DESY 95–112
hep-th/9510011

hep-th/9510011 4 OCT 95

Non–Commutative Geometry on Quantum Phase–Space

M. Reuter

Deutsches Elektronen–Synchrotron DESY
Notkestrasse 85
D–22603 Hamburg
Germany

Abstract

A non–commutative analogue of the classical differential forms is constructed on the phase–space of an arbitrary quantum system. The non–commutative forms are universal and are related to the quantum mechanical dynamics in the same way as the classical forms are related to classical dynamics. They are constructed by applying the Weyl–Wigner symbol map to the differential envelope of the linear operators on the quantum mechanical Hilbert space. This leads to a representation of the non–commutative forms considered by A. Connes in terms of multiscalar functions on the classical phase–space. In an appropriate coincidence limit they define a quantum deformation of the classical tensor fields and both commutative and non–commutative forms can be studied in a unified framework. We interprete the quantum differential forms in physical terms and comment on possible applications.

1 Introduction

Non–commutative geometry is a fascinating new field with a wide range of possible applications in physics and mathematics. Among many other developments, both the approach of Connes [1] and of Dubois–Violette [2] were used to construct particle physics models [3, 4] and unified models including gravity [5]. Also the scheme of Coquereaux et al. [6] has been worked out in detail [7]. Though one of the main inspirations of non–commutative geometry are the operator algebras of quantum mechanics, whose non–commutative nature is due to the non–commutativity of the *phase–space* variables p and q, it is space–time which is turned into a non–commutative manifold in these papers. A similar remark also applies to most of the work done on non–commutative geometries based on quantum groups [8, 9].

In the present paper we are instead interested in the non–commutative geometry on the phase–space of an arbitrary quantum mechanical system with N degrees of freedom. As we shall see, there is a very natural way of introducing a non–commutative version of differential forms on phase–space. The resulting generalized p–forms have very interesting properties both from a geometrical and a dynamical point of view. Our approach has two basic ingredients:

(I) Given an arbitrary algebra A, non–commutative geometry [1, 10] tells us how to associate the algebra ΩA of universal differential forms to it. One of the possible constructions of ΩA proceeds by identifying the universal p–forms with special elements of the $(p+1)$–fold tensor product $A \otimes A \otimes \cdots \otimes A$. For A we shall take the algebra of linear operators (observables, unitary transformations, etc.) acting on the quantum mechanical Hilbert space $\mathcal{H}$ of the system under consideration. The universal p–forms are operators on $\mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ then.

(II) Following the ideas of what is known as deformation or star–product quantization [11] we describe the quantum system at hand not by operators and Hilbert space vectors but rather by functions over the classical phase space, henceforth denoted $\mathcal{M}_{2N}$. In this manner the vector $\psi \in \mathcal{H}$ and the operator $\hat{a} \in A$ become replaced by the Wigner function W_ψ and the operator symbol $a \equiv \text{symb}\,(\hat{a})$, respectively. Both $W_\psi \equiv W_\psi(\phi^a)$ and $a \equiv a(\phi^a)$ are scalar c–number functions over $\mathcal{M}_{2N}$ whose local coordinates are denoted as ϕ^a. Similarly the non–commutative p–forms in $A \otimes \cdots \otimes A$ are turned into functions $F_p(\phi_0, \phi_1, \cdots, \phi_p)$ which depend

on $p+1$ phase–space points. We shall refer to them as multiscalars.

The multiscalars provide a link between the classical and the non–commutative tensor calculus. At the classical level ($\hbar = 0$) the functions $F_p(\phi_0, \phi_1, \cdots, \phi_p)$ define tensor fields (and in particular differential forms) if we perform the coincidence limit where all points ϕ_i are very "close" to each other. Setting $\phi_i^a = \phi_0^a + \eta_i^a, i = 1, \cdots, p$, and expanding to first order in η_i yields terms of the type $\partial \, {}^{(1)}_{a_1} \cdots \partial \, {}^{(p)}_{a_p} F_p \, (\phi_0, \phi_0, \cdots, \phi_0) \cdot \eta_1^{a_1} \eta_2^{a_2} \cdots \eta_p^{a_p}$, where $\partial_a^{(i)}$ is the derivative with respect to ϕ_i^a. All derivatives act on different arguments of F_p. Therefore the coefficients of $\eta_1^{a_1} \cdots \eta_p^{a_p}$ define a classical p–form upon antisymmetrization. In the non–commutative case ($\hbar > 0$) it is well known [10] that the abstract construction of $\Omega^p A$ has a concrete realization in terms of functions of $p+1$ arguments which vanish if two neighboring arguments are equal. The pseudoscalars F_p resulting from the construction (II) are very similar to these functions. For $\hbar = 0$ they, too, vanish if two adjacent arguments are equal, and for $\hbar > 0$ they satisfy a deformed version of this condition.

The main virtue of our deformation theory approach is that it represents the non–commutative p–forms in the same setting as classical p–forms, namely as c–number functions on the classical phase–space. The deformation parameter $\hbar$ allows for a smooth interpolation between the classical and the non–commutative case.

In order to fix our notation and to collect some formulas which we shall need later on let us recall some elements of the phase–space formulation of quantum mechanics [11, 12, 13, 14]. We consider a set of operators $\hat{f}, \hat{g}, \cdots$ on some Hilbert space $\mathcal{H}$, and we set up a one–to–one correspondence between these operators and the complex–valued functions $f, g, \cdots \in \mathrm{Fun}(\mathcal{M})$ defined over a suitable manifold $\mathcal{M}$. We write $f = \mathrm{symb}\,(\hat{f})$, and we refer to the function f as the symbol of the operator $\hat{f}$. The symbol map "symb" is linear and has a well–defined inverse. An important notion is the "star product" which implements the operator multiplication at the level of symbols:

$$\mathrm{symb}\,(\hat{f}\ \hat{g}) = \mathrm{symb}\,(\hat{f})\ *\ \mathrm{symb}\,(\hat{g}) \tag{1.1}$$

The star product is non–commutative, but associative, because "symb" provides an algebra homomorphism between the operator algebra and the symbols. In the

physical applications we have in mind, the Hilbert space $\mathcal{H}$ is the state space of a quantum mechanical system, and the manifold $\mathcal{M} \equiv \mathcal{M}_{2N}$ is the $2N$–dimensional classical phase–space of this system. Quantum mechanical operators $\hat{f}$ are then represented by functions $f = f(\phi)$, where $\phi^a = (p^1, \cdots, p^N, q^1, \cdots, q^N), a = 1, \cdots, 2N$ are canonical coordinates on $\mathcal{M}_{2N}$. Here and in the following we assume that the phase–space has the topology of $\mathbf{R}^{2N}$. Hence, by Darboux's theorem, we may assume that the symplectic 2–form of $\mathcal{M}_{2N}$, $\omega = \frac{1}{2}\omega_{ab} d\phi^a \wedge d\phi^b$, has constant components:

$$\omega_{ab} = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{1.2}$$

The inverse of this matrix,

$$\omega^{ab} = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \tag{1.3}$$

defines the Poisson bracket

$$\{f, g\}_{\rm pb}(\phi) = \partial_a f(\phi) \omega^{ab} \partial_b g(\phi) \tag{1.4}$$

where $\partial_a \equiv \frac{\partial}{\partial \phi^a}$. In the language of quantum mechanics, specifying the symbol map means fixing an ordering prescription, because it associates a unique operator $\hat{f}(\hat{p}, \hat{q}) = \mathrm{symb}^{-1}(f(p, q))$ to any classical phase function $f(p, q)$. A typical example is the Weyl symbol. It associates the symmetrically, or Weyl–ordered operator $\hat{f}(\hat{p}, \hat{q})$ to the polynomial $f(p, q)$. For instance, $\mathrm{symb}^{-1}(pq) = \frac{1}{2}(\hat{p}\hat{q} + \hat{q}\hat{p})$. Formally the Weyl symbol f of the operator $\hat{f}$ is given by

$$f(\phi^a) = \int \frac{d^{2N}\phi_0}{(2\pi\hbar)^N} \exp\left[\frac{i}{\hbar}\phi_0^a \omega_{ab} \phi^b\right] \mathrm{Tr}\left[\widehat{T}(\phi_0)\hat{f}\right] \tag{1.5}$$

with the operators

$$\widehat{T}(\phi_0) = \exp\left[\frac{i}{\hbar}\phi_0^a \omega_{ab} \hat{\phi}^b\right] = \exp\left[\frac{i}{\hbar}(p_0\hat{q} - q_0\hat{p})\right] \tag{1.6}$$

which generate translations on phase–space.

For any pair of Weyl symbols, $f, g \in \mathrm{Fun}(\mathcal{M}_{2N})$, the star product reads

$$\begin{aligned} (f * g)(\phi) &= f(\phi) \exp\left[i\frac{\hbar}{2} \overleftarrow{\partial}_a \, \omega^{ab} \, \overrightarrow{\partial}_b\right] g(\phi) \\ &\equiv \exp\left[i\frac{\hbar}{2}\omega^{ab}\partial_a^{(1)}\partial_b^{(2)}\right] f(\phi_1) g(\phi_2)\Big|_{\phi_1=\phi_2=\phi} \end{aligned} \tag{1.7}$$

where $\partial_a^{(1,2)} \equiv \partial/\partial\phi^a_{1,2}$. In the classical limit $\hbar \to 0$ one has $(f*g)(\phi) = f(\phi)g(\phi) + 0(\hbar)$, i.e. the star product is a "deformation" of the pointwise product of two functions. The Moyal bracket of two symbols is defined by

$$\{f,g\}_{\rm mb} = \frac{1}{i\hbar}(f*g - g*f) = \text{symb}\,\left(\frac{1}{i\hbar}[\hat{f},\hat{g}]\right) \tag{1.8}$$

In the classical limit it reduces to the Poisson bracket: $\{f,g\}_{\rm mb} = \{f,g\}_{\rm pb} + O(\hbar^2)$. The most remarkable property of the star product is its associativity. As a consequence of it, the Moyal bracket obeys the Jacobi identity.

Any density matrix $\hat{\rho}(t)$ time–evolves according to von Neumann's equation

$$i\hbar\ \partial_t\hat{\rho} = \left[\hat{H},\hat{\rho}\right] \tag{1.9}$$

Applying the symbol map to both sides of this equation we obtain Moyal's equation for the "pseudodensity" $\rho = \text{symb}(\hat{\rho})$:

$$\partial_t\rho(\phi) = \{H,\rho\}_{\rm mb} \equiv V_H(\phi)\rho(\phi) \tag{1.10}$$

In the classical limit, (1.10) reduces to Liouville's equation $\partial_t\rho = \{H,\rho\}_{\rm pb}$. In eq. (1.10) we have introduced the pseudo–differential operator [15, 16]

$$V_H(\phi) = \frac{2}{\hbar}H(\phi)\sin\left[\frac{\hbar}{2}\ \overleftarrow{\partial}_a\ \omega^{a\,b}\ \overrightarrow{\partial}_b\right] \tag{1.11}$$

which reduces in the classical limit to minus the hamiltonian vector field: $V_H = -\omega^{ba}\partial_a H\partial_b + O(\hbar^2)$. The operators (1.11) form a closed algebra:

$$[V_{H_1}, V_{H_2}] = V_{\{H_1,H_2\}_{\rm mb}} \tag{1.12}$$

This relation is most easily established by noting that, at the operatorial level, V_H corresponds to $(L_H - R_H)/i\hbar$ where $L_H(R_H)$ denotes the left(right) multiplication by $\hat{H}$. (See [16] for details).

