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Abelian versus Non-Abelian Higgs Model in Three Dimensions

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Abstract

We study the phase structure of the abelian Higgs model in three dimensions based on perturbation theory and a set of gauge independent gap equations for Higgs boson and vector boson masses. Contrary to the non-abelian Higgs model, the vector boson mass vanishes in the symmetric phase. In the Higgs phase the gap equations yield masses consistent with perturbation theory. The phase transition is first-order for small values of the scalar self-coupling λ, where the employed loop expansion is applicable.

The "free-energy functional" of the Ginzburg-Landau theory of superconductivity is given by the action of the abelian Higgs model in three dimensions. Its phase structure has first been analyzed by Halperin, Lubensky and Ma [1]. For a type-I superconductor, where the scalar self-coupling λ is small compared to the gauge coupling g, the phase transition from the normal "symmetric" phase to the superconducting "Higgs" phase is weakly first-order. The case of a type-II superconductor, where λ/g^2 is large, is more complicated and has been studied by various methods, in particular the ϵ-expansion and renormalization group techniques [1].

The three-dimensional abelian Higgs model also describes the corresponding four-dimensional theory at high temperatures. As a model for the cosmological electroweak phase transition, this case was studied by Kirzhnits and Linde [2], who also found a first-order transition from the symmetric phase to the Higgs phase for $\lambda/g^2 \ll 1$. In recent years the abelian Higgs model at high temperatures has been studied in more detail [3, 4] using resummed perturbation theory, and the effective potential has been determined to order g^4, λ^2 by a complete two-loop calculation [5].

In the electroweak phase transition non-perturbative effects are expected to be important, at least for large values of λ/g^2. They are related to the infrared behaviour of the non-abelian SU(2) Higgs model in three dimensions. So far, the nature of the symmetric phase and the order of the phase transition for large λ/g^2 have not been firmly established. In a recent paper [6] we have studied some non-perturbative aspects of the SU(2) Higgs model by means of gap equations. Complementing the mass resummation by a vertex resummation a gauge independent set of gap equations was obtained for Higgs boson and vector boson masses, defined on the respective mass shells. The analysis led to the conclusion that the symmetric phase is again a Higgs phase, just with different parameters. The first-order phase transition, found for $\lambda/g^2 < 1$, changes to a crossover at a critical scalar coupling λ_c, whose value is correlated with the magnitude of the vector boson mass in the symmetric phase.

In this letter we apply the same resummation method to the abelian Higgs model in three dimensions. Due to the absence of gauge boson self-couplings the abelian Higgs model does not suffer from the same infrared problems as the non-abelian theory. It may therefore serve as a testing ground for the method employed in [6]. Much work has been done on the compact and non-compact versions of the abelian Higgs model on the lattice[1]. Monte Carlo simulations provide evidence for a phase transition from a Higgs phase to a symmetric Coulomb phase with zero-mass photon for all values of λ/g^2 [8].

[1] For a review and references, see [7].

Let us first recall the results of ordinary perturbation theory. Naively, one may expect that g^2/m, λ/m, g^2/M and λ/M appear as expansion parameters, where m and M denote vector boson and Higgs boson mass, respectively. In this case the perturbative expansion would fail in the symmetric phase, for vanishing vector boson mass, $m = 0$. However, due to the absence of gauge boson self-couplings the infrared behaviour in the abelian theory is much simpler than in the non-abelian theory. As shown by Hebecker [5], the effective potential does not develop a term linear in the Higgs field to all orders of perturbation theory in the case of non-vanishing Higgs mass. The proof immediately implies that the effective potential is finite to all orders also in the symmetric phase, with $m = 0$ and $M > 0$. The convergence of the perturbative expansion is determined by g^2/M and λ/M. Following [5], one also easily verifies that $m = 0$ to all orders in the symmetric phase. Hence, perturbation theory in the symmetric phase is free of infrared divergencies for $M > 0$, and its results are consistent with non-perturbative studies on the lattice.

