o,\bar{o})=\zeta'^{Q}{}_A(s,o,\bar{o})=\frac{Q}{t\cdot l}\, t_A^{C'}o_{C'}. \tag{2.52}$$

Using the identity

$$\int \delta(y\cdot l)\,\frac{d^2l}{t\cdot l}=\frac{2\pi}{((y\cdot t)^2-y^2)^{1/2}}$$

we find the relation

$$\frac{1}{2\pi}\int \delta'(y\cdot l)o_{(A}\zeta^{Q}_{B)}(-\infty,o,\bar{o})d^2l=\lim_{R\to\infty} R^2\varphi^{Q}{}_{AB}(a+Ry).$$

Comparison with the free field case and the use of the trick described above allow us to write in general case for any point $a$ and spacelike vector $y$

$$\lim_{R\to\infty} R^2\varphi_{AB}(a+Ry)=\frac{1}{2\pi}\int \delta'(y\cdot l)o_{(A}\zeta_{B)}(-\infty,o,\bar{o})d^2l. \tag{2.53}$$

Formulas (2.50) and (2.53) furnish the required interpretation of the limit values of asymptotics. The limit value $\zeta_A(-\infty,o,\bar{o})=\zeta'_A(+\infty,o,\bar{o})$ describes the long-range degrees of freedom of the total field; $\zeta^{\rm out}(-\infty,o,\bar{o})$ and $\zeta'^{\,\rm in}(+\infty,o,\bar{o})$ furnish the characterization of the asymptotic infrared degrees of freedom of the outgoing and incoming free fields respectively; $\zeta_A(+\infty,o,\bar{o})$ and $\zeta'_A(-\infty,o,\bar{o})$ describe the asymptotics of the Coulomb field of the outgoing and incoming asymptotic currents, respectively.

Further transformation of these long-range variables will prove useful. From the homogeneity (2.41) it follows $o^{A'}\partial_{A'}\zeta_A(s,o,\bar{o})+s\zeta_A(s,o,\bar{o})=0$. For the limit points $s=\pm\infty$ we obtain $o^{A'}\partial_{A'}\zeta_A(\pm\infty,o,\bar{o})=0$, or $\partial_{A'}\zeta_A(\pm\infty,o,\bar{o})\propto l_a$ [as at the same time $o^A\partial_{A'}\zeta_A(s,o,\bar{o})=0$ for all $s$]. The same holds true for $\zeta'_A(\pm\infty,o,\bar{o})$. The rhs of the following equations introduce new variables

$$\partial_{A'}\zeta_A(+\infty,o,\bar{o})=-l_a q(o,\bar{o}), \tag{2.54}$$

$$\partial_{A'}\zeta'_A(-\infty,o,\bar{o})=-l_a q'(o,\bar{o}), \tag{2.55}$$

$$\partial_{A'}(\zeta_A(-\infty,o,\bar{o})-\zeta_A(+\infty,o,\bar{o}))=o_A\partial_{A'}\zeta^{\text{out}}(-\infty,o,\bar{o})=-l_a\sigma(o,\bar{o}), \tag{2.56}$$

$$\partial_{A'}(\zeta'_A(+\infty,o,\bar{o})-\zeta'_A(-\infty,o,\bar{o}))=o_A\partial_{A'}\zeta'^{\text{in}}(+\infty,o,\bar{o})=-l_a\sigma'(o,\bar{o}). \tag{2.57}$$

As a consequence of (2.47) one has a constraint

$$q+\sigma=q'+\sigma'. \tag{2.58}$$

All of the new variables are spinor functions of the homogeneity

$$f(\alpha o,\bar{\alpha}\bar{o})=(\alpha\bar{\alpha})^{-2}f(o,\bar{o}), \tag{2.59}$$

where $f$ stands for any of $q$, $q'$, $\sigma$ or $\sigma'$. Moreover they satisfy

$$\frac{1}{2\pi}\int q(l)d^2l=\frac{1}{2\pi}\int q'(l)d^2l=Q, \tag{2.60}$$

$$\frac{1}{2\pi}\int \sigma(l)d^2l=\frac{1}{2\pi}\int \sigma'(l)d^2l=0. \tag{2.61}$$

One calculates these means by contracting (2.54)–(2.57) with a timelike, unit, future-pointing vector, integrating by parts (see Appendix A) and using (2.42), e.g., for $q$ one has

$$-\frac{1}{2\pi}\int t^a\partial_{A'}\zeta_A(+\infty,o,\bar{o})\,\frac{d^2l}{t\cdot l}=-\frac{1}{4\pi}\int o^A\zeta_A(+\infty,o,\bar{o})\,\frac{d^2l}{(t\cdot l)^2}=Q.$$

Conversely, the conditions (2.59)–(2.61) are the only ones following from (2.54)–(2.57) and the functions $q$, $q'$, $\sigma$, and $\sigma'$ satisfying them determine the long-range variables $\zeta_A(+\infty,o,\bar{o})$, $\zeta'_A(-\infty,o,\bar{o})$, $\zeta^{\text{out}}(-\infty,o,\bar{o})$ and $\zeta'^{\text{ in}}(+\infty,o,\bar{o})$ uniquely. For the last two of them this follows directly from (A5). To prove the statement for the other two, we choose a vector $t^a$ and denote $\iota^A=(t\cdot l)^{-1}t^{AA'}o_{A'}$. Then $\zeta_A(+\infty,o,\bar{o})=Q\iota_A+\iota^B\zeta_B(+\infty,o,\bar{o})o_A$ and (2.54) is equivalent to $\partial_{A'}(\iota^A\zeta_A(+\infty,o,\bar{o}))=-o_{A'}(q(o,\bar{o})-[Q/2(t\cdot l)^2])$. The proof now ends as for $\sigma$'s. Similarly for the primed quantities.

The vanishing of means (2.61) implies also that there exist homogeneous functions of degree zero

$$\Phi(\alpha o,\bar{\alpha}\bar{o})=\Phi(o,\bar{o}),\quad \Phi'(\alpha o,\bar{\alpha}\bar{o})=\Phi'(o,\bar{o}), \tag{2.62}$$

such that

$$\partial_A \partial_{A'} \Phi(o,\bar{o}) = l_a \sigma(o,\bar{o}), \quad \partial_A \partial_{A'} \Phi'(o,\bar{o}) = l_a \sigma'(o,\bar{o}). \tag{2.63}$$

These conditions determine $\Phi$'s up to additive constants. In view of (2.56) and (2.57), Eqs. (2.63) are equivalent to

$$\partial_A \Phi(o,\bar{o}) = -o_A \zeta^{\mathrm{out}}(-\infty,o,\bar{o}), \quad \partial_A \Phi'(o,\bar{o}) = -o_A \zeta'^{\mathrm{in}}(+\infty,o,\bar{o}). \tag{2.64}$$

Physical meaning of the additive constant in $\Phi$ (and $\Phi'$) comes from the fact, that gauges of potentials can be divided into equivalence classes, each class remaining in one to one correspondence with a choice of this constant. Classes differ by their infrared contributions to symplectic form; see Ref. 9.

With the use of the variables $q,\ldots,\sigma'$ further insight into the meaning of (2.53) is possible. We observe first that the result of (2.53), considered as a function of $y$, is a free electromagnetic field in the region $y^2<0$, which is homogeneous of degree $-2$. We denote this field $\varphi^{\mathrm{l.r.}}{}_{AB}(y)$ (l.r. standing for long range). Using the identity $o_A = o^C \epsilon_{CA} = (2/y^2) o^C y_{CD'} y_A^{D'}$ we can represent $\delta'(y\cdot l) o_A = (2/y^2) y_A^{D'} \partial_{D'} \delta(y\cdot l)$. Setting this into (2.53) and integrating by parts we get

$$\varphi^{\mathrm{l.r.}}{}_{AB}(y) = y_{(A}^{C'} K_{B)C'}(y), \tag{2.65}$$

where

$$K_a(y) = \frac{1}{2\pi y^2} \nabla_a \int \operatorname{sgn}(y\cdot l)(q+\sigma)(o,\bar{o}) d^2 l. \tag{2.66}$$

The tensor form of Eq. (2.65) gives the electromagnetic field $F^{\mathrm{l.r.}}_{ab}$ corresponding to the spinor $\varphi^{\mathrm{l.r.}}{}_{AB}(y)$

$$F^{\mathrm{l.r.}}{}_{ab} = F^{\mathrm{l.r.E}}{}_{ab} + F^{\mathrm{l.r.M}}{}_{ab}, \tag{2.67}$$

where the fields on the rhs are determined by

$$F^{\mathrm{l.r.E}}{}_{ab}(y) = \operatorname{Re} K_a(y) y_b - \operatorname{Re} K_b(y) y_a, \tag{2.68}$$

$$*F^{\mathrm{l.r.M}}{}_{ab}(y) = \operatorname{Im} K_a(y) y_b - \operatorname{Im} K_b(y) y_a. \tag{2.69}$$

The above equations imply $*F^{\mathrm{l.r.E}}{}_{ab}(y) y^b = 0$ and $F^{\mathrm{l.r.M}}{}_{ab}(y) y^b = 0$. This can be interpreted as follows: in any Minkowski frame the radial components of the magnetic part of the field $F^{\mathrm{l.r.E}}{}_{ab}$ and of the electric part of the field $F^{\mathrm{l.r.M}}{}_{ab}$ vanish. Equivalently formulated: $F^{\mathrm{l.r.E}}{}_{ab}$ has no magnetic multipole contributions and $F^{\mathrm{l.r.M}}{}_{ab}$ has no electric multipole contributions in any Minkowski frame. Accordingly, $F^{\mathrm{l.r.E}}{}_{ab}$ and $F^{\mathrm{l.r.M}}{}_{ab}$ will be called the electric- and the magnetic-type long-range fields respectively (cf. Ref. 5). The whole above discussion applies also to the long-range part of free asymptotic fields [$q+\sigma$ replaced by $\sigma$ or $\sigma'$ in (2.66)] and of the Coulomb fields of the asymptotic currents ($q+\sigma$ replaced by $q$ or $q'$). The real (imaginary) parts of $q,\ldots,\sigma'$ describe the electric (magnetic) parts of the respective fields.

We end this section by testing our assumptions on admissible currents in two cases: a system of charged point particles and a free Dirac field. A system of $N$ point charges moving along trajectories $z_i^a(\tau)$, $i=1,\ldots,N$, each parametrized by its proper time, corresponds to an obviously spacelike finitely extended current density

$$J^a(x) = \sum_{i=1}^{N} Q_i \int \delta(x - z_i(\tau)) v_i{}^a(\tau) d\tau, \tag{2.70}$$

where $v^a(\tau)\equiv(dz^a/d\tau)(\tau)$. The asymptotic characteristic (2.38) is easily obtained

$$c^A(s,o,\bar{o})=\sum_{i=1}^{N}\left[\frac{Q_i}{v_i(\tau)\cdot l}\,v_i^{AA'}(\tau)o_{A'}\right]\Bigg|_{\tau\,:\,s=z_i(\tau)\cdot l}. \tag{2.71}$$

If the asymptotic behavior of four-velocities satisfies $|(dv^a/d\tau)(\tau)|<\text{const.}/|\tau|^{1+\epsilon}$ for $|\tau|\to\infty$, then $c^A(s,o,\bar{o})$ has limits

$$c^A(\pm\infty,o,\bar{o})=\sum_{i=1}^{N}\frac{Q_i}{v_i(\pm\infty)\cdot l}\,v_i^{AA'}(\pm\infty)o_{A'} \tag{2.72}$$

and the assumptions on the fall-off of $|c^A(s,o,\bar{o})-c^A(\pm\infty,o,\bar{o})|$ for $s\to\pm\infty$ are satisfied. The class of thus admitted asymptotic motions includes the typical behavior of the Coulomb scattering, where $\epsilon=1$. From (2.72) we get

$$q(o,\bar{o})=\sum_{i=1}^{N}\frac{Q_i}{2(v_i(+\infty)\cdot l)^2},\quad q'(o,\bar{o})=\sum_{i=1}^{N}\frac{Q_i}{2(v_i(-\infty)\cdot l)^2}. \tag{2.73}$$

If no magnetic monopoles are present, the $q$'s are real.

For the discussion of a free Dirac field we use its Fourier representation in the following form

$$\psi(x)=\left(\frac{m}{2\pi}\right)^{3/2}\int(e^{-imx\cdot v}P_+f(v)-e^{+imx\cdot v}P_-f(v))d\mu(v), \tag{2.74}$$

in the notation explained in the second paragraph of Sec. IV. If we assume that $f(v)$ is an infinitely differentiable function of compact support, then $\psi(x)$ is a regular wave packet, so it is a function of fast decrease in all spacelike and lightlike directions.[13] The current density has infinite extension in spacelike directions, but its exponential fall-off is sufficient for the extension of our results to this case.

The definition (2.38) can be rewritten in the Fourier transformed form

$$\tilde{c}^A(\omega,o,\bar{o})=\hat{J}^{AA'}(\omega l)o_{A'}, \tag{2.75}$$

with the conventions introduced in (2.16), (2.17). From (2.74) we get

$$\begin{aligned}\hat{\psi}(mv)&=\left(\frac{2\pi}{m}\right)^{3/2}\frac{2}{m}\,\delta(v^2-1)(\theta(v^0)P_+f(v)-\theta(-v^0)P_-f(-v))\\&\equiv\left(\frac{2\pi}{m}\right)^{3/2}\frac{2}{m}\,\delta(v^2-1)h(v),\end{aligned}$$

so

$$\begin{aligned}\hat{J}^a(mu)&=\frac{em^4}{(2\pi)^3}\int\bar{\hat{\psi}}\left(m\left(r-\frac{u}{2}\right)\right)\gamma^a\hat{\psi}\left(m\left(r+\frac{u}{2}\right)\right)d^4r\\&=\frac{2e}{m}\int\delta(u\cdot r)\,\delta\left(r^2+\frac{u^2}{4}-1\right)\bar{h}\left(r-\frac{u}{2}\right)\gamma^a h\left(r+\frac{u}{2}\right)d^4r.\end{aligned}$$

Hence

$$\hat{J}^a(\omega l)=e\,\delta(\omega)\int v^a\bar{f}\gamma\cdot vf(v)\,\frac{d\mu(v)}{v\cdot l}. \tag{2.76}$$

The asymptotic characteristic $c_A(s,o,\bar{o})$ does not depend on $s$, as was to be expected (a free charged field sends no radiation field)

$$c^A(s,o,\bar{o})=e\int v^{AA'}\bar{f}\gamma\cdot vf(v)\,\frac{d\mu(v)}{v\cdot l}\,o_{A'}\,. \tag{2.77}$$

Not only the charge of the field

$$Q=c^Ao_A=e\int \bar{f}\gamma\cdot vf(v)d\mu(v) \tag{2.78}$$

but also the asymptotic variables

$$q(o,\bar{o})=q'(o,\bar{o})=e\int \bar{f}\gamma\cdot vf(v)\,\frac{d\mu(v)}{2(v\cdot l)^2} \tag{2.79}$$

are obviously real. The long-range electromagnetic field produced by a free Dirac field is therefore of purely electric type

$$F^{\mathrm{l.r.}}{}_{ab}=e\int \bar{f}\gamma\cdot vf(v)\,\frac{y_av_b-y_bv_a}{((v\cdot y)^2-y^2)^{3/2}}\,d\mu(v). \tag{2.80}$$

The absence of magnetic-type long-range fields is a typical feature of the scattering processes involving no magnetic monopoles. To produce a long-range magnetic-type field without the use of magnetic monopoles one would need an asymptotic current of infinitely increasing magnetic multipoles, a magnetic dipole linearly growing with time giving the simplest possibility. As the infrared singular free fields are typically produced as radiation fields of some scattering processes they also yield the long-range fields of electric type only.