The remaining sections of this paper are organized as follows. In section 2 we introduce the multiscalar functions F_p, define an exterior and a Lie derivative for them, and establish their relation to classical differential forms. In section 3 we provide some details about the non–commutative universal differential forms which will be needed in order to relate the abstract algebraic construction to the multiscalars. In section 4 the symbol map is applied to the operatorial construction of ΩA, and the quantum mechanical multiscalar formalism is obtained. In

section 5 the coincidence limit of the quantum mechanical multiscalars is investigated, and the physical meaning of the quantum deformation from classical to non–commutative forms is discussed.

2 Classical Differential Forms from Multiscalars

In this section we construct the exterior algebra of antisymmetric tensor fields by starting from multiscalar functions defined on the manifold under consideration. This section serves as a preparation for an analogous treatment of the quantum deformed case.

Let $\bigwedge^p_{\rm MS}$ denote the set of "multiscalar" functions $F_p(\phi_0, \phi_1, \cdots, \phi_p)$ which depend on $p+1$ arguments $\phi_i, i = 0, 1, \cdots, p$ and which vanish whenever two neighboring arguments are equal [10]:

$$F_p\left(\phi_0, \phi_1, \cdots, \phi_{i-1}, \phi_i, \phi_i, \phi_{i+2}, \cdots, \phi_p\right) = 0 \tag{2.1}$$

Now we define a map

$$\delta : \bigwedge^p_{\rm MS} \longrightarrow \bigwedge^{p+1}_{\rm MS}$$

by

$$\left(\delta F_p\right)\left(\phi_0, \cdots, \phi_{p+1}\right) = \sum_{i=0}^{p+1} (-1)^i F_p\left(\phi_0, \cdots, \phi_{i-1}, \widehat{\phi}_i, \phi_{i+1}, \cdots, \phi_{p+1}\right) \tag{2.2}$$

where the caret over ϕ_i means that this argument is omitted. The δ–operation maps a function of $p+1$ arguments onto a function of $p+2$ arguments. It is easy to verify that the image δF_p is in $\bigwedge^{p+1}_{\rm MS}$, i.e. that it vanishes if two adjacent arguments are equal. The first few examples are

$$\left(\delta F_0\right)\left(\phi_0, \phi_1\right) = F_0\left(\phi_1\right) - F_0\left(\phi_0\right) \tag{2.3}$$

$$\left(\delta F_1\right)\left(\phi_0, \phi_1, \phi_2\right) = F_1\left(\phi_1, \phi_2\right) - F_1\left(\phi_0, \phi_2\right) + F_1\left(\phi_0, \phi_1\right) \tag{2.4}$$

$$\begin{aligned}\left(\delta F_2\right)\left(\phi_0, \phi_1, \phi_2, \phi_3\right) &= F_2\left(\phi_1, \phi_2, \phi_3\right) - F_2\left(\phi_0, \phi_2, \phi_3\right) \\ &\quad + F_2\left(\phi_0, \phi_1, \phi_3\right) - F_2\left(\phi_0, \phi_1, \phi_2\right)\end{aligned} \tag{2.5}$$

Remarkably enough, δ turns out to be nilpotent:

$$\delta^2 = 0 \tag{2.6}$$

This can be checked explicitly, but it is most easily seen by noting that if we interpret $F_p(\phi_0, \cdots, \phi_p)$ as a p–simplex with vertices $\phi_0, \cdots, \phi_p$ then δ acts exactly like the nilpotent boundary operator of conventional simplicial homology [17].

Henceforth we shall refer to a function $F_p \in \bigwedge^p_{\mathrm{MS}}$ as a "p–form". On the direct sum

$$\bigwedge^*_{\mathrm{MS}} = \bigoplus_{p=0}^{\infty} \bigwedge^p_{\mathrm{MS}}$$

there exists a natural product of a p–form F_p with a q–form G_q yielding a $(p+q)$–form $F_p \bullet G_q$:

$$\begin{aligned} &(F_p \bullet G_q)(\phi_0, \phi_1, \cdots, \phi_p, \phi_{p+1}, \cdots, \phi_{p+q}) \\ &= F_p(\phi_0, \phi_1, \cdots, \phi_p)\, G_q(\phi_p, \phi_{p+1}, \cdots, \phi_{p+q}) \end{aligned} \tag{2.7}$$

For later use we note that δ obeys a kind of Leibniz rule with respect to this product:

$$\delta(F_0 \bullet G_0) = (\delta F_0) \bullet G_0 + F_0 \bullet (\delta G_0) \tag{2.8}$$

This relation is easily proven by using (2.3). It has no analogue for higher p–forms.

Multiscalars

Up to now we have not yet specified the precise nature of the functions F_p and their arguments ϕ_i. From now on we assume that the $\phi_i \equiv (\phi_i^a)$ are local coordinates on some manifold $\mathcal{M}$, and that the F_p's are smooth functions which transform as multiscalars under general coordinate transformations (diffeomorphisms) on $\mathcal{M}$. This means in particular that F_p evolves under the flow generated by some vector field $h = h^a(\phi)\partial_a, \partial_a \equiv \frac{\partial}{\partial \phi^a}$, according to

$$\partial_t F_p(\phi_0, \cdots, \phi_p; t) = \sum_{i=0}^{p} \mathrm{V}(\phi_i)\, F_p(\phi_0, \cdots, \phi_p; t) \tag{2.9}$$

where

$$\mathrm{V}(\phi_i) = -h^a(\phi_i)\, \partial_a^{(i)} \equiv -h^a(\phi_i)\, \frac{\partial}{\partial \phi_i^a} \tag{2.10}$$

acts only on the $i-$ th argument of F_p. Here the "time" t parametrizes points along the flow lines of the vector field h.

In this paper we restrict our attention to symplectic manifolds $\mathcal{M} \equiv \mathcal{M}_{2N} = \mathbf{R}^{2N}$ and to vector fields which are hamiltonian, i.e. we assume that (locally)

$$h^a(\phi) = \omega^{ab}\partial_b H(\phi) \tag{2.11}$$

for some generating function ("Hamiltonian") H. Let us introduce the operators

$$\mathcal{L}_p[H] = -\sum_{i=0}^{p} \mathrm{V}_H(\phi_i), \tag{2.12}$$

$$\mathrm{V}_H(\phi_i) \equiv \omega^{ab}\partial_a H(\phi_i)\frac{\partial}{\partial \phi_i^b} \tag{2.13}$$

and let us determine their commutation relations for different generating functions. Using

$$[\mathrm{V}_{H_1}(\phi_i), \mathrm{V}_{H_2}(\phi_j)] = \delta_{ij}\mathrm{V}_{\{H_1,H_2\}_{\mathrm{pb}}}(\phi_i) \tag{2.14}$$

we see that the $\mathcal{L}_p$'s form a closed algebra:

$$[\mathcal{L}_p[H_1], \mathcal{L}_p[H_2]] = -\mathcal{L}_p[\{H_1, H_2\}_{\mathrm{pb}}] \tag{2.15}$$

This is the Lie algebra of infinitesimal symplectic diffeomorphisms (canonical transformations) on $\mathcal{M}_{2N}$. It is therefore natural to look at $\mathcal{L}_p$ as the Lie derivative appropriate for the generalized p–forms F_p. It is easy to see that $\mathcal{L}_p$ commutes with the "exterior derivative" on $\bigwedge^p_{\mathrm{MS}}$, $\delta \equiv \delta_p$:

$$\delta_p \mathcal{L}_p = \mathcal{L}_{p+1}\,\delta_p \tag{2.16}$$

Later on we shall replace the generator $\mathrm{V}_H(\phi_i)$, eq. (2.13), by its quantum deformed (Moyal) analogue (1.11). Even then the relations (2.15) and (2.16) remain valid, provided one replaces the Poisson bracket in (2.15) by the corresponding Moyal bracket.

The coincidence limit

We are now going to show how in the limit when the arguments of the multiscalar $F_p(\phi_0, \phi_1, \cdots, \phi_p)$ are very "close" to each other, the generalized p–form

$F_p \in \bigwedge^p_{\rm MS}$ gives rise to a conventional p–form. We set

$$\begin{aligned} \phi^a_0 &= \phi^a \\ \phi^a_i &= \phi^a + \eta^a_i \, , \; i = 1, \cdots, p \end{aligned} \tag{2.17}$$

and expand $\left(\partial \, {}^{(i)}_{a} \equiv \partial / \partial \phi^a_i \right)$

$$F_p \left(\phi, \phi + \eta_1, \cdots, \phi + \eta_p \right) = \exp \left[\sum_{i=1}^{p} \eta^a_i \partial^{(i)}_a \right] F_p(\phi, \phi, \cdots, \phi) \tag{2.18}$$

to lowest order in $\eta^a_1, \cdots, \eta^a_p$. We keep only terms in which all η_i's are different, e.g. $\eta^a_1 \eta^b_2$, but not $\eta^a_1 \eta^b_1$, say. In this manner we obtain a sum of terms of the type

$$\eta^{a_1}_{i_1} \eta^{a_2}_{i_2} \cdots \eta^{a_l}_{i_l} \; \partial^{(i_1)}_{a_1} \cdots \partial^{(i_l)}_{a_l} F_p(\phi, \phi, \cdots, \phi) \tag{2.19}$$

for $0 \leq l \leq p$. An important, though trivial, observation is that, after "stripping off" the η's, the quantities $\partial^{(i_1)}_{a_1} \cdots \partial^{(i_l)}_{a_l} F_p(\phi, \phi, \cdots, \phi)$, for $i_1, \cdots, i_l$ fixed, transform as the components of a covariant tensor field of rank l, because on each ϕ–argument there acts at most one derivative. Upon explicit antisymmetrization in the indices $a_1, \cdots, a_l$ we obtain the components of an l–form:

$$\partial^{(i_1)}_{[a_1} \cdots \partial^{(i_l)}_{a_l]} F_p \left(\phi, \phi, \cdots, \phi \right)$$

These remarks also apply to any generic multiscalar F_p. What is special about the generalized forms $F_p \in \wedge^p_{\rm MS}$ is that for them the expansion of (2.18) does not contain any term with $l < p$, but only the one with the maximal rank $l = p$:

$$\begin{aligned} & F_p \left(\phi, \phi + \eta_1, \cdots, \phi + \eta_p \right) \\ & = \eta^{a_1}_1 \eta^{a_2}_2 \cdots \eta^{a_p}_p \; \partial^{(1)}_{a_1} \cdots \partial^{(p)}_{a_p} F_p(\phi, \phi, \cdots, \phi) + O\left(\eta^2_i \right) \end{aligned} \tag{2.20}$$

In appendix A we show that eq. (2.20) follows from the fact that F_p vanishes if two neighboring arguments coincide.