We now use the method developed in [6] to derive the gap equations for the abelian Higgs model. The action of the three-dimensional theory is given by

$$S = \int d^3x \, \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \Phi)^\dagger D_\mu \Phi + \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 \right] , \tag{1}$$

with

$$\Phi = \frac{1}{\sqrt{2}} (\varphi + i\chi) , \quad D_\mu \Phi = (\partial_\mu - i \frac{g}{2} A_\mu) \Phi \ . \tag{2}$$

We perform a perturbative calculation in the Higgs phase, i.e., we shift the scalar field φ around its vacuum expectation value v, $\varphi = v + \varphi'$, and add the R_ξ-gauge fixing term

$$L_{GF} = \frac{1}{2\xi} (\partial_\mu A_\mu - \xi \frac{g}{2} v \chi)^2 \tag{3}$$

and the corresponding ghost term to the lagrangian (1). The shifted lagrangian contains the usual cubic and quartic couplings between vector field, Higgs field, Goldstone field and ghost field.

At tree level the vector boson, Goldstone boson, ghost and Higgs boson masses are, respectively,

$$m_0^2 = \frac{g^2}{4} v^2 , \quad m_{\chi 0}^2 = \mu^2 + \lambda v^2 + \xi m_0^2 , \quad m_{c0}^2 = \xi m_0^2 , \quad M_0^2 = \mu^2 + 3\lambda v^2 \ . \tag{4}$$

Expanding around the asymmetric tree level minimum one has $\mu^2 + \lambda v^2 = 0$, and thus $m_{\chi 0}^2 = m_{c0}^2 = \xi m_0^2$. These mass relations aquire corrections in higher orders, and they do not hold for an expansion around the symmetric minimum $v = 0$.

Following the approach of [6], we now perform a mass resummation. The tree level masses are expressed as

$$m_0^2 = m^2 - \delta m^2 \ , \ M_0^2 = M^2 - \delta M^2 \ , \ m_{c0}^2 = \xi m^2 - \delta m_c^2 \ , \ m_{\chi 0}^2 = \xi m^2 - \delta m_\chi^2 \ , \qquad (5)$$

where the full masses m, M and $\sqrt{\xi} m$ enter the propagators in loop diagrams, and δm^2, δM^2, δm_c^2 and δm_χ^2 are treated as counter terms perturbatively. Note, that the full ghost and Goldstone boson masses are chosen such that the tree level mass relations are preserved. Calculation of the vector boson and Higgs boson self-energies with full propagators then leads to the coupled set of gap equations

$$\begin{aligned} \delta m^2 + \Pi_T(p^2 = -m^2, m, M, \xi) = 0 \ , \\ \delta M^2 + \Sigma(p^2 = -M^2, m, M, \xi) = 0 \ , \end{aligned} \qquad (6)$$

where $\Pi_T(p^2)$ is the transverse part of the vacuum polarization tensor.

As in the non-abelian case, in order to obtain a gauge independent result for the gap equations (6), it is necessary to also perform the following vertex resummations [6],

$$\begin{aligned} \frac{g^2}{2} v &= gm - \delta V^g_{\phi\phi\phi} \ , \ \phi = A, \ c, \ \varphi' \, , \chi \ , \\ \lambda v &= \frac{gM^2}{4m} - \delta V^\lambda_{\phi\phi\phi} \ , \ \phi = \varphi', \ \chi \ , \\ \lambda &= \frac{g^2 M^2}{8m^2} - \delta V^\lambda_{\phi\phi\phi\phi} \ , \ \phi = \varphi' \ , \ \chi \ . \end{aligned} \qquad (7)$$

After these manipulations the lagrangian (1) takes the form

$$L = L_R + L_1 + L_0 \quad . \qquad (8)$$

Here, the first term $L_R \equiv L_{RT} + L_{RGF}$ contains the full masses and vertices which enter the loop graphs. It is given by the sum of the gauge invariant lagrangian [6]

$$L_{RT} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \Phi)^\dagger D_\mu \Phi - \frac{1}{2} M^2 \Phi^\dagger \Phi + \frac{g^2 M^2}{4m^2} (\Phi^\dagger \Phi)^2 \ , \qquad (9)$$

with the Higgs field shifted by the "classical" minimum, $\varphi = \varphi' + 2m/g$, and the gauge fixing term

$$L_{RGF} = \frac{1}{2\xi} (\partial_\mu A_\mu - \xi m \chi)^2 \ , \qquad (10)$$

supplemented by the corresponding ghost lagrangian. L_1 in eq. (8) stands for the difference between tree level and resummed quadratic, cubic and quartic vertices, and L_0 is the constant term of the shifted lagrangian (1). L_1 and L_0 are identical to the expressions given in [6].