## III. ENERGY-MOMENTUM AND ANGULAR MOMENTUM TENSOR OF THE ASYMPTOTIC ELECTROMAGNETIC FIELD

Consider now a closed dynamical system, part of which constitute the (generalized) Maxwell equations (2.27) with the current $J_a$ satisfying the assumptions of the previous section. Finite spacelike extension of $J_a$, which we assume for simplicity, could be replaced by some fast decrease condition, more appropriate in the case where $J_a$ is due to some charged massive field. Suppose further that the system is equipped with a locally conserved, symmetric energy-momentum tensor $T_{ab}$, which outside the electromagnetic sources reduces to the usual symmetric electromagnetic tensor $T^{\mathrm{elm}}{}_{ab}=-(1/4\pi)(F_{ac}F_b{}^c-\frac{1}{4}g_{ab}F_{cd}F^{cd})$, and the amount of energy-momentum and angular momentum passing through a hypersurface $\mathscr{S}$ is given as usual respectively by

$$P_a[\mathscr{S}]=\int_{\mathscr{S}}T_{ac}(x)d\sigma^c(x), \tag{3.1}$$

$$M_{ab}[\mathscr{S}]=\int_{\mathscr{S}}(x_aT_{bc}(x)-x_bT_{ac}(x))d\sigma^c(x). \tag{3.2}$$

In the context of electromagnetic fields it proves convenient to use the spinor version of Eq. (3.2). If the symmetric angular momentum spinor $\mu_{AB}$ is defined by

$$M_{ab}=\mu_{AB}\epsilon_{A'B'}+\bar{\mu}_{A'B'}\epsilon_{AB}, \tag{3.3}$$

then (3.2) is equivalent to

$$\mu_{AB}[\mathscr{S}]=\int_{\mathscr{S}}\mu_{ABc}(x)d\sigma^c(x), \tag{3.4}$$

where

$$\mu_{ABc}(x)=x_{D'(A}T^{D'}_{B)c}(x). \tag{3.5}$$

We want to consider now the total energy-momentum and angular momentum of the system and express these quantities in terms of asymptotic fields. The usual straightforward expressions, obtained by setting for $\mathscr{S}$ in (3.1) and (3.2) a Cauchy surface $\Sigma$, are not appropriate for our purpose for two reasons.

(i) The asymptotic electromagnetic fields discussed in the preceding section are defined in the null asymptotic region, whereas one should expect the asymptotic massive fields to be defined in timelike asymptotic regions (we shall return to this question in Sec. 4). This physical picture suggests that separation of the contributions to the conserved quantities could be possible. We shall see that this is almost true, the reservation representing a physically important term in the total angular momentum involving long-range Coulomb and infrared degrees of freedom. For the demonstration of this separation the Cauchy surface integration is not well suited, as this surface contains the whole information on the system, even if it is pushed to infinite past or future.

(ii) In the case of angular momentum an even more serious obstacle arises: the integrand of (3.2) is not absolutely integrable for a Cauchy surface, so, strictly speaking, the integral does not exist. This is easily seen from the spacelike asymptotic behavior of electromagnetic field, discussed in the preceding section. On a hypersurface $x^0=$const. the field is $O(|\mathbf{x}|^{-2})$, so the integrand is $O(|\mathbf{x}|^{-3})$, while the measure is $d^3x$.

With our purpose in mind we consider first the energy-momentum and the angular momentum radiated with the electromagnetic field into future null directions. Let $a$ be a point vector of arbitrary point in Minkowski space and $t$ a timelike, unit, future-pointing vector. Choose the line $a+\tau t$, $\tau\in R$, as the time-axis of the origin of three-space orthogonal to $t$. Consider the timelike tube given by $x=a+\tau t+Rl$, $R=$const., $\tau\in R,l$ going over the set of all future null vectors in the $t$-gauge $t\cdot l=1$. The energy-momentum and the angular momentum passing through a bounded portion of this tube are given by

$$\int_B \rho_c(a+\tau t+Rl)(t^c-l^c)d\tau\, R^2 d\Omega_t(l), \tag{3.6}$$

where $\rho_c=T_{ac}$ for energy-momentum and $\rho_c=\mu_{ABc}$ for angular momentum spinor, and integration extends over a bounded interval of retarded time $T$ and a solid angle $\Theta$ of $l$ directions. The limit of the above expressions when $R\rightarrow\infty$, if it exists, gives the respective quantities radiated into the solid angle $\Theta$ over the time-span $T$. More general bounded measurable sets of integration $B$ are possible. For sufficiently large $R$ we move into the region where sources vanish and $T_{ab}=T^{\rm elm}{}_{ab}$. In the spinor language the latter takes the form

$$T^{\rm elm}{}_{ab}(x)=\frac{1}{2\pi}\,\bar{\varphi}_{A'B'}(x)\varphi_{AB}(x), \tag{3.7}$$

which yields

$$\mu^{\mathrm{elm}}{}_{ABc}(x) = -\frac{1}{2\pi}\,\bar{\varrho}_{C'(A}(x)\varphi_{B)C}(x), \tag{3.8}$$

with $\varrho_{AA'}$ defined in (2.28). We see now, that for the energy momentum and the angular momentum radiated over finite time intervals to be well defined, both $\varphi_{AB}(x)$ and $\varrho_{AA'}(x)$ must have the null asymptotics of the assumed type. Then

$$\lim_{R\to\infty} R^2 T^{\mathrm{elm}}{}_{ac}(a+\tau t+Rl) = \frac{1}{2\pi}\,\bar{\dot{\zeta}}_{A'}\dot{\zeta}_A(a\cdot l+\tau,o,\bar{o})l_c, \tag{3.9}$$

$$\lim_{R\to\infty} R^2 \mu^{\mathrm{elm}}{}_{ABc}(a+\tau t+Rl) = -\frac{1}{2\pi}\,\bar{\nu}_{(A}\dot{\zeta}_{B)}(a\cdot l+\tau,o,\bar{o})l_c, \tag{3.10}$$

where (2.43) and (2.44) have been used (in $t$-gauge). This justifies our assumptions on the asymptotic behavior of electromagnetic field (Sec. II). Using the trick described in Sec. II after Eq. (2.50) and a bound analogous to that following Eq. (2.9) one easily shows that for large $R$ the quantities under the limits on the lhs's of (3.9) and (3.10) are bounded by const. $R^{-1}$ on any bounded set $B$. The limits may be thus performed under the integral sign in (3.6), which yields the radiated quantities

$$P^{\mathrm{out-n}}{}_a[B] = \frac{1}{2\pi}\int_B \bar{\dot{\zeta}}_{A'}\dot{\zeta}_A(a\cdot l+\tau,o,\bar{o})d\tau\; d\Omega_t(l), \tag{3.11}$$

$$\mu^{\mathrm{out-n}}{}_{AB}[B] = -\frac{1}{2\pi}\int_B \bar{\nu}_{(A}\dot{\zeta}_{B)}(a\cdot l+\tau,o,\bar{o})d\tau\; d\Omega_t(l), \tag{3.12}$$

n standing for null. The fall-off of asymptotics is sufficient for the integrals in (3.11) and (3.12) to be absolutely integrable over any measurable set $B$, not necessarily bounded. Thus extension of the range of integration $B$ to all times and full solid angle is possible. The result is

$$P^{\mathrm{out-n}}{}_a = \frac{1}{2\pi}\int \bar{\dot{\zeta}}_{A'}\dot{\zeta}_A(s,o,\bar{o})ds\; d^2l, \tag{3.13}$$

$$\mu^{\mathrm{out-n}}{}_{AB} = -\frac{1}{2\pi}\int \bar{\nu}_{(A}\dot{\zeta}_{B)}(s,o,\bar{o})ds\; d^2l, \tag{3.14}$$

where no assumption on the gauge of spinors is needed any more and any reference to the time-axis (the point vector $a$ and the vector $t$) has been lost. Thus the total radiated quantities are unambiguously defined.

In the next step we turn to the energy-momentum and angular momentum going out with the massive part of the system in timelike directions. We choose again the time-axis $a+\tau t$, fix a time parameter $\tau=\tau_1$ and consider the future lightcone $\mathscr{C}^{\mathrm{fut}}(\tau_1)$ with the vertex in $a+\tau_1 t$. The amount of energy-momentum and angular momentum passing through that cone is that contained in the system when the quantities radiated prior to $\tau=\tau_1$ are disregarded. For this interpretation to make sense the appropriate integrals over the cone should be absolutely convergent. That this is the case here, can be seen with the use of the trick described in the preceding section, the estimates for a free field discussed in Appendix B and properties of the Coulomb field (2.51). In this way we obtain quantities $P_a[\mathscr{C}^{\mathrm{fut}}(\tau_1)]$ and $\mu_{AB}[\mathscr{C}^{\mathrm{fut}}(\tau_1)]$. (We have tacitly assumed that there are no nonintegrable singularities in the region of nonvanishing sources.) Due to the assumed local conservation of the energy-momentum tensor the difference of quantities calculated on two different cones $\mathscr{C}^{\mathrm{fut}}(\tau_2)$ and $\mathscr{C}^{\mathrm{fut}}(\tau_1)$ is given by the integrals (3.11) and (3.12), with $B=|\tau_2-\tau_1|\times\{\text{full solid angle}\}$. The convergence of integrals (3.13) and (3.14) implies now the existence of the limits

$$P^{\mathrm{out-t}}{}_a \equiv \lim_{\tau\to\infty} P_a[\mathscr{C}^{\mathrm{fut}}(\tau)], \tag{3.15}$$

$$\mu^{\mathrm{out-t}}{}_{AB} \equiv \lim_{\tau\to\infty} \mu_{AB}[\mathscr{C}^{\mathrm{fut}}(\tau)], \tag{3.16}$$

t standing for timelike. By a similar argument, with the use of two different time-axis, the quantities thus obtained are independent of the choice of the axis. Moreover, instead of lightcones any timelike hypersurfaces tending to them asymptotically can be used. In the case of free electromagnetic field one should expect that all energy momentum and angular momentum are radiated into null directions. This is indeed the case, as shown in Appendix B, i.e., we have

$$P^{\mathrm{out-t}}{}_a(\mathrm{free})=0, \quad \mu^{\mathrm{out-t}}{}_{AB}(\mathrm{free})=0. \tag{3.17}$$

In this way we are led to unambiguous interpretation of (3.15) and (3.16) as quantities going out with the massive part of the system. For an explicit representation in terms of dynamical asymptotic variables one needs more detailed knowledge of the system. We discuss this question in the following sections.

The preceding discussion strongly suggests the identification of the total energy-momentum and angular momentum of the system by

$$P^{\mathrm{tot}}{}_a = P^{\mathrm{out-n}}{}_a + P^{\mathrm{out-t}}{}_a, \tag{3.18}$$

$$\mu^{\mathrm{tot}}{}_{AB} = \mu^{\mathrm{out-n}}{}_{AB} + \mu^{\mathrm{out-t}}{}_{AB}. \tag{3.19}$$

Two points in this connection have to be clarified.

(i) The connection of (3.18) and (3.19) to the quantities obtained by the Cauchy surface integration, if it can be performed, should be understood.

(ii) The picture lying at the base of our discussion can be reflected in time, with the subsequent change of orientation of hypersurfaces. The radiated quantities are then replaced by the respective quantities incoming from the past null directions, given by

$$P^{\mathrm{in-n}}{}_a = \frac{1}{2\pi}\int \overline{\dot{\zeta}'}_{A'}\dot{\zeta}'_A(s,o,\bar{o})ds\, d^2l, \tag{3.20}$$

$$\mu^{\mathrm{in-n}}{}_{AB} = -\frac{1}{2\pi}\int \overline{\nu'}_{(A}\dot{\zeta}'_{B)}(s,o,\bar{o})ds\, d^2l. \tag{3.21}$$

Similarly, the past timelike limits of integrals over past lightcones, $P^{\mathrm{in-t}}{}_a$ and $\mu^{\mathrm{in-t}}{}_{AB}$, replace (3.15) and (3.16) respectively, and again

$$P^{\mathrm{in-t}}{}_a(\mathrm{free})=0, \quad \mu^{\mathrm{in-t}}{}_{AB}(\mathrm{free})=0. \tag{3.22}$$

For the consistence of physical interpretation formulas (3.18) and (3.19) should yield the same results with in-quantities on the rhs.

To clarify the above raised points consider the situation depicted in Fig. 1. We choose an arbitrary spacelike hyperplane $\Sigma$ and a time-axis with the unit vector $t$ orthogonal to $\Sigma$, crossing the plane at the point $a$. $\mathscr{C}^{\mathrm{fut}}(\tau)$, $\tau>0$, is the future lightcone with the vertex in $a+\tau t$; $\mathscr{C}'^{\mathrm{fut}}(-r)$, $r>0$, is the unbounded portion of the future lightcone with the vertex in $a-rt$, cut off by the plane

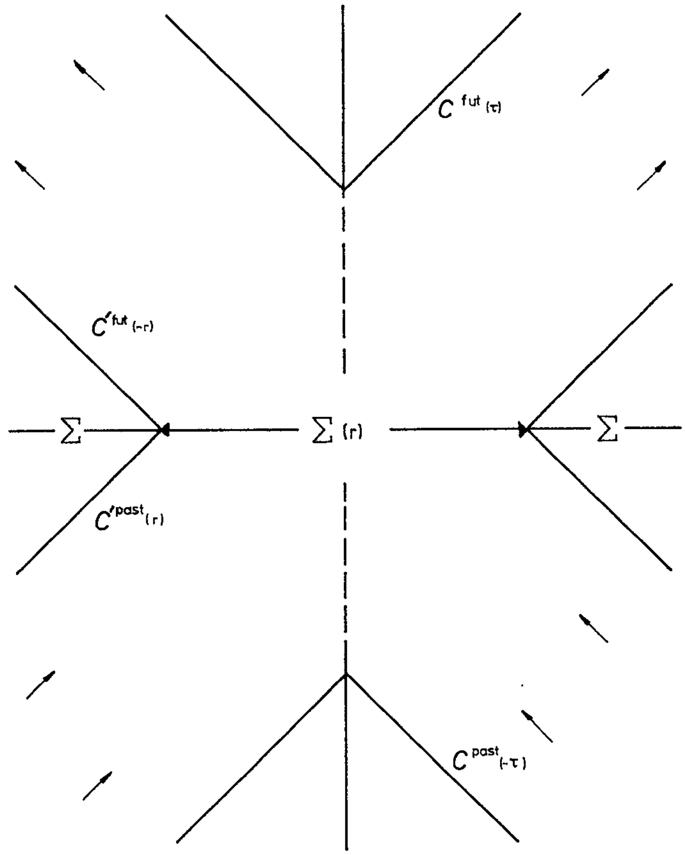


FIG. 1. The choice of hypersurfaces for the derivation of Eqs. (3.23) and (3.24) (two spacelike dimensions are suppressed).