It is convenient to look at the ordinary differential form induced by a certain multiscalar $F_p \in \bigwedge^p_{\rm MS} \equiv \bigwedge^p_{\rm MS} \left(\mathcal{M}_{2N} \right)$ as the image of the so-called "classical map" [10]

$$\mathrm{Cl} : \bigwedge^p_{\rm MS} \left(\mathcal{M}_{2N} \right) \longrightarrow \bigwedge^p \left(\mathcal{M}_{2N} \right)$$

which is defined by

$$[\mathrm{Cl}\,(F_p)]\,(\phi) = \partial^{(1)}_{a_1} \cdots \partial^{(p)}_{a_p} F(\phi, \cdots, \phi)\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p} \tag{2.21}$$

Under the Cl–map the $\bullet$–product (2.7) becomes the standard wedge product of the exterior algebra $\bigwedge^* (\mathcal{M}_{2N}) = \bigoplus_{p=0}^{2N} \bigwedge^p (\mathcal{M}_{2N})$:

$$\mathrm{Cl}\,(F_p \bullet G_q) = \mathrm{Cl}(F_p) \wedge \mathrm{Cl}(G_q) \tag{2.22}$$

Similarly, the operator δ goes over into the exterior derivative,

$$\mathrm{Cl}\,(\delta F_p) = d\,\mathrm{Cl}(F_p)\,, \tag{2.23}$$

and $\mathcal{L}_p[H]$ of (2.12) with (2.13) becomes the Lie derivative along h,

$$\mathrm{Cl}\,(\mathcal{L}_p[H] F_p) = l_h \mathrm{Cl}(F_p) \tag{2.24}$$

It acts on the components of $\alpha \in \bigwedge^p (\mathcal{M}_{2N})$ in the usual way:

$$\begin{aligned} l_h \alpha a_1 \cdots a_p &= h^b \partial_b \alpha a_1 \cdots a_p + \partial a_1 h^b \alpha_{b a_2 \cdots a_p} \\ &\quad + \partial a_2 h^b \alpha_{a_1 b a_3 \cdots a_p} + \cdots \end{aligned} \tag{2.25}$$

The proofs of eqs. (2.23) and (2.24) can be found in appendix B. Here we only illustrate (2.24) for $p = 1$. In this case the "Lie transport" of the biscalar $F_1\,(\phi_0, \phi_1)$ is described by the equation

$$\begin{aligned} -\,\partial_t F_1\,(\phi_0, \phi_1; t) &= \mathcal{L}_1 F_1\,(\phi_0, \phi_1; t) \\ &= \left[h^a\,(\phi_0)\, \frac{\partial}{\partial \phi_0^a} + h^a\,(\phi_1)\, \frac{\partial}{\partial \phi_1^a} \right] F_1\,(\phi_0, \phi_1; t) \end{aligned} \tag{2.26}$$

Now we set $\phi_0 = \phi, \phi_1 = \phi + \eta$, and compare the coefficients of $(\eta)^0$ and $(\eta)^1$. By using $\frac{\partial}{\partial \phi^a} F_1(\phi, \phi; t) = 0$, with the derivative acting on both arguments, one finds that the coefficient of $(\eta)^0$ vanishes on both sides of the equation. The result at order $(\eta)^1$ is

$$-\,\partial_t \alpha_a(\phi) = h^b \partial_b \alpha_a(\phi) + \partial_a h^b \alpha_b(\phi) \equiv l_h \alpha_a(\phi) \tag{2.27}$$

where

$$\alpha_a(\phi) = \frac{\partial}{\partial \phi_1^a} F_1\,(\phi_0, \phi_1)\,\Big|_{\phi_0 = \phi_1 = \phi} = [\mathrm{Cl}\,(F_1)]\,(\phi) \tag{2.28}$$

A special class of multiscalars

Let us assume we are given a set of scalar functions on $\mathcal{M}_{2N}, f_i(\phi), i = 0, 1, \cdots, p$. Then we can construct the following $F_p \in \bigwedge^p_{\rm MS}$ out of them:

$$\begin{aligned} F_p\left(\phi_0, \phi_1, \cdots, \phi_p\right) = f_0\left(\phi_0\right)\left[f_1\left(\phi_1\right) - f_1\left(\phi_0\right)\right] \\ \left[f_2\left(\phi_2\right) - f_2\left(\phi_1\right)\right] \cdots \left[f_p\left(\phi_p\right) - f_p\left(\phi_{p-1}\right)\right] \end{aligned} \tag{2.29}$$

Generalized forms of this type will play an important role later on. Inserting (2.17) and expanding in η, one finds that they have a particularly simple classical limit:

$$\left[\mathrm{Cl}(F_p)\right](\phi) = f_0(\phi) df_1(\phi) \wedge df_2(\phi) \wedge \cdots \wedge df_p(\phi) \tag{2.30}$$

By virtue of (2.23) we know that

$$d\,\mathrm{Cl}(F_p) = df_0 \wedge df_1 \wedge \cdots \wedge df_p = \mathrm{Cl}(\delta F_p) \tag{2.31}$$

It is instructive to verify the second equality directly. For $p = 1$, say, we can apply eq. (2.4) to $F_1\left(\phi_0, \phi_1\right) = f_0\left(\phi_0\right)\left[f_1\left(\phi_1\right) - f_1\left(\phi_0\right)\right]$ and find

$$\begin{aligned} \left(\delta F_1\right)\left(\phi_0, \phi_1, \phi_2\right) = \; & f_0\left(\phi_1\right)\left[f_1\left(\phi_2\right) \;-\; f_1\left(\phi_1\right)\right] \\ & -f_0\left(\phi_0\right)\left[f_1\left(\phi_2\right) \;-\; f_1\left(\phi_0\right)\right] \\ & +f_0\left(\phi_0\right)\left[f_1\left(\phi_1\right) \;-\; f_1\left(\phi_0\right)\right] \\ = \; & \left[f_0\left(\phi_1\right) - f_0\left(\phi_0\right)\right]\;\left[\, f_1\left(\phi_2\right) \;-\; f_1\left(\phi_1\right)\,\right], \end{aligned} \tag{2.32}$$

as it should be. In the next section we shall introduce a set of algebraic tools which render manipulations of this type much more transparent.

3 Non–Commutative Universal Differential Forms

Now we turn to a different subject whose relation to the multiscalars of the previous section will become clear later. In this section we briefly review some properties of the universal differential forms in A. Connes' non–commutative geometry [1]. We partly follow the presentation of ref. [10].

To any algebra A we can associate its universal differential envelope ΩA, the algebra of "universal differential forms". Later on we shall identify A with the linear operators acting on a quantum mechanical Hilbert space, but for the time being we make no assumptions about A. The construction of ΩA proceeds as follows. To each element $a \in A$ we associate a new object δa. As a vector space, ΩA is defined to be the linear space of words built from the symbols $a_i \in A$ and δa_i, e.g., $a_1 \delta a_2 a_3 a_4 \delta a_3$. The multiplication in ΩA is defined to be associative and distributive over the addition $+$. The product of two elementary words is obtained by simply concatenating the two factors. For instance,

$$(a_1 \delta a_2) \cdot (\delta a_3 a_4 \delta a_1) = a_1 \delta a_2 \delta a_3 a_4 \delta a_1$$

One imposes the following relation (a kind of Leibniz rule) between the elements $a_1, a_2, \cdots \in A$ and $\delta a_1, \delta a_2, \cdots$:

$$\delta (a_1 a_2) = (\delta a_1)\, a_2 + a_1 \delta a_2 \tag{3.1}$$

By virtue of this relation, any element of ΩA can be rewritten as a sum of monomials of the form

$$a_0 \delta a_1 \delta a_2 \cdots \delta a_p \tag{3.2}$$

or

$$\delta a_1 \delta a_2 \cdots \delta a_p \tag{3.3}$$

This form can be achieved by repeatedly applying the trick

$$(\delta a_1)\, a_2 = \delta (a_1 a_2) - a_1 \delta a_2 \tag{3.4}$$

In order to put the two types of monomials, (3.2) and (3.3), on an equal footing it is convenient to add a new unit "1" to A, which is different from the unit A might have had already. We require $\delta 1 = 0$. As a consequence, we have to consider only words of the type (3.2), because (3.3) reduces to (3.2) for $a_0 = 1$.

Finally one defines an operator δ by the rules

$$\begin{aligned} \delta^2 &= 0\,, \\ \delta (a_0 \delta a_1 \delta a_2 \cdots \delta a_p) &= \delta a_0 \delta a_1 \delta a_2 \cdots \delta a_p \end{aligned} \tag{3.5}$$

By linearity, the action of δ is extended to all elements of ΩA. We define $\Omega^p A$ to be the linear span of the words $a_0 \delta a_1 \cdots \delta a_p$, referred to as "universal p–forms". Then

$$\Omega A = \bigoplus_{p=0}^{\infty} \Omega^p A\,, \quad \Omega^0 A \equiv A\,,$$

is a graded differential algebra with the "exterior derivative" $\delta : \Omega^p A \to \Omega^{p+1} A$.

Defining ΩA via tensor products

For our purposes it is important to realize that the space of universal p–forms, $\Omega^p A$, can be identified with a certain subspace of the tensor product

$$\underbrace{A \otimes A \otimes \cdots \otimes A}_{(p+1)\text{ factors}} \equiv A^{\otimes(p+1)}$$

Let us start with a few definitions. The $\otimes_A$–product of elements from $A^{\otimes(p+1)}$ with elements from $A^{\otimes(q+1)}$ yields elements in $A^{\otimes(p+q+1)}$. It is defined by

$$\begin{aligned} &[a_0 \otimes a_1 \otimes \cdots \otimes a_p] \ \otimes_A \ [b_0 \otimes b_1 \otimes \cdots \otimes b_q] \\ &= a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1} \otimes a_p b_0 \otimes b_1 \otimes \cdots \otimes b_q \end{aligned} \tag{3.6}$$

where $a_p b_0$ is an ordinary algebra product. Obviously $\otimes_A$–multiplication is associative. Furthermore, it is convenient to introduce the multiplication maps $\mathrm{m}_i : A^{\otimes(p+1)} \to A^{\otimes p}\,, i = 1, 2, \cdots, p$. They are linear and act as

$$\begin{aligned} &\mathrm{m}_i [a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i \otimes \cdots \otimes a_p] \\ &= a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1} a_i \otimes a_{i+1} \otimes \cdots \otimes a_p \end{aligned} \tag{3.7}$$

In this language, the construction of ΩA is as follows. Again we set $\Omega^0 A \equiv A$, and we identify $\delta a \in \Omega^1 A$ with the following element of $A \otimes A$:

$$\delta a = 1 \otimes a - a \otimes 1 \tag{3.8}$$

"Words" are formed by taking $\otimes_A$–products of a's and δa's:

$$a(\delta b) c(\delta d) \cdots = a \otimes_A [1 \otimes b - b \otimes 1] \otimes_A c \otimes_A [1 \otimes d - d \otimes 1] \otimes_A \cdots \tag{3.9}$$

A generic element of $\Omega^1 A$ has the structure

$$a\delta b = a \otimes_A [1 \otimes b - b \otimes 1] = a \otimes b - ab \otimes 1 \tag{3.10}$$

Obviously it is in the kernel of the multiplication map m_1: $\mathrm{m}_1(a\delta b) = ab - ab = 0$. More generally one defines

$$\begin{aligned} \Omega^1 A &= \mathrm{Ker}\,(\mathrm{m}_1) \\ \Omega^p A &= \underbrace{\Omega^1 A \otimes_A \Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A}_{p \text{ factors}} \end{aligned} \tag{3.11}$$

In this formalism the Leibniz rule (3.1) becomes a relation in $A \otimes A$. It is easy to see that the identification (3.8) is consistent with it:

$$\begin{aligned} (\delta a)b + a(\delta b) &= [1 \otimes a - a \otimes 1] \otimes_A b + a \otimes_A [1 \otimes b - b \otimes 1] \\ &= 1 \otimes ab - ab \otimes 1 \\ &= \delta(ab) \end{aligned} \tag{3.12}$$

Let us assume that it is possible to enumerate the elements of A as $A = \{a_n, n \in \Im\}$ where $\Im$ is some index set whose precise nature we shall not specify here. Though formal, the following discussion is a useful preparation for the construction of "quantum forms".