For the one-loop self-energies of vector boson and Higgs boson, as evaluated from the lagrangian L_R, we obtain the result (cf. fig. 1)

$$\begin{aligned}\Pi_T(p^2) &= g^2\left[\frac{m}{gM^2}v(\mu^2+\lambda v^2)+\left(\frac{5}{8}-\frac{M^2}{8p^2}+\frac{m^2}{8p^2}\right)A_0(M^2)\right.\\ &+\left(\frac{1}{8p^2}(p^2+M^2-m^2)+\frac{m^2}{M^2}\right)A_0(m^2)\\ &\left.+\left(\frac{m^2}{2}-\frac{1}{8p^2}(p^2+M^2-m^2)^2\right)B_0(p^2,m^2,M^2)\right]\ , \qquad (11)\end{aligned}$$

$$\begin{aligned}\Sigma(p^2) &= \frac{g^2}{4}\left[\frac{6}{gm}v(\mu^2+\lambda v^2)+3\frac{M^2}{m^2}A_0(M^2)+\left(\frac{M^2}{m^2}+\frac{p^2}{m^2}\right)A_0(\xi m^2)\right.\\ &+\left(4-\frac{p^2}{m^2}\right)A_0(m^2)+\left(\frac{M^4}{2m^2}-\frac{p^4}{2m^2}\right)B_0(p^2,\xi m^2,\xi m^2)\\ &\left.+\frac{9M^4}{2m^2}B_0(p^2,M^2,M^2)+\left(4m^2+2p^2+\frac{p^4}{2m^2}\right)B_0(p^2,m^2,m^2)\right]\ , \qquad (12)\end{aligned}$$

with the three-dimensional integrals

$$\begin{aligned}A_0(m^2) &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m^2}\ ,\\ B_0(p^2,m_1^2,m_2^2) &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{(k^2+m_1^2)((k+p)^2+m_2^2)}\ . \qquad (13)\end{aligned}$$

The linear divergence of A_0 can be cancelled by a counterterm generated by renormalizing the mass parameter μ^2. The divergence is absent in dimensional regularization, which we shall use.

Contrary to the non-abelian case, the photon self-energy (11) is gauge independent already off the mass shell. However, the Higgs boson self-energy (12) has to be evaluated on the mass shell in order to get a gauge independent result. Using eqs. (6) and (11)-(13), one obtains the following gap equations for vector boson and Higgs boson masses,

$$m^2 = m_0^2-\frac{gz}{M}v(\mu^2+\lambda v^2)+mg^2\bar{f}(z)\ , \qquad (14)$$

$$M^2 = M_0^2-\frac{3g}{2m}v(\mu^2+\lambda v^2)+Mg^2\bar{F}(z)\ , \qquad (15)$$

with

$$\begin{aligned}\bar{f}(z) &= \frac{1}{4\pi}\left[\frac{1}{4}+\frac{1}{8z^3}-\frac{1}{8z^2}+\frac{1}{2z}\right.\\ &\left.+z^2-\left(\frac{1}{16z^4}-\frac{1}{4z^2}+\frac{1}{2}\right)\ln(1+2z)\right]\ , \qquad (16)\\ \bar{F}(z) &= \frac{1}{4\pi}\left[(\frac{3}{4}-\frac{9}{16}\ln 3)\frac{1}{z^2}+\frac{1}{4z}+z\right.\\ &\left.-\left(\frac{1}{2}z^2-\frac{1}{4}+\frac{1}{16z^2}\right)\ln\frac{2z+1}{2z-1}\right]\ , \qquad (17)\end{aligned}$$

and $z = m/M$. For $M > 2m$ the equation for M becomes complex, since in this case the Higgs boson can decay into two vector bosons.