$\Sigma$. $\mathscr{C}^{\text{past}}(-\tau)$ and $\mathscr{C}'^{\text{ past}}(r)$ are obtained from the former two cones by reflection with respect to $\Sigma$. $\Sigma(r)$ is the portion of $\Sigma$ (a ball) closing the cut of $\mathscr{C}'^{\text{ fut}}(-r)$ and $\mathscr{C}'^{\text{ past}}(r)$. We consider conservation of energy momentum and angular momentum for two infinite regions: the first contained between $\mathscr{C}^{\text{fut}}(\tau)$, $\mathscr{C}'^{\text{ fut}}(-r)$ and $\Sigma(r)$, the second contained between $\mathscr{C}^{\text{past}}(-\tau)$, $\mathscr{C}'^{\text{ past}}(r)$ and $\Sigma(r)$. Taking the limits $\tau\rightarrow\infty$ and $r\rightarrow\infty$ we arrive at

$$G^{\text{out-n}}+G^{\text{out-t}}=\lim_{r\rightarrow\infty}(G[\Sigma(r)]+G[\mathscr{C}'^{\text{ fut}}(-r)]), \tag{3.23}$$

$$G^{\text{in-n}}+G^{\text{in-t}}=\lim_{r\rightarrow\infty}(G[\Sigma(r)]+G[\mathscr{C}'^{\text{ past}}(r)]), \tag{3.24}$$

where $G$ stands for $P_a$ or $\mu_{AB}$. Conventions of hypersurface orientations are such, that positive direction of crossing the surface is from past to future. The limits on the rhs's of Eqs. (3.23) and (3.24) exist, since the lhs's exist. We shall show that those limits exist indeed for each term on the rhs's separately and the following explicit formulas hold true

$$\lim_{r\rightarrow\infty} P_a[\mathscr{C}'^{\text{ fut}}(-r)]=\lim_{r\rightarrow\infty} P_a[\mathscr{C}'^{\text{ past}}(r)]=0, \tag{3.25}$$

$$\lim_{r\to\infty} \mu_{AB}[\mathscr{C}'^{\,\mathrm{fut}}(-r)] = -\lim_{r\to\infty} \mu_{AB}[\mathscr{C}'^{\,\mathrm{past}}(r)] = \frac{1}{4\pi}\int \bar{\nu}_{(A}\zeta_{B)}(-\infty,o,\bar{o})d^2l. \tag{3.26}$$

It is easy to see, that if these formulas are shown for the special choice of the point $a=0$, then they remain true for other time axes, so we consider this special case. We observe first that for sufficiently large $r$ both $\mathscr{C}'^{\,\mathrm{fut}}(-r)$ and $\mathscr{C}'^{\,\mathrm{past}}(r)$ enter the region where we can use the trick of the preceding section. Contributions of the quadratic free field terms, Coulomb terms and mixed terms can be considered separately. For the free field one easily shows with the use of Lemma C.1 that

$$\lim_{r\to\infty} P_a[\Sigma(r)]|_{\mathrm{free}} = P^{\mathrm{out-n}}{}_a|_{\mathrm{free}} = P^{\mathrm{in-n}}{}_a|_{\mathrm{free}}, \tag{3.27}$$

$$\lim_{r\to\infty} \mu_{AB}[\Sigma(r)]|_{\mathrm{free}} = \tfrac{1}{2}(\mu^{\mathrm{out-n}}{}_{AB} + \mu^{\mathrm{in-n}}{}_{AB})|_{\mathrm{free}}. \tag{3.28}$$

Setting this into (3.23) and (3.24), taking into account (3.17) and (3.22) and using (3.13), (3.14), (3.20), and (3.21) for free fields one arrives at (3.25) and (3.26). For mixed terms one shows by a direct calculation demonstrated in Appendix B that (3.25) holds and the rhs of (3.26) takes the required form

$$\frac{1}{4\pi}\int (\overline{\nu^Q}_{(A}\zeta^{\mathrm{free}}{}_{B)}(-\infty,o,\bar{o}) + \overline{\nu^{\mathrm{free}}}_{(A}\zeta^Q{}_{B)}(-\infty,o,\bar{o})d^2l.$$

For the Coulomb terms all the terms in (3.25) and (3.26) vanish. This ends the proof of (3.25) and (3.26).

We return to the physical interpretation. Consistent identification of the total energy momentum by

$$P^{\mathrm{in}}{}_a = P^{\mathrm{out}}{}_a = P_a[\Sigma] \tag{3.29}$$

is always correct due to (3.25); on the rhs the proper integral replaces the limit, as the integrand is absolutely integrable. For angular momentum we have to impose a (Poincaré covariant) condition on the long-range variables

$$\frac{1}{4\pi}\int \bar{\nu}_{(A}\zeta_{B)}(-\infty,o,\bar{o}) = 0 \tag{3.30}$$

to be able to conclude

$$\mu^{\mathrm{in}}{}_{AB} = \mu^{\mathrm{out}}{}_{AB} = \lim_{r\to\infty} \mu_{AB}[\Sigma(r)]. \tag{3.31}$$

Crucial for the interpretation is the first equality. The Cauchy integral is not absolutely integrable, but the second equality gives its finite regularization, which, however, has no independent direct physical justification. [The mechanism of this regularization is the antisymmetry of the leading asymptotic term of the integrand with respect to reflection of three-space $\Sigma$—this is easily seen from (2.65) and (2.66).] In the absence of the condition (3.30) no well-founded identification of angular momentum seems to be possible—angular momentum leaks out into the spacelike infinity. Condition (3.30) imposes constraints on the long-range field (2.65). We recall that $o_A\nu_{A'}(-\infty,o,\bar{o}) = \partial_{A'}\zeta_A(-\infty,o,\bar{o}) = -(q+\sigma)l_a$, and decompose $\zeta_A(-\infty,o,\bar{o})$ and $\nu_{A'}(-\infty,o,\bar{o})$ into their electric- and magnetic-type parts: $o_A\nu^{\mathrm{E}}{}_{A'}(-\infty,o,\bar{o}) = \partial_{A'}\zeta^{\mathrm{E}}{}_A(-\infty,o,\bar{o}) = -\mathrm{Re}(q+\sigma)l_a$ and $o_A\nu^{\mathrm{M}}{}_{A'}(-\infty,o,\bar{o}) = \partial_{A'}\zeta^{\mathrm{M}}{}_A(-\infty,o,\bar{o})$

$=-i\,\mathrm{Im}(q+\sigma)l_a$. Then also $o_{A'}\overline{\nu^{\mathrm{E}}}_A(-\infty,o,\bar{o})=\partial_{A'}\zeta^{\mathrm{E}}_A(-\infty,o,\bar{o})$ but $o_{A'}\overline{\nu^{\mathrm{M}}}_A(-\infty,o,\bar{o})$ $=-\partial_{A'}\zeta^{\mathrm{M}}_A(-\infty,o,\bar{o})$. Choose an arbitrary timelike, unit, future-pointing vector $t$ and contract the two last equations with the associated spinor $\iota^{A'}$. Setting the result into (3.30) and integrating by parts we have

$$\int \iota^{C'}\partial_{C'}\zeta^{\mathrm{E}}{}_{(A}\zeta^{\mathrm{M}}{}_{B)}(-\infty,o,\bar{o})d^2l=0.$$

The electric and the magnetic parts are independent, and we know that electric-type fields may be present. Therefore, we demand that magnetic-type fields do not occur in the theory. This condition, as shown in the preceding section, is satisfied in the known situations of scattering phenomena. On the other hand, for the field of freely moving charged (possibly both electrically and magnetically) particles the lhs of (3.30) takes the form $\Sigma_{i<k}(Q_i\overline{Q}_k - Q_k\overline{Q}_i)h(v_i\cdot v_k)v_{iC'(A}v_{kB)}{}^{C'}$, where $h$ is a real function [use (2.72) as asymptotic]. This vanishes identically only if the ratios of the magnetic to the electric charge are equal for all particles. This excludes presence of magnetic charges, as pure electric charges have to be admitted. Finally, we extend the condition of no magnetic part also to free fields, and assume accordingly from now on

$$\bar{q}=q,\quad \overline{q'}=q',\quad \bar{\sigma}=\sigma,\quad \overline{\sigma'}=\sigma',\quad \bar{\Phi}=\Phi,\quad \overline{\Phi'}=\Phi'. \tag{3.32}$$

With the knowledge gained on the long-range degrees of freedom we turn again to the quantities radiated into or coming from null directions. With the usual definitions of outgoing and incoming free fields ($\varphi^{\mathrm{out}}{}_{AB}=\varphi_{AB}-\varphi^{\mathrm{ret}}{}_{AB}$, $\varphi^{\mathrm{in}}{}_{AB}=\varphi_{AB}-\varphi^{\mathrm{adv}}{}_{AB}$) the asymptotics split according to (2.48) and (2.49). Using these splittings in (3.13), (3.14), (3.20), and (3.21), we obtain

$$P^{\mathrm{out-n}}{}_a=\frac{1}{2\pi}\int l_a\overline{\dot{\zeta}^{\mathrm{out}}}\dot{\zeta}^{\mathrm{out}}(s,o,\bar{o})ds\; d^2l, \tag{3.33}$$

$$\mu^{\mathrm{out-n}}{}_{AB}=-\frac{1}{2\pi}\int o_{(A}\partial_{B)}\overline{\zeta^{\mathrm{out}}}\dot{\zeta}^{\mathrm{out}}(s,o,\bar{o})ds\; d^2l+\frac{1}{2\pi}\int q o_{(A}\partial_{B)}\Phi(o,\bar{o})d^2l, \tag{3.34}$$

$$P^{\mathrm{in-n}}{}_a=\frac{1}{2\pi}\int l_a\overline{\dot{\zeta}'^{\mathrm{in}}}\dot{\zeta}'^{\mathrm{in}}(s,o,\bar{o})ds\; d^2l, \tag{3.35}$$

$$\mu^{\mathrm{in-n}}{}_{AB}=-\frac{1}{2\pi}\int o_{(A}\partial_{B)}\overline{\zeta'^{\mathrm{in}}}\dot{\xi}'^{\mathrm{in}}(s,o,\bar{o})ds\; d^2l-\frac{1}{2\pi}\int q' o_{(A}\partial_{B)}\Phi'(o,\bar{o})d^2l, \tag{3.36}$$

The first terms on the rhs's of all above equations are the pure free field quantities. However, there are additional angular momentum contributions due to the long-range tail of the electromagnetic field. These terms mix free electromagnetic field characteristics $\sigma$, $\sigma'$ with the Coulomb characteristics of asymptotic currents $q,q'$. To illustrate a possible observational consequence of these additional terms we present a very heuristic argument based on a guess on possible asymptotic states of the theory.

Suppose that the timelike in-asymptotic of a scattering process is characterized by a single massive spinless particle, carrying charge $Q$, energy-momentum $P^{\mathrm{in-t}}{}_a=mv_a$ and angular momentum $M^{\mathrm{in-t}}{}_{ab}\doteq m(y^{\mathrm{in}}{}_a v_b - y^{\mathrm{in}}{}_b v_a)$, where $y^{\mathrm{in}}{}_a$ is a point-vector of any point on the trajectory of the particle. Suppose further that the dynamics of the theory supports an "adiabatic limit" characterized by the following statements on the scattering states. We assume that the electromagnetic in-field is infinitely low energetic, that is $\int t\cdot l|\dot{\zeta}'^{\mathrm{in}}(s,o,\bar{o})|^2 ds\; d^2l\to 0$, with the infrared characteristic $\sigma'$, however, remaining finite. We guess that there is then no particle production, no energy transfer, so $P^{\mathrm{out-t}}{}_a=mv_a$, and no radiation field, so the free electromagnetic out-field is identical with the in-field. In consequence there is $\sigma=\sigma'$ and $q=q'=Q/2(v\cdot l)^2$. The scattering process, nevertheless, is not completely trivial, since according to (3.31), (3.34) and (3.36) there is an angular momentum change of the particle

$$\mu^{\text{out}-t}{}_{AB}-\mu^{\text{in}-t}{}_{AB}=-\frac{1}{\pi}\int q o_{(A}\partial_{B)}\Phi(o,\bar{o})d^2l=\frac{1}{\pi}\int \Phi o_{(A}\partial_{B)}q(o,\bar{o})d^2l$$
$$=\frac{Q}{\pi}\int o_{(A}v_{B)}^{C'}o_{C'}\Phi(o,\bar{o})\,\frac{d^2l}{(v\cdot l)^3},$$

or in the tensor language

$$M^{\text{out}-t}{}_{ab}=m(y^{\text{out}}{}_a v_b-y^{\text{out}}{}_b v_a),$$

where

$$y^{\text{out}}{}_a=y^{\text{in}}{}_a+\frac{Q}{\pi m}\int l_a\Phi(o,\bar{o})\,\frac{d^2l}{(v\cdot l)^3}, \tag{3.37}$$

determined up to a multiple of $v_a$ (this freedom corresponding to $\Phi\to\Phi+\text{const.}$). The effect of the scattering is thus an adiabatic translation of the trajectory of the particle. This kind of effect will not show up in the usual scattering cross-section measurements. To get an idea about the size of the effect let us assume that the infrared characteristic of the free incoming field is that of a field radiated by a charge $Q_0$ if it changes its four-velocity from $u_1$ to $u_2$. In that case

$$\zeta'^{\text{free}}{}_A(+\infty,o,\bar{o})=Q_0\left(\frac{u_{2AC'}}{u_2\cdot l}-\frac{u_{1AC'}}{u_1\cdot l}\right)o^{C'},$$

so that up to an arbitrary constant $\Phi(o,\bar{o})=Q_0\ln[(u_1\cdot l)/(u_2\cdot l)]$. One calculates then $\Delta y\equiv y^{\text{out}}-y^{\text{in}}=(QQ_0/m)(f(\operatorname{arcosh} u_2\cdot v)u_2-f(\operatorname{arcosh} u_1\cdot v)u_1)$, where $f(\beta)=(\sinh\beta\cosh\beta-\beta)/\sinh^3\beta$. Assume for simplicity the following experimental arrangement: In the laboratory system the energy of the particle producing the incoming free field remains constant and its three-velocity is adiabatically reflected $\mathbf{u}_2=-\mathbf{u}_1$; the test particle is almost at rest in laboratory, so we neglect $|\mathbf{v}|$; both particles are taken to be "electrons" (but with spin neglected). Then $\Delta y_a$ is a translation in the three-space of the laboratory and $\|\Delta y_a\|=r_{\text{cl}}(\sinh\xi\cosh\xi-\xi)/\sinh^2\xi$, where $\cosh\xi=u_2\cdot v=u_1\cdot v$ and $r_{\text{cl}}=e^2/m\approx2.8\times10^{-13}$ cm is "the classical radius of electron." The maximum value of the displacement is of the order of $r_{\text{cl}}$, which is not too impressive. However, the effect cumulates by multiple sending of identical incoming fields.

The first calculation of an observable effect produced by a free zero frequency field is due to Staruszkiewicz.[14] He calculates, in the quasiclassical approximation, the change of the phase of the wave function of a particle in an external electromagnetic field. The plane wave $e^{-imv\cdot x}$ ($v$ is a four-velocity) undergoes in the complete process of scattering by the field (2.36) the change of phase

$$\delta(v)=-\frac{Q}{2\pi}\int\frac{v\cdot V(-\infty,l)}{v\cdot l}\,d^2l$$

[this is Eq. (6) of Ref. 14 in our notation]. Every gauge of $V_a(-\infty,l)$ can be represented by $V_a(-\infty,l)=\partial_A h_{A'}(o,\bar{o})+\partial_{A'}\bar{h}_A(o,\bar{o})$ with some choice of the function $h_{A'}$ satisfying $h_{A'}(\alpha o,\bar{\alpha}\bar{o})=\bar{\alpha}^{-1}h_{A'}(o,\bar{o})$, $h_{A'}(o,\bar{o})o^{A'}=\Phi(o,\bar{o})$ (cf. Ref. 9). Using this representation and integrating by parts we obtain

$$\delta(v) = -\frac{Q}{2\pi}\int \frac{\Phi(l)}{(v\cdot l)^2}\, d^2 l. \tag{3.38}$$

If a wave packet is formed, then this phase induces exactly the shift of the trajectory given by (3.37).

## IV. DIRAC EQUATION IN THE FORWARD LIGHTCONE

Our aim in this section is to reformulate the Dirac equation for an electron in electromagnetic field as an evolution equation on the set of hyperboloids $x^2=\lambda^2$, $x^0>0$, with $\lambda$ taking the role of evolution parameter. We show next that, under certain assumptions, scattering states exist. The class of admitted potentials includes those Coulomb-like as well, if an appropriate gauge transformation is performed.