We consider two 1–forms α and β. As $\Omega^1 A$ is contained in $A \otimes A$, they have expansions of the form

$$\alpha = \sum_{n,m} \alpha_{nm} a_n \otimes a_m \,,\; \beta = \sum_{k,l} \beta_{kl} a_k \otimes a_l \tag{3.13}$$

Because $\Omega^1 A = \mathrm{Ker}(\mathrm{m}_1)$ the coefficients α_{nm} must be chosen such that

$$\mathrm{m}_1(\alpha) = \sum_{n,m} \alpha_{nm} a_n a_m = 0 \tag{3.14}$$

and similarly for β. The product $\alpha\beta \equiv \alpha \otimes_A \beta = \Sigma \alpha_{nm}\beta_{kl}\, a_n \otimes a_m a_k \otimes a_l$ indeed lies in $\Omega^2 A$, because

$$\mathrm{m}_1(\alpha\beta) = \sum_{k,l} \beta_{kl} \left[\sum_{n,m} \alpha_{nm} a_n a_m \right] a_k \otimes a_l = 0 \tag{3.15}$$

vanishes by virtue of (3.14), and similarly $\mathrm{m}_2(\alpha\beta) = 0$. The coefficients of a generic p–form $\alpha_p \in \Omega^p A$ given by the expansion

$$\alpha_p = \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \; a_{m_0} \otimes a_{m_1} \otimes \cdots \otimes a_{m_p} \tag{3.16}$$

are subject to the conditions

$$\begin{aligned} 0 = \mathrm{m}_i(\alpha_p) = \sum_{m_0 \cdots m_p} & \alpha_{m_0 \cdots m_p} \; a_{m_0} \otimes \cdots \otimes a_{m_{i-2}} \\ & \otimes a_{m_{i-1}} a_{m_i} \otimes a_{m_{i+1}} \otimes \cdots \otimes a_{m_p} \end{aligned} \tag{3.17}$$

for $i = 1, \cdots, p$. Therefore (3.16) can be rewritten as

$$\begin{aligned} \alpha_p \;\; &= \;\; \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \; a_{m_0} \otimes_A \left[1 \otimes a_{m_1} - a_{m_1} \otimes 1\right] \otimes_A \\ & \qquad \cdots \otimes_A \Big[1 \otimes a_{m_p} - a_{m_p} \otimes 1\Big] \\ &= \;\; \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \; a_{m_0} \delta a_{m_1} \delta a_{m_2} \cdots \delta a_{m_p} \end{aligned} \tag{3.18}$$

When written in this fashion, each term in the sum is a p–form, and it is now easy to apply the differential δ to (3.18). It follows directly from the definition (3.5) that $\delta\alpha_p \in \Omega^{p+1} A$ is represented by the following element in $A^{\otimes(p+2)}$:

$$\begin{aligned} \delta\alpha_p \;\; &= \;\; \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \; \delta a_{m_0} \delta a_{m_1} \cdots \delta a_{m_p} \\ &= \;\; \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p} \bigotimes_{j=0}^{p} {}_A \Big[1 \otimes a_{m_j} - a_{m_j} \otimes 1\Big] \end{aligned} \tag{3.19}$$

Now we need the following relation, which is easily proven by induction

$$\begin{aligned} & \bigotimes_{j=0}^{p} {}_A \Big[1 \otimes a_{m_j} - a_{m_j} \otimes 1\Big] \\ = \sum_{i=0}^{p+1} (-1)^i & a_{m_0} \otimes a_{m_1} \otimes \cdots \otimes a_{m_{i-1}} \otimes 1 \otimes a_{m_1} \otimes \cdots \otimes a_{m_p} \\ & + \text{ irrelevant} \end{aligned} \tag{3.20}$$

Here "irrelevant" stands for terms containing algebra products $a_{m_{i-1}} a_{m_i}$ inside at least one factor of the tensor product. These terms vanish upon contraction with

$\alpha_{m_0\cdots m_p}$. Inserting (3.20) into (3.19) we arrive at the final result

$$\delta\alpha_p = \sum_{i=0}^{p+1}(-1)^i \sum_{m_0\cdots m_p} \alpha_{m_0\cdots m_p}\ a_{m_0}\otimes a_{m_1}\otimes\cdots \\ \cdots\otimes a_{m_{i-1}}\otimes 1\otimes a_{m_i}\otimes\cdots\otimes a_{m_p} \tag{3.21}$$

In the next section this representation of $\delta\alpha_p$ will allow us to make contact with the operator δ defined on multiscalars.

4 Quantum Forms on Phase–Space

In this section we apply the symbol map described in section 1 to ΩA, where A is now taken to be the algebra of operators acting on the Hilbert space $\mathcal{H}$. Again we think of $\mathcal{H}$ as the space of states of a certain quantum system with classical phase–space $\mathcal{M}_{2N}$. In this manner we shall arrive at a representation of the non–commutative universal forms which, for $\hbar\to 0$, connects smoothly to the standard exterior algebra. In this way, the non–commutative forms are seen to be a "deformation" of the classical ones in the same sense as the Moyal bracket is a deformation of the Poisson bracket.

In the introduction we discussed the symbol map $\mathrm{symb}: A\to \mathrm{Fun}\,(\mathcal{M}_{2N}), \hat{a}\to \mathrm{symb}\,(\hat{a})\equiv a$ which relates operators on $\mathcal{H}$ to complex functions over $\mathcal{M}_{2N}$. We generalize its definition by including operators $\hat{a}_0\otimes\hat{a}_1\otimes\cdots\otimes\hat{a}_p\in A^{\otimes(p+1)}$ which act on $\mathcal{H}^{\otimes(p+1)}$. The (linear) map

$$\mathrm{symb}: A\otimes A\otimes\cdots\otimes A\to \mathrm{Fun}\,(\mathcal{M}_{2N}\times\mathcal{M}_{2N}\times\cdots\times\mathcal{M}_{2N})$$

represents operators in $A^{\otimes(p+1)}$ by functions of $(p+1)$ arguments:

$$[\mathrm{symb}\,(\hat{a}_0\otimes\hat{a}_1\otimes\cdots\otimes\hat{a}_p)]\,(\phi_0,\phi_1,\cdots,\phi_p) \\ =[\mathrm{symb}\,(\hat{a}_0)]\ (\phi_0)\ [\mathrm{symb}\,(\hat{a}_1)](\phi_1)\cdots[\mathrm{symb}\,(\hat{a}_p)]\,(\phi_p) \tag{4.1}$$

In this manner we establish a one–to–one correspondence between the abstract non–commutative differential forms in $\Omega^p A$ and functions of $p+1$ arguments. For instance, if $\alpha_p\in\Omega^p A$ is given by

$$\alpha_p = \sum_{m_0\cdots m_p}\alpha_{m_0\cdots m_p}\ \hat{a}_{m_0}\otimes\hat{a}_{m_1}\otimes\cdots\otimes\hat{a}_{m_p} \tag{4.2}$$

then its symbol $\mathrm{symb}\,(\alpha_p) \equiv F_p$ reads

$$F_p\,(\phi_0, \phi_1, \cdots, \phi_p) = \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p}\; a_{m_0}\,(\phi_0)\, a_{m_1}\,(\phi_1) \cdots a_{m_p}\,(\phi_p) \tag{4.3}$$

where we have set $a_m \equiv \mathrm{symb}\,(\hat{a}_m)$. At the level of symbols, the $\otimes_A$–product becomes

$$\begin{aligned} &\left[\mathrm{symb}\left(\,[\hat{a}_0 \otimes\; \hat{a}_1 \otimes \cdots \otimes \hat{a}_p] \otimes_A \left[\hat{b}_0 \otimes \hat{b}_1\; \otimes \cdots \otimes \hat{b}_q\right]\right)\right](\phi_0, \cdots, \phi_{p+q}) \\ &\quad = a_0\,(\phi_0)\, a_1\,(\phi_1) \cdots a_{p-1}\,(\phi_{p-1})\,[a_p * b_0]\,(\phi_p)\, b_1\,(\phi_{p+1}) \cdots b_q\,(\phi_{p+q}) \end{aligned} \tag{4.4}$$

with $a_i \equiv \mathrm{symb}\,(\hat{a}_i)$ and $b_i \equiv \mathrm{symb}\left(\hat{b}_i\right)$. The multiplication maps m_i act on symbols of the type (4.3) according to

$$\begin{aligned} &(\mathrm{m}_i F_p)\left(\phi_0, \phi_1, \cdots, \hat{\phi}_{i-1}, \phi_i, \cdots, \phi_p\right) \\ &= \sum_{m_0 \cdots m_p} \alpha_{m_0 \cdots m_p}\; a_{m_0}\,(\phi_0) \cdots a_{m_{i-2}}\,(\phi_{i-2})\left[a_{m_{i-1}} * a_{m_i}\right](\phi_i) \\ &\qquad \cdot a_{m_{i+1}}\,(\phi_{i+1}) \cdots a_{m_p}\,(\phi_p) \end{aligned} \tag{4.5}$$

If $F_p = \mathrm{symb}(\alpha_p)$, the condition $\alpha_p \in \Omega^p A$ turns into $\mathrm{m}_i F_p = 0, i = 1, \cdots, p$.

In the light of these rules we can look at the algebraic structures of section 3 in either of two ways. We can identify A with the algebra of operators $\hat{a}, \hat{b}, \cdots$ (with the notational change $a \to \hat{a}$, etc., in section 3 and the algebra product with the operator multiplication, or we can identify A with the algebra of symbols equipped with the $*$–product. The most interesting object to compare in both formulations is the differential δ. We define its action on symbols in the obvious way: $\delta\,\mathrm{symb}\,(\alpha_p) = \mathrm{symb}\,(\delta\alpha_p)$. For α_p's of the type (4.2), $\delta\alpha_p$ has been given in eq. (3.21). Using (4.3) and noting that

$$\begin{aligned} &\left(\hat{a}_{m_0} \otimes \hat{a}_{m_1} \otimes \cdots \otimes \hat{a}_{m_{i-1}} \otimes 1 \otimes \hat{a}_{m_i} \otimes \cdots \otimes \hat{a}_{m_p}\right)(\phi_0, \cdots, \phi_{p+1}) \\ &\qquad = a_{m_0}\,(\phi_0) \cdots a_{m_{i-1}}\,(\phi_{i-1})\, a_{m_i}\,(\phi_{i+1}) \cdots a_{m_p}\,(\phi_{p+1}) \end{aligned} \tag{4.6}$$

we arrive at an explicit representation of $\delta\;:\;\mathrm{Fun}\left(\mathcal{M}_{2N}^{(p+1)}\right) \to \mathrm{Fun}\left(\mathcal{M}_{2N}^{(p+2)}\right)$, namely

$$(\delta F_p)\,(\phi_0, \phi_1, \cdots, \phi_{p+1}) = \sum_{i=0}^{p+1} (-1)^i\; F_p\left(\phi_0, \cdots, \phi_{i-1}, \hat{\phi}_i, \phi_{i+1}, \cdots, \phi_{p+1}\right) \tag{4.7}$$

In (4.6) we assumed that symb(1) is the constant function with value 1, which is true for the Weyl symbol. It is quite remarkable, that the formula (4.7) coincides exactly with eq. (2.2), which was at the heart of our multiscalar approach to the classical exterior algebra. There remains an important difference however. The forms $F_p \in \bigwedge^p_{\rm MS}(\mathcal{M}_{2N})$ studied in section 2 were supposed to vanish when two adjacent arguments are equal. The symbols $F_p = \mathrm{symb}\,(\alpha_p)$, instead, obey a deformed version of this condition, namely $\mathrm{m}_i F_p = 0$. In fact, in the classical (i.e. commutative) limit $\hbar \to 0$, the star–product becomes the ordinary point–wise product of functions, and (4.5) yields

$$(\mathrm{m}_i F_p)\left(\phi_0, \cdots, \hat{\phi}_{i-1}, \phi_i, \cdots, \phi_p\right) = F_p\left(\phi_0, \cdots, \phi_{i-2}, \phi_i, \phi_i, \cdots, \phi_p\right) + O(\hbar) \quad (4.8)$$

so that the conditions are the same in both cases. Similary, the product (4.4) reduces to the product (2.7) for $\hbar \to 0$. Therefore we may conclude that in the classical limit $\Omega^p A$ and $\bigwedge^p_{\rm MS}(\mathcal{M}_{2N})$ are equivalent. Symbolically,

$$\lim_{\hbar \to 0} \mathrm{symb}\,(\Omega^p A) = \bigwedge^p_{\rm MS}(\mathcal{M}_{2N})$$

As shown in section 2, the ordinary exterior algebra $\bigwedge(\mathcal{M}_{2N})$ is obtained from $\bigwedge_{\rm MS}(\mathcal{M}_{2N})$ by an appropriate coincidence limit. The interesting question which we will address in section 5, is what happens if we perform this coincidence limit for $\hbar \neq 0$.