Solutions of the gap equations depend on the vacuum expectation value v, which is defined by the requirement that the expectation value of the shifted field vanishes,

$$\langle \varphi' \rangle = 0 \; . \tag{18}$$

This condition on the sum of tadpole graphs yields at one-loop order in resummed perturbation theory,

$$\begin{aligned} v(\mu^2 + \lambda v^2) &= -\frac{1}{4} g m \left(4A_0(m^2) + \frac{M^2}{m^2} A_0(\xi m^2) + 3\frac{M^2}{m^2} A_0(M^2) \right) \\ &= \frac{1}{16\pi} g \left(4m^2 + \sqrt{\xi} M^2 + 3\frac{M^3}{m} \right) \; . \end{aligned} \tag{19}$$

As in ordinary perturbation theory the vacuum expectation value v, which is not a physical observable, is gauge dependent. This also implies a weak gauge dependence $\mathcal{O}(M^2/m^2)$ for the solutions m and M of the gap equations. Since the masses are observables, this gauge dependence must be cancelled by higher order corrections. Numerically, the gauge dependence of a particular solution of the gap equations can be used as an indication for the importance of higher order corrections. In the following we shall work in Landau gauge, $\xi = 0$.

For any solution v of eq. (19) the gap equations (14), (15) can be written as

$$m^2 = \frac{g^2}{4} v^2 + m g^2 f(z) \; , \tag{20}$$

$$M^2 = \mu^2 + 3\lambda v^2 + M g^2 F(z) \, , \tag{21}$$

with functions $f(z)$ and $F(z)$ which can easily be obtained from eqs. (16), (17) and (19). This form of the gap equations is the same in the abelian and the non-abelian Higgs model, and it is particularly useful to study the solutions. In the non-abelian case the function $f(z)$ is positive, except for very large values of z. In particular, for $z = \mathcal{O}(1)$, one has $f(z) \approx (63 \ln 3 - 12)/(64\pi) \equiv C$ [6]. This contribution to $f(z)$ is due to a gauge invariant subset of graphs corresponding to the gauged non-linear SU(2) σ-model. As a consequence, one finds two solutions of the gap equations for small values of μ^2/g^4 and scalar self-couplings λ below a critical coupling λ_c. One solution, with $v/g > 1$, corresponds to the usual Higgs phase. The second solution, with $v/g < 1$, can be interpreted as "symmetric" phase, which thus appears as another Higgs phase with different parameters. To good approximation the vector boson mass in the symmetric

phase is $m = Cg^2$. The range of μ^2/g^4, where two solutions of the gap equations exist, defines the metastability region where a first-order phase transition occurs.

In the case of the abelian Higgs model the situation is very different. Here, the function $f(z)$ is negative for all values of z. This is related to the fact that in the abelian case the non-linear σ-model is a free theory. Hence, no solution with $v/g < 1$ exists, and one is left with a unique solution of the gap equations corresponding to the familiar Higgs phase with $v/g > 1$. This reassures us that the non-trivial values for v and m found in the symmetric phase of the non-abelian model are not stipulated by our resummation scheme. It is also consistent with the fact that in the abelian case the values $v = 0$, $m = 0$ correspond to a stationary point of the effective potential to all orders of perturbation theory, contrary to the non-abelian case! We conclude that in the abelian Higgs model the trivial vacuum with a massless photon represents indeed the symmetric phase.

The one-loop results of ordinary perturbation theory can be recovered from eqs. (14), (15) and (19) by substituting the tree level masses $m_0 = gv/2$ and $M_0 = \sqrt{2\lambda}v$, with fixed ratio $z = \sqrt{g^2/8\lambda}$, into the one-loop expressions. This yields

$$v(\mu^2 + \lambda v^2) = \frac{v^2}{4\pi}\left(\frac{1}{4}g^3 + 3\sqrt{2}\lambda^{3/2} + \frac{1}{2}\sqrt{\xi}\lambda g\right) , \tag{22}$$

$$m^2 = -\frac{g^2}{4\lambda}\mu^2 + \frac{g^3}{2}v\,\bar{f}\left(\sqrt{\frac{g^2}{8\lambda}}\right) , \tag{23}$$

$$M^2 = -2\mu^2 + g^2\sqrt{2\lambda}v\,\bar{F}\left(\sqrt{\frac{g^2}{8\lambda}}\right) . \tag{24}$$

These equations determine the perturbative results for v, m and M in the Higgs phase.