To fix our notation we rewrite some standard facts about free Dirac field. The Cauchy problem for the free Dirac equation

$$(i\gamma\cdot\nabla - m)\psi(x) = 0$$

is solved by

$$\psi(x) = \int_{\mathscr{S}} S(x-y)\gamma^a \psi(y)\, d\sigma_a(y), \tag{4.1}$$

where $\mathscr{S}$ is a spacelike hypersurface and the Fourier representation of $S(x)$ can be written as

$$S(x) = \left(\frac{m}{2\pi}\right)^3 \int e^{-imx\cdot v\gamma\cdot v}\gamma\cdot v\ d\mu(v),$$

where $d\mu(v) = 2\delta(v^2-1)\theta(v^0)d^4v$ is the invariant measure on the unit hyperboloid. If $\mathscr{S}$ is not a Cauchy surface, then $\psi(x)$ is still uniquely determined by (4.1) in the domain of causal dependence of $\mathscr{S}$. Similarly the Fourier representation of the free Dirac field can be written as

$$\psi(x) = \left(\frac{m}{2\pi}\right)^{3/2} \int e^{-imx\cdot v\gamma\cdot v}\gamma\cdot v f(v)\, d\mu(v), \tag{4.2}$$

with $f$ some complex four-component function on the unit hyperboloid. If we set $x=\lambda z$, with $z^2=1$, $z^0>0$ then the leading asymptotic term when $\lambda\to\infty$ is

$$\psi(\lambda z) \sim -i\lambda^{-3/2} e^{-i(m\lambda+\pi/4)\gamma\cdot z} f(z).$$

To see this one only has to observe that $e^{-i\alpha\gamma\cdot v} = e^{-i\alpha}P_+(v) + e^{+i\alpha}P_-(v)$, with $P_\pm(v) = \frac{1}{2}(1\pm\gamma\cdot v)$, and use the standard stationary phase method. We note for later use that $P_\pm^2 = P_\pm$, $P_+P_- = P_-P_+ = 0$, $P_+ + P_- = 1$. The above asymptotic behavior of free field will guide us to the reformulation mentioned at the beginning of this section.

We start with some geometric preliminaries. Let $x=\lambda z$, with $z^2=1$, $z^0>0$, and let $\nabla_a$ denote the flat derivative with respect to $x^a$. We denote $\delta_a = \lambda(\nabla_a - z_a\partial_\lambda)$. $\delta_a$ is the derivative in the directions tangent to the hyperboloid, and $[\delta_a, \partial_\lambda]=0$. Moreover, $\delta_a z^b = h_a^b$, where $h_a^b = g_a^b - z_a z^b$ is the projection tensor. Every vector (and tensor) can be decomposed according to $\xi^a = z^a \xi\cdot z + \xi_{\mathrm{T}}{}^a$, $\xi_{\mathrm{T}}{}^a z_a = 0$. In particular, the algebra of the Dirac matrices is given by

$$(\gamma\cdot z)^2 = 1, \quad \gamma\cdot z\gamma_{\mathrm{T}}{}^a + \gamma_{\mathrm{T}}{}^a\gamma\cdot z = 0, \quad \gamma_{\mathrm{T}}{}^a\gamma_{\mathrm{T}}{}^b + \gamma_{\mathrm{T}}{}^b\gamma_{\mathrm{T}}{}^a = 2h^{ab}.$$

For all differentiable functions which fall off fast enough for the surface term in the Stokes theorem

$$0=\int_{x^2=1}\nabla_c\{(x^c g_a^b - x^b g_a^c)f(x)\}d\sigma_b(x)$$

to vanish, one has the integral identity involving only the $\delta_a$ derivative:

$$\int(\delta_a - 3z_a)f(z)d\mu(z)=0. \tag{4.3}$$

The Dirac equation

$$[\gamma\cdot(i\nabla - eA(x)) - m]\psi(x)=0$$

written in terms of the variables $\lambda$ and $z^a$ reads now

$$i\partial_\lambda\chi(\lambda,z)=\{-\lambda^{-1}\gamma\cdot z\gamma_{\rm T}\cdot p + m\gamma\cdot z + e\gamma\cdot z\gamma\cdot A(\lambda z)\}\chi(\lambda,z), \tag{4.4}$$

where $\chi(\lambda,z)=\lambda^{3/2}\psi(\lambda z)$, and the operator

$$p_a=i(\delta_a+\tfrac{1}{2}\gamma\cdot z\gamma_{{\rm T}a}-\tfrac{3}{2}z_a) \tag{4.5}$$

has been introduced. The conserved current of the Dirac equation is now $\bar\psi(x)\gamma^a\psi(x)=\lambda^{-3}\bar\chi\gamma^a\chi(\lambda,z)$, which, when integrated over the hyperboloid $x^2=\lambda^2$, $x^0>0$, gives the conserved quantity $\int\overline{\chi(\lambda,z)}\gamma\cdot z\ \chi(\lambda,z)d\mu(z)$; bar over a four-component spinor function denotes the usual Dirac conjugation. The integrand is easily shown to be $(1/z^0)(\chi_+^\dagger\chi_+ + \chi_-^\dagger\chi_-)$, where the dagger denotes the matrix hermitian conjugation and $\chi_\pm=P_\pm\chi$. The quantity is thus positive definite, which suggests the precise formulation of the problem as a unitary evolution in the Hilbert space $\mathscr{H}$ of the equivalence classes of $\mathbf{C}^4$-valued functions on the unit hyperboloid $z^2=1$, $z^0>0$, with the scalar product

$$(g,f)=\int\overline{g(z)}\gamma\cdot zf(z)d\mu(z). \tag{4.6}$$

(We note that this cannot be achieved by a simple evolution-independent unitary transformation within the usual formulation on hypersurface of constant time $x^0$, as the change to the hyperboloid mixes the space and time aspects.) Special classes (dense in $\mathscr{H}$) of such functions such as $k$ times continuously differentiable functions of compact support $C_0^k$ and the Schwartz test functions $\mathscr{S}$ are defined as those $f(z^0,\mathbf{z})$ for which the respective properties hold for $f(\sqrt{1+\mathbf{z}^2},\mathbf{z})$ with respect to $\mathbf{z}$; the identification is time-axis independent. In the Hilbert space $\mathscr{H}$ the operator of multiplication by $\gamma\cdot z$ is easily seen to be a self-adjoint unitary operator and $P_\pm$ become projection operators. The operators $i\gamma_{\rm T}^a$, and $p_a$ defined in (4.5), are not bounded, but they are symmetric on each of the special class of functions mentioned above.

The discussion of the free field case is best carried through with the use of Fourier-type transformation on the unit hyperboloid. For functions in $C_0^\infty$ we define two integral transformations

$$F_\kappa f(u)=\left(\frac{\kappa}{2\pi}\right)^{3/2}\int e^{-i\kappa u\cdot z\gamma\cdot z}\gamma\cdot zf(z)d\mu(z), \tag{4.7}$$

$$F_\kappa^* f(u)=\left(\frac{\kappa}{2\pi}\right)^{3/2}\int e^{+i\kappa u\cdot z\gamma\cdot u}\gamma\cdot zf(z)d\mu(z). \tag{4.8}$$

By $F_\kappa f(u)$ and $F^*_\kappa f(u)$ we mean functions defined on the unit hyperboloid, but the above integrals are valid outside the hyperboloid as well.

*Proposition 4.1: $F_\kappa$ and $F^*_\kappa$ are isometric operators from $C_0^\infty$ into $\mathscr{S}$, so they both can be extended to isometries of $\mathscr{H}$. $F_\kappa$ and $F^*_\kappa$ are then mutually conjugated, hence they are unitary.*

*Proof:* If $f\in C_0^\infty$ then $F_\kappa f(u)$ and $F^*_\kappa f(u)$ are infinitely differentiable. Denote $\chi(u)$ the extension of $F_\kappa f(u)$ outside the hyperboloid. $\chi(u)$ is a regular wave packet, in the sense of Ref. 13, so it vanishes rapidly between (and on) the surfaces $u^0=0$ and $u^2=1$, $u^0>0$, in particular $F_\kappa f\in\mathscr{S}$. Moreover, the Dirac equation $(i\gamma\cdot\nabla-\kappa)\chi(u)=0$ is satisfied. Hence, the current conservation gives

$$\int \overline{F_\kappa f(u)}\gamma\cdot uF_\kappa f(u)d\mu(u)=\int \chi^\dagger(0,\mathbf{u})\chi(0,\mathbf{u})d^3u.$$

By the usual Fourier transform properties the rhs is easily transformed into

$$\int \frac{d^3z}{(z^0)^2}\,(f_+^\dagger(z)f_+(z)+f_-^\dagger(z)f_-(z)-f_+^\dagger(z)f_-(z^0,-\mathbf{z})-f_-^\dagger(z)f_+(z^0,-\mathbf{z})),$$

which, after some manipulation with projectors $P_\pm$, gives $\int\overline{f(z)}\gamma\cdot z\, f(z)d\mu(z)$. This shows that $F_\kappa:C_0^\infty\rightarrow\mathscr{S}$ isometrically, so it extends to isometry of $\mathscr{H}$. To prove the same result for $F^*_\kappa f$ we assume that the support of $f$ lies in $z^0<a$. For $z^0>a+\epsilon$ deform the hyperboloid smoothly in such a way, that for large $|\mathbf{z}|$ it tends to $z^0=a+2\epsilon$, and regard $f(z)$ as initial data on this surface for $k(z)$ satisfying $(i\gamma\cdot\nabla-\kappa)k(z)=0$. Then $k(z)$ has compact support between this surface and $z^0=0$, and moreover $\nabla_a^{(z)}(e^{i\kappa u\cdot z\gamma\cdot u}\gamma^a k(z))=0$. Therefore, changing the surface of integration in (4.8) to $z^0=0$ one obtains $F^*_\kappa f(u)=P_+(u)G(\mathbf{u})+P_-(u)G(-\mathbf{u})$, where

$$G(\mathbf{u})=\left(\frac{\kappa}{2\pi}\right)^{3/2}\int e^{-i\kappa\mathbf{u}\cdot\mathbf{z}}\gamma^0k(0,\mathbf{z})d^3z.$$

Hence $F^*_\kappa f(u)$ is a function of fast decrease and one finds

$$\int \overline{F^*_\kappa f(u)}\gamma\cdot uF^*_\kappa f(u)d\mu(u)=\int G^\dagger(\mathbf{u})G(\mathbf{u})d^3u.$$

This, by standard Fourier transformation properties and then by current conservation for $k(u)$, is again $\|f\|^2$. Finally, one easily proves $(F_\kappa f,g)=(f,F^*_\kappa g)$ for $f,g\in C_0^\infty$, which extends to $\mathscr{H}$. This ends the proof.

The operator $U_0(\lambda_2,\lambda_1)=F_{m\lambda_2}F^*_{m\lambda_1}$ can be now identified as the evolution operator of the free Dirac field in $\mathscr{H}$. Indeed, the following proposition holds.

*Proposition 4.2: The families of operators $F_\kappa$, $F^*_\kappa$ and $U_0(\lambda_2,\lambda_1)$ are strongly continuous in their parameters. For $f\in C_0^\infty$ the vectors $F^*_\kappa f$ and $U_0(\lambda_2,\lambda_1)f$ are strongly differentiable in $\kappa$, $\lambda_2$ and $\lambda_1$ according to the following formulas*

$$-i\,\frac{d}{d\kappa}F^*_\kappa f=F^*_\kappa H_\kappa f, \tag{4.9}$$

$$i\partial_{\lambda_2}U_0(\lambda_2,\lambda_1)f=mH_{m\lambda_2}U_0(\lambda_2,\lambda_1)f, \tag{4.10}$$

$$-i\partial_{\lambda_1}U_0(\lambda_2,\lambda_1)f=U_0(\lambda_2,\lambda_1)mH_{m\lambda_1}f, \tag{4.11}$$

*where $H_\kappa=\Gamma(-(1/\kappa)\gamma_{\mathrm{T}}\cdot p+1)$, $\Gamma f(z)=\gamma\cdot zf(z)$.*

*Proof:* If $f \in C_0^\infty$ then $F^*_{m\lambda_1} f \in \mathscr{S}$ and $F_{m\lambda_2}$ applied to the latter is therefore expressible in the integral form (4.7). The function $F_{m\lambda_2}F^*_{m\lambda_1}f(z)$ is continuously differentiable in $\lambda_2$ and $z$, satisfies in these variables the free version of Eq. (4.4), and for $\lambda_2=\lambda_1$ is equal to $f(z)$. Formulated in terms of the original Dirac equation this means that $F_{m\lambda_2}F^*_{m\lambda_1}f(z) = \lambda_2^{3/2}\psi(\lambda_2 z)$, where $\psi(x)$ is the solution (4.1) of the initial data problem for the Dirac equation with the initial data $\psi(x)=\lambda_1^{-3/2}f(z)$ on $x=\lambda_1 z$. Both $F_{m\lambda_2}F^*_{m\lambda_1}f(z)$ and its derivative on $\lambda_2$ are therefore jointly continuous in $\lambda_2$ and $z$, and have compact support in $z$ for $\lambda_2$ in some neighborhood of $\lambda_1$. The strong continuity of $U_0(\lambda_2,\lambda_1)$, differentiability of $U_0(\lambda_2,\lambda_1)f$ in $\lambda_2$ and Eq. (4.10) now easily follow. $F_\kappa = U_0(\kappa/m,\kappa_0/m)F_{\kappa_0}$ is then strongly continuous as well, so as is $U_0(\lambda_2,\lambda_1)$ in $\lambda_1$. Strong differentiability of $U_0(\lambda_2,\lambda_1)f$ in $\lambda_1$ for $f\in C_0^\infty$ and Eq. (4.11) follow from

$$\left\| \frac{U_0(\lambda_2,\lambda_1')f-U_0(\lambda_2,\lambda_1)f}{\lambda_1'-\lambda_1} - U_0(\lambda_2,\lambda_1)imH_{m\lambda_1}f \right\|$$

$$= \left\| \frac{U_0(\lambda_1',\lambda_1)f-f}{\lambda_1'-\lambda_1} + U_0(\lambda_1',\lambda_1)imH_{m\lambda_1}f \right\| \to 0$$

for $\lambda_1' \to \lambda_1$. In consequence (4.9) follows as well.

We are now ready to discuss evolution in the presence of the electromagnetic potential $A_a(x)$. Let $R(\lambda_2,\lambda_1)$ be the unitary propagator generated by the family of operators $V_R(\lambda) = F^*_{m\lambda}V(\lambda)F_{m\lambda}$, where $V(\lambda)$ is the operator of multiplication by $e\gamma\cdot z\gamma\cdot A(\lambda z)$. For $R(\lambda_2,\lambda_1)$ to be well defined it suffices to assume (which is sufficient for our purposes), that $V(\lambda)$ is a strongly continuous family of bounded operators. Then $R(\lambda_2,\lambda_1)$ is jointly strongly continuous in $\lambda_2$ and $\lambda_1$ and

$$i\partial_{\lambda_2}R(\lambda_2,\lambda_1)f=V_R(\lambda_2)R(\lambda_2,\lambda_1)f, \tag{4.12}$$

$$-i\partial_{\lambda_1}R(\lambda_2,\lambda_1)f=R(\lambda_2,\lambda_1)V_R(\lambda_1)f \tag{4.13}$$

for any $f\in\mathscr{H}$. If $a_a(z)$ is a measurable vector function then $\|\gamma^{\mathrm{T}}\cdot af\|=\|\sqrt{-a_{\mathrm{T}}^2}f\|$, hence $\|\gamma\cdot af\|\leq\|a\cdot z\, f\|+\|\sqrt{-a_{\mathrm{T}}^2}f\|$. To satisfy the conditions on $V(\lambda)$ we assume therefore that $A_a(x)$ is continuous and both $|z\cdot A(\lambda z)|$ and $|A2_{\mathrm{T}}^2(\lambda z)|$ have bounds independent of $z$.