The quantum–deformed Lie derivative

Let us fix a certain $\alpha_p \in \Omega^p A$ with an expansion of the form (4.2). It is an operator on the $(p+1)^{st}$ tensor power of the Hilbert space $\mathcal{H}$, $\mathcal{H}^{\otimes(p+1)}$. Let us now perform the same unitary transformation on all factors of $\mathcal{H}^{\otimes(p+1)}$. We assume that the $\hat{a}_m$'s in (4.2) are transformed according to

$$\hat{a}_{m_j}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}\,t\right)\hat{a}_{m_j}\exp\left(\frac{i}{\hbar}\hat{H}t\right) \quad (4.9)$$

where $\hat{H}$ is a hermitian generator and t is a real parameter. Thus the family of operators $\hat{a}_{m_j}(t)$ obeys the von Neumann–type equation

$$i\hbar\,\partial_t\,\hat{a}_{m_j}(t) = \left[H, \hat{a}_{m_j}(t)\right] \quad (4.10)$$

and the evolution of α_p as a whole is governed by

$$\begin{aligned} i\hbar\,\partial_t\,\alpha_p(t) \quad &= \quad \sum_{j=0}^{p}\sum_{m_0\cdots m_p}\alpha_{m_0\cdots m_p}\;\hat{a}_{m_0}(t)\otimes\cdots \\ &\cdots \quad \otimes\left[\hat{H},\hat{a}_{m_j}(t)\right]\otimes\cdots\otimes\hat{a}_{m_p}(t) \end{aligned} \tag{4.11}$$

If we apply the Weyl symbol map to both sides of this equation and use (1.9) and (1.10) with (1.11) we arrive at

$$-\,\partial_t F_p\,(\phi_0,\cdots,\phi_p;t) = \mathcal{L}_p^{\hbar}\,[H]\,F_p\,(\phi_0,\cdots,\phi_p;t) \tag{4.12}$$

with the "quantum deformed Lie derivative"

$$\mathcal{L}_p^{\hbar}\,[H] = -\sum_{i=0}^{p}\,\frac{2}{\hbar}\,H\,(\phi_i)\sin\left[\frac{\hbar}{2}\;\overleftarrow{\partial}_a^{(i)}\;\omega^{ab}\;\overrightarrow{\partial}_b^{(i)}\right] \tag{4.13}$$

and $F_p \equiv \mathrm{symb}(\alpha_p)$, $H \equiv \mathrm{symb}\left(\hat{H}\right)$.

In the limit $\hbar \to 0$, $\mathcal{L}_p^{\hbar}$ reduces to the classical Lie derivative (2.12) appropriate for multiscalars. This suggests the interpretation of the symbols $\mathrm{symb}(\alpha), \alpha \in \Omega A$, as quantum deformed multiscalars. When a classical multiscalar is subject to a canonical transformation, the hamiltonian vector field $-V_H = \omega^{ba}\partial_a H \partial_b$ acts on any of its arguments. In the non–commutative case, V_H is replaced by its Moyal analogue, eq. (1.11). Like the classical one, the quantum Lie derivative commutes with the differential δ :

$$\delta_p\,\mathcal{L}_p^{\hbar} = \mathcal{L}_{p+1}^{\hbar}\,\delta_p \tag{4.14}$$

As a consequence of the algebra (1.12) for the deformed hamiltonian vector fields, the deformed Lie derivatives form a closed algebra as well:

$$\left[\mathcal{L}_p^{\hbar}\,[H_1],\mathcal{L}_p^{\hbar}\,[H_2]\right] = -\mathcal{L}_p\,[\{H_1,H_2\}_{\mathrm{mb}}] \tag{4.15}$$

This is the algebra of *quantum* canonical transformations, which is closely related to the W_∞ algebra [15, 16]. In ref. [16] it was shown that, under the symbol map, the algebra of infinitesimal unitary transformations on Hilbert space translates to an algebra of the above type. Choosing a basis on the space of all generating functions H one arrives at the more familiar forms of the W_∞-algebra [15] in which the structure constants are given by the Moyal brackets of the basis functions.

5 Coincidence Limit in the Non-Commutative Case

In this section we investigate the coincidence limit of the Moyal multiscalars $F_p = \mathrm{symb}\,(\alpha_p)$. Differences relative to the classical discussion in section 2 will occur because the pointwise product of functions on $\mathcal{M}_{2N}$ is now replaced by the star product. The impact of this deformation on the properties of differential forms is best illustrated by means of a few examples.

First we consider a 1–form $\alpha_1 \in \Omega^1 A$ represented by

$$\alpha_1 = \hat{a}_0 \delta \hat{a}_1 = \hat{a}_0 \otimes \hat{a}_1 - \hat{a}_0 \hat{a}_1 \otimes 1 \tag{5.1}$$

Writing as usual $a_i = \mathrm{symb}\,(\hat{a}_i)$, the symbol of α_1 reads

$$\begin{aligned} F_1\,(\phi_0, \phi_1) &= a_0\,(\phi_0)\, a_1\,(\phi_1) - (a_0 * a_1)\,(\phi_0) \\ &= F_1^{\mathrm{class}}\,(\phi_0, \phi_1) + (a_0 a_1 - a_0 * a_1)\,(\phi_0) \end{aligned} \tag{5.2}$$

with

$$F_1^{\mathrm{class}}\,(\phi_0, \phi_1) \equiv a_0\,(\phi_0)\,[\,a_1\,(\phi_1) - a_1\,(\phi_0)] \tag{5.3}$$

In the second line of (5.2) we decomposed F_1 in the classical part F_1^{class} and a quantum correction which vanishes for $\hbar \to 0$. Obviously F_1^{class} is of the type (2.29), and eq. (2.30) tells us that in the coincidence limit $F_1^{\mathrm{class}}\,(\phi, \phi + \eta) = \eta^b a_0 \partial_b a_1 + 0\,(\eta^2)$. Therefore the full F_1 yields

$$F_1\,(\phi, \phi + \eta) = \eta^b a_0\,(\phi)\,\partial_b a_1\,(\phi) + (a_0 a_1 - a_0 * a_1)\,(\phi) + O\left(\eta^2\right) \tag{5.4}$$

Contrary to the classical multiscalars in $\bigwedge^p_{\mathrm{MS}}(\mathcal{M}_{2N})$, the Moyal multiscalars F_p are not simply proportional to $\eta_1^{a_1} \cdots \eta_p^{a_p}$ in the coincidence limit: there are also terms with $\eta_1^{a_1} \cdots \eta_l^{a_l}, l < p$. In (5.4) this general rule is illustrated by the η–independent quantum correction $a_0 a_1 - a_0 * a_1$. To be more explicit, let us choose $\hat{a}_1 = \hat{\phi}^b$ for some fixed index b. Hence $\hat{a}_1$ is one of the canonical operators $\left(\hat{p}^1, \cdots, \hat{p}^N; \hat{q}^1, \cdots, \hat{q}^N\right)$, and the associated Weyl symbol is simply $a_1\,(\phi) = \phi^b$. In this case the series expansion for the star product terminates after the second term:

$$\left[\mathrm{symb}\left(\hat{a}_0 \delta \hat{\phi}^b\right)\right](\phi, \phi + \eta) = a_0\,(\phi)\,\eta^b + \frac{1}{2} i\hbar \omega^{bc} \partial_c a_0\,(\phi) \tag{5.5}$$

The most unusual property of the non–commutative 1–form $\hat{a}_0\delta\hat{\phi}^b$ is that its symbol does not vanish even at coinciding points:

$$\left[\text{symb}\left(\hat{a}_0\delta\hat{\phi}^b\right)\right](\phi,\phi) = \frac{1}{2}i\hbar\omega^{bc}\partial_c a_0(\phi) \tag{5.6}$$

Note that the RHS of (5.6) is purely imaginary and that it is proportional to be b–component of the hamiltonian vector field generated by a_0. The nonvanishing RHS of (5.6) is a typical quantum effect. It seems to contradict our "classical" intuition about the meaning of a differential $d\phi^b$. Loosely speaking, given two "nearby" points ϕ_0 and ϕ_1, we would like to visualize $d\phi^b$ as the "displacement" $\phi_1^b - \phi_0^b \equiv \eta^b$. Consequently we expect that, in an appropriate sense, $d\phi^b = 0$ if $\phi_0 = \phi_1$. *Eq.(5.6) shows that this is not necessarily the case for quantum 1–forms.* Though it is true that symb$(\delta\hat{\phi}^b)$ vanishes in the coincidence limit, this is not the case anymore as soon as we multiply $\delta\hat{\phi}^b$ by some nontrivial operator $\hat{a}_0$.