From the gap equations (19)-(21) the vacuum expectation value v/g can be obtained as function of the dimensionless parameters λ/g^2 and μ^2/g^4. In fig. 2 the result is plotted as function of μ^2/g^4, with $\lambda/g^2 = 1/128$. It agrees well with the perturbative solution obtained from (22). Also shown is the value $v = 0$, corresponding to the symmetric phase. For $\mu^2/g^4 < 0$ the system is in the Higgs phase with a large vacuum expectation value. This solution of the gap equations persists up to a small positive value of μ^2/g^4, where it terminates. The range of small positive μ^2/g^4 with a Higgs solution à la Coleman-Weinberg [9] corresponds to the metastability region of the theory. Compared to perturbation theory, the gap equations predict a smaller range in μ^2/g^4 with metastability. In fig. 3 vector boson and Higgs boson masses are shown for the same parameters as in fig. 2. In the symmetric phase, for positive μ^2, the perturbative masses are $m = 0$ and $M = \mu(1 + \mathcal{O}(g^2, \lambda))$.

Ordinary perturbation theory and also the gap equations are only reliable for type-I superconductors, where λ/g^2 and M^2/m^2 are small. As eqs. (19) and (22) show, the results become strongly gauge dependent otherwise. This indicates that the one-loop results are no longer trustworthy. For type-II superconductors other methods have to be used. Particularly interesting is the use of coarse grained effective actions [10, 11] where high frequency modes are integrated out.

For type-I superconductors, with small λ/g^2, the gap equations confirm the conventional picture of a first-order phase transition between a perturbative Higgs phase and a symmetric Coulomb phase, which is familiar from ordinary perturbation theory. This result is also in agreement with non-perturbative numerical simulations on a lattice. On the contrary, in the non-abelian SU(2) Higgs model a non-vanishing vector boson mass in the symmetric phase is expected on general grounds, and it is also found by explicit non-perturbative solutions of the gap equations. The difference between abelian and non-abelian Higgs models with respect to the symmetric phase is also reflected in the nature of the transition. In the abelian Higgs model one expects a phase transition for all values of λ, with a possible change from first-order to second-order at some critical coupling λ_c. For the non-abelian Higgs model, on the other hand, the gap equations predict a change from a first-order transition to a smooth crossover already at a rather small value of λ. Further studies of the symmetric phase of the non-abelian Higgs model are crucial in order to achieve a full understanding of the electroweak phase transition.

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References

[1] B.I. Halperin, T.C. Lubensky and S.-K. Ma, Phys. Rev. Lett. 32 (1974) 292

[2] D. A. Kirzhnits and A. D. Linde, Ann. Phys. 101 (1976) 195

[3] P. Arnold, Phys. Rev. D46 (1992) 2628

[4] W. Buchmüller, T. Helbig and D. Walliser, Nucl. Phys. B407 (1993) 387

[5] A. Hebecker, Z. Phys. C60 (1993) 271

[6] W. Buchmüller and O. Philipsen, DESY preprint DESY-94-202 (1994), Nucl. Phys. B, to appear

[7] I. Montvay and G. Münster, *Quantum fields on a Lattice*, Cambridge University Press (Cambridge, 1994)

[8] J. Bartholomew, Phys. Rev. B28 (1983) 5378

[9] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888

[10] M. Reuter and C. Wetterich, Nucl. Phys. B408 (1993) 91;
B. Bergerhoff, F. Freire, D. Litim, S. Lola and C. Wetterich, preprint HD-THEP-95-05 (1995)

[11] G. Mack, T. Kalkreuter, G. Palma and M. Speh, Lecture Notes in Physics 409 (1992) 205;
U. Kerres, G. Mack and G. Palma, preprints DESY 94-226, 94-254 (1994)

Figure captions

Fig.1a One-loop contributions to the vector boson propagator.
Fig.1b One-loop contributions to the Higgs boson propagator.
Fig.2 The vacuum expectation value v/g as function of the mass parameter μ^2/g^4. Full line: solution of gap equations, dash-dotted line: perturbation theory. $\lambda/g^2 = 1/128$.
Fig.3 Vector boson and Higgs boson masses for $\lambda/g^2 = 1/128$. Gap equations: m (full line), M (dashed line); perturbation theory: m (dash-dotted line), M (dotted line).