The unitary propagator $U(\lambda_2,\lambda_1)=F_{m\lambda_2}R(\lambda_2,\lambda_1)F^*_{m\lambda_1}$ gives the Dirac evolution at least in the weak sense

$$i\partial_{\lambda_2}(f,U(\lambda_2,\lambda_1)g)=([mH_{m\lambda_2}+V(\lambda_2)]f,U(\lambda_2,\lambda_1)g),$$

where $g$ is any vector in $\mathscr{H}$ and $f\in C_0^\infty$. Moreover, for any $f\in C_0^\infty$

$$-i\partial_{\lambda_1}U(\lambda_2,\lambda_1)f=U(\lambda_2,\lambda_1)(mH_{m\lambda_1}+V(\lambda_1))f.$$

The scattering states of the evolution so determined are easily obtained by a simple unitary transformation, as suggested by the asymptotics of $\psi(\lambda z)$ discussed at the beginning of this section. Let us denote $G_\kappa=e^{i\kappa\Gamma}$, where $\Gamma$ is the operator defined in Proposition 4.2. $G_\kappa$ is strongly continuous, differentiable on every $f\in\mathscr{H}$, family of unitary operators. Denote further $T_\kappa=iG_{\kappa+\pi/4}F_\kappa$ and $W(\lambda_2,\lambda_1) = G_{m\lambda_2+\pi/4}U(\lambda_2,\lambda_1)G^*_{m\lambda_1+\pi/4} = T_{m\lambda_2}R(\lambda_2,\lambda_1)T^*_{m\lambda_1}$.

*Lemma 4.3: If $f\in C_0^2$ then $\|T_\kappa f - f\| = \|T^*_\kappa f - f\| \leq (1/\kappa)(\|hf\| + \|h^2f\|)$, where $h=-i\gamma_{\mathrm{T}}\cdot p$.*

*Proof:* Let $f\in C_0^\infty$ first. Then $G^*_\kappa f \in C_0^\infty$ as well, so the differentiations in $(d/d\kappa)T^*_\kappa f$ can be performed. Using the fact that $p$ commutes and $\gamma^{\mathrm{T}}$ anticommutes with $\Gamma$ one finds

$$\frac{d}{d\kappa} T_\kappa^* f = T_\kappa^* \frac{i}{\kappa} G_{2\kappa} h f = T_\kappa^* \frac{d}{d\kappa} g_{2\kappa} h f,$$

where

$$g_\kappa = -i \int_\kappa^\infty G_u \frac{du}{u} = \Gamma\left(\frac{G_\kappa}{\kappa} - \int_\kappa^\infty G_u \frac{du}{u^2}\right)$$

(the first form as an improper integral). From the latter form one has $\|g_\kappa\| \leqslant (2/\kappa)$. $g_{2\kappa} h f$ is again in $C_0^\infty$, so

$$\frac{d}{d\kappa}(T_\kappa^* f - T_\kappa^* g_{2\kappa} h f) = T_\kappa^* \frac{i}{\kappa} G_{2\kappa} g_{2\kappa}^* h^2 f.$$

Integration from $\kappa_1$ to $\kappa_2$ leads to

$$T_{\kappa_2}^* f - T_{\kappa_1}^* f = T_{\kappa_2} g_{2\kappa_2} h f - T_{\kappa_1} g_{2\kappa_1} h f + i \int_{\kappa_1}^{\kappa_2} T_u^* G_{2u} g_{2u}^* h^2 f \frac{du}{u}$$

and

$$\|T_{\kappa_2}^* f - T_{\kappa_1}^* f\| \leqslant \left(\frac{1}{\kappa_2} + \frac{1}{\kappa_1}\right)(\|hf\| + \|h^2 f\|).$$

This shows that $T_\kappa^* f$ has a limit; this limit has to be $f$, as for any $g \in C_0^\infty$ there is $(g, T_\kappa^* f - f) \to 0$, which is easily seen e.g. by stationary phase method. Taking the limit $\kappa_1 \to \infty$ one obtains the stated result for $f \in C_0^\infty$. Any $f \in C_0^2$ can be uniformly approximated together with its derivatives by functions from $C_0^\infty$ vanishing outside a common compact set. This ends the proof.

The above lemma reduces the problem of asymptotics of $W(\lambda_2,\lambda_1)$ to that of $R(\lambda_2,\lambda_1)$

$$\|W(\lambda,\lambda_2)f - W(\lambda,\lambda_1)f\| \leqslant \|R(\lambda,\lambda_2)f - R(\lambda,\lambda_1)f\| + \left(\frac{1}{m\lambda_2} + \frac{1}{m\lambda_1}\right)(\|hf\| + \|h^2 f\|) \quad (4.14)$$

for $f \in C_0^\infty$. The generator of the propagator $R(\lambda_2,\lambda_1)$ can be written in the form $V_R(\lambda) = T_{m\lambda}^*(G_{2m\lambda} v_1(\lambda) + v_2(\lambda)) T_{m\lambda}$, where $v_1(\lambda)$ and $v_2(\lambda)$ are the operators of multiplication by $ie\gamma_{\mathrm{T}} \cdot A(\lambda z)$ and $ez \cdot A(\lambda z)$ respectively. Transforming (4.13) with the use of the method applied in the proof of Lemma 4.3 one obtains

$$\begin{aligned}
&-i\partial_u\{R(\lambda,u)f - R(\lambda,u) g_{2mu} u v_1(u) f\} \\
&\quad = R(\lambda,u) T_{mu}^* v_2(u) T_{mu} f - R(\lambda,u) V_R(u) g_{2mu} u v_1(u) f + R(\lambda,u) T_{mu}^* G_{2mu} v_1(u) \\
&\quad\quad \times (T_{mu} - 1) f + R(\lambda,u)(T_{mu}^* - 1) G_{2mu} v_1(u) f + R(\lambda,u) g_{2mu} i \frac{d}{du}(u v_1(u) f).
\end{aligned} \quad (4.15)$$

The strong differentiation in the last term will be allowed under the assumptions of the following theorem.

*Proposition 4.4: Let $A_a(x)$ be a vector function twice continuously differentiable and for* $\lambda > \lambda_0 > 0$ *subject to the following bounds for some* $\epsilon > 0$

$$|z\cdot A(\lambda z)|<\frac{\text{const.}}{\lambda^{1+\epsilon}},\quad |A_{\mathrm{T}}^2(\lambda z)|<\frac{\text{const.}}{\lambda^{1+\epsilon}},$$

$$|[\partial_\lambda(\lambda A_{\mathrm{T}}(\lambda z))]^2|<\frac{C(z)}{\lambda^{2\epsilon}}, \tag{4.16}$$

$$|\delta_a A_{\mathrm{T}b}(\lambda z)|<\frac{D(z)}{\lambda^{\epsilon}},\quad |\delta_a\delta_b A_{\mathrm{T}c}(\lambda z)|<\frac{D(z)}{\lambda^{\epsilon}},$$

*where $C(z)$ and $D(z)$ are continuous functions and the last two estimates hold component-wise in arbitrary fixed Lorentz frame [change of the frame results only in the change of $D(z)$].*

*Then for all $f\in C_0^\infty$*

$$\|W(\lambda,\lambda_2)f-W(\lambda,\lambda_1)f\|\leq c(f)\left(\frac{1}{\lambda_2^\alpha}+\frac{1}{\lambda_1^\alpha}\right), \tag{4.17}$$

*where $\alpha=\min\{\epsilon,1\}$ and $c(f)$ is a constant depending on $f$. Hence for every $\lambda>\lambda_0$ the strong limit $\lim_{u\to\infty} W(\lambda,u)f=f_\lambda$ exists, is strongly continuous in $\lambda$ and*

$$\|f_\lambda-f\|\leq c(f)/\lambda^\alpha. \tag{4.18}$$

*Proof:* To prove (4.17) it remains to estimate various terms in (4.15). The successive terms (i),...,(v) on the rhs of (4.15) are bounded in norm respectively by

$$\|(\mathrm{i})\|\leq\|v_2(u)\|\|f\|,$$

$$\|(\mathrm{ii})\|\leq\frac{1}{m}(\|v_1(u)\|+\|v_2(u)\|)\|v_1(u)\|\ \|f\|,$$

$$\|(\mathrm{iii})\|\leq\|v_1(u)\|\frac{1}{mu}(\|hf\|+\|h^2f\|),$$

$$\|(\mathrm{iv})\|\leq\frac{1}{mu}(\|hv_1(u)f\|+\|h^2v_1(u)f\|),$$

$$\|(\mathrm{v})\|\leq\frac{1}{mu}\left\|\frac{d}{du}uv_1(u)f\right\|;$$

for (iv) the fact was used, that $v_1(u)f\in C_0^2$. The assumed estimates of the potential force all these bounds below some constant depending on $f$ times $\lambda^{-1-\alpha}$. The integration of (4.15) leads therefore to

$$\|[R(\lambda,u)f-R(\lambda,u)g_{2mu}uv_1(u)f]|_{u=\lambda_1}^{u=\lambda_2}\|\leq\text{const.}(f)\left(\frac{1}{\lambda_2^\alpha}+\frac{1}{\lambda_1^\alpha}\right).$$

The form of this inequality remains unchanged, if we omit the second term inside the brackets on the lhs (this term is bounded by const.$\|f\|u^{-1-\epsilon}$). Taking into account (4.14) one arrives at (4.17). The continuity of $f_\lambda$ is evident from $f_{\lambda'}=W(\lambda',\lambda)f_\lambda$, and (4.18) is obtained by putting $\lambda_1=\lambda$ and letting $\lambda_2\to\infty$ in (4.17). This ends the proof.

*Corollary 4.5: For every $f\in\mathscr{H}$ the strong limit $\lim_{u\to\infty}W(\lambda,u)f=f_\lambda$ exists. $f_\lambda$ is strongly continuous and $\|f_\lambda-f\|\to0$ for $\lambda\to\infty$.*

The crucial point of our discussion is the fact, that the long-range electromagnetic fields of the Coulomb type are admitted by the premises of Proposition 4.4, provided one chooses the potential in an appropriate gauge. Let us observe first, that if the electromagnetic field is represented as a superposition, it suffices to satisfy (4.16) for the potentials of the superposed fields separately. Suppose that one of the superposed fields is the asymptotic Coulomb-type field homogeneous of degree $-2$: $F_{ab}(\kappa x)=\kappa^{-2}F_{ab}(x)$. The simplest choice of the potential inside the lightcone is of the form $A_a(\lambda z)=\lambda^{-1}a_a(z)$. This potential breaks the first of the bounds (4.16). Assume, however, that $a_a(z)$ is three times continuously differentiable and satisfies the bounds

$$|a_{\mathrm{T}}^2(z)|<\text{const.},\quad |[\delta(z\cdot a(z))]^2|<\text{const.}\ . \tag{4.19}$$

Choose the new gauge by $A_{\mathrm{tr}}(x)=A(x)-\nabla S(x)$ with $S(x)$ given by $S(\lambda z)=\ln\lambda\ z\cdot a(z)$ inside the lightcone. Then

$$A_{\mathrm{tr}b}(\lambda z)=\lambda^{-1}\{a_{\mathrm{T}b}(z)-\ln\lambda\,\delta_b(z\cdot a(z))\} \tag{4.20}$$

and $z\cdot A_{\mathrm{tr}}(\lambda z)=0$. The other bounds of (4.16) are satisfied for any $\epsilon<1$ (with constants depending on $\epsilon$).

Another class of potentials admitted by Proposition (4.4) consists of Lorentz-gauge potentials (2.36) of free fields discussed in Sec. II. With the use of (B6) and (B7) we get

$$|A_a(\lambda z)|<\frac{\text{const.}}{\lambda z^0},\quad |\nabla_a A_b(\lambda z)|<\text{const.}\ \frac{(z^0)^\epsilon}{\lambda^{2+\epsilon}},\quad |\nabla_a\nabla_b A_c(\lambda z)|<\text{const.}\ \frac{(z^0)^\epsilon}{\lambda^{2+\epsilon}}.$$

These bounds imply the third, fourth, and fifth of the estimates (4.16), while the first bound above is sufficient for the second estimate in (4.16) to hold, if the first one is satisfied. To prove this remaining estimate we observe first that, as follows from (2.37), $V_a(s,l)=o_A k_{A'}(s,o,\bar{o})$ +compl.conj., where $o_{C'}k^{C'}(s,o,\bar{o})=\zeta(s,o,\bar{o})$. Inserting this into (2.36) we get by (A.8)

$$x\cdot A(x)=\frac{1}{2\pi}\int\partial'_{A'}k^{A'}(x\cdot l,o,\bar{o})d^2l+\text{compl. conj.}\ .$$

Hence $|z\cdot A(\lambda z)|<\text{const.}/\lambda^{1+\epsilon}(z^0)^\epsilon$ by (B5), which ends the proof.

We stress that the transformation used here to compensate the asymptotic behavior of the Dirac field is interaction independent, unlike in the usual Dollard treatment of the Coulomb potential,[15,13] or in a recent discussion of the Cauchy problem for the classical spinor electrodynamics.[7]

The Dirac field is expressed in terms of $f_\lambda(z)$ as

$$\psi(\lambda z)=-i\lambda^{-3/2}e^{-i(m\lambda+\pi/4)\gamma\cdot z}f_\lambda(z). \tag{4.21}$$

If $f_\lambda^0(z)$ is a solution of the free evolution, with the corresponding Dirac field $\psi^0(\lambda z)$, then

$$\int_{x^2=\lambda^2}\bar{\psi}^0(x)\gamma^a\psi(x)d\sigma_a(x)=(f_\lambda^0,f_\lambda)\rightarrow(f^0,f\ ),$$

for $\lambda\rightarrow\infty$. This suggests that the precise formulation of the asymptotic Dirac field in the quantum electrodynamics be looked for as a limit of the expression on the lhs, with $\psi^0$ being a test field.

## V. TOTAL CONSERVED QUANTITIES

We want to return now to the consideration of a closed system with electromagnetic interaction, which has been taken up in Sec. III. The results should not depend essentially on what kind of massive field one couples minimally to the electromagnetic field, but we consider for definiteness the Maxwell–Dirac system. The discussion of the present section will make use of the results of the preceding sections, but in the full theory we lack rigorous results along the lines presented here. Rigorous results on the Cauchy problem and scattering properties of the Maxwell–Dirac theory were recently reported by Flato *et al.*,[7] but the method used by these authors is quite different, and the relation of the present work with Ref. 7 remains to be clarified. What is clear, however, is the difference in the choice of transformation leading to asymptotic states: Our transformation is interaction-independent, which is made possible by a special choice of gauge, while the transformation of Flato *et al.* is a Dollard-type treatment (cf. Ref. 15), consisting of extraction of a phase in momentum space, thus not constituting a gauge transformation in the usual sense. Moreover, the method used in the present work aims at appropriate description of the spacetime separation of asymptotic matter and radiation, so far as it can be achieved. We stress, however, that no results on the Cauchy problem or asymptotic completeness are given here.

Proceeding heuristically we shall assume that the asymptotics of fields of the interacting theory are of the type described in Sec. II for the electromagnetic and in Sec. IV for the Dirac field respectively. When needed we shall add further assumptions on how these asymptotics are achieved. These extrapolations seem plausible, provided (i) the full electromagnetic potential falls into the class admitted by Proposition 4.4 and (ii) the current of the Dirac field vanishes in spacelike directions sufficiently fast for the discussion of Sec. 2 to remain valid. Basing the intuitions on the free Dirac field case we regard the second point as unproblematic, but for its rigorous justification more control over the limit $f_\lambda \to f$, and also the solution of the Dirac equation outside the cone would be needed. As to the first point, we can only present a very simplified argument of self-consistency type, which, however, takes care of the Coulomb term, the most troublesome from the point of view of asymptotics of the matter field.