Next we look at a non–commutative 2–form $\alpha_2 = \delta\hat{a}_0\delta\hat{a}_1 \in \Omega^2 A$. Its tensor product representation

$$\begin{aligned}\alpha_2 &= [1\otimes\hat{a}_0 - \hat{a}_0\otimes 1]\otimes_A[1\otimes\hat{a}_1 - \hat{a}_1\otimes 1] \\ &= 1\otimes\hat{a}_0\otimes\hat{a}_1 - 1\otimes\hat{a}_0\hat{a}_1\otimes 1 - \hat{a}_0\otimes 1\otimes\hat{a}_1 + \hat{a}_0\otimes\hat{a}_1\otimes 1\end{aligned} \tag{5.7}$$

translates into the symbol

$$F_2(\phi_0,\phi_1,\phi_2) = F_2^{\text{class}}(\phi_0,\phi_1,\phi_2) + (a_0a_1 - a_0 * a_1)(\phi_1) \tag{5.8}$$

with

$$F_2^{\text{class}}(\phi_0,\phi_1,\phi_2) = [a_0(\phi_1) - a_0(\phi_0)][a_1(\phi_2) - a_1(\phi_1)] \tag{5.9}$$

The non–classical piece in (5.8) is due to the operator product $\hat{a}_0\hat{a}_1$ in the second line of (5.7). Using (2.17) and (2.30) for the expansion of F_2^{class}, we obtain

$$\begin{aligned}F_2(\phi,\phi+\eta_1,\phi+\eta_2) &= (a_0a_1 - a_0*a_1)(\phi) + \eta_1^b\partial_b(a_0a_1 - a_0*a_1)(\phi) \\ &\quad + \eta_1^b\eta_2^c\,\partial_b a_0(\phi)\,\partial_c a_1(\phi) + O\left(\eta_1^2,\eta_2^2\right)\end{aligned} \tag{5.10}$$

Apart from the classical term proportional to $\eta_1\eta_2$, we find a term linear in η and a constant piece which survives the limit $\eta_1,\eta_2 \to 0$. Eq. (5.10) becomes

particularly transparent for the choice $\hat{a}_0 = \hat{\phi}^a$, $\hat{a}_1 = \hat{\phi}^b$ with fixed indices a and b:

$$\begin{aligned}\left[\text{symb}\left(\delta\hat{\phi}^a\delta\hat{\phi}^b\right)\right](\phi, \phi+\eta_1, \phi+\eta_2) \\ = -\frac{1}{2}\, i\hbar\omega^{ab} + \eta_1^a\eta_2^b + O\left(\eta_1^2, \eta_2^2\right)\end{aligned} \tag{5.11}$$

Upon antisymmetrization, the term $\eta_1^a\eta_2^b$ gives rise to the classical analogue of $\delta\hat{\phi}^a\delta\hat{\phi}^b$, namely $d\phi^a \wedge d\phi^b$. In a symbolic notation the modified wedge product reads

$$\delta\hat{\phi}^a \wedge \delta\hat{\phi}^b \equiv \delta\hat{\phi}^a\delta\hat{\phi}^b - \delta\hat{\phi}^b\delta\hat{\phi}^a \sim -i\hbar\omega^{ab} + d\phi^a \wedge d\phi^b \tag{5.12}$$

A natural candidate for a "quantum deformed symplectic 2–form" is

$$\hat{\omega}_q = \frac{1}{2}\,\omega_{ab}\,\delta\hat{\phi}^a \wedge \delta\hat{\phi}^b \tag{5.13}$$

so that $\hat{\omega}_q \sim \omega_{\text{class}} + iN\hbar$. Using (5.7) and $\omega_{ab}\hat{\phi}^a\hat{\phi}^b = -i\hbar N$, which follows from the canonical commutation relations, $\hat{\omega}_q$ can be written as

$$\begin{aligned}\hat{\omega}_q \;\; = \;\; & \omega_{ab}\left[1\otimes\hat{\phi}^a\otimes\hat{\phi}^b - \hat{\phi}^a\otimes 1\otimes\hat{\phi}^b + \hat{\phi}^a\otimes\hat{\phi}^b\otimes 1\,\right] \\ & + i\hbar N\,(1\otimes 1\otimes 1)\end{aligned} \tag{5.14}$$

Even without invoking the coincidence limit its symbol ω_q has a rather transparent structure:

$$\omega_q\,(\phi_0, \phi_1, \phi_2) = \omega_{ab}\left[\phi_1^a\phi_2^b - \phi_0^a\phi_2^b + \phi_0^a\phi_1^b\right] + iN\hbar \tag{5.15}$$

For $\hbar = 0$ the function $\omega_q\,(\phi_0, \phi_1, \phi_2)$ is precisely (twice) the symplectic area of the triangle with vertices [1] ϕ_0, ϕ_1 and ϕ_2. Clearly this area vanishes when the edges of the triangle shrink and its vertices merge in a single point. The situation is dramatically different for $\hbar \neq 0$. If we use (5.15) to define a symplectic area also in the quantum case, we find that this "area" is nonzero even for degenerate triangles with coincident vertices:

$$\omega_q\,(\phi, \phi, \phi) = iN\hbar \tag{5.16}$$

Accepting this definition of a "quantum area" we see that in the non–commutative case there is always a mininal symplectic area of order $\hbar$. It is tempting to relate this minimum area to the well-known statement that "it is impossible to localize

[1] Recall that we are using Darboux canonical coordinates on $\mathcal{M}_{2N} = \mathbf{R}^{2N}$.

a quantum state in a phase–space volume smaller than $(2\pi\hbar)^N$". In fact, for $N = 1$, $\hat{\omega}_q$ is the volume form, and from the derivation of (5.16) it is clear that both phenomena have the same origin, namely the non–commutative nature of $\hat{p}$ and $\hat{q}$ or, equivalently, of the star–product. However, as the quantum mechanical term in the 2–form $\delta\hat{p} \wedge \delta\hat{q} \sim i\hbar + dp \wedge dq$ is purely imaginary, the naive picture of phase–space being partitioned into "cells" of volume $\hbar$ cannot be taken too literally.

6 Discussion and Conclusion

In this paper we obtained a representation of the universal algebra of non–commutative differential forms on the phase–space of an arbitrary quantum system by applying the Weyl–Wigner symbol map to the operatorial construction. The resulting quantum p–forms are multiscalar functions of $p+1$ phase–space variables. Their coincidence limit yields a deformation of the classical exterior algebra. At the operatorial level there exists a natural definition of an exterior derivative and of a Lie derivative for non–commutative forms. Their image under the symbol map leads to the corresponding operations acting on multiscalars. In the coincidence limit these derivations yield a deformation of the classical exterior derivative and of the Lie derivative, to which they reduce for $\hbar \to 0$. For $\hbar > 0$ the Lie derivative is a generalization of the Moyal bracket. Lie derivatives belonging to different hamiltonian vector fields form a closed algebra of the W_∞–type.

A priori, the quantum p–forms, seen as elements of $A \otimes A \otimes \cdots \otimes A$, seem to be of a very different nature than the classical tensor fields. In our approach both of them can be represented within the same setting, and the non–commutative case can be studied as a smooth deformation of the classical one.

The non–commutative exterior algebra developped in this paper contains the standard phase–space formulation of quantum mechanics as its zero–form sector. The same is also true for an earlier, different model [18] of a non–commutative symplectic geometry. However, in [18] a slightly ad hoc definition of a quantum differential form was used, which led to the unusual feature that the algebra of Lie derivatives closed only on a space larger than that of the classical Hamiltonians. The non–commutative exterior algebra obtained in the present paper is

also different from the ones studied in refs. [19] and [20]. Also Segal's "quantized deRham complex" [21] is based upon a different notion of a quantum differential form.

Before closing we would like to emphasize that the non–commutative forms studied here are interesting objects also from a dynamical point of view. In physical terms the transition from the algebra A to $\Omega^p A$ means that we go over from a one–particle theory living on the Hilbert space $\mathcal{H}$ to a $(p+1)$–particle theory which lives on the tensor product $\mathcal{H}\otimes\mathcal{H}\otimes\cdots\otimes\mathcal{H}$ with $p+1$ factors. The dynamics for all $p+1$ particles is exactly the same, and no explicit interactions between the particles are introduced, see eq. (4.11). Nevertheless, by the very definition of the non–commutative tensor product, particles belonging to different factors of $\mathcal{H}\otimes\cdots\otimes\mathcal{H}$ actually "know" about each other. To make this more explicit we first consider Hamilton's equation $\partial_t\phi^a(t) = h^a(\phi(t))$ and linearize it about a given solution $\phi^a(t)$. This leads to Jacobi's equation $\partial_t\Delta^a(t) = \partial_b h^a(\phi)\Delta^b(t)$ which tells us how the "displacement" Δ^a between the classical trajectories $\phi^a(t)$ and $\phi^a(t)+\Delta^a(t)$ evolves with time. The "Jacobi field" $\Delta^a(t)$ defines a family of classical 1–forms along $\phi^a(t)$. The corresponding non–commutative construction is as follows. From the canonical operators $\hat{\phi}^a(t)$ in the Heisenberg picture we define the quantum 1–form

$$\delta\hat{\phi}^a(t) = 1\otimes\hat{\phi}^a(t) - \hat{\phi}^a(t)\otimes 1 \tag{6.1}$$

whose symbol is $\phi_1^a(t) - \phi_0^a(t) \equiv \Delta_q^a(t)$. Under the symbol map the Heisenberg equation for $\hat{\phi}^a(t)$ becomes Hamilton's equation for the symbols $\phi^a(t) \equiv \text{symb}\left(\hat{\phi}^a(t)\right)$. Therefore

$$\begin{aligned}\partial_t\Delta_q^a(t) &= h^a\left(\phi_1(t)\right) - h^a\left(\phi_0(t)\right) \\ &= \partial_b h^a\left(\phi_0\right)\Delta_q^b(t) + 0\left(\Delta_q^2\right)\end{aligned} \tag{6.2}$$

and Δ_q^a can be identified with the classical Jacobi field Δ^a in the coincidence limit $\phi_1 \to \phi_0$. This simple observation illustrates why a non–commutative 1–form is naturally related to a quantum system on the doubled Hilbert space $\mathcal{H}\otimes\mathcal{H}$: in order to define $\delta\hat{\phi}^a(t)$ via some kind of "displacement" we have to know the position of *two* particles at each instant of time.

The 1–form (6.1) is still an essentially classical object. This is different for the higher p–forms

$$\delta\hat{\phi}^{a_1}(t) \wedge \delta\hat{\phi}^{a_2}(t) \wedge \cdots \wedge \delta\hat{\phi}^{a_p}(t) \tag{6.3}$$

which represent p–volumes transported along the hamiltonian flow. They can be visualized as parallelepipeds with vertices $(\phi_0, \phi_1, \cdots, \phi_p)$ which should be thought of as the arguments of the symbol of (6.3). Because of the non–commutative tensor product involved in (6.3), the coincidence limit of this symbol contains the typical $\hbar\omega^{ab}$–terms studied in section 5. For instance, if we have three nearby trajectories $\phi_0(t)$, $\phi_1(t)$ and $\phi_2(t)$ the above construction leads to a two–dimensional area which is dragged along the hamiltonian flow, and in the symbolic notation of eq. (5.12) we obtain the time–dependent area element

$$\delta\hat{\phi}^{a}(t) \wedge \delta\hat{\phi}^{b}(t) \sim i\hbar\omega^{ab} + d\phi^{a}(t) \wedge d\phi^{b}(t) \tag{6.4}$$

Here we see very clearly that, though individually each trajectory is governed by the standard one–particle dynamics, their tensorization leads to the quantum correction $i\hbar\omega^{ab}$ as a collective effect. In physical terms it expresses the fact that the (imaginary) area of the parallelogram with vertices $\phi_0(t), \phi_1(t)$ and $\phi_2(t)$ is bounded below by $\hbar$.

In a classical context, time–dependent p–volume elements of the type (6.3) play an important rôle in the study of the chaotic behavior of a system [22]. The exponential growth rate of (6.3) defines the $p-th$ Lyapunov exponent. If it is non–zero, the classical evolution of the system shows a strong form of stochasticity. It is very natural to ask whether the non–commutative p–forms play a similar rôle for "quantum chaos". We shall come back to this point elsewhere.

The work contained in this paper can be generalized and extended in various directions. Here we assumed, for instance, that the phase space under consideration is $\mathbf{R}^{2N}$. Using recent results by Fedosov [23] it seems possible to generalize the construction to arbitrary symplectic manifolds. Another point which has to be explored further is whether there are natural candidates for vectors dual to the non–commutative forms discussed here. (Contrary to the approach of Dubois–Violette [19] this is not self–evident if one starts from [1].) As for applications, it is clear that in order to better understand the dynamical and the geometrical meaning of the non–commutative forms concrete examples should be worked

out. The important question is whether, given a conventional quantum system on $\mathcal{H}$, the dynamics of the higher differential forms on $\mathcal{H} \otimes \cdots \otimes \mathcal{H}$ gives us useful information about the original system. In the classical theory the Lyapunov exponents are a typical example where this actually happens. The higher p–form sectors encode information about the dynamics of the zero–form sector in a very transparent way.

Investigations along these lines are not necessarily restricted to systems where $\mathcal{M}_{2N}$ is the true phase–space. For example, one could also consider planar fermion systems in strong magnetic fields where the configuration space is effectively turned into a phase–space [24], or the model of Doplicher et al. [25] in which quantum gravity effects induce a kind of symplectic structure in space–time.

Acknowledgements

I would like to thank E. Gozzi for many stimulating discussions and the Department of Theoretical Physics, University of Trieste, for its hospitality while this work was in progress. I am also grateful to F. Benatti, M. Dubois–Violette and N. Papadopoulos for several helpful conversations. This work was supported in part by INFN and by a NATO travelling grant.

References

[1] A. Connes, Publ. Math. IHES 62 (1985) 41.

[2] M. Dubois–Violette, C.R. Acad. Sci., Paris, I, 307 (1988) 403.

[3] A. Connes, J. Lott, Nucl. Phys. Proc. Suppl. B18 (1989) 29; D. Kastler, T. Schücker, Theor. Math. Phys. 92 (1992) 1075; D. Kastler, Rev. Math. Phys. 5 (1993) 477; T. Schücker, J.-M. Zylinski, Preprint CPT–93/P.2960 (1993).

[4] M. Dubois–Violette, R. Kerner, J. Madore, Phys. Lett. 217B (1989) 485; Class. Quant. Grav. 6 (1989) 1709 and 8 (1991) 1077; J. Math. Phys. 31 (1990) 316 and 323; H. Grosse, J. Madore, Phys. Lett. 283B (1992) 218.

[5] A. Chamseddine, G. Felder, J. Fröhlich, Commun. Math. Phys. 155 (1993) 205; A. Chamseddine, J. Fröhlich, Phys. Lett. 314B (1993) 308; G. Landi, N.A. Viet, K.C. Wali, Phys. Lett. 326B (1994) 45; D. Kastler, Preprint CPT–93–P–2970 (1993); W. Kalau, M. Waltze, Preprint MZ–TH/93–38 (1993); A. Sitarz, Preprint TPJU–1/94 (1994).

[6] R. Coquereaux, G. Esposito–Farèse, G. Vaillant, Nucl. Phys. B353 (1991) 689.

[7] R. Coquereaux, G. Esposito–Farèse, F. Scheck, Int. J. Mod. Phys. A7 (1992) 6555; R. Häussling, N.A. Papadopoulos, F. Scheck, Phys. Lett. 260B (1991) 125 and 303B (1993) 265; R. Coquereaux, R. Häussling, N.A. Papadopoulos, F. Scheck, Int. J. Mod. Phys. A7 (1992) 2809; N.A. Papadopoulos, J. Plass, F. Scheck, Phys. Lett. 324B (1994) 380.

[8] S.L. Woronowicz, Publ. RIMS 23 (1987) 117; J. Wess, B. Zumino, Nucl. Phys. Proc. Suppl. 18B (1990) 302; G. Mack, V. Schomerus, Commun. Math. Phys. 149 (1992) 513.

[9] W.B. Schmidke, J. Wess, B. Zumino, Z. Phys. C52 (1991) 471; O. Ogievetskii, W.B. Schmidke, J. Wess, B. Zumino, Commun. Math. Phys. 150 (1992) 495; P. Aschieri, L. Castellani, Int. J. Mod. Phys. A8 (1993) 1667.

[10] R. Coquereaux, J. Geom. Phys. 11 (1993) 307.

[11] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Ann. Phys. 111 (1978) 61 and 111.

[12] J.E. Moyal, Proc. Cambridge Phil. Soc. 45 (1949) 99.

[13] F.A. Berezin, Sov. Phys. Usp. 23 (1980) 1981.

[14] R.G. Littlejohn, Phys. Rep. 138 (1986) 193; I. Bakas, A.C. Kakas, J. Phys. A20 (1987) 3713; G.V. Dunne, J. Phys. A21 (1988) 2321; G.V. Dunne, A.C. Kakas, Phys. Lett. 211B (1988) 445; M. Flato, D. Sternheimer, Preprint hepth–9409052.

[15] D.B. Fairlie, C.K. Zachos, Phys. Lett. 224B (1989) 101; D.B. Fairlie, P. Fletcher, C.K. Zachos, Phys. Lett. 218B (1989) 203; I. Bakas, Phys. Lett. 228B (1989) 57; Commun. Math. Phys. 134 (1990) 487; I.M. Gelfand, D.B. Fairlie, Commun. Math. Phys. 136 (1991) 487.

[16] E. Gozzi, M. Reuter, Int. J. Mod. Phys. A9 (1994) 5801; D.B. Fairlie, J. Nuyts, Commun. Math. Phys. 134 (1990) 413.

[17] M. Nakahara, *Geometry, Topology and Physics*, Adam Hilger, Bristol, 1990.

[18] E. Gozzi, M. Reuter, Mod. Phys. Lett. A8 (1993) 1433; Int. J. Mod. Phys. A9 (1994) 2191.

[19] M. Dubois–Violette in: *Differential Geometric Methods in Theoretical Physics*, C. Bartocci, U. Bruzzo, R. Cianci (Eds.), Springer, Berlin, 1991.

[20] C. Chryssomalakos, P. Schupp, B. Zumino, Berkeley Preprint LBL–35034 (1994).

[21] I. Segal, Topology 7 (1968) 147; Proc. Symp. Pure Math. 16 (1970) 205.

[22] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion*, Springer, New York, 1983;
E. Gozzi, M. Reuter, Chaos, Solitons and Fractals 4 (1994) 1117.

[23] B.A. Fedosov, Harvard Preprint, 1993.

[24] G. Dunne, R. Jackiw in *Common Trends in Condensed Matter and High–Energy Physics*, L. Alvarez–Gaume et al. (Eds.), North–Holland, Amsterdam, 1994.

[25] S. Doplicher, K. Fredenhagen, J.E. Roberts, Phys. Lett. 331B (1994) 39; Rome preprint 94/6 (1994).

Appendix A

In this appendix we derive eq. (2.20) of section 2. We shall exploit the identity

$$\partial_{a_1}^{(l_1)} \cdots \partial_{a_M}^{(l_M)} \left[\partial_b^{(i)} + \partial_b^{(i+1)}\right] F_p\,(\phi, \cdots, \phi) = 0 \tag{A.1}$$

which holds for any $F_p \in \bigwedge^p_{\rm MS}$ provided $l_k \neq i, i+1$ for all $k = 1, \cdots, M$. Eq. (A.1) follows simply from the fact that the function defined by

$$G(\varphi) = F_p\,(\phi_0, \phi_1, \cdots, \phi_{i-1}, \varphi, \varphi, \phi_{i+2}, \cdots, \phi_p)$$

vanishes identically in φ, and therefore also all its partial derivatives are zero:

$$\frac{\partial}{\partial \varphi^b} G(\varphi) = \left[\partial_b^{(i)} + \partial_b^{(i+1)}\right] F\,(\phi_0, \cdots, \varphi, \varphi, \cdots, \phi_p) = 0$$

Taking derivatives with respect to the "parameters" $\left(\phi_0, \phi_1, \cdots, \hat{\phi}_i, \hat{\phi}_{i+1}, \cdots, \phi_p\right)$ we still get zero on the RHS, which proves (A.1).

In order to prove (2.20) we rewrite (2.18) according to

$$\begin{aligned} & F_p\,(\phi, \phi + \eta_1, \cdots, \phi + \eta_p) \\ &= \prod_{i=1}^{p} \exp\left[\eta_i^a \partial_a^{(i)}\right] F_p\,(\phi, \phi, \cdots, \phi) \\ &= \prod_{i=1}^{p} \left[1 + \eta_i^a \partial_a^{(i)}\right] F_p\,(\phi, \phi, \cdots, \phi) + O\,(\eta_i^2) \end{aligned} \tag{A.2}$$

In the expansion of the exponential we omitted terms with two or more equal factors of η_i. If we perform the product in the last line of (A.2) we seem to obtain tensors of any rank between zero (p factors of "1") and p (p factors of $\eta_i^a \partial_a^{(i)}$). However, for multiscalars $F_p \in \bigwedge^p_{\rm MS}$ we shall now show that

$$\partial_{a_1}^{(l_1)} \cdots \partial_{a_r}^{(l_r)} F_p\,(\phi, \phi, \cdots, \phi) = 0 \qquad \forall r = 1, \cdots, p-1 \tag{A.3}$$

with all l_i's different. This means that only the p–form piece in (A.2) is non–zero.

The function F_p in eq. (A.3) has $p+1$ arguments which are acted upon by at most $p-1$ derivatives. All of them act on different arguments of F_p. Hence F_p has $(p+1) - r$ arguments which are not differentiated. If two of these arguments are next to each other, we obtain zero immediately. In order to illustrate the situation when no undifferentiated arguments are next to each other, let us consider the

most "dangerous" case $r = p-1$ with only two undifferentiated arguments, ϕ_0 and ϕ_p, say. By repeatedly applying (A.1) we can write

$$\partial^{(1)}_{a_1}\partial^{(2)}_{a_2}\cdots\partial^{(p-2)}_{a_{p-2}}\partial^{(p-1)}_{a_{p-1}}F_p\left(\phi,\phi,\cdots,\phi\right)$$

$$=-\partial^{(1)}_{a_1}\partial^{(2)}_{a_2}\cdots\partial^{(p-3)}_{a_{p-3}}\partial^{(p-2)}_{a_{p-2}}\partial^{(p)}_{a_{p-1}}F_p\left(\phi,\phi,\cdots,\phi\right)$$

$$=+\partial^{(1)}_{a_1}\partial^{(2)}_{a_2}\cdots\partial^{(p-3)}_{a_{p-3}}\partial^{(p-1)}_{a_{p-2}}\partial^{(p)}_{a_{p-1}}F_p\left(\phi,\phi,\cdots,\phi\right)$$

$$=\mp\cdots \tag{A.4}$$

The argument which is not differentiated moves from the right to the left. This process can be continued until it reaches the position adjacent to ϕ_0. At this point (A.4) is seen to vanish. In the same way one concludes that (A.3) is true in general.