More explicitly, we represent the Dirac field inside the lightcone as in (4.21) and assume that $f_\lambda \to f$ as in Corollary 4.5. For any current density denote inside the lightcone $j_a(\lambda,z)=\lambda^3 J_a(\lambda z)$, $z^2=1$, $z^0>0$. For the Dirac field

$$j_a(\lambda,z)=z_a\rho(z)+(e^{-2im\lambda}\kappa_a(z)+\text{compl. conj.})+r_a(\lambda,z), \tag{5.1}$$

where

$$\rho(z)=e\overline{f(z)}\gamma\cdot z f(z),$$

$$\kappa_a(z)=-ie\overline{P_-f(z)}\gamma_{\mathrm{T}a}P_+f(z),$$

$$\begin{aligned}r_a(\lambda,z)=&e\overline{e^{-i(m\lambda+\pi/4)\gamma\cdot z}f_\lambda(z)}\gamma_a e^{-i(m\lambda+\pi/4)\gamma\cdot z}f_\lambda(z)\\&-e\overline{e^{-i(m\lambda+\pi/4)\gamma\cdot z}f(z)}\gamma_a e^{-i(m\lambda+\pi/4)\gamma\cdot z}f(z).\end{aligned}$$

The electromagnetic potential in the Lorentz gauge can be split into the free outgoing and advanced parts. As for the free part, its admissibility in Proposition 4.4 has been proved already in the preceding section. The advanced field of the current $J_a(x)$ can be written inside the future lightcone as

$$A^{\mathrm{adv}}{}_a(x)=\int j_a(x\cdot v+\sqrt{(x\cdot v)^2-x^2},v)\,\frac{d\mu(v)}{\sqrt{(x\cdot v)^2-x^2}}.$$

For the Dirac density the first term of (5.1) yields a Coulomb potential

$$A^{\mathrm{Coul}}{}_b(\lambda z)=\frac{a_b(z)}{\lambda}, \tag{5.2}$$

with

$$a_b(z)=\int v_b\rho(v)\,\frac{d\mu(v)}{\sqrt{(z\cdot v)^2-1}}. \tag{5.3}$$

This is a homogeneous potential of the type discussed after Corollary 4.5. All we have to show for admissibility of its gauged form $A^{\rm Coul}_{\rm tr}$ (4.20) is the threefold differentiability of $a_b(z)$ and the bounds (4.19). The differentiability of $a_b(z)$ follows easily by the use of the identity

$$\delta_b\int h(v)\,\frac{d\mu(v)}{\sqrt{(z\cdot v)^2-1}}=\int [z\cdot v(\delta_b-3v_b)+z_b]h(v)\,\frac{d\mu(v)}{\sqrt{(z\cdot v)^2-1}}$$

and suitable assumptions on the regularity and fall-off of $\rho(v)$. [The identity follows by multiplication of

$$\delta_b^{(z)}\,\frac{1}{\sqrt{(z\cdot v)^2-1}}=\frac{z_b}{\sqrt{(z\cdot v)^2-1}}-\delta_b^{(v)}\,\frac{z\cdot v}{\sqrt{(z\cdot v)^2-1}}$$

by $h(v)$ and integration by parts according to (4.3).] From (D1) we have

$$|a_b(z)|<\frac{\rm const.}{z^0},\quad |z\cdot a(z)|<\text{const.}, \tag{5.4}$$

so that the first of the bounds (4.19) is satisfied. To obtain the other one we observe first that the components of any unit vector orthogonal to a timelike unit vector $z^a$ are bounded by $z^0$, in particular $|(v^a-z\cdot vz^a)/[\sqrt{(z\cdot v)^2-1}]|\leqslant z^0$. [*Proof:* if $w\cdot z=0$, then $|z^0w^0|\leqslant|\mathbf{z}||\mathbf{w}|$, or, using $z^2=-w^2=1$, $(|\mathbf{w}|^2-1)(z^0)^2\leqslant|\mathbf{w}|^2((z^0)^2-1)$, hence $z^0\geqslant|\mathbf{w}|\geqslant|w^0|$.] Hence, by (D.1),

$$\delta_b(z\cdot a(z))=-\int\rho(v)\,\frac{v_b-z\cdot vz_b}{\sqrt{(z\cdot v)^2-1}}\,\frac{d\mu(v)}{(z\cdot v)^2-1}, \tag{5.5}$$

and $|\delta_b(z\cdot a(z))|<\text{const.}/z^0$, which implies the second of the inequalities (4.19). From now on $A^{\rm Coul}_{tr}$ replaces $A^{\rm Coul}$ in the Dirac equation.

The remaining contributions to $A^{\rm adv}{}_b(\lambda z)$ will not be discussed in detail, but we assume, what could be achieved with some additional assumptions on uniformness of the limit $f_\lambda(z)\rightarrow f(z)$ and on regularity of $f(z)$, that

$$|A^{\rm adv}{}_b(\lambda z)-A^{\rm Coul}{}_b(\lambda z)|<\frac{\rm const.}{(\lambda z^0)^{1+\epsilon}} \tag{5.6}$$

and

$$|F^{\rm adv}{}_{ab}(\lambda z)-F^{\rm Coul}{}_{ab}(\lambda z)|<\frac{\rm const.}{(\lambda z^0)^{2+\epsilon}}, \tag{5.7}$$

where

$$F^{\rm Coul}{}_{ab}(\lambda z)=\frac{f_{ab}(z)}{\lambda^2}, \tag{5.8}$$

$$f_{ab}(z)=\int \rho(v)\,\frac{z_a v_b - z_b v_a}{\sqrt{(z\cdot v)^2-1}}\,\frac{d\mu(v)}{(z\cdot v)^2-1}. \tag{5.9}$$

Since $z_{[a}v_{b]}=(z_{[a}-z\cdot v v_{[a})v_{b]}$ we have $|(z_a v_b - z_b v_a)/[\sqrt{(z\cdot v)^2-1}]| < 2(v^0)^2$ and from (D1)

$$|f_{ab}(z)|<\frac{\text{const.}}{(z^0)^2}, \tag{5.10}$$

if $|\rho(v)|<\text{const.}/(v^0)^{4+\epsilon}$.

The new gauge of the electromagnetic potential, which we use here for its simplicity, is a nonlocal one, being reached from a Lorentz gauge by a transformation depending on the asymptotic current. However, the same asymptotic effect can be achieved by a local gauge transformation $A(x)\rightarrow A(x)-\nabla S(x)$, with $S(x)=\ln\sqrt{x^2}\;x\cdot A(x)$ inside the lightcone.

We come now to our principal aim in this section. We want to complete the discussion of Sec. II by supplying the up to now lacking expressions for energy-momentum and angular momentum going out in timelike directions with the massive part of the system. We recall, that these quantities are determined by (3.15) and (3.16) respectively, and they do not depend on the choice of the time axis along which the limits in those formulas are achieved. We take advantage of this independence to chose an axis going through the origin of Minkowski space (with arbitrary time-vector $t$). The total energy momentum tensor of the theory is given by

$$T_{ab}=T^{\mathrm{D}}{}_{ab}+T^{\mathrm{elm}}{}_{ab},$$

where $T^{\mathrm{elm}}{}_{ab}$ is the tensor of the total electromagnetic field (3.7) and

$$T^{\mathrm{D}}{}_{ab}=\tfrac{1}{4}\{\bar\psi\gamma_a(i\nabla_b-eA_b)\psi+\text{compl. conj.}\}+(a\leftrightarrow b),$$

where $(a\leftrightarrow b)$ stands for terms with interchanged indices. Recalling result (3.17) we see that the contribution to the rhs's of (3.15) and (3.16) coming from the out field vanish. Also the contributions coming from the mixed adv-out terms in $T^{\mathrm{elm}}{}_{ab}$ vanish, as shown in Appendix B.

We are left with the task of calculating the rhs's of (3.15) and (3.16) for

$$T'_{ab}=T^{\mathrm{D}}{}_{ab}+T^{\mathrm{adv}}{}_{ab},$$

where $T^{\mathrm{adv}}{}_{ab}$ is the electromagnetic tensor of advanced field. We want to show first that the limits of the integrals over $\mathscr{C}^{\mathrm{fut}}(\tau)$ for $\tau\rightarrow\infty$ may be replaced by the limits for $\lambda\rightarrow\infty$ of the integrals over hyperboloids $\mathscr{H}(\lambda)=\{x|x^2=\lambda^2,x^0>0\}$. To this end consider integrals over the region contained between $\mathscr{C}^{\mathrm{fut}}(\tau)$ and $\mathscr{H}(\lambda)$ of the quantities

$$\nabla^c T'_{ac}=-F^{\mathrm{out}}{}_{ac}J^c \tag{5.11}$$

and

$$\nabla^c(x_a T'_{bc}-x_b T'_{ac})=(-x_a F^{\mathrm{out}}{}_{bc}+x_b F^{\mathrm{out}}{}_{ac})J^c. \tag{5.12}$$

Since $T'_{ab}$ gives no flow of energy momentum or angular momentum to null infinity, these integrals give the differences of energy momentum and angular momentum passing through $\mathscr{C}^{\mathrm{fut}}(\tau)$ and $\mathscr{H}(\lambda)$. If the above divergencies are absolutely integrable over the region $x^2>1$, $x^0>0$, then these differences vanish in the limit and the replacement of $\mathscr{C}^{\mathrm{fut}}(\tau)$ by $\mathscr{H}(\lambda)$ is justified. If we assume that $|j_a(\lambda,z)|<h(z)$, then by (B7) the rhs of (5.11) is bounded by

const.$[h(z)(z^0)^{1+\epsilon}/\lambda^4 z^0(\lambda+s_t z^0)^{1+\epsilon}]$ and the rhs of (5.12) by a similar quantity multiplied by $\lambda z^0$. For $x=\lambda z$ there is $d^4x=\lambda^3 d\lambda\ d\mu(z)$, so both these expressions are integrable over $\lambda>1$ for $h(z)$ such that $\int h(z)(z^0)^{1+\epsilon}\, d\mu(z)<\infty$.

The preceding discussion brings us to the following representations

$$P^{\text{out-t}}{}_a=\lim_{\lambda\to\infty}\int \lambda^3 T'_{ac}(\lambda z)z^c\, d\mu(z),$$

$$M^{\text{out-t}}{}_{ab}=\lim_{\lambda\to\infty}\int \lambda^4(z_a T'_{bc}(\lambda z)-z_b T'_{ac}(\lambda z))z^c\, d\mu(z).$$

The limits here will be treated rather formally, by assuming that for large $\lambda$ only the leading (constant at least) terms of the integrands contribute. In this way there is no contribution from $T^{\text{adv}}{}_{ab}$ to $P^{\text{out-t}}{}_a$ and contribution to $M^{\text{out-t}}{}_{ab}$ comes from $-(1/16\pi)(z_a f_{bd}(z)-z_b f_{ad}(z))f_c{}^d z^c$. This term, however, vanishes identically, since $z_{[a}f_{bc]}=0$.

Consider finally $T^{\text{D}}{}_{ab}$, which gives the only nonvanishing contributions. Writing $\psi(\lambda z)=\lambda^{-3/2}\chi(\lambda,z)$ and using the Dirac equation (4.4) we have

$$(i\nabla^a-eA^a)\psi(\lambda z)=\lambda^{-3/2}\left\{z^a\gamma\cdot z\left(m-\frac{1}{\lambda}\gamma_{\text{T}}\cdot p+e\gamma\cdot A\right)+\frac{1}{\lambda}p^a-eA^a-\frac{i}{2\lambda}\gamma\cdot z\gamma_{\text{T}}{}^a\right\}\chi(\lambda,z),$$

where $p^a$ is the operator defined in (4.5). Now, $\chi(\lambda,z)=-ie^{-i(m\lambda+\pi/4)\gamma\cdot z}f_\lambda(z)$ and we treat $\chi$ as $O(\lambda^0)$. Then

$$\lambda^3 T^{\text{D}}{}_{ac}z^c=mz_a\bar{\chi}\chi+O(\lambda^{-\epsilon}),$$

$$\lambda^4(z_a T^{\text{D}}{}_{bc}-z_b T^{\text{D}}{}_{ac})z^c=\tfrac{1}{2}\{z_a\bar{\chi}\gamma\cdot z(p_b-e\lambda A_b+\tfrac{1}{2}\gamma_{\text{T}[b}\gamma_{\text{T}c]}p^c)\chi-(a\leftrightarrow b)\}+\text{compl. conj.}\,.$$

The result of integration over the hyperboloid can be written in terms of the scalar product of Sec. IV

$$\int \lambda^3 T^{\text{D}}{}_{ac}(\lambda z)z^c\, d\mu(z)=m(\chi,\gamma\cdot z z_a\chi)+O(\lambda^{-\epsilon}), \tag{5.13}$$

$$\begin{aligned}\int \lambda^4(z_a T^{\text{D}}{}_{bc}(\lambda z)-z_b T^{\text{D}}{}_{ac}(\lambda z))z^c\, d\mu(z)\\ =(\chi,(z_a p_b-z_b p_a)\chi)-e(\chi,(z_a\lambda A_b(\lambda z)-z_b\lambda A_a(\lambda z))\chi)\\ +\tfrac{1}{4}(\chi,[z_a\gamma_{\text{T}[b}\gamma_{\text{T}c]}-z_b\gamma_{\text{T}[a}\gamma_{\text{T}c]},p^c]\chi),\end{aligned} \tag{5.14}$$

where the symmetry of operators was taken into account. The operators appearing in the averages commute with $\gamma\cdot z$, so $\chi$ can be replaced by $f_\lambda$, and further, up to $O(\lambda^{-\epsilon})$, by $f(z)$. Using $[\gamma_{\text{T}[b}\gamma_{\text{T}c]},p^c]=0$ and $[p_c,z_a]=ih_{ca}$ we transform the third term in (5.14) to the form $(i/4)(f,[\gamma_{\text{T}a}\gamma_{\text{T}b}]f)+O(\lambda^{-\epsilon})$. Contributions to the second term up to $O(\lambda^{-\epsilon})$ could only come from $A_{\text{tr}}^{\text{Coul}}$. However,

$$\begin{aligned}-2e\Big(f,z_{[a}(a_{b]}(z)-\ln\lambda\,\delta_{b]}(z\cdot a(z))f\Big)\\ =\int\rho(z)\int\rho(v)\frac{z_a v_b-z_b v_a}{\sqrt{(z\cdot v)^2-1}}\left(1+\frac{\ln\lambda}{(z\cdot v)^2-1}\right)d\mu(v)d\mu(z)=0,\end{aligned}$$

due to antisymmetry of the integrand with respect to interchange of integration variables $z \leftrightarrow v$. Taking now the limit $\lambda \to \infty$ we finally obtain

$$P^{\mathrm{out-t}}{}_a = m(f, \gamma \cdot z z_a f) = m \int z_a \bar{f} f(z) d\mu(z), \tag{5.15}$$

$$\begin{aligned} M^{\mathrm{out-t}}{}_{ab} &= \left( f, \left( z_a p_b - z_b p_a + \frac{i}{4} [\gamma_{\mathrm{T}a}, \gamma_{\mathrm{T}b}] \right) f \right) \\ &= \int \bar{f} \gamma \cdot z \left( z_a i \delta_b - z_b i \delta_a + \frac{i}{4} [\gamma_a, \gamma_b] \right) f(z) d\mu(z), \end{aligned} \tag{5.16}$$

which are the desired formulas for the quantities going out in timelike directions. If we define the free outgoing Dirac field by [cf. (4.2)]

$$\psi^{\mathrm{out}}(x) = \left( \frac{m}{2\pi} \right)^{3/2} \int e^{-imx \cdot v \gamma \cdot v} \gamma \cdot v f(v) d\mu(v),$$

then the above expressions give the Fourier representations of the conserved quantities of this field (in a somewhat unusual but most compact form). Similar expressions could be obtained for the timelike past infinity.

The task of expressing the total energy momentum and angular momentum of the interacting theory in terms of asymptotic fields has been now completed. As anticipated, the contributions of electromagnetic and massive free fields almost separate, except for a term in the radiated angular momentum due to the long-range part of the electromagnetic field. This term [the second one on the rhs of (3.34)] can be now rewritten by the use of matter asymptotics. $q(o,\bar{o})$ is now given by formula (2.79), hence

$$\begin{aligned} \Delta\mu_{AB} &\equiv \frac{1}{2\pi} \int q o_{(A} \partial_{B)} \Phi(o,\bar{o}) d^2 l = -\frac{e}{4\pi} \int \bar{f} \gamma \cdot z f(z) \int \Phi(o,\bar{o}) o_{(A} \partial_{B)} \frac{1}{(z \cdot l)^2} d^2 l \, d\mu(z) \\ &= -\int \bar{f} \gamma \cdot z f(z) z_{C'(A} \delta^{C'}_{B)} H(z) d\mu(z), \end{aligned}$$

or in the tensor form

$$\Delta M_{ab} \equiv \Delta\mu_{AB} \epsilon_{A'B'} + \text{compl. conj.} = -(f, (z_a \delta_b H - z_b \delta_a H) f),$$

where

$$H(z) = \frac{e}{4\pi} \int \frac{\Phi(l)}{(z \cdot l)^2} d^2 l. \tag{5.17}$$

If we now change the phase of $f(z)$ by introducing

$$g(z) = e^{iH(z)} f(z), \tag{5.18}$$

then $M^{\mathrm{out-t}}{}_{ab} + \Delta M_{ab}$ has again the form (5.16) but with $f$ replaced by $g$, while $P^{\mathrm{out-t}}{}_a$ retains its form under this replacement. With this final representation the total quantities (3.18) and (3.19) look formally like sums of two free fields contributions. The very nonlocal transformation (5.18) has now accommodated the mixing aspects of the asymptotics.

It is interesting to note that $H(z)$ acquires here the role of a phase in a very natural way. This is rather satisfying, since the same conclusion has been reached earlier in a different way, by considering a quantum version of an "adiabatic approximation," see Ref. 9 [for "quantum field" $\Phi(l)$ definition (5.17) gives $\Phi(g_z)$ of this reference]. Moreover, $-2H(z)$ is identical with the change of phase $\delta(z)$ (3.38) in the external field problem calculated by Staruszkiewicz. On the other hand $H(z)$ is distinct from a phase variable considered by Staruszkiewicz[5] in his theory of quantum Coulomb field. We discuss the difference in some detail. A phase field of Ref. 5 is a homogeneous of degree 0 field in the region $x^2<0$, satisfying there the homogeneous wave equation. Such a field can be represented by

$$S(x)=\int\left\{\operatorname{sgn} x\cdot l\, f_1(l)+\ln\frac{|x\cdot l|}{t\cdot l}\, f_2(l)\right\}d^2l+c_t,$$

where $f_1, f_2$ are homogeneous of degree $-2$ functions of $l$ and $\int f_2(l)d^2l=0$, $t$ is a timelike, unit, future-pointing vector and $c_t$ is a constant; this constant changes for another choice of vector $t$ according to $c_{\tilde{t}}=c_t+\int\ln[(\tilde{t}\cdot l)/(t\cdot l)]f_2(l)d^2l$. Consider the spherically symmetric term $S_z(x)$ in the expansion of $S(x)$ in spherical harmonics in a coordinate system in which $z$ points in the direction of the time-axis. One easily shows that

$$S_z(x)=\int f_1(l)d^2l\,\frac{x\cdot z}{\sqrt{(x\cdot z)^2-x^2}}+c_z.$$

Identifications of Staruszkiewicz are

$$-\frac{1}{e}\int f_1(l)d^2l=\text{charge},\qquad c_z=\text{phase variable}.$$

To compare this with our identifications we use the relation of $S(x)$ to the long-range field of Ref. 5

$$-eF^{\text{l.r.}}{}_{ab}(x)x^b=\nabla_a S(x).$$

One easily shows using (2.68) and (2.66) that in our description a field $S(x)$ which can be formed out of the long-range variables and which satisfies this relation is given by

$$S(x)=-\frac{e}{2\pi}\int \operatorname{sgn} x\cdot l\,(q(l)+\sigma(l))d^2l.$$

For this field

$$S_z(x)=-eQ\,\frac{x\cdot z}{\sqrt{(x\cdot z)^2-x^2}}.$$

The charge part agrees with that of Staruszkiewicz, but the analog of his phase variable is absent. Our phase variable, which is the null spherical harmonic in $e\Phi(l)$, does not appear in $S_z(x)$. [The absence of logarithmic terms in our version of $S(x)$ is due to the conditions on null asymptotics of fields.]

## VI. CONCLUSIONS

The main results of our analysis can be summarized as follows.

(i) Despite nonintegrability of the angular momentum tensor density over a Cauchy surface, the total angular momentum (four dimensional) can be unambiguously identified, provided (a) angular momentum radiated (or incoming from null directions) over finite time intervals is well defined, and (b) the magnetic part of the spacelike asymptotic of the electromagnetic field vanishes.

(ii) Asymptotic Dirac field can be identified by a special choice of gauge and consideration of the asymptotic behavior of the Dirac field on the hyperboloid $x^2=\lambda^2$ for $\lambda\rightarrow\infty$.

(iii) The total energy momentum of the system can be expressed as a sum of independent contributions from the asymptotic free electromagnetic field and the asymptotic Dirac field. However, in the analogous representation of the angular momentum an additional term survives, which mixes the asymptotic Dirac field characteristic with the infrared characteristic of the free asymptotic electromagnetic field. This effect persists in the limit of the energy tending to zero. The additional term can be accommodated into the matter part by a redefinition of the asymptotic Dirac field. This is a very nonlocal transformation mixing the matter aspects with the spacelike asymptotics of radiation.

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## APPENDIX A: HOMOGENEOUS FUNCTIONS OF A SPINOR VARIABLE

We reproduce here some facts about the invariant measure over the null directions in Minkowski space[14,16,10] and on spin-weighted spherical harmonics.[10]

Let $u$ denote a vector on the future lightcone. The measure $d^3u/u^0$ is known to be Lorentz invariant. If we define a measure over the null directions $d^2u$ by $d^3u/u^0=(du^0/u^0)d^2u$ (in the notation of Ref. 14), then the new measure is Lorentz invariant in the following sense: The result of integration of a homogeneous of degree $-2$ function of $u$ is manifestly Lorentz invariant.

The invariant measure has a very simple and elegant representation in the spinor language.[10] If $\xi^A$ is a spinor of the null vector $u$, then

$$d^2u=i\xi^{A'}d\xi_{A'}\wedge\xi^A d\xi_A\,.$$

Here any parametrization of spinors is implied for which every null direction is represented by exactly one spinor. The scaling behavior of $d^2u$ is now explicit:

$$\text{if}\quad \xi\rightarrow\alpha\xi,\quad \text{then}\quad d^2u\rightarrow(\alpha\bar{\alpha})^2\, d^2u.$$

Some special scalings of spinors are useful. We say that a spinor $o^A$ is chosen in a $t$-gauge, if its null vector $l$ satisfies $t\cdot l=1$, where $t$ is a fixed unit timelike vector. In this scaling the measure $d^2u$ is the rotationally invariant measure on the unit sphere in the three-space orthogonal to $t$,[14,16] which we denote $d\Omega_t(u)$.

Let us choose a fixed spinor $o^A$ in a $t$-gauge and denote $\iota^A=t^{AA'}o_{A'}$. Then $\{o^A,\iota^A\}$ is a normalized spinor basis: $o_A\iota^A=1$. Parametrize $\xi^A$ with complex numbers $\kappa$ from the closed unit circle by the formula

$$\xi^A=(1-|\kappa|^2)^{1/2}o^A+\sqrt{2}\kappa\iota^A. \tag{A1}$$

Then $\xi^A$ is in a $t$-gauge and $d\Omega_t(u)=2i\, d\bar{\kappa}\wedge d\kappa$. Setting $\kappa=\rho e^{-i\varphi}$, $\rho\in\langle 0,1\rangle$, $\varphi\in\langle 0,2\pi\rangle$ we obtain $d\Omega_t(u)=2d\rho^2\wedge d\varphi$. Finally, substituting $\rho=\sin(\vartheta/2)$, $\vartheta\in\langle 0,\pi\rangle$ we obtain the spherical angles parametrization:

$$\xi^A=\cos\frac{\vartheta}{2}\, o^A+\sqrt{2}\,\sin\frac{\vartheta}{2}\, e^{-i\varphi}\iota^A, \tag{A2}$$

$$u^a=t^a+\sin\vartheta(\cos\varphi\, X^a+\sin\varphi\, Y^a)+\cos\vartheta\, Z^a, \tag{A3}$$

where $X^a=(1/\sqrt{2})(o^A\iota^{A'}+\iota^A o^{A'})$, $Y^a=(i/\sqrt{2})(o^A\iota^{A'}-\iota^A o^{A'})$, $Z^a=l^a-t^a$ is a Cartesian basis, and $d\Omega_t(u)=\sin\vartheta\, d\vartheta\wedge d\varphi$.

The invariant integral is an important tool in the theory of homogeneous functions of a spinor variable, known as the theory of spin-weighted spherical harmonics.[10] We reproduce some results of the theory needed in the present paper.

A function $f(o,\bar{o})$ is said to be of type $\{p,q\}$ if

$$f(\alpha o,\bar{\alpha}\bar{o})=\alpha^p\bar{\alpha}^q f(o,\bar{o}). \tag{A4}$$

Choose any timelike versor $t^a$ and denote $\iota^A=t^{AA'}o_{A'}/t\cdot l$. Denote also $\partial_A=\partial/\partial o^A$, $\partial_{A'}=\partial/\partial o^{A'}$. Then one has

$$\begin{aligned}&\text{(i) if } p-q>0 \text{ then } \iota^A\partial_A f=0\Rightarrow f=0;\\ &\text{(ii) if } p-q<0 \text{ then } \iota^{A'}\partial_{A'}f=0\Rightarrow f=0.\end{aligned} \tag{A5}$$

For $f_1$: $\{0,q_1\}$, $f_2$: $\{p_2,0\}$ one has by Euler theorem

$$\partial_A f_1=o_A g_1,\quad \partial_{A'}f_2=o_{A'}g_2. \tag{A6}$$

If $q_1<0$, $p_2<0$ then (i) and (ii) imply that $f_i$ are uniquely determined by $g_i$ ($i=1,2$). Moreover, if $q_1=p_2=-2$, so that $g_1$ and $g_2$ are of type $\{-2,-2\}$, then

$$\int g_1\, d^2l=\int g_2\, d^2l=0. \tag{A7}$$

Using this, one also easily shows that

$$\int \partial_A h_1\, d^2l=0,\quad \int \partial_{A'}h_2\, d^2l=0, \tag{A8}$$

for $h_1$: $\{-1,-2\}$, and $h_2$: $\{-2,-1\}$.

## APPENDIX B: SOME ESTIMATES AND LIMITS

We prove here various estimates of asymptotic behavior of fields and quantities appearing in this paper. Our tool is the following simple lemma. Let $a>0$, $b\geq 0$, $c>0$, $\alpha>0$ (all real). Then

$$\int_0^c (a+bu)^{-\alpha}\, du<\begin{cases}\dfrac{\alpha}{\alpha-1}\dfrac{c}{a^{\alpha-1}(a+cb)}, & \alpha>1\\[2mm] \dfrac{1}{1-\alpha}\dfrac{c}{(a+cb)^{\alpha}}, & \alpha<1.\end{cases} \tag{B1}$$

To see this, represent the result of integration by $[c/(a+bc)^\alpha]h_\alpha(cb/a)$ for $\alpha<1$ and by $[c/a^{\alpha-1}(a+bc)]h_{2-\alpha}(cb/a)$ for $\alpha>1$, where $h_\beta(x)=[1/(1-\beta)x][x+1-(x+1)^\beta]$ for $\beta<1$.

$h_\beta(x)$ is a continuous function of $x \geqslant 0$, $h_\beta(0)=1$, $\lim_{x\to\infty} h_\beta(x)=1/(1-\beta)$. Moreover, it is easy to see that $x^2 h'_\beta(x)=\beta\int_0^x y(y+1)^{\beta-2}\,dy$, so $h_\beta(x)$ is monotonous. Hence $h_\beta(x)<\max\{1,1/(1-\beta)\}$, which ends the proof of (B1).

Let $C_k(x)$, $k=0,1,2$, be free fields

$$C_k(x)=-\frac{1}{2\pi}\int f_k(x\cdot u,u)d^2u,$$

such that for $s>0$ there is $|f_k(s,u)|<\text{const.}/(s+s_t)^{k+\epsilon}$ in some $t$ gauge ($s_t>0$). Then, using the above lemma we obtain immediately in $t$ gauge

$$|C_0(st+Rl)|<\frac{\text{const.}}{(s+s_t+2R)^\epsilon}, \tag{B2}$$

$$|C_1(st+Rl)|<\frac{\text{const.}}{(s+s_t)^\epsilon(s+s_t+2R)}, \tag{B3}$$

$$|C_2(st+Rl)|<\frac{\text{const.}}{(s+s_t)^{1+\epsilon}(s+s_t+2R)}. \tag{B4}$$

For $x^a$ inside the forward lightcone set $x^a=\lambda z^a$, with $z^2=1$, $z^0>0$. The above bounds imply then

$$|C_0(\lambda z)|<\frac{\text{const.}}{(s_t+\lambda z^0)^\epsilon}, \tag{B5}$$

$$|C_1(\lambda z)|<\text{const.}\ \frac{(z^0)^\epsilon}{(\lambda+s_t z^0)^\epsilon(s_t+\lambda z^0)}, \tag{B6}$$

$$|C_2(\lambda z)|<\text{const.}\ \frac{(z^0)^{1+\epsilon}}{(\lambda+s_t z^0)^{1+\epsilon}(s_t+\lambda z^0)}. \tag{B7}$$

All bounds (B2)–(B7) hold also in other reference systems, with $s_t$ and other constants depending on vector $t$.