Appendix B

In this appendix we prove eqs. (2.23) and (2.24) of section 2. Let us start by showing that

$$\mathrm{Cl}\left(\delta F_p\right)=d\,\mathrm{Cl}\left(F_p\right) \tag{B.1}$$

We have to extract the monomial with p different factors of η from

$$D\left(\eta_1,\cdots,\eta_{p+1}\right)\equiv\left(\delta F_p\right)\left(\phi,\phi+\eta_1,\cdots,\phi+\eta_{p+1}\right)$$

Applying the definition of δ, eq. (2.2), yields $D=D_1+D_2$ with

$$\begin{aligned} D_1\left(\eta_1,\cdots,\eta_{p+1}\right) &\equiv F_p\left(\phi+\eta_1,\phi+\eta_2,\cdots,\phi+\eta_{p+1}\right) \\ D_2\left(\eta_1,\cdots,\eta_{p+1}\right) &\equiv \sum_{i=1}^{p+1}\left(-1\right)^i F_p\left(\phi,\phi+\eta_1,\cdots,\widehat{\phi+\eta_i},\cdots,\phi+\eta_{p+1}\right) \end{aligned} \tag{B.2}$$

Here we have separated the term with $i=0$. Ignoring irrelevant terms with higher powers of η_i, its contribution reads

$$\begin{aligned} D_1\left(\eta_1,\cdots,\eta_{p+1}\right) &= \exp\left[\sum_{i=1}^{p+1}\left(\eta_i\;\partial^{(i-1)}\right)\right]F_p(\phi,\phi,\cdots,\phi) \\ &= \prod_{i=1}^{p+1}\left[1+\left(\eta_i\;\partial^{(i-1)}\right)\right]F_p(\phi,\phi,\cdots,\phi)+\cdots \end{aligned} \tag{B.3}$$

where $\left(\eta_i\,\partial^{(i-1)}\right) \equiv \eta_i^a \partial_a^{(i-1)}$. By eq. (A.3) of appendix A the coincidence limit of the monomials $\partial\partial\cdots\partial\, F_p$ is zero if there are not at least p (different) derivatives acting on F_p. This means that when we expand the product in the last line of (B.3) we have to keep only the terms with $p+1$ derivatives and with p derivatives:

$$\begin{aligned} D_1 &= D_1^p + D_1^{p-1} \\ D_1^p &\equiv \left(\eta_1\partial^{(0)}\right)\left(\eta_2\partial^{(1)}\right)\cdots\left(\eta_{p+1}\partial^{(p)}\right)\, F_p\,(\phi,\phi,\cdots,\phi) \\ D_1^{p-1} &\equiv \sum_{i=1}^{p+1}\left(\eta_1\partial^{(0)}\right)\left(\eta_2\partial^{(1)}\right)\cdots\left(\eta_{i-1}\partial^{(i-2)}\right)\left(\eta_{i+1}\partial^{(i)}\right)\cdots \\ &\qquad\qquad \cdots\left(\eta_{p+1}\partial^{(p)}\right)\, F_p\,(\phi,\cdots,\phi) \end{aligned} \tag{B.4}$$

The relevant piece in D_2 of (B.2) is

$$\begin{aligned} D_2\,(\eta_1,\cdots,\eta_p) = \sum_{i=1}^{p+1}(-1)^i\left(\eta_1\partial^{(1)}\right)\left(\eta_2\partial^{(2)}\right)\cdots \\ \cdots\left(\eta_{i-1}\partial^{(i-1)}\right)\left(\eta_{i+1}\partial^{(i)}\right)\cdots\left(\eta_{p+1}\partial^{(p)}\right) \\ F_p(\phi,\phi,\cdots,\phi) \end{aligned} \tag{B.5}$$

Now we take advantage of the identity (A.1). It allows us to make the replacements

$$\partial^{(1)} \to -\partial^{(0)}, \partial^{(2)} \to -\partial^{(1)}, \cdots, \partial^{(i-1)} \to -\partial^{(i-2)}.$$

Thus

$$\begin{aligned} D_2\,(\eta_1,\cdots,\eta_p) = -\sum_{i=1}^{p+1}\left(\eta_1\partial^{(0)}\right)\left(\eta_2\partial^{(1)}\right)\cdots\left(\eta_{i-1}\partial^{(i-2)}\right)\left(\eta_{i+1}\partial^{(i)}\right) \\ \cdots\left(\eta_{p+1}\partial^{(p)}\right) F_p(\phi,\cdots,\phi) \end{aligned} \tag{B.6}$$

and therefore $D_2 + D_1^{p-1} = 0$, i.e. $D = D_1^p$:

$$\begin{aligned} &(\delta F_p)\,(\phi,\phi+\eta_1,\cdots,\phi+\eta_{p+1}) \\ &= \eta_1^{a_1}\eta_2^{a_2}\cdots\eta_{p+1}^{a_{p+1}}\;\; \partial_{a_1}^{(0)}\partial_{a_2}^{(1)}\cdots\partial_{a_{p+1}}^{(p)} F_p(\phi,\cdots,\phi) \end{aligned} \tag{B.7}$$

The antisymmetrized components of this tensor are

$$\begin{aligned} &\partial_{[a_1}^{(0)}\partial_{a_2}^{(1)}\cdots\partial_{a_{p+1}]}^{(p)} F_p(\phi,\cdots,\phi) \\ =\;& (\partial^{(0)}+\partial^{(1)}+\cdots+\partial^{(p)})_{[a_1}\partial_{a_2}^{(1)}\cdots\partial_{a_{p+1}]}^{(p)} F_p(\phi,\cdots,\phi) \\ =\;& \partial_{\,[a_1}\mathrm{Cl}\,(F_p)_{\,a_2 a_3\cdots a_{p+1}]} \\ =\;& (d\,\mathrm{Cl}(F_p))_{a_1\cdots a_{p+1}} \end{aligned} \tag{B.8}$$

The derivative

$$\partial_a \equiv \partial_a^{(0)} + \partial_a^{(1)} + \cdots + \partial^{(p)} \tag{B.9}$$

acts on all arguments of F_p, but only $\partial_a^{(0)}$ survives the antisymmetrization. Eq. (B.7) with (B.8) proves our claim (B.1).

Next we derive eq. (2.24)

$$\mathrm{Cl}\left(\mathcal{L}_p F_p\right) = l_h \mathrm{Cl}\left(F_p\right), \tag{B.10}$$

where

$$\mathcal{L}_p = \sum_{i=0}^{p} h^a\left(\phi_i\right) \partial_a^{(i)}. \tag{B.11}$$

This time we have to expand

$$\mathrm{E}\left(\eta_1, \cdots, \eta_p\right) \equiv \left(\mathcal{L}_p F_p\right)\left(\phi, \phi + \eta_1, \cdots, \phi + \eta_p\right). \tag{B.12}$$

Formally setting $\eta_0 \equiv 0$, $\mathcal{L}_p$, with the arguments shifted, reads

$$\begin{aligned} &\sum_{i=0}^{p} h^a\left(\phi + \eta_i\right) \partial_a^{(i)} \\ &= h^a(\phi) \sum_{i=0}^{p} \partial_a^{(i)} + \sum_{i=1}^{p} \eta_i^b \, \partial_b \, h^a(\phi) \;\; \partial_a^{(i)} + \cdots \end{aligned} \tag{B.13}$$

Inserting (B.13) into (B.12) and expanding F_p itself leads to $\mathrm{E} = \mathrm{E}_1 + \mathrm{E}_2$ with

$$\mathrm{E}_1\left(\eta_1, \cdots, \eta_p\right) = h^a(\phi) \sum_{i=0}^{p} \partial_a^{(i)} \;\; \exp\left[\sum_{j=1}^{p} \left(\eta_j \partial^{(j)}\right)\right] F_p\left(\phi, \cdots, \phi\right) \tag{B.14}$$

$$\mathrm{E}_2\left(\eta_1, \cdots, \eta_p\right) = \partial_b h^a(\phi) \sum_{i=1}^{p} \eta_i^b \partial_a^{(i)} \;\; \exp\left[\sum_{j=1}^{p} \left(\eta_j \partial^{(j)}\right)\right] F_p(\phi, \cdots, \phi) \tag{B.15}$$

Let us first look at E_1, (B.14). We would like to conclude that the term with p different η's is the only one which survives the expansion, and that there are no terms with a smaller number of η's. Because of the presence of the operator $\sum_i \partial_a^{(i)}$, we cannot apply eq. (A.3) directly, but it is easy to derive an appropriate generalization of it. To every $F_p \in \bigwedge_{\mathrm{MS}}^p$ we can associate a new $\widetilde{F}_p \in \bigwedge_{\mathrm{MS}}^p$ by shifting all $p+1$ arguments by a constant vector v:

$$\begin{aligned} \widetilde{F}_p\left(\phi_0, \cdots, \phi_p\right) &= F_p\left(\phi_0 + v, \cdots, \phi_p + v\right) \\ &= \exp\left[v^b \sum_{i=0}^{p} \partial_b^{(i)}\right] F_p\left(\phi_0, \phi_1, \cdots, \phi_p\right) \end{aligned} \tag{B.16}$$

Now we apply (A.3) to $\widetilde{F}_p$ and take the derivative $\frac{\partial}{\partial v^b}$ at $v=0$. This yields the desired relation:

$$\left(\sum_{i=0}^{p}\partial_b^{(i)}\right)\ \partial_{a_1}^{(l_1)}\cdots\partial_{a_r}^{(l_r)}F_p(\phi,\phi,\cdots,\phi)=0 \tag{B.17}$$

It holds for all $r=1,2,\cdots,p-1$ provided all the l_i's are different from each other. Therefore, applying (B.17) to (B.14) leads to

$$\mathrm{E}_1\left(\eta_1,\cdots,\eta_p\right)=\eta_1^{a_1}\cdots\eta_p^{a_p}\ h^a(\phi)\partial_a\ \partial_{a_1}^{(1)}\cdots\partial_{a_p}^{(p)}F_p(\phi,\cdots,\phi) \tag{B.18}$$

where also eq. (B.9) has been used.

Let us now turn to E_2, (B.15), which may be represented as

$$\begin{aligned}\mathrm{E}_2\left(\eta_1,\cdots,\eta_p\right)&=\partial_b h^a(\phi)\sum_{i=1}^{p}\eta_i^b\frac{\partial}{\partial\eta_i^a}\\ &\exp\left[\sum_{j=1}^{p}\eta_j^c\partial_c^{(j)}\right]F_p(\phi,\cdots,\phi)\end{aligned} \tag{B.19}$$

Because the operator $\sum\eta\frac{\partial}{\partial\eta}$ can be applied after the $\partial^{(j)}$–derivatives have been taken and the coincidence limit has been performed, eq. (A.3) immediately implies that only the term with p factors of η survives from $\exp[\cdots]\,F_p\ (\phi,\cdots,\phi)$. Therefore

$$\begin{aligned}\mathrm{E}_2\left(\eta_1,\cdots,\eta_p\right)&=\eta_1^{a_1}\cdots\eta_p^{a_p}\sum_{i=1}^{p}\partial_{a_i}h^b(\phi)\\ &\partial_{a_1}^{(1)}\cdots\partial_{a_{i-1}}^{(i-1)}\,\partial_b^{(i)}\,\partial_{a_{i+1}}^{(i+1)}\cdots\partial_{a_p}^{(p)}\,F_p\left(\phi,\cdots,\phi\right)\end{aligned} \tag{B.20}$$

This argument justifies also the truncation of the series (B.13) at order η. Combining (B.18) with (B.20) leads to

$$\begin{aligned}&\left(\mathcal{L}_pF_p\right)\left(\phi,\phi+\eta_1,\cdots,\phi+\eta_p\right)\\ &=\eta_1^{a_1}\cdots\eta_p^{a_p}\left[h^b(\phi)\partial_b\alpha_{a_1\cdots a_p}\left(\phi\right)\right.\\ &\qquad\left.+\sum_{i=1}^{p}\partial_{a_i}h^b\left(\phi\right)\alpha_{a_1\cdots b\cdots a_p}(\phi)\right]\end{aligned} \tag{B.21}$$

with

$$\begin{aligned}\alpha_{a_1\cdots a_p}(\phi)&\equiv\partial_{a_1}^{(1)}\cdots\partial_{a_p}^{(p)}F_p\left(\phi,\cdots,\phi\right)\\ &=\left[\mathrm{Cl}\left(F_p\right)\right]_{a_1\cdots a_p}\left(\phi\right)\end{aligned} \tag{B.22}$$

Recalling (2.25) we see that (B.21) is exactly what we wanted to prove, viz. eq. (B.10).