We turn now to the energy momentum and angular momentum of electromagnetic field passing through the cone $\mathscr{C}^{\text{fut}}(\tau)$:

$$P^{\text{elm}}{}_a[\mathscr{C}^{\text{fut}}(\tau)]=\int T^{\text{elm}}{}_{ac}(\tau t+Rl)l^cR^2\,dR\,d\Omega_t(l), \tag{B8}$$

$$\mu^{\text{elm}}{}_{AB}[\mathscr{C}^{\text{fut}}(\tau)]=\int \mu^{\text{elm}}{}_{ABc}(\tau t+Rl)l^cR^2\,dR\,d\Omega_t(l), \tag{B9}$$

where $l^c$ is in a $t$ gauge, $T^{\text{elm}}{}_{ac}$ and $\mu^{\text{elm}}{}_{ABc}$ are given by (3.7) and (3.8) respectively. We shall show that both quantities vanish in the limit $\tau\to\infty$ for the free field (2.32) and for the mixed terms of the free outgoing field and the advanced field of the asymptotic Dirac current. To this end estimates of $\varphi_{AC}(\tau t+Rl)o^C$ and $\varrho_{A'C}(\tau t+Rl)o^C$ are needed. For the free field we use the representations (2.32), (2.33) with the spinor variable $\xi^A$ and integrate by the use of identity

$$\left(\frac{\tau}{2}+R\right)o_A\xi^A\dot{f}((\tau t+Rl)\cdot u,\xi,\bar{\xi})=\iota^{A'}\left(\frac{\partial}{\partial\xi^{A'}}-\frac{\partial'}{\partial\xi^{A'}}\right)f((\tau t+Rl)\cdot u,\xi,\bar{\xi}),$$

$(\iota^{A'}=t^{A'A}o_A)$. Then

$$\varphi_{AC}(\tau t+Rl)o^C=-\frac{1}{\pi(\tau+2R)}\int \xi_A\iota^{B'}\partial'_{B'}\dot{\zeta}((\tau t+Rl)\cdot u,\xi,\bar{\xi})d^2u,$$

$$\varrho_{A'C}(\tau t+Rl)o^C=-\frac{1}{\pi(\tau+2R)}\int \iota^{B'}\partial'_{B'}\partial'_{A'}\zeta((\tau t+Rl)\cdot u,\xi,\bar{\xi})d^2u.$$

The estimates (B3), (B2) give

$$|\varphi_{AC}(\tau t+Rl)o^C|<\frac{\text{const.}}{\tau^{\epsilon}(\tau+2R)^2}, \tag{B10}$$

$$|\varrho_{A'C}(\tau t+Rl)o^C|<\frac{\text{const.}}{(\tau+2R)^{1+\epsilon}}. \tag{B11}$$

Using these bounds in (B8) and (B9) one gets $|P_a[\mathscr{S}^{\text{fut}}(\tau)]|<\text{const.}/\tau^{1+2\epsilon}$, $|\mu_{AB}[\mathscr{S}^{\text{fut}}(\tau)]|<\text{const.}/\tau^{2\epsilon}$ for the free field, which proves (3.17). For the advanced field we use (5.7)–(5.9). This yields

$$|\varphi^{\text{adv}}{}_{AB}(\tau t+Rl)|<\frac{\text{const.}}{(\tau+R)^2}, \tag{B12}$$

$$|(\varrho^{\text{adv}}{}_{AA'}-\varrho^{\text{Coul}}{}_{AA'})(\tau t+Rl)|<\frac{\text{const.}}{(\tau+R)^{1+\epsilon}}. \tag{B13}$$

For estimation of $\varrho^{\text{Coul}}{}_{A'C}(\tau t+Rl)o^C$ one has to use more specific algebraic property of the Coulomb field. One shows with the use of (5.8) and (5.9) that

$$\varphi^{\text{Coul}}{}_{AB}(x)x^A_{A'}x^B_{B'}=\frac{x^2}{2}\int \rho(v)\,\frac{v_{C(A'}x^C_{B')}}{\sqrt{(x\cdot v)^2-x^2}}\,\frac{d\mu(v)}{(x\cdot v)^2-x^2}.$$

The integral on the rhs is estimated as the Coulomb field itself. Taking into account that

$$\varphi^{\text{Coul}}{}_{AB}(x)x^A_{A'}x^B_{B'}\iota^{B'}|_{x=\tau t+Rl}=\left(\frac{\tau}{2}+R\right)\varrho^{\text{Coul}}{}_{A'C}(\tau t+Rl)o^C$$

we get

$$|\varrho^{\text{Coul}}{}_{A'C}(\tau t+Rl)o^C|<\text{const.}\,\frac{\tau}{(\tau+R)^2}. \tag{B14}$$

A straightforward calculation shows now that (B12)–(B14) together with (B10) and (B11) are sufficient for vanishing of the mixed terms contributions to (B8) and (B9) in the limit $\tau\to\infty$.

The last point in this appendix is the demonstration of (3.25) and (3.26) for the mixed contributions of the free field and the (generalized) Coulomb field (2.51) to the electromagnetic energy-momentum tensor. We give explicit calculations for the case of $\mathscr{S}'^{\text{fut}}(-r)$, where

$$P^{\text{mix}}{}_a[\mathscr{S}'^{\text{fut}}(-r)]=\int d\Omega_t(u)\int_r^\infty T^{\text{mix}}{}_{ac}(-rt+Ru)u^cR^2\,dR,$$

$$\mu^{\text{mix}}{}_{AB}[\mathscr{S}'^{\text{fut}}(-r)]=\int d\Omega_t(u)\int_r^\infty \mu^{\text{mix}}{}_{ABc}(-rt+Ru)u^cR^2\,dR.$$

We use (2.51) (with $a=0$) and (2.32) and (2.33) in mixed terms of (3.7) and (3.8). Integration over $R$ is then explicitly carried out with the use of identities

$$\varphi_{AC}(-rt+Ru)\xi^C=\partial_R\left(-\frac{1}{2\pi}\right)\int o_A\dot{\zeta}(-r+Rl\cdot u,o,\bar{o})\,\frac{d\Omega_t(l)}{o_{C'}\xi^{C'}},$$

$$\varrho_{A'C}(-rt+Ru)\xi^C=\partial_R\left(-\frac{1}{2\pi}\right)\int \partial'_{A'}\zeta(-r+Rl\cdot u,o,\bar{o})\,\frac{d\Omega_t(l)}{o_{C'}\xi^{C'}}.$$

We get

$$P^{\text{mix}}{}_a[\mathscr{C}^{\text{fut}}(-r)]=-\frac{\bar{Q}}{8\pi^2}\int d\Omega_t(u)d\Omega_t(l)\,\frac{o_A\xi_{A'}}{o_{C'}\xi^{C'}}\,\dot{\zeta}(r(l\cdot u-1),o,\bar{o})+\text{compl. conj.},$$

$$\mu^{\text{mix}}{}_{AB}[\mathscr{C}^{\text{fut}}(-r)]=\frac{1}{8\pi^2}\int d\Omega_t(u)d\Omega_t(l)\left\{\frac{Q}{o_C\xi^C}\,\xi_{(A}\partial'_{B)}\overline{\zeta(r(l\cdot u-1),o,\bar{o})}\right.$$
$$\left.-\frac{\bar{Q}r}{o_{C'}\xi^{C'}}\,\xi_{D'}t^{D'}_{(A}o_{B)}\dot{\zeta}(r(l\cdot u-1),o,\bar{o})\right\}.$$

The energy-momentum expression vanishes in the limit $r\to\infty$ by the Lebesgue theorem, while the first term in the angular momentum expression yields

$$\frac{Q}{8\pi^2}\int d\Omega_t(l)d\Omega_t(u)\,\frac{\theta(1-l\cdot u)}{o_C\xi^C}\,\xi_{(A}\partial_{B)}\overline{\zeta(-\infty,o,\bar{o})}=\frac{1}{4\pi}\int \zeta^Q{}_{(A}\bar{\nu}_{B)}(-\infty,o,\bar{o})d^2l,$$

where $\xi$ integration in the parametrization (A2) was performed. The second term in this parametrization after $\varphi$-integration is

$$\frac{\bar{Q}r}{8\pi}\int d\Omega_t(l)d\vartheta\,\sin\vartheta\,o_Ao_B\dot{\zeta}(-r\cos\vartheta,o,\bar{o})=\frac{\bar{Q}}{8\pi}\int d\Omega_t(l)o_Ao_B[\zeta(r,o,\bar{o})-\zeta(-r,o,\bar{o})]$$
$$\to\frac{1}{4\pi}\int \zeta_{(A}\bar{\nu}^Q{}_{B)}(-\infty,o,\bar{o})d^2l,$$

for $r\to\infty$, which ends the proof.

## APPENDIX C: THREE-SPACE INTEGRALS

We prove here a lemma, from which the formulas (3.28) and (3.28) for conserved quantities of a free electromagnetic field follow by a simple computation.

*Lemma C.1: Let* $f_1(s,o,\bar{o})$ *and* $f_2(s,o,\bar{o})$ *be continuously differentiable functions satisfying scaling law* $f(\alpha\bar{\alpha}s,\alpha o,\bar{\alpha}\bar{o})=\alpha^{-2}\bar{\alpha}^{-1}f(s,o,\bar{o})$, *such that* $|f_2(s,o,\bar{o})|$ *is bounded,* $|f_1(s,o,\bar{o})|$, $|\partial_Af_1(s,o,\bar{o})|$ *and* $|\partial_{A'}f_1(s,o,\bar{o})|$ *are bounded by an integrable function (in any fixed gauge), and there exist limits* $\lim_{s\to+\infty}f_2(s,o,\bar{o})=-\lim_{s\to-\infty}f_2(s,o,\bar{o})$.
*Then*

$$\lim_{r\to\infty}\int_{\{t\cdot x=c,(t\cdot x)^2-x^2\leq r^2\}}\left\{t^a\frac{1}{2\pi}\int\overline{\dot f_1(x\cdot u,\xi,\bar\xi)\xi_{A'}}\,d^2u\,\frac{1}{2\pi}\int\dot f_2(x\cdot l,o,\bar o)o_A d^2l\right\}d^3x$$

$$=\int\overline{f_1(s,o,\bar o)}f_2(s,o,\bar o)ds\,d^2l. \tag{C1}$$

Note that the rhs is explicitly hyperplane independent.

*Proof:* Choose a $t$-gauge $t\cdot l=t\cdot u=1$, fix $o^A$ and parametrize $\xi^A$ by (A2) and $x^a$ by

$$x^a=x^0t^a-y_1Z^a-y_2(\cos\varphi\,X^a+\sin\varphi\,Y^a)-y_3(\sin\varphi\,X^a-\cos\varphi\,Y^a).$$

Then $x\cdot l=x^0+y_1$, $x\cdot u=x^0+y_2\sin\vartheta+y_1\cos\vartheta$. Hence,

$$\int_{\{x^0=c,|\mathbf{x}|\leq r\}}t^a\int\overline{\dot f_1(x\cdot u,\xi,\bar\xi)\xi_{A'}}\,d^2u\dot f_2(x\cdot l,o,\bar o)o_A\,d^3x$$

$$=2\int d\Omega(\vartheta,\varphi)\int_{\{y_1^2+y_2^2\leq r^2\}}d^2y\sqrt{r^2-y_1^2-y_2^2}$$

$$\times\cos\frac{\vartheta}{2}\overline{\dot f_1(y_1\cos\vartheta+y_2\sin\vartheta+c,\xi,\bar\xi)}\dot f_2(y_1+c,o,\bar o)$$

$$=2\int d\vartheta\,d\varphi\int_{\{y_1^2+y_2^2\leq r^2\}}d^2y\frac{y_2}{\sqrt{r^2-y_1^2-y_2^2}}$$

$$\times\cos\frac{\vartheta}{2}\overline{f_1(y_1\cos\vartheta+y_2\sin\vartheta+c,\xi,\bar\xi)}\dot f_2(y_1+c,o,\bar o).$$

The effect of the constant $c$ is a translation of both functions in the first argument, so if the lemma is proved for $c=0$, then it is true for all $c$. We set $c=0$ for simplicity. By the change of variables $s=y_1\cos\vartheta+y_2\sin\vartheta$, $v=-y_1\sin\vartheta+y_2\cos\vartheta$ we get

$$-2\int d\vartheta\,d\varphi\int_{\{s^2+v^2\leq r^2\}}\frac{ds\,dv}{\sqrt{r^2-s^2-v^2}}\cos\frac{\vartheta}{2}\overline{f_1(s,\xi,\bar\xi)}\partial_\vartheta f_2(s\cos\vartheta-v\sin\vartheta,o,\bar o)$$

$$=-2\int_{-r}^{r}ds\int_{-1}^{1}\frac{d\kappa}{\sqrt{1-\kappa^2}}\int d\vartheta\,d\varphi\cos\frac{\vartheta}{2}\overline{f_1(s,\xi,\bar\xi)}\partial_\vartheta f_2(s\cos\vartheta-\sqrt{r^2-s^2}\kappa\sin\vartheta,o,\bar o).$$

Integrating by parts over $\vartheta$ we obtain

$$(2\pi)^2\int_{-r}^{r}\overline{f_1(s,o,\bar o)}f_2(s,o,\bar o)ds+2\int_{-r}^{r}ds\int_0^1\frac{d\kappa}{\sqrt{1-\kappa^2}}\int d\vartheta\,d\varphi\,\partial_\vartheta\left(\cos\frac{\vartheta}{2}\overline{f_1(s,\xi,\bar\xi)}\right)$$

$$\times[f_2(s\cos\vartheta+\sqrt{r^2-s^2}\kappa\sin\vartheta,o,\bar o)+f_2(s\cos\vartheta-\sqrt{r^2-s^2}\kappa\sin\vartheta,o,\bar o)].$$

By the Lebesgue theorem the second integral vanishes in the limit, which ends the proof.

## APPENDIX D: AN ESTIMATE

We show here, that if $2\geqslant\beta\geqslant0$, $\gamma\geqslant0$, $\alpha>|\beta+\gamma-1|$ and $|G(z,v)|<\text{const.}/(v^0)^{\alpha+1}$ for both $z$ and $v$ on the unit four-velocity hyperboloid, then

$$\left|\int\frac{G(z,v)}{(\sqrt{(z\cdot v)^2-1})^\beta(z\cdot v+\sqrt{(z\cdot v)^2-1})^\gamma}\,d\mu(v)\right|<\frac{\text{const.}}{(z^0)^{\beta+\gamma}}. \tag{D1}$$

The bound is then valid in any other reference system (with some other constant).

Let $t^a$ be the time-axis versor of the reference system. We show that

$$I\equiv\int\frac{d\mu(v)}{(\sqrt{(z\cdot v)^2-1})^\beta(z\cdot v+\sqrt{(z\cdot v)^2-1})^\gamma(t\cdot v)^{\alpha+1}}<\frac{\text{const.}}{(t\cdot z)^{\beta+\gamma}}.$$

Choose the time-axis of the coordinate system in which integration is performed along $z$ and set $v^0=\sqrt{|\mathbf{v}|^2+1}$. Then

$$I=2\pi\int\frac{|\mathbf{v}|^{2-\beta}d|\mathbf{v}|}{v^0(v^0+|\mathbf{v}|)^\gamma}\int_0^2\frac{d\xi}{(\cosh\chi v^0-\sinh\chi|\mathbf{v}|+\sinh\chi|\mathbf{v}|\xi)^{\alpha+1}},$$

where $\cosh\chi=t\cdot z$, $\chi\geqslant0$. By (B1) the inside integral is bounded by

$$\frac{\text{const.}}{(\cosh\chi v^0-\sinh\chi|\mathbf{v}|)^\alpha(\cosh\chi v^0+\sinh\chi|\mathbf{v}|)}.$$

By change of integration variable $|\mathbf{v}|=\sinh\psi$ we get

$$\begin{aligned}I&<\text{const.}\int_0^\infty\frac{(\sinh\psi)^{2-\beta}\,d\psi}{(e^\psi)^\gamma\cosh(\psi+\chi)(\cosh(\psi-\chi))^\alpha}\\&<\frac{\text{const.}}{(\cosh\chi)^{\beta+\gamma}}\int_{-\chi}^\infty\frac{d\psi}{(e^\psi)^{\beta+\gamma-1}(\cosh\psi)^\alpha}\\&<\frac{\text{const.}}{(\cosh\chi)^{\beta+\gamma}}\int_{-\infty}^{+\infty}\frac{d\psi}{(e^\psi)^{\beta+\gamma-1}(\cosh\psi)^\alpha}.\end{aligned}$$


[1] R. Haag, *Local Quantum Physics* (Springer, Berlin, 1992).
[2] G. Morchio and F. Strocchi, in *Fundamental Problems in Gauge Field Theory*, edited by G. Velo and A. S. Wightman (Plenum, New York, 1986).
[3] D. Buchholz, Phys. Lett. B **174**, 331 (1986).
[4] D. Buchholz, Commun. Math. Phys. **85**, 49 (1982).
[5] A. Staruszkiewicz, Ann. Phys. (NY) **190**, 354 (1989).
[6] A. Staruszkiewicz, Acta Phys. Pol. B **23**, 591 (1992).
[7] M. Flato, J. Simon, and E. Taflin, "Asymptotic completeness, global existence and the infrared problem for the Maxwell–Dirac equations," preprint hep-th/9502061 (1995).
[8] A. Herdegen, J. Phys. A **26**, L449 (1993).
[9] A. Herdegen, Phys. Lett. B **321**, 205 (1994).
[10] R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge U.P., Cambridge, 1984), Vol. 1.
[11] B. D. Bramson, in *Asymptotic Structure of Space-Time*, edited by F. P. Esposito and L. Witten (Plenum, New York, 1977), pp. 273–359.
[12] A. Ashtekar, *Asymptotic Quantization* (Bibliopolis, Naples, 1986).
[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1979), Vol. III.
[14] A. Staruszkiewicz, Acta Phys. Pol. B **12**, 327 (1981).
[15] J. D. Dollard, J. Math. Phys. **5**, 729 (1964).
[16] D. Zwanziger, Phys. Rev. D **14**, 2570 (1